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Central extension of mapping class group via Chekhov–Fock quantization

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ABSTRACT

The central extension of mapping class groups of punctured surfaces of finite type that arises in Chekhov–Fock quantization is 12 times of the Meyer class plus the Euler classes of the punctures, which agree with the one arising in the Kashaev quantization.

1. Introduction

The quantum theory of Teichmüller spaces of punctured surfaces was developed in [1] and [2] independently, and then generalized to higher rank Lie groups and cluster algebras in [3] and [4]. The main ingredient of both constructions is Faddeev's quantum dilogarithm introduced in [5]. This theory leads to one parameter families of projective unitary representations of Ptolemy groupoids associated to ideal triangulations of punctured surfaces. These projective unitary representations moreover induce central extensions of mapping class groups of associated punctured surfaces.

The main goal of this paper is to study these central extensions arising for the Chekhov–Fock model of the quantum Teichmüller space and to identify them in terms of cohomology 2-classes of corresponding mapping class groups. This problem was first considered for the universal Teichmüller space where the mapping class group is replaced by the Thompson group. In [6], Funar and Sergiescu computed the cohomology class of central extensions of the Thompson group arising for the Chekhov–Fock model of the quantum universal Teichmüller space. In [7], Hyun Kyu Kim made further computation for the Kashaev model of the quantum universal Teichmüller space and found a different class. This suggests that the relationship between two models is subtler than thought before. Coming back to Teichmüller spaces of punctured surfaces of finite type, in [8], Funar and Kashaev computed the class arising in the Kashaev model. The present paper settles the last case open and shows that the class arising in the Chekhov–Fock model agrees with that arising in the Kashaev model.

Let V be a vector space and G be a group. A *projective representation* of G on V is a homomorphism from G to $PGL(V)$. It is well-known that a projective representation of a group gives rise to a linear representation of some central extension of the given group. More precisely, let h be a projective representation of G on V. Let $\widetilde{G}$ be a central extension of G by $\mathbb{C}^*$ which is the pullback of $GL(V) \to PGL(V)$ by h. Then one can associate a representation $\widetilde{h}$ of $\widetilde{G}$ on V, such that the following commutative diagram holds:

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$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL(V) \arrow[r] & PGL(V) \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[u] & \widetilde{G} \arrow[r] \arrow[u, "\widetilde{h}"] & G \arrow[r] \arrow[u, "h"] & 1 \end{tikzcd}$$

A *reduction* $\widetilde{G}_1$ of $\widetilde{G}$ is a central extension of G by a subgroup A_1 of $\mathbb{C}^*$, such that $\widetilde{G}_1$ is a subgroup of $\widetilde{G}$ and the associated representation $\widetilde{h}_1 : \widetilde{G}_1 \to GL(V)$ is the restriction of $\widetilde{h}$. We say that $\widetilde{G}_1$ is the *minimal reduction* of $\widetilde{G}$ if $\widetilde{G}_1$ is a smallest possible reduction of $\widetilde{G}$.

Suppose that G has a presentation F/R where F is a free group and R is the normal subgroup of F generated by a set of relations. A projective representation of G on V is induced by a linear representation $\bar{h}$ of F on V such that R is sent to the center of $GL(V)$. The homomorphism $\bar{h}$ will be called an *almost linear representation* of G on V, in order to distinguish it from the projective representation.

Let Σ_g^s be an oriented compact surface of genus g with s punctures where $s > 0$. Let $\Gamma(\Sigma_g^s)$ be its mapping class group. An element of $\Gamma(\Sigma_g^s)$ is a mapping class of homeomorphisms from Σ_g^s to itself fixing the punctures setwise. Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbb{R}^{2n})$ where n is the number of arcs in an ideal triangulation of Σ_g^s. The Chekhov–Fock quantization constructs a family of projective representations $\rho_z : \Gamma(\Sigma_g^s) \to PGL(\mathcal{H})$ depending on one parameter z with $|z| = 1$. Denote by $\widetilde{\boldsymbol{\Gamma}(\Sigma_g^s)} \to \Gamma(\Sigma_g^s)$ the central extension $\rho_z^*\eta$ where η is the central extension $GL(\mathcal{H}) \to PGL(\mathcal{H})$.

Central extensions of a group G by an abelian group A are classified, up to isomorphism, by elements of the 2-cohomology group $H^2(G, A)$. For a mapping class group $\Gamma(\Sigma_g^s)$ with $g \geq 4$ and $s \geq 2$, by the work of Harer in [9] and Korkmaz and Stipsicz in [10], we have:

$$H^2(\Gamma(\Sigma_g^s), \mathbb{Z}) = \mathbb{Z}^{1+s},$$

where the generators are given by one fourth of the Meyer signature class χ and s Euler classes e_i that are associated to s punctures p_i respectively. For $g = 3$, by the work of Sakasai in [11], the group $H^2(\Gamma(\Sigma_g^s), \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{s+1} \oplus \mathbb{Z}/2\mathbb{Z}$. For $g = 2$, it has been proved that $H^2(\Gamma(\Sigma_g^s), \mathbb{Z})$ contains the subgroup $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$ whose torsion part is generated by χ and whose free part is generated by Euler classes.

The main result in this paper is the following theorem:

Theorem 1.1. *Let $g \geq 2$ and $s \geq 4$. A minimal reduction $\widetilde{\Gamma(\Sigma_g^s)}$ of $\widetilde{\boldsymbol{\Gamma}(\Sigma_g^s)}$ is a central extension of $\Gamma(\Sigma_g^s)$ by a cyclic subgroup $A \subset \mathbb{C}^*$ generated by z^{-12}, whose cohomology class is*

$$c_{\widetilde{\Gamma(\Sigma_g^s)}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2(\Gamma(\Sigma_g^s), A).$$

The organization of this paper is as follows. In Section 2, we will recall the Chekhov–Fock quantization of the Teichmüller space of a puncture surface of finite type and describe the almost linear representation of the corresponding Ptolemy groupoid. In Section 3 we will give the proof of our main theorem.

2. Preliminaries

2.1. Ptolemy groupoids

Let $\Sigma = \Sigma_g^s$ and $\Gamma = \Gamma(\Sigma_g^s)$.

Definition 2.1. An **arc** is the homotopy class of a simple curve on Σ connecting punctures which is non homotopic to a point or a puncture of Σ.

Definition 2.2. An **ideal triangulation** of Σ is a maximal collection of distinct arcs which have pairwise disjoint representatives. A **labeled ideal triangulation** is obtained from an ideal triangulation by adding labels to its arcs.

We denote by $|\mathbf{T}(\Sigma)|$ the set of ideal triangulations of Σ and by $\mathbf{T}(\Sigma)$ the set of labeled ideal triangulations of Σ.

Let T be a labeled ideal triangulation. Let α be an arc of T which is the common boundary of two distinct ideal triangles whose union is an embedded quadrilateral in Σ.

Definition 2.3. A **flip** on α is to get a new ideal triangulation T' from T by substituting α by the other diagonal α' of the same ideal quadrilateral.

We will use $F_\alpha(T)$ to denote the flip on the arc α in T.

A **groupoid** is a category such that all morphisms are invertible and for each pair of objects there exists at least one morphism between them. The set of automorphisms of an object forms a group. Reciprocally if a group G acts freely on a set X, we can define an associated groupoid whose objects are G-orbits in X and whose morphisms are orbits of the diagonal G-action on $X \times X$.

The mapping class group Γ acts freely on $\mathbf{T}(\Sigma)$. The corresponding groupoid is called the **Ptolemy groupoid**.

Remark 2.1. When $s > 1$, the Γ-action on $|\mathbf{T}(\Sigma)|$ is not free.

The Ptolemy groupoid can also be defined by using actions of flips and the symmetry group on $\mathbf{T}(\Sigma)$ where the symmetry group acts as permutations of labels. Then the Ptolemy groupoid has the following presentation due to Harer in [12] and Penner in [13] and [14]:

Theorem 2.1. *If Σ is different from the three-punctured sphere and the one-punctured torus, then any pair of labeled ideal triangulations can be connected by a chain of flips.*

The Ptolemy groupoid is generated by the action of flips and the symmetry group. The relations between them are the following:

(1) *For any arc α in T, we have $F_\alpha^2 = 1$;*
(2) *If α and β are two arcs in T having no common end point, we have $F_\alpha F_\beta = F_\beta F_\alpha$;*
(3) *For any two arcs α and β contained in an ideal pentagon, the pentagon relation holds:*

$$F_\alpha F_\beta F_\alpha F_\beta F_\alpha = \sigma(\alpha, \beta),$$

where σ is the permutation exchanging labels of α and β;
(4) *Let $\sigma \in S_n$ and let α be a labeled arc, then we have $F_\alpha \sigma = \sigma F_{\sigma(\alpha)}$.*

2.2. Shearing coordinates

In this section, we first recall shearing coordinates of the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ, and then recall the Poisson structure on $\mathcal{T}(\Sigma)$ in term of these coordinates.

We choose and fix an orientation on Σ. Given a hyperbolic structure on Σ, each arc has a unique geodesic representative. In the following, the word "arc" will denote its geodesic representative. Let α be an arc in an ideal triangulation T of Σ. Fix an orientation on α. Let $\widetilde{\alpha}$ be one of its lifts in the hyperbolic plane. Then $\widetilde{\alpha}$ will be an oriented diagonal in an ideal quadrilateral Q with one ideal triangle Δ_l on its left side and another ideal triangle Δ_r on its right side. In each triangle, the vertex not lying on $\widetilde{\alpha}$ can be orthogonally projected on $\widetilde{\alpha}$. We denote by v_l and v_r the images of the vertices of Δ_l and Δ_r respectively.

Definition 2.4. The **shearing coordinate** $t(\alpha)$ on α is the directed hyperbolic distance from v_l to v_r with respect to the orientation of $\widetilde{\alpha}$.

Remark 2.2. Notice that lifts of α are different from each other by orientation preserving isometries of the hyperbolic plane, thus $t(\alpha)$ does not depend on the choice of $\widetilde{\alpha}$. At the same time, reversing the orientation of α also exchanges the left side and the right side of $\widetilde{\alpha}$. This implies that $t(\alpha)$ is also independent of the choice of the orientation of α. So the shearing coordinate $t(\alpha)$ is well defined.

The explicit formula of $t(\alpha)$ can be given by using cross-ratios of vertices of the associated ideal quadrilateral. Denote by $\{p_1, p_2, p_3, p_4\}$ four vertices of the associated ideal quadrilateral Q with a counter-clockwise order. Assuming that p_1 and p_3 are vertices of $\widetilde{\alpha}$ with the orientation such that p_2 is on the left of $\widetilde{\alpha}$ and p_4 is on its right. Then the shearing coordinate on α is defined by:

$$t(\alpha) = \log(-[p_1, p_3; p_2, p_4]) = \log\left(-\frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_4)(p_3 - p_2)}\right).$$

The shearing coordinates system depends on the choice of the ideal triangulation. By doing a flip we obtain a different shearing coordinates system. Let T' be the ideal triangulation obtained from T by flipping α to α'. By comparing the cross-ratios before and after the flip, we obtain the change formula:

$$t'(\beta) = \begin{cases} -t(\alpha) & \text{if } \beta = \alpha' \\ t(\beta) + \epsilon(\alpha, \beta)\phi(\operatorname{sign}(\epsilon(\alpha, \beta))t(\alpha)) & \text{if } \beta \text{ and } \alpha' \text{ are adjacent but } \beta \neq \alpha' \\ t(\beta) & \text{otherwise,} \end{cases}$$

where $\phi(z) = \log(1 + \exp(z))$. The function ϵ_T is defined in the following way. Let Δ be an ideal triangle on Σ which is a connected component of $\Sigma \setminus T$. Let $E(\Delta)$ be the set of its edges. The orientation of Σ induces an orientation of Δ which induces a cyclic order among elements in $E(\Delta)$, thus we can define an anti-symmetric map

$$\epsilon_{T,\Delta} : E(\Delta) \times E(\Delta) \to \{0, \pm 1\}$$

by the following formula:

$$\epsilon_{T,\Delta}(\alpha,\beta)=\begin{cases}-1 & \text{if } \beta \text{ comes after } \alpha' \text{ counter-clockwisely}\\ 1 & \text{if } \beta \text{ comes after } \alpha' \text{ clockwisely}\\ 0 & \text{otherwise,}\end{cases}$$

where (α,β) is in $E(\Delta)\times E(\Delta)$. By taking the sum of $\epsilon_{T,\Delta}$ over all ideal triangles Δ, we obtain the following anti-symmetric map ϵ_T:

$$\epsilon_T : E(T)\times E(T)\to\{0,\pm 1,\pm 2\},$$

where $E(T)$ is the set of arcs in T.

This anti-symmetric map ϵ_T also induces the Poisson structure on the Teichmüller space by the following bi-vector field:

$$P(T)=\sum_{\alpha,\beta}\epsilon_T(\alpha,\beta)\frac{\partial}{\partial t(\alpha)}\wedge\frac{\partial}{\partial t(\beta)}.$$

2.3. Quantum Teichmüller space

The quantization of a Poisson manifold equivariant with respect to a discrete group G-action is a family of $*$-algebras $A^\hbar$ depending smoothly on a positive real parameter $\hbar$ satisfying the following properties:

(1) All $A^\hbar$ are isomorphic to each other as linear spaces ;
(2) The group G acts as outer automorphisms on each $*$-algebra ;
(3) For $\hbar=0$, the algebra A^0 is isomorphic as a G-module to the $*$-algebra of the complex-valued function on the Poisson manifold ;
(4) The Poisson bracket $\{,\}$ on A^0 is the limit of $\{,\}_\hbar/(2\pi i\hbar)$ as $\hbar$ going to zero. It coincides with the one on the original Poisson manifold.

For each $\hbar>0$, we associate one $*$-algebra $A^\hbar(T)$ to each ideal triangulation T on Σ, generated by $\{Z_\hbar(\alpha):\alpha\in T\}$ with the $*$-structure:

$$(Z_\hbar(\alpha))^*=Z_\hbar(\alpha).$$

The Poisson bracket on $A^\hbar(T)$ is obtained by deforming the Poisson bracket for shearing coordinates associated to T by the following formula:

$$\{Z_\hbar(\alpha),Z_\hbar(\beta)\}_\hbar=2\pi i\hbar\{t(\alpha),t(\beta)\}.$$

A flip acts on the $*$-algebras $A^\hbar(T)$ by the formula:

$$Z'_\hbar(\beta)=\begin{cases}-Z_\hbar(\alpha) & \text{if } \beta=\alpha'\\ Z_\hbar(\beta)+\epsilon(\alpha,\beta)\phi^\hbar(\mathrm{sign}(\epsilon(\alpha,\beta))Z_\hbar(\alpha)) & \text{if } \beta \text{ and } \alpha' \text{ are adjacent but } \beta\neq\alpha'\\ Z_\hbar(\beta) & \text{otherwise,}\end{cases}$$

where

$$\phi^\hbar(z)=-\frac{\pi\hbar}{2}\int_\Omega\frac{\exp(-iuz)}{\sinh(\pi u)\sinh(\pi\hbar u)}\mathrm{d}u,$$

where Ω is the path going along the real axis from $-\infty$ to $+\infty$ and passing the origin from above. The symmetric group acts as permutations of labels.

For each $\hbar$, the construction above gives us a projective functor $\mathcal{Q}_\hbar$ from the Ptolemy groupoid to the category of $*$-algebra.

Definition 2.5. The family of projective functors $\mathcal{Q}_\hbar$ are called the **quantization** of Teichmüller space.

2.4. Almost linear representation of Ptolemy groupoid

A Heisenberg $*$-algebra H_n is generated by $2n+1$ generators $P_1,\dots,P_n,Q_1,\dots,Q_n$ and C satisfying the following relations:

(1) The generator C is the central element ;
(2) For any two index j and k, we have relations:

$$\{P_j,P_k\}=\{Q_j,Q_k\}=0;$$

(3) For any two index j and k, we have relations:

$$\{P_j,Q_k\}=C\delta_{jk}.$$

It has a irreducible integrable representation in the Hilbert space $\mathcal{H}$ described as follows: consider the canonical complex structure on $\mathbb{R}^{2n}$ and denote by $z_1, \dots, z_n$ the complex coordinates and by $x_1, \dots, x_n, y_1, \dots, y_n$ the real coordinates such that $z_j = x_j + iy_j$, then we represent generators as following operators:

$$\rho(P_j)(f)(z_1, \dots, z_n) = z_j f(z_1, \dots, z_n),$$
$$\rho(Q_j)(f)(z_1, \dots, z_n) = -2\pi i\hbar \frac{\partial f}{\partial z_j}(z_1, \dots, z_n),$$
$$\rho(C)(f)(z_1, \dots, z_n) = 2\pi i\hbar f(z_1, \dots, z_n).$$

The representation of the $*$-algebra $A^\hbar(T)$ in $\mathcal{H}$ is generated by the linear combinations of $\rho(P_j)$ and $\rho(Q_k)$ above.

The Stone von Neumann theorem holds true for $A^\hbar(T)$. In particular, let T and T' be two labeled ideal triangulations different from each other by a flip F_α. Consider the representations $\rho(A^\hbar(T))$ and $\rho(A^\hbar(T'))$ of $A^\hbar(T)$ and $A^\hbar(T')$ respectively. The uniqueness of representations yields the existence of an intertwinner $K(F_\alpha)$ between these two representations. It acts in the following way:

$$\rho(A^\hbar(T')) = K(F_\alpha)^{-1}\rho(A^\hbar(T))K(F_\alpha).$$

This intertwinner functor K induces an almost linear representation of the Ptolemy groupoid which induces an almost linear representation of Γ. The following result is proved in [4]:

Proposition 2.1. *The almost linear representation K has the following property:*

(1) *For any pair of disjoint arcs α and β in the labeled ideal triangulation T, the associated operators $K(F_\alpha)$ and $K(F_\beta)$ commute with each other;*
(2) $K(F_\alpha)^2 = 1$;
(3) *The pentagon relation:*

$$K(F_\alpha)K(F_\beta)K(F_\alpha)K(F_\beta)K(F_\alpha) = e^{2\pi i\hbar}\sigma,$$

where σ is the permutation exchanging labels α and β.

3. Proof of the theorem

In [8], authors gave the following presentation of Γ which is a consequence of Gervais' result in [15].

Lemma 3.1. *For any oriented surface S of genus $g \geq 2$ and $s \geq 4$ punctures, the mapping class group has the following presentation:*

(1) *The generators are the Dehn twists D_a along all non separating simple close geodesics a in S;*
(2) *The relation between them are the following:*
  (a) *The type-0 braid relation: for each pair of disjoint non-separating simple closed geodesics a and b, we have:*
  $D_aD_b = D_bD_a$;
  (b) *The type-1 braid relation: for each pair of non-separating simple closed geodesics a and b with the geometric intersection number $i(a, b) = 1$, we have:*
  $D_aD_bD_a = D_bD_aD_b$;
  (c) *The lantern relation: for each four-holes sphere embedded in the surface whose boundary a_0, a_1, a_2, a_3 are the non-separating simple closed geodesics, we have:*
  $D_{a_0}D_{a_1}D_{a_2}D_{a_3} = D_{a_{12}}D_{a_{23}}D_{a_{13}}$;
  (d) *The chain relation: for each two-holed torus embedded in the surface, we have:*
  $(D_aD_bD_c)^4 = D_eD_f$;
  (e) *The puncture relation: for each sphere with three holes and one puncture embedded in the surface, we have:*
  $D_{a_1}D_{a_2}D_{a_3} = D_{a_{12}}D_{a_{13}}D_{a_{23}}$.

By using this presentation, we are able to prove the following proposition which is the main step of the proof of Theorem 1.1:

Proposition 3.1. *By using the Chekhov–Fock quantization, we obtain a central extension of Γ with the following presentation:*

(1) *Generators:*
  (a) *One central element: $w = z^{-12}$, where z is the constant coming from the Chekhov–Fock quantization;*
  (b) *One element $\widetilde{D}_a$ associated to each Dehn twist D_a for each non-separating simple close geodesic a in S.*
(2) *Relations:*
  (a) *The type-0 braid relation: $\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a$;*
  (b) *The type-1 braid relation: $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$;*
  (c) *The Lantern relation: $\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = \widetilde{D}_{a_{12}}\widetilde{D}_{a_{23}}\widetilde{D}_{a_{13}}$;*

(d) *The chain relation:* $(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = w^{12}\widetilde{D}_e\widetilde{D}_f$;
(e) *The puncture relation:* $\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = w\widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}}$.

We prove Proposition 3.1 by proving a sequence of lemmas.

Lemma 3.2. *For the type-0 braid relation, we have* $\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a$.

Proof. The lifts of two commutative elements are commutative in the central extension. □

Lemma 3.3. *For the type-1 braid relation, by choosing the lifts, we have:*

$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b.$$

Proof. Let a and b be two non-separating simple closed geodesics in Σ with $i(a, b) = 1$, that is with one intersection point. We choose and fix one lift $\widetilde{D}_a$ of D_a. For any lift $\widetilde{D}_b$ of D_b, there exists an integer k such that

$$\widetilde{D}_b\widetilde{D}_a\widetilde{D}_b = z^k\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a.$$

Then by changing the lift $\widetilde{D}_b$ to $\widetilde{D}'_b = z^k\widetilde{D}_b$ we have:

$$\widetilde{D}_a\widetilde{D}'_b\widetilde{D}_a = \widetilde{D}'_b\widetilde{D}_a\widetilde{D}'_b.$$

Notice that for a fixed lift $\widetilde{D}_a$, there is a unique lift $\widetilde{D}'_b$ having trivial type-1 braid relation with it.

Let x and y be any other pair of non-separating simple closed geodesics with $i(x, y) = 1$. Then there is a homeomorphism ϕ of Σ sending a and b to x and y respectively. We choose and fix one lift $\widetilde{\phi}$ of ϕ. Lifts of D_x and D_y are chosen to be as follows:

$$\widetilde{D}_x = \widetilde{\phi}^{-1}\widetilde{D}_a\widetilde{\phi},$$
$$\widetilde{D}_y = \widetilde{\phi}^{-1}\widetilde{D}'_b\widetilde{\phi}.$$

Then the corresponding type-1 braid relation is

$$\widetilde{D}_x\widetilde{D}_y\widetilde{D}_x = \widetilde{D}_y\widetilde{D}_x\widetilde{D}_y.$$

In this way, we can find lifts for all Dehn twists along non-separating curves. Moreover all type-1 braid relations among them are trivial. □

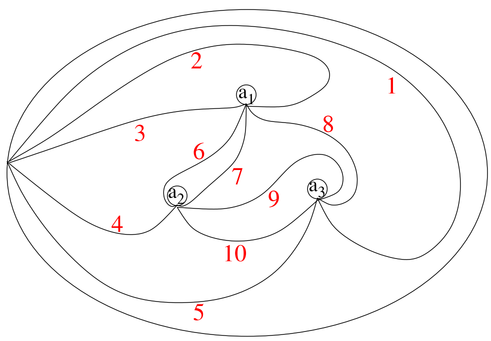
Let us consider Dehn twists as automorphisms in Ptolemy groupoid. Then they can be expressed as compositions of morphisms corresponding to arc flips and label permutations. By using the almost linear representation K, we obtain lifts of Dehn twists. Notice that a Dehn twist may have several expressions different from each other by the relations in Theorem 2.1. Different expressions may induce different lifts. In the following part, we will prove that by choosing the expression carefully the induced lifted Dehn twists satisfy the relations in Proposition 3.1.

Remark 3.1. We will use $\widetilde{F}_\alpha$ to denote $K(F_\alpha)$. We will use $\mathrm{Ad}(F_\alpha)$ to denote the adjoint action of F_α. For underlined parts in the following proofs, we use either pentagon relations or commutation relations. All computation will be read from left to right.

Lemma 3.4. *By choosing carefully the lifts for each Dehn twist in the lantern relation, we have:*

$$\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = z^{-12}\widetilde{D}_{a_{12}}\widetilde{D}_{a_{23}}\widetilde{D}_{a_{13}}.$$

Proof. The proof is similar to the one in [8]. Consider the four-holed sphere with one puncture on each boundary component. The ideal triangulation and the labels are given as follows:



Then D_0, D_1, D_2 and D_3 have the following expression:

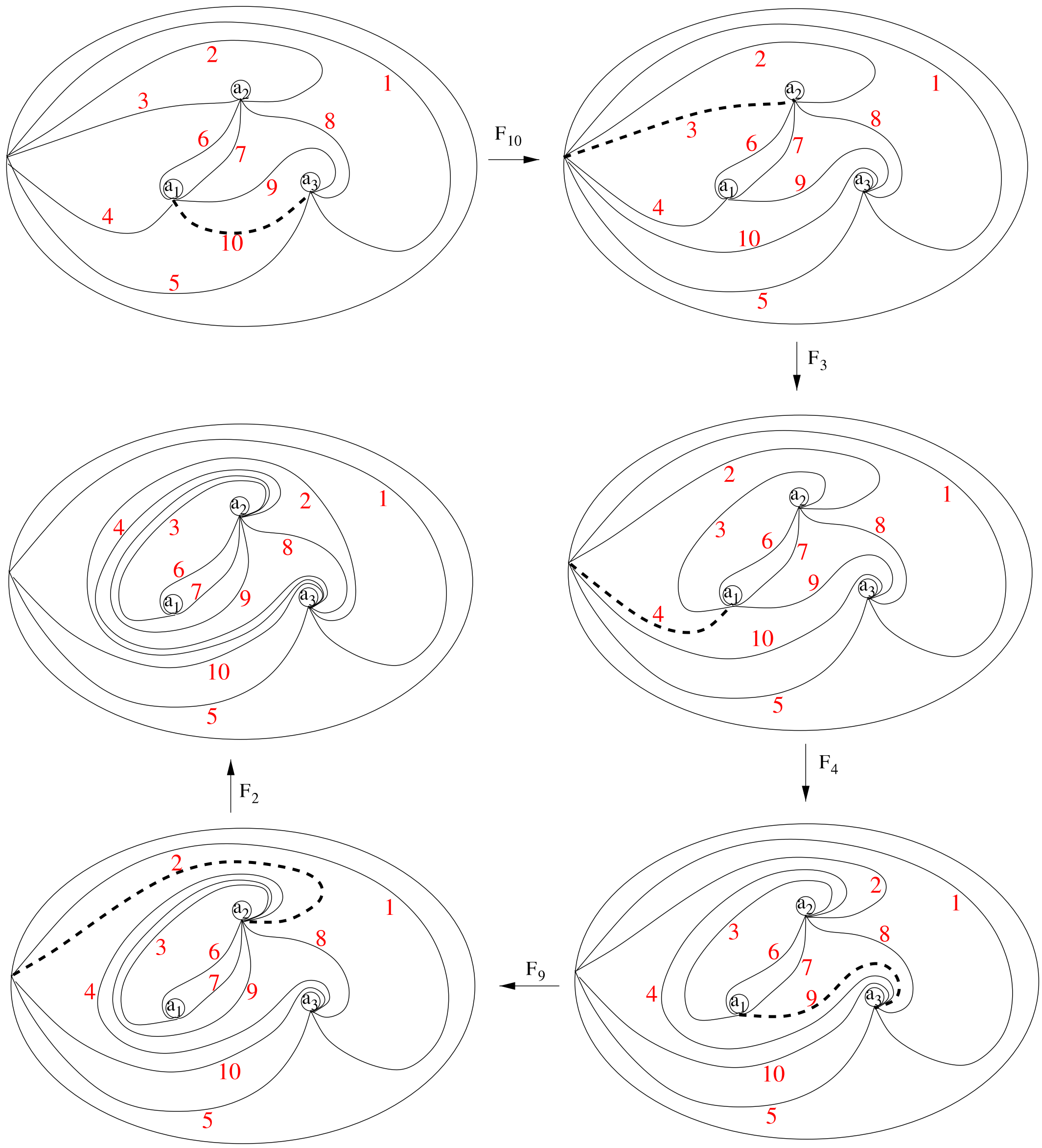
$$D_0 = F_5F_4F_3F_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix},$$

$$D_1 = F_6F_4F_{10}F_9 \begin{pmatrix} 4 & 6 & 7 & 9 & 10 \\ 10 & 4 & 6 & 7 & 9 \end{pmatrix},$$

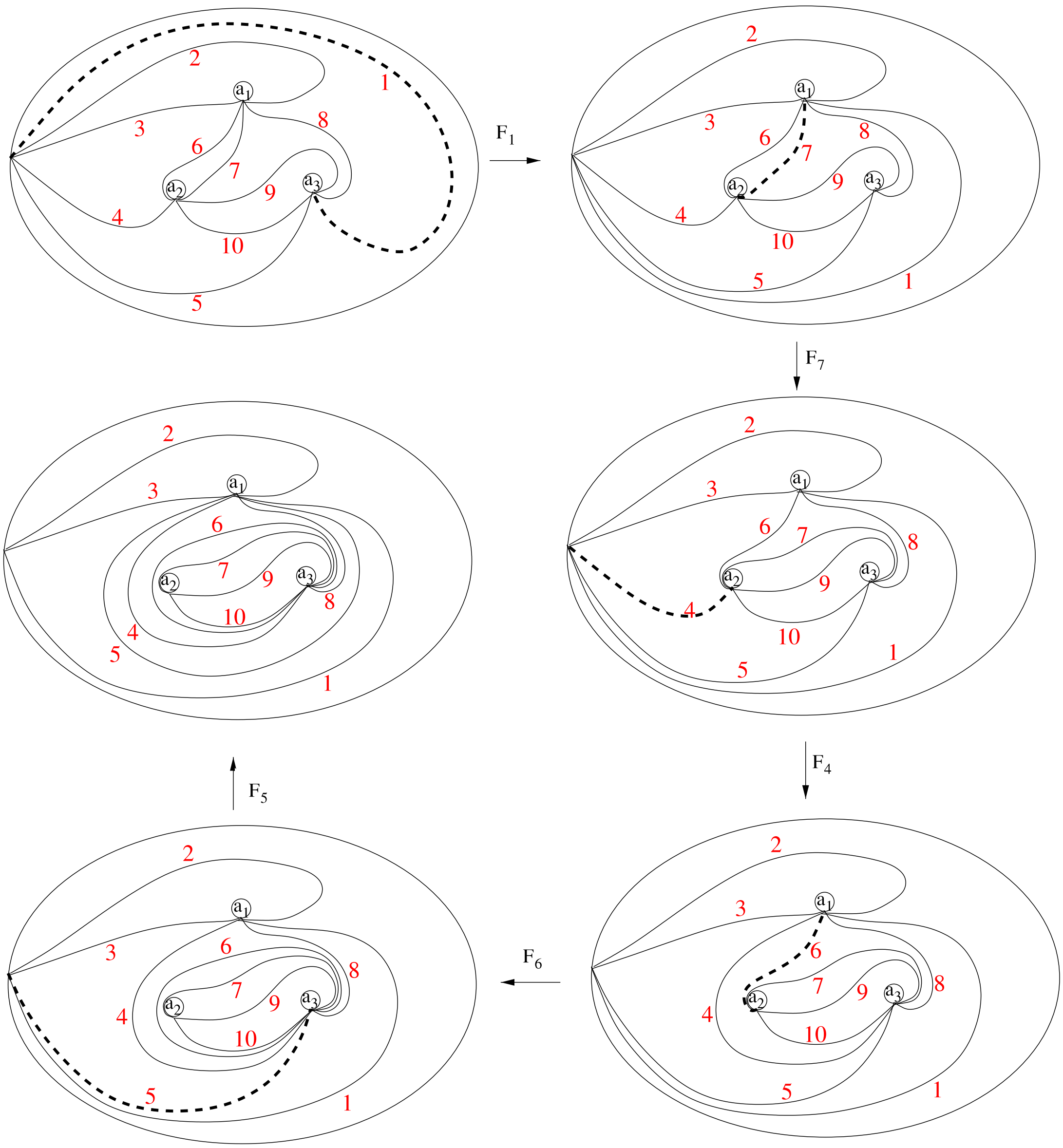
$$D_2 = F_3F_6F_7F_8 \begin{pmatrix} 2 & 3 & 6 & 7 & 8 \\ 3 & 6 & 7 & 8 & 2 \end{pmatrix},$$

$$D_3 = F_{10}F_5F_1F_8 \begin{pmatrix} 1 & 5 & 8 & 9 & 10 \\ 8 & 1 & 9 & 10 & 5 \end{pmatrix}.$$

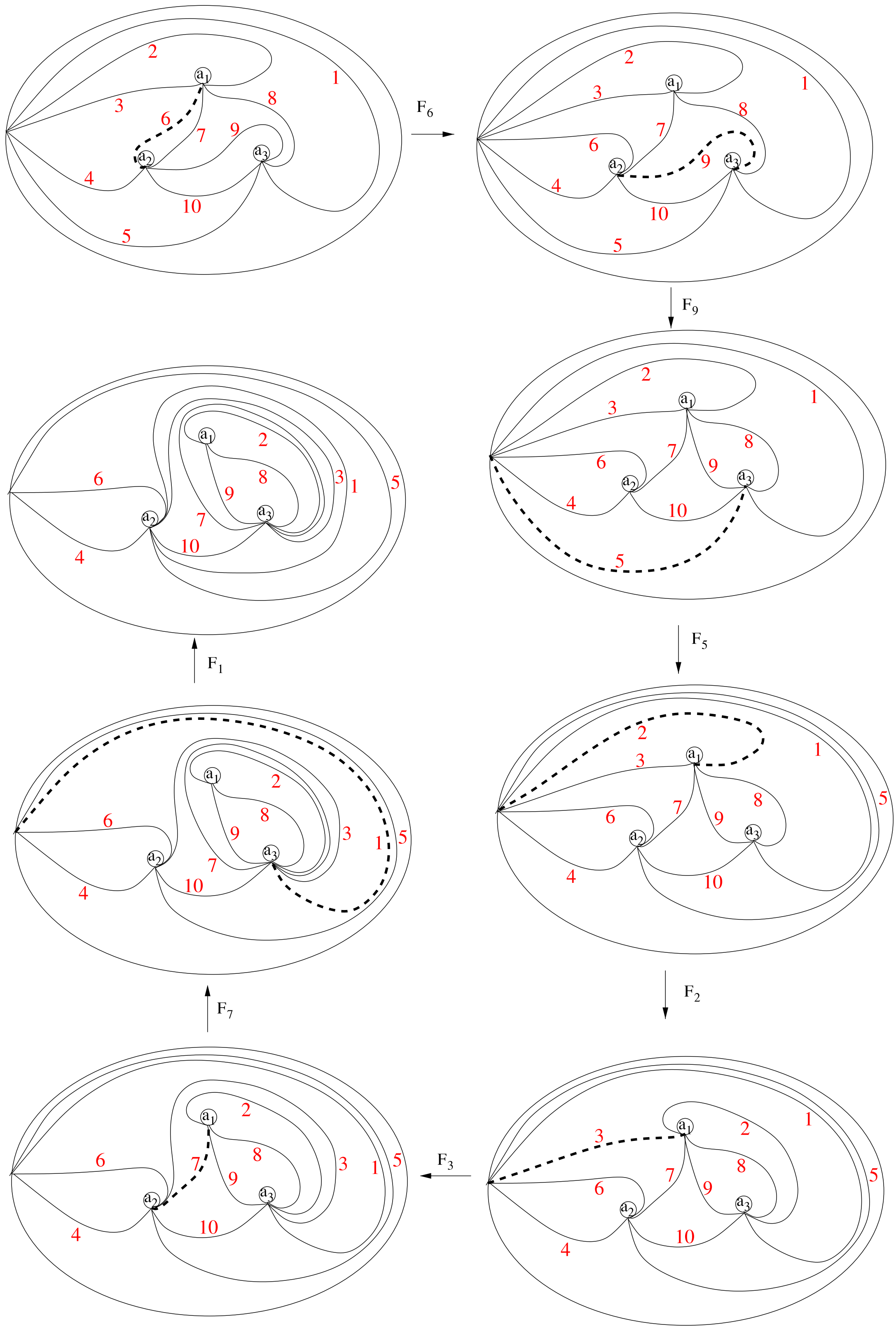
For D_{12}, D_{23} and D_{13}, we use a sequence of flips to transform the triangulation to another one in which there are only two arcs intersecting the geodesic associated to the Dehn twist. Then we get the following formulas:



$$D_{12} = \mathrm{Ad}(F_{10}F_3F_4F_9F_2)F_8 \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix},$$



$$D_{13} = \mathrm{Ad}(F_1F_7F_4F_6F_5)F_4\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix},$$



$$D_{23} = \mathrm{Ad}(F_6F_9F_5F_2F_3F_7F_1)F_{10}\begin{pmatrix} 3 & 10 \\ 10 & 3 \end{pmatrix}.$$

They can be simplified as follows:

$$\begin{aligned}
D_{12} &= F_{10}F_3F_4F_9F_2F_8\begin{pmatrix}4&8\\8&4\end{pmatrix}F_2F_9F_4F_3F_{10} = F_{10}F_3F_4F_2\underline{F_9F_8F_9}F_2F_8F_3F_{10}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= F_{10}F_3F_4F_2F_8F_9\begin{pmatrix}8&9\\9&8\end{pmatrix}F_2F_8F_3F_{10}\begin{pmatrix}4&8\\8&4\end{pmatrix} = F_{10}F_3F_4F_2F_8F_2F_9\begin{pmatrix}2&9\\9&2\end{pmatrix}F_3F_{10}\begin{pmatrix}8&9\\9&8\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= F_{10}F_3F_4F_8F_2\begin{pmatrix}2&8\\8&2\end{pmatrix}F_9F_3F_{10}\begin{pmatrix}2&9\\9&2\end{pmatrix}\begin{pmatrix}8&9\\9&8\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix} = \underline{F_{10}F_3F_4F_8}F_2\underline{F_9F_3F_{10}}\begin{pmatrix}2&8\\8&2\end{pmatrix}\begin{pmatrix}2&9\\9&2\end{pmatrix}\begin{pmatrix}8&9\\9&8\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= F_3\underline{F_{10}F_8}F_4F_2F_{10}F_9F_3\begin{pmatrix}2&8\\8&2\end{pmatrix}\begin{pmatrix}2&9\\9&2\end{pmatrix}\begin{pmatrix}8&9\\9&8\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix} = F_3F_8\underline{F_{10}F_4F_{10}F_{10}F_2F_{10}}F_9F_3\begin{pmatrix}2&9\\9&2\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= F_3F_8F_4F_{10}\begin{pmatrix}4&10\\10&4\end{pmatrix}F_2F_{10}\begin{pmatrix}2&10\\10&2\end{pmatrix}F_9F_3\begin{pmatrix}2&9\\9&2\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix} = F_3F_8F_4F_{10}F_2F_4F_9F_3\begin{pmatrix}2&4&8&9&10\\10&9&4&2&8\end{pmatrix},\\
D_{13} &= F_1F_7F_4\underline{F_6F_5}F_4\begin{pmatrix}4&8\\8&4\end{pmatrix}\underline{F_5F_6}F_4F_7F_1 = F_1F_7F_4F_5\underline{F_6F_4F_6}F_5F_8F_7F_1\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= F_1F_7\underline{F_4F_5F_4}F_6\begin{pmatrix}4&6\\6&4\end{pmatrix}F_5F_8F_7F_1\begin{pmatrix}4&8\\8&4\end{pmatrix} = F_1F_7F_5F_4\begin{pmatrix}4&5\\5&4\end{pmatrix}F_6F_5F_8F_7F_1\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= F_1\underline{F_7F_5}F_6F_4\begin{pmatrix}4&6\\6&4\end{pmatrix}\underline{F_8F_7}F_1\begin{pmatrix}4&5\\5&4\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix} = F_1F_5\underline{F_7F_6F_7F_7F_4F_7}F_8F_1\begin{pmatrix}4&8\\8&4\end{pmatrix}\begin{pmatrix}5&6\\6&5\end{pmatrix}\\
&= F_1F_5F_6F_7F_4F_6F_8F_1\begin{pmatrix}4&5&6&7&8\\7&6&8&5&4\end{pmatrix},\\
D_{23} &= F_6F_9F_5F_2F_3\underline{F_7F_1}F_{10}\begin{pmatrix}3&10\\10&3\end{pmatrix}\underline{F_1F_7}F_3F_2F_5F_9F_6\\
&= F_6F_9F_5F_2F_3F_1\underline{F_7F_{10}F_7}F_1F_{10}F_2F_5F_9F_6\begin{pmatrix}3&10\\10&3\end{pmatrix}\\
&= F_6F_9F_5F_2F_3F_1F_{10}\underline{F_7F_1F_7}F_2F_5F_9F_6\begin{pmatrix}7&10\\10&7\end{pmatrix}\begin{pmatrix}3&10\\10&3\end{pmatrix}\\
&= F_6F_9F_5F_2F_3\underline{F_1F_{10}F_1}F_7F_2F_5F_9F_6\begin{pmatrix}1&7\\7&1\end{pmatrix}\begin{pmatrix}7&10\\10&7\end{pmatrix}\begin{pmatrix}3&10\\10&3\end{pmatrix}\\
&= F_6F_9F_5F_2F_3F_{10}F_1F_7F_2F_5F_9F_6\underline{\begin{pmatrix}1&10\\10&1\end{pmatrix}\begin{pmatrix}1&7\\7&1\end{pmatrix}\begin{pmatrix}7&10\\10&7\end{pmatrix}\begin{pmatrix}3&10\\10&3\end{pmatrix}}\\
&= F_6F_9F_5F_2F_3F_{10}F_1F_7F_2F_5F_9F_6\begin{pmatrix}1&7\\7&1\end{pmatrix}\begin{pmatrix}3&10\\10&3\end{pmatrix}.
\end{aligned}$$

The expressions which we use to lift Dehn twists are as follows:

$$\begin{aligned}
D_0 &= F_5F_4F_3F_2\begin{pmatrix}1&2&3&4&5\\5&1&2&3&4\end{pmatrix},\\
D_1 &= F_6F_4F_{10}F_9\begin{pmatrix}4&6&7&9&10\\10&4&6&7&9\end{pmatrix},\\
D_0 &= F_3F_6F_7F_8\begin{pmatrix}2&3&6&7&8\\3&6&7&8&2\end{pmatrix},\\
D_3 &= F_{10}F_5F_1F_8\begin{pmatrix}1&5&8&9&10\\8&1&9&10&5\end{pmatrix},\\
D_{12} &= F_3F_8F_4F_{10}F_2F_4F_9F_3\begin{pmatrix}2&4&8&9&10\\10&9&4&2&8\end{pmatrix},\\
D_{13} &= F_1F_5F_6F_7F_4F_6F_8F_1\begin{pmatrix}4&5&6&7&8\\7&6&8&5&4\end{pmatrix},\\
D_{23} &= F_6F_9F_5F_2F_3F_{10}F_1F_7F_2F_5F_9F_6\begin{pmatrix}1&7&3&10\\7&1&10&3\end{pmatrix}.
\end{aligned}$$

Then we compose lifts $\widetilde{D}_{12}$, $\widetilde{D}_{13}$ and $\widetilde{D}_{23}$ together:

$$\begin{aligned}
\widetilde{D}_{12}\widetilde{D}_{23}\widetilde{D}_{13} = {}& \widetilde{F}_3\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_9\widetilde{F}_3\begin{pmatrix}2&4&8&9&10\\10&9&4&2&8\end{pmatrix}\widetilde{F}_6\widetilde{F}_9\widetilde{F}_5\widetilde{F}_2\widetilde{F}_3\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_2\widetilde{F}_5\widetilde{F}_9\widetilde{F}_6\begin{pmatrix}1&7\\7&1\end{pmatrix}\begin{pmatrix}3&10\\10&3\end{pmatrix}\\
&\times\underline{\widetilde{F}_1\widetilde{F}_5\widetilde{F}_6}\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_1\begin{pmatrix}4&5&6&7&8\\7&6&8&5&4\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{F}_3\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_9\widetilde{F}_3 \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \widetilde{F}_6\widetilde{F}_9\widetilde{F}_5\widetilde{F}_2\widetilde{F}_3\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_2\underline{\widetilde{F}_5\widetilde{F}_9 \begin{pmatrix} 1&7\\ 7&1 \end{pmatrix} \begin{pmatrix} 3&10\\ 10&3 \end{pmatrix}} \\
&\quad \times \widetilde{F}_1\widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_1 \begin{pmatrix} 4&5&6&7&8\\ 7&6&8&5&4 \end{pmatrix} \\
&= z^{-1}\widetilde{F}_3\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_9\widetilde{F}_3 \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \widetilde{F}_6\widetilde{F}_9\widetilde{F}_5\widetilde{F}_2\widetilde{F}_3\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\underline{\widetilde{F}_2\widetilde{F}_9 \begin{pmatrix} 1&7\\ 7&1 \end{pmatrix} \begin{pmatrix} 3&10\\ 10&3 \end{pmatrix} \widetilde{F}_1} \\
&\quad \times \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-2}\widetilde{F}_3\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_9\widetilde{F}_3 \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \widetilde{F}_6\widetilde{F}_9\widetilde{F}_5\widetilde{F}_2\widetilde{F}_3\underline{\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_2}\widetilde{F}_7\widetilde{F}_9 \begin{pmatrix} 2&7\\ 7&2 \end{pmatrix} \begin{pmatrix} 1&7\\ 7&1 \end{pmatrix} \begin{pmatrix} 3&10\\ 10&3 \end{pmatrix} \\
&\quad \times \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-3}\widetilde{F}_3\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_9\widetilde{F}_3 \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \widetilde{F}_6\underline{\widetilde{F}_9\widetilde{F}_5\widetilde{F}_3}\widetilde{F}_2\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_9 \begin{pmatrix} 2&3\\ 3&2 \end{pmatrix} \begin{pmatrix} 2&7\\ 7&2 \end{pmatrix} \begin{pmatrix} 1&7\\ 7&1 \end{pmatrix} \begin{pmatrix} 3&10\\ 10&3 \end{pmatrix} \\
&\quad \times \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-4}\widetilde{F}_3\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\underline{\widetilde{F}_9 \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \widetilde{F}_6\widetilde{F}_3} \begin{pmatrix} 3&6\\ 6&3 \end{pmatrix} \underline{\widetilde{F}_9\widetilde{F}_5\widetilde{F}_2}\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_9 \begin{pmatrix} 1&2&3&7&10\\ 7&10&1&2&3 \end{pmatrix} \\
&\quad \times \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-5}\widetilde{F}_3\underline{\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_6}\widetilde{F}_3\widetilde{F}_9 \begin{pmatrix} 3&9\\ 9&3 \end{pmatrix} \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \begin{pmatrix} 3&6\\ 6&3 \end{pmatrix} \widetilde{F}_9\widetilde{F}_5\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_9 \begin{pmatrix} 1&2&3&7&10\\ 7&10&1&2&3 \end{pmatrix} \\
&\quad \times \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-5}\underline{\widetilde{F}_3\widetilde{F}_6}\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_3\widetilde{F}_9 \begin{pmatrix} 3&9\\ 9&3 \end{pmatrix} \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \begin{pmatrix} 3&6\\ 6&3 \end{pmatrix} \widetilde{F}_9\widetilde{F}_5\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_9 \begin{pmatrix} 1&2&3&7&10\\ 7&10&1&2&3 \end{pmatrix} \\
&\quad \times \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-5}D_2 \begin{pmatrix} 2&3&6&7&8\\ 8&2&3&6&7 \end{pmatrix} \underline{\widetilde{F}_8\widetilde{F}_7\widetilde{F}_8}\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_3\widetilde{F}_9 \begin{pmatrix} 3&9\\ 9&3 \end{pmatrix} \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \begin{pmatrix} 3&6\\ 6&3 \end{pmatrix} \widetilde{F}_9\widetilde{F}_5\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_9 \\
&\quad \times \begin{pmatrix} 1&2&3&7&10\\ 7&10&1&2&3 \end{pmatrix} \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-6}\widetilde{D}_2 \begin{pmatrix} 2&3&6&7&8\\ 8&2&3&6&7 \end{pmatrix} \widetilde{F}_7\widetilde{F}_8 \begin{pmatrix} 7&8\\ 8&7 \end{pmatrix} \widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_3\widetilde{F}_9 \begin{pmatrix} 3&9\\ 9&3 \end{pmatrix} \begin{pmatrix} 2&4&8&9&10\\ 10&9&4&2&8 \end{pmatrix} \begin{pmatrix} 3&6\\ 6&3 \end{pmatrix} \widetilde{F}_9\widetilde{F}_5\widetilde{F}_{10}\widetilde{F}_1\widetilde{F}_7\widetilde{F}_9 \\
&\quad \times \begin{pmatrix} 1&2&3&7&10\\ 7&10&1&2&3 \end{pmatrix} \widetilde{F}_5\widetilde{F}_7\widetilde{F}_4\widetilde{F}_6\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8\\ 6&7&1&8&5&4 \end{pmatrix} \\
&= z^{-6}\widetilde{D}_2\widetilde{F}_8\widetilde{F}_2\underline{\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_6}\widetilde{F}_3\widetilde{F}_4\widetilde{F}_9\widetilde{F}_4\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_9\widetilde{F}_{10}\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8&9&10\\ 6&9&1&10&6&7&8&4 \end{pmatrix} \\
&= z^{-6}\widetilde{D}_2\widetilde{F}_8\widetilde{F}_2\widetilde{F}_6\underline{\widetilde{F}_6\widetilde{F}_4\widetilde{F}_6\widetilde{F}_6\widetilde{F}_{10}\widetilde{F}_6}\widetilde{F}_3\widetilde{F}_4\widetilde{F}_9\widetilde{F}_4\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_9\widetilde{F}_{10}\widetilde{F}_5 \begin{pmatrix} 1&4&5&6&7&8&9&10\\ 6&9&1&10&6&7&8&4 \end{pmatrix} \\
&= z^{-8}\widetilde{D}_2\widetilde{F}_8\widetilde{F}_2\widetilde{F}_6\widetilde{F}_4\underline{\widetilde{F}_6 \begin{pmatrix} 4&6\\ 6&4 \end{pmatrix} \widetilde{F}_{10}\widetilde{F}_6} \begin{pmatrix} 6&10\\ 10&6 \end{pmatrix} \widetilde{F}_3\widetilde{F}_4\widetilde{F}_9\widetilde{F}_4\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_9\widetilde{F}_{10}\widetilde{F}_5 \\
&\quad \times \begin{pmatrix} 1&4&5&6&7&8&9&10\\ 6&9&1&10&6&7&8&4 \end{pmatrix} \\
&= z^{-8}\widetilde{D}_2\underline{\widetilde{F}_6\widetilde{F}_4\widetilde{F}_{10}}\widetilde{F}_8\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix} 4&6\\ 6&4 \end{pmatrix} \widetilde{F}_6 \begin{pmatrix} 6&10\\ 10&6 \end{pmatrix} \widetilde{F}_3\widetilde{F}_4\widetilde{F}_9\widetilde{F}_4\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_9\widetilde{F}_{10}\widetilde{F}_5 \\
&\quad \times \begin{pmatrix} 1&4&5&6&7&8&9&10\\ 6&9&1&10&6&7&8&4 \end{pmatrix} \\
&= z^{-8}\widetilde{D}_2\widetilde{D}_1 \begin{pmatrix} 4&6&7&9&10\\ 6&7&9&10&4 \end{pmatrix} \widetilde{F}_9\widetilde{F}_8\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix} 4&6\\ 6&4 \end{pmatrix} \widetilde{F}_6 \begin{pmatrix} 6&10\\ 10&6 \end{pmatrix} \widetilde{F}_3\widetilde{F}_4\widetilde{F}_9\widetilde{F}_4\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_9\widetilde{F}_{10}\widetilde{F}_5 \\
&\quad \times \begin{pmatrix} 1&4&5&6&7&8&9&10\\ 6&9&1&10&6&7&8&4 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= z^{-8}\widetilde{D}_2\widetilde{D}_1\underline{\widetilde{F}_7\widetilde{F}_8}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_3\underline{\widetilde{F}_7}\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_7\widetilde{F}_{10}\widetilde{F}_5 \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix}\begin{pmatrix} 4 & 9 & 10 \\ 9 & 10 & 4 \end{pmatrix}\\
&= z^{-7}\widetilde{D}_2\widetilde{D}_1\widetilde{F}_8\widetilde{F}_7 \begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_{10}\underline{\widetilde{F}_3\widetilde{F}_5\widetilde{F}_3}\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8\widetilde{F}_7\widetilde{F}_{10}\widetilde{F}_5 \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix}\begin{pmatrix} 4 & 9 & 10 \\ 9 & 10 & 4 \end{pmatrix}\\
&= z^{-8}\widetilde{D}_2\widetilde{D}_1\widetilde{F}_8\widetilde{F}_7 \begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix}\widetilde{F}_2\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_3 \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}\underline{\widetilde{F}_1\widetilde{F}_2}\widetilde{F}_4\underline{\widetilde{F}_5\widetilde{F}_1}\widetilde{F}_8\widetilde{F}_7\widetilde{F}_{10}\widetilde{F}_5 \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix}\begin{pmatrix} 4 & 9 & 10 \\ 9 & 10 & 4 \end{pmatrix}\\
&= z^{-9}\widetilde{D}_2\widetilde{D}_1\widetilde{F}_8\widetilde{F}_7\underline{\widetilde{F}_2\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5}\widetilde{F}_3\widetilde{F}_2\widetilde{F}_4\widetilde{F}_1 \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}\widetilde{F}_3\widetilde{F}_7\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_3 \begin{pmatrix} 1 & 3 & 4 & 5 & 9 & 10 \\ 5 & 1 & 9 & 3 & 10 & 4 \end{pmatrix}\\
&= z^{-10}\widetilde{D}_2\widetilde{D}_1\widetilde{F}_8\widetilde{F}_7\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_3\underline{\widetilde{F}_2} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\widetilde{F}_4\widetilde{F}_1\underline{\widetilde{F}_3}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_3 \begin{pmatrix} 1 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 5 & 3 & 10 & 4 \end{pmatrix}\\
&= z^{-10}\widetilde{D}_2\widetilde{D}_1\widetilde{F}_8\underline{\widetilde{F}_7}\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_3\widetilde{F}_4\widetilde{F}_1\underline{\widetilde{F}_7}\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{pmatrix}\\
&= z^{-10}\widetilde{D}_2\widetilde{D}_1\underline{\widetilde{F}_8\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_3\widetilde{F}_4}\widetilde{F}_1\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{pmatrix}\\
&= z^{-11}\widetilde{D}_2\widetilde{D}_1\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_3\widetilde{F}_4\widetilde{F}_1\widetilde{F}_8 \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix}\widetilde{F}_{10}\widetilde{F}_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{pmatrix}\\
&= z^{-11}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}\widetilde{F}_2\widetilde{F}_3\widetilde{F}_4\widetilde{F}_5\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_3\widetilde{F}_4\widetilde{F}_1\widetilde{F}_8 \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix}\widetilde{F}_{10}\widetilde{F}_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{pmatrix}\\
&= z^{-12}D_2D_1D_0\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\underline{\widetilde{F}_3} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\widetilde{F}_{10}\underline{\widetilde{F}_4}\widetilde{F}_2\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_1 \begin{pmatrix} 3 & 4 & 5 & 8 & 9 & 10 \\ 5 & 3 & 8 & 9 & 10 & 4 \end{pmatrix}\\
&= z^{-12}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\widetilde{F}_1\underline{\widetilde{F}_2}\widetilde{F}_4\widetilde{F}_{10}\underline{\widetilde{F}_2}\widetilde{F}_4\widetilde{F}_5\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_1 \begin{pmatrix} 4 & 5 & 8 & 9 & 10 \\ 5 & 8 & 9 & 10 & 4 \end{pmatrix}\\
&= z^{-12}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\widetilde{F}_1\underline{\widetilde{F}_4\widetilde{F}_{10}\widetilde{F}_4}\widetilde{F}_5\widetilde{F}_8\widetilde{F}_{10}\widetilde{F}_1 \begin{pmatrix} 4 & 5 & 8 & 9 & 10 \\ 5 & 8 & 9 & 10 & 4 \end{pmatrix}\\
&= z^{-11}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\widetilde{F}_1\widetilde{F}_{10}\underline{\widetilde{F}_4} \begin{pmatrix} 4 & 10 \\ 10 & 4 \end{pmatrix}\widetilde{F}_5\widetilde{F}_8\underline{\widetilde{F}_{10}}\widetilde{F}_1 \begin{pmatrix} 4 & 5 & 8 & 9 & 10 \\ 5 & 8 & 9 & 10 & 4 \end{pmatrix}\\
&= z^{-11}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\underline{\widetilde{F}_1\widetilde{F}_{10}}\widetilde{F}_5\underline{\widetilde{F}_8\widetilde{F}_1} \begin{pmatrix} 5 & 8 & 9 & 10 \\ 8 & 9 & 10 & 5 \end{pmatrix}\\
&= z^{-12}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_1 \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\widetilde{F}_8 \begin{pmatrix} 5 & 8 & 9 & 10 \\ 8 & 9 & 10 & 5 \end{pmatrix}\\
&= z^{-12}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\widetilde{F}_{10}\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8 \begin{pmatrix} 1 & 5 & 8 & 9 & 10 \\ 8 & 1 & 9 & 10 & 5 \end{pmatrix}\\
&= z^{-12}\widetilde{D}_2\widetilde{D}_1\widetilde{D}_0\widetilde{D}_3. \quad \square
\end{aligned}
$$

Lemma 3.5. *By choosing carefully the lifts of the Dehn twists in the puncture relation, we have:*

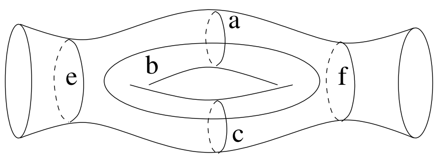
$$\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = z^{-12}\widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}}.$$

Proof. As this is a degenerated case of the lantern relation, the proof is the same as above. □

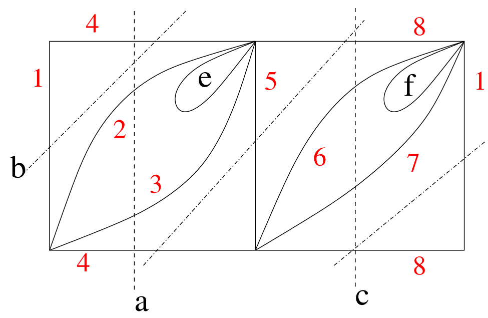
Lemma 3.6. *By choosing carefully the lifts of the Dehn twists in the chain relation, we have:*

$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = z^{-24}\widetilde{D}_e\widetilde{D}_f.$$

Proof. Consider the two-holed torus with one puncture on each boundary component. The simple closed geodesics a, b, c, e and f are as follows:



We consider the following ideal triangulation of the two-holed torus:



The Dehn twists corresponding to a, b, c, e and f can be written as follows:

$$\begin{aligned}
D_a &= F_3F_4F_3\begin{pmatrix}2&4\\4&2\end{pmatrix},\\
D_b &= \mathrm{Ad}(F_8F_1F_5)F_8\begin{pmatrix}4&8\\8&4\end{pmatrix},\\
D_c &= F_7F_8F_7\begin{pmatrix}6&8\\8&6\end{pmatrix},\\
D_e &= \mathrm{Ad}(F_5F_4F_8F_6F_7)F_3\begin{pmatrix}2&3\\3&2\end{pmatrix},\\
D_f &= \mathrm{Ad}(F_1F_8F_4F_2F_3)F_7\begin{pmatrix}6&7\\7&6\end{pmatrix}.
\end{aligned}$$

The Dehn twist D_e can be simplified as follows:

$$\begin{aligned}
D_e &= F_5F_4F_8F_6F_7F_3\begin{pmatrix}2&3\\3&2\end{pmatrix}F_7F_6F_8F_4F_5\\
&= F_5F_4F_8F_6\underline{F_7F_3F_7}F_6F_8F_4F_5\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= F_5F_4F_8\underline{F_6F_3}F_7\begin{pmatrix}3&7\\7&3\end{pmatrix}F_6F_8F_4F_5\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= F_5\underline{F_4F_8F_3}F_7F_6\underline{F_8F_4}F_5\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= F_5F_8F_3\underline{F_7F_6F_4F_8}F_5\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= \underline{F_5}F_3F_8F_7F_6F_4F_5\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= F_3F_5\underline{F_3F_8F_7F_6}\underline{F_4F_3}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= F_3F_5F_8F_7F_6F_3F_4\begin{pmatrix}3&6\\6&3\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= F_3F_5F_8F_7F_6F_3F_4\begin{pmatrix}2&3&4&5&6&7&8\\3&4&2&8&5&6&7\end{pmatrix}.
\end{aligned}$$

By sending (2, 3, 4, 5, 6, 7, 8) to (6, 7, 8, 1, 2, 3, 4), we can simplify D_f in a similar way:

$$D_f = F_7F_1F_4F_3F_2F_7F_8\begin{pmatrix}1&2&3&4&6&7&8\\4&1&2&3&7&8&6\end{pmatrix}.$$

We will use the following expression of these Dehn twists to obtain their lifts:

$$\begin{aligned}
D_a &= F_3F_4F_3\begin{pmatrix}2&4\\4&2\end{pmatrix},\\
D_b &= \mathrm{Ad}(F_8F_1F_5)F_8\begin{pmatrix}4&8\\8&4\end{pmatrix},\\
D_c &= F_7F_8F_7\begin{pmatrix}6&8\\8&6\end{pmatrix},\\
D_e &= F_5F_3F_8F_7F_6F_4F_5\begin{pmatrix}2&3&4&6&7&8\\3&8&2&4&6&7\end{pmatrix},\\
D_f &= F_1F_7F_4F_3F_2F_8F_1\begin{pmatrix}2&3&4&6&7&8\\8&2&3&7&4&6\end{pmatrix}.
\end{aligned}$$

We need to check that the induced lifts of the Dehn twists satisfy the trivial type-1 braid relation. As there is a symmetry between D_a and D_c with respect to D_b, it is sufficient to verify the trivial type-1 braid relation for $\widetilde{D}_a$ and $\widetilde{D}_b$.

We rewrite $\widetilde{D}_b$ as follows:

$$\begin{aligned}\widetilde{D}_b &= \widetilde{F}_8\widetilde{F}_1\widetilde{F}_5\widetilde{F}_8\begin{pmatrix}4&8\\8&4\end{pmatrix}\widetilde{F}_5\widetilde{F}_1\widetilde{F}_8 = z^{-1}\underline{\widetilde{F}_8\widetilde{F}_1\widetilde{F}_8}\widetilde{F}_5\begin{pmatrix}5&8\\8&5\end{pmatrix}\widetilde{F}_1\widetilde{F}_4\begin{pmatrix}4&8\\8&4\end{pmatrix} = z^{-2}\widetilde{F}_1\underline{\widetilde{F}_8}\begin{pmatrix}1&8\\8&1\end{pmatrix}\underline{\widetilde{F}_5\widetilde{F}_1}\widetilde{F}_4\begin{pmatrix}5&8\\8&5\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\ &= z^{-3}\widetilde{F}_1\widetilde{F}_5\widetilde{F}_8\begin{pmatrix}5&8\\8&5\end{pmatrix}\widetilde{F}_4\begin{pmatrix}1&8\\8&1\end{pmatrix}\begin{pmatrix}5&8\\8&5\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix} = z^{-3}\widetilde{F}_1\widetilde{F}_5\widetilde{F}_8\widetilde{F}_4\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}.\end{aligned}$$

By using once the pentagon relation, we have the following equality:

$$\widetilde{D}_a = \widetilde{F}_3\widetilde{F}_4\widetilde{F}_3\begin{pmatrix}2&4\\4&2\end{pmatrix} = z^{-1}\widetilde{F}_4\widetilde{F}_3\begin{pmatrix}2&3&4\\4&2&3\end{pmatrix}.$$

Then the trivial type-1 braid relation is verified as follows:

$$\begin{aligned}\widetilde{D}_b\widetilde{D}_a\widetilde{D}_b &= z^{-7}\widetilde{F}_1\widetilde{F}_5\widetilde{F}_8\widetilde{F}_4\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\widetilde{F}_4\widetilde{F}_3\begin{pmatrix}2&3&4\\3&4&2\end{pmatrix}\widetilde{F}_1\widetilde{F}_5\widetilde{F}_8\widetilde{F}_4\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\\
&= z^{-7}\widetilde{F}_1\widetilde{F}_5\underline{\widetilde{F}_8}\widetilde{F}_4\underline{\widetilde{F}_8}\widetilde{F}_3\widetilde{F}_5\widetilde{F}_1\widetilde{F}_4\widetilde{F}_2\begin{pmatrix}2&3&8\\8&2&3\end{pmatrix}\\
&= z^{-7}\underline{\widetilde{F}_1\widetilde{F}_5}\widetilde{F}_4\underline{\widetilde{F}_3\widetilde{F}_5\widetilde{F}_1}\widetilde{F}_4\widetilde{F}_2\begin{pmatrix}2&3&8\\8&2&3\end{pmatrix}\\
&= z^{-6}\widetilde{F}_5\widetilde{F}_4\underline{\widetilde{F}_1}\begin{pmatrix}1&4\\4&1\end{pmatrix}\underline{\widetilde{F}_3}\widetilde{F}_5\widetilde{F}_4\widetilde{F}_2\begin{pmatrix}2&3&8\\8&2&3\end{pmatrix}\\
&= z^{-5}\underline{\widetilde{F}_5\widetilde{F}_4\widetilde{F}_3\widetilde{F}_5}\widetilde{F}_1\begin{pmatrix}1&5\\5&1\end{pmatrix}\widetilde{F}_2\begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&3&8\\8&2&3\end{pmatrix}\\
&= z^{-5}\underline{\widetilde{F}_5\widetilde{F}_4\widetilde{F}_5\widetilde{F}_5\widetilde{F}_3\widetilde{F}_5}\widetilde{F}_1\begin{pmatrix}1&5\\5&1\end{pmatrix}\widetilde{F}_2\begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&3&8\\8&2&3\end{pmatrix}\\
&= z^{-5}\widetilde{F}_4\underline{\widetilde{F}_5}\begin{pmatrix}4&5\\5&4\end{pmatrix}\underline{\widetilde{F}_3}\widetilde{F}_5\begin{pmatrix}3&5\\5&3\end{pmatrix}\widetilde{F}_1\widetilde{F}_2\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&3&8\\8&2&3\end{pmatrix}\\
&= z^{-5}\widetilde{F}_4\widetilde{F}_3\widetilde{F}_5\widetilde{F}_4\widetilde{F}_1\widetilde{F}_2\begin{pmatrix}1&2&3&4&5&8\\5&8&4&2&1&3\end{pmatrix}\\
&= z^{-4}\widetilde{D}_a\begin{pmatrix}2&3&4\\3&4&2\end{pmatrix}\widetilde{F}_5\widetilde{F}_4\widetilde{F}_1\widetilde{F}_2\begin{pmatrix}1&2&3&4&5&8\\5&8&4&2&1&3\end{pmatrix}\\
&= z^{-4}\widetilde{D}_a\underline{\widetilde{F}_5\widetilde{F}_3\widetilde{F}_1}\widetilde{F}_4\begin{pmatrix}1&2&3&4&5&8\\5&4&2&8&1&3\end{pmatrix}\\
&= z^{-4}\widetilde{D}_a\underline{\widetilde{F}_1\widetilde{F}_5}\widetilde{F}_3\widetilde{F}_4\begin{pmatrix}1&2&3&4&5&8\\5&4&2&8&1&3\end{pmatrix}\\
&= z^{-1}\widetilde{D}_a\widetilde{D}_b\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\underline{\widetilde{F}_4}\widetilde{F}_8\widetilde{F}_3\underline{\widetilde{F}_4}\begin{pmatrix}1&2&3&4&5&8\\5&4&2&8&1&3\end{pmatrix}\\
&= \widetilde{D}_a\widetilde{D}_b\widetilde{D}_a.\end{aligned}$$

The following computation will be used in the later proof:

$$\begin{aligned}\widetilde{D}_c &= \widetilde{F}_7\widetilde{F}_8\widetilde{F}_7\begin{pmatrix}6&8\\8&6\end{pmatrix} = z^{-1}\widetilde{F}_8\widetilde{F}_7\begin{pmatrix}6&7&8\\8&6&7\end{pmatrix},\\
\widetilde{D}_e &= \widetilde{F}_5\widetilde{F}_3\widetilde{F}_8\widetilde{F}_7\widetilde{F}_6\widetilde{F}_4\widetilde{F}_5\begin{pmatrix}2&3&4&6&7&8\\3&8&2&4&6&7\end{pmatrix}\\
&= \underline{\widetilde{F}_5\widetilde{F}_3}\widetilde{F}_8\widetilde{F}_7\widetilde{F}_6\widetilde{F}_4\widetilde{F}_5\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= \widetilde{F}_3\widetilde{F}_5\underline{\widetilde{F}_3\widetilde{F}_8\widetilde{F}_7}\widetilde{F}_6\underline{\widetilde{F}_4\widetilde{F}_3}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= \widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_7\underline{\widetilde{F}_3\widetilde{F}_6\widetilde{F}_3}\widetilde{F}_4\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= z^{-1}\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_7\widetilde{F}_6\widetilde{F}_3\widetilde{F}_4\begin{pmatrix}3&6\\6&3\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}\begin{pmatrix}6&7\\7&6\end{pmatrix}\begin{pmatrix}3&7\\7&3\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\\
&= z^{-1}\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_7\widetilde{F}_6\widetilde{F}_3\widetilde{F}_4\begin{pmatrix}2&3&4&5&6&7&8\\3&4&2&8&5&6&7\end{pmatrix},\\
\widetilde{D}_f &= \widetilde{F}_1\widetilde{F}_7\widetilde{F}_4\widetilde{F}_3\widetilde{F}_2\widetilde{F}_8\widetilde{F}_1\begin{pmatrix}2&3&4&6&7&8\\8&2&3&7&4&6\end{pmatrix} = z^{-1}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3\widetilde{F}_2\widetilde{F}_7\widetilde{F}_8\begin{pmatrix}1&2&3&4&6&7&8\\4&1&2&3&7&8&6\end{pmatrix}.\end{aligned}$$

Taking the composition of $\widetilde{D}_c$, $\widetilde{D}_b$ and $\widetilde{D}_a$, we have:

$$\begin{aligned}\widetilde{D}_c\widetilde{D}_b\widetilde{D}_a &= z^{-5}\widetilde{F}_8\widetilde{F}_7\begin{pmatrix}6&7&8\\8&6&7\end{pmatrix}\widetilde{F}_1\widetilde{F}_5\widetilde{F}_8\widetilde{F}_4\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}\widetilde{F}_4\widetilde{F}_3\begin{pmatrix}2&3&4\\4&2&3\end{pmatrix}\\
&= z^{-5}\widetilde{F}_8\widetilde{F}_7\widetilde{F}_1\widetilde{F}_5\underline{\widetilde{F}_6}\widetilde{F}_4\underline{\widetilde{F}_6}\widetilde{F}_3\begin{pmatrix}2&3&4\\4&2&3\end{pmatrix}\\
&= z^{-5}\widetilde{F}_8\widetilde{F}_7\widetilde{F}_1\widetilde{F}_5\widetilde{F}_4\widetilde{F}_3\begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}2&3&4&6&7&8\\4&2&8&3&6&7\end{pmatrix}.\end{aligned}$$

By taking the square, we have:

$$
\begin{aligned}
(\widetilde{D}_c\widetilde{D}_b\widetilde{D}_a)^2 &= z^{-10}\widetilde{F}_8\widetilde{F}_7\widetilde{F}_1\widetilde{F}_5\underline{\widetilde{F}_4}\widetilde{F}_3 \begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}2&3&4&6&7&8\\4&2&8&3&6&7\end{pmatrix}\underline{\widetilde{F}_8}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_5\widetilde{F}_4\widetilde{F}_3 \begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}2&3&4&6&7&8\\4&2&8&3&6&7\end{pmatrix}\\
&= z^{-10}\widetilde{F}_8\widetilde{F}_7\widetilde{F}_1\underline{\widetilde{F}_5\widetilde{F}_3\widetilde{F}_8}\widetilde{F}_5\widetilde{F}_1\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}2&3&4&6&7&8\\8&4&7&2&3&6\end{pmatrix}\\
&= z^{-10}\widetilde{F}_8\widetilde{F}_7\underline{\widetilde{F}_1\widetilde{F}_8\widetilde{F}_5\widetilde{F}_3\widetilde{F}_5}\widetilde{F}_1\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}2&3&4&6&7&8\\8&4&7&2&3&6\end{pmatrix}\\
&= z^{-12}\underline{\widetilde{F}_8\widetilde{F}_7\widetilde{F}_8}\widetilde{F}_1\widetilde{F}_8\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}1&8\\8&1\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}2&3&4&6&7&8\\8&4&7&2&3&6\end{pmatrix}\\
&= z^{-11}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_1\underline{\widetilde{F}_7}\widetilde{F}_3\widetilde{F}_5\underline{\widetilde{F}_7}\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}7&8\\8&7\end{pmatrix}\begin{pmatrix}1&8\\8&1\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}2&3&4&6&7&8\\8&4&7&2&3&6\end{pmatrix}\\
&= z^{-11}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}.
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
(\widetilde{D}_c\widetilde{D}_b\widetilde{D}_a)^4 &= z^{-22}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\underline{\widetilde{F}_2}\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\underline{\widetilde{F}_7\widetilde{F}_8}\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\\
&= z^{-22}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\underline{\widetilde{F}_8\widetilde{F}_7\widetilde{F}_8}\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_2\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\\
&= z^{-23}\widetilde{F}_7\underline{\widetilde{F}_8\widetilde{F}_1\widetilde{F}_3}\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_1\widetilde{F}_3\underline{\widetilde{F}_5\widetilde{F}_2}\widetilde{F}_6 \begin{pmatrix}7&8\\8&7\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\\
&= z^{-23}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_3\underline{\widetilde{F}_8\widetilde{F}_5}\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_1\widetilde{F}_3\underline{\widetilde{F}_5\widetilde{F}_2}\widetilde{F}_6 \begin{pmatrix}7&8\\8&7\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\\
&= z^{-24}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix}5&8\\8&5\end{pmatrix}\underline{\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\widetilde{F}_7\widetilde{F}_8}\underline{\widetilde{F}_1\widetilde{F}_3}\widetilde{F}_2\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}7&8\\8&7\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\\
&= z^{-24}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix}5&8\\8&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\widetilde{F}_7\widetilde{F}_8\underline{\widetilde{F}_2\widetilde{F}_3}\widetilde{F}_1\widetilde{F}_2\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}7&8\\8&7\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&1&3\end{pmatrix}\\
&= z^{-23}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&3&2&1&4\end{pmatrix}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_3\widetilde{F}_2\widetilde{F}_3 \begin{pmatrix}2&3\\3&2\end{pmatrix}\widetilde{F}_1\widetilde{F}_2\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&4&2&3&1\end{pmatrix}\\
&= z^{-23}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_5 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&3&2&1&4\end{pmatrix}\widetilde{F}_7\widetilde{F}_8\widetilde{F}_3\widetilde{F}_2\underline{\widetilde{F}_3\widetilde{F}_1\widetilde{F}_3}\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&5&8&7&4&2&3&1\end{pmatrix}\\
&= z^{-24}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\underline{\widetilde{F}_5} \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&3&2&1&4\end{pmatrix}\widetilde{F}_7\widetilde{F}_8\underline{\widetilde{F}_3}\widetilde{F}_2\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&3\\3&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\6&5&8&7&4&2&3&1\end{pmatrix}\\
&= z^{-24}\widetilde{F}_7\widetilde{F}_1\underline{\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&3&2&1&4\end{pmatrix}\widetilde{F}_7}\widetilde{F}_8\widetilde{F}_2\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\8&5&6&7&4&2&3&1\end{pmatrix}\\
&= z^{-24}\widetilde{F}_7\widetilde{F}_1\underline{\widetilde{F}_3\widetilde{F}_4}\widetilde{F}_5\widetilde{F}_8 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&3&2&1&4\end{pmatrix}\widetilde{F}_8\widetilde{F}_2\widetilde{F}_1\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\8&5&6&7&4&2&3&1\end{pmatrix}\\
&= z^{-25}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3\widetilde{F}_4 \begin{pmatrix}3&4\\4&3\end{pmatrix}\widetilde{F}_5\underline{\widetilde{F}_8 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&5&7&3&2&1&4\end{pmatrix}\widetilde{F}_8}\underline{\widetilde{F}_2\widetilde{F}_1}\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\8&5&6&7&4&2&3&1\end{pmatrix}\\
&= z^{-25}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3\widetilde{F}_4\widetilde{F}_5\widetilde{F}_2 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&7&5&3&2&1&4\end{pmatrix}\underline{\widetilde{F}_4\widetilde{F}_1\widetilde{F}_2\widetilde{F}_3}\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\8&5&6&7&4&2&3&1\end{pmatrix}\\
&= z^{-27}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3\widetilde{F}_4\underline{\widetilde{F}_5\widetilde{F}_2 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&7&5&3&2&1&4\end{pmatrix}\widetilde{F}_1}\underline{\widetilde{F}_4\widetilde{F}_1 \begin{pmatrix}1&4\\4&1\end{pmatrix}\widetilde{F}_3}\widetilde{F}_2\widetilde{F}_3 \begin{pmatrix}2&3\\3&2\end{pmatrix}\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\8&5&6&7&4&2&3&1\end{pmatrix}\\
&= z^{-27}\widetilde{F}_7\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3\widetilde{F}_4\widetilde{F}_2\widetilde{F}_7 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&7&5&3&2&1&4\end{pmatrix}\underline{\widetilde{F}_3\widetilde{F}_4\widetilde{F}_3}\widetilde{F}_1\widetilde{F}_2\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\7&6&5&8&4&2&3&1\end{pmatrix}\\
&= z^{-26}\underline{\widetilde{F}_7\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3}\widetilde{F}_4\underline{\widetilde{F}_2\widetilde{F}_7} \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&7&5&3&2&1&4\end{pmatrix}\underline{\widetilde{F}_4}\widetilde{F}_3 \begin{pmatrix}3&4\\4&3\end{pmatrix}\widetilde{F}_1\widetilde{F}_2\widetilde{F}_3\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\7&6&5&8&4&2&3&1\end{pmatrix}\\
&= z^{-25}\widetilde{D}_f \begin{pmatrix}6&7&8\\8&6&7\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\widetilde{F}_4 \begin{pmatrix}1&2&3&4&5&6&7&8\\6&8&7&5&3&2&1&4\end{pmatrix}\widetilde{F}_3\widetilde{F}_1\widetilde{F}_2\widetilde{F}_4\widetilde{F}_5\widetilde{F}_6 \begin{pmatrix}1&2&3&4&5&6&7&8\\7&6&8&5&4&2&3&1\end{pmatrix}\\
&= z^{-25}\widetilde{D}_f\widetilde{F}_3\widetilde{F}_5\widetilde{F}_8\widetilde{F}_7\widetilde{F}_6\widetilde{F}_3\widetilde{F}_4 \begin{pmatrix}2&3&4&5&6&7&8\\3&4&2&8&5&6&7\end{pmatrix}\\
&= z^{-24}\widetilde{D}_f\widetilde{D}_e.
\end{aligned}
$$

The chain relation becomes:

$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = z^{-24}\widetilde{D}_e\widetilde{D}_f. \quad \square$$

We say that the lifts of Dehn twists are **normalized** if all braid type relations and all lantern relations are lifted in the trivial way.

By the lemmas that we proved above, we normalize the lift of each Dehn twist from $\widetilde{D}$ to $z^{-12}\widetilde{D}$. Then we get the presentation in Proposition 3.1.

The cohomological arguments from [8] settle our theorem.

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