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On the Chern–Gauss–Bonnet theorem for the noncommutative 4-sphere

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ABSTRACT

We construct a differential calculus over the noncommutative 4-sphere in the framework of pseudo-Riemannian calculi, and show that for every metric in a conformal class of perturbations of the round metric, there exists a unique metric and torsion-free connection. Furthermore, we find a localization of the projective module corresponding to the space of vector fields, which allows us to formulate a Chern–Gauss–Bonnet type theorem for the noncommutative 4-sphere.

1. Introduction

Over the last years, there have been increasing interests in understanding the curvature of noncommutative manifolds. Starting from seminal work on the scalar curvature and Gauss–Bonnet type theorems for the noncommutative torus [1–3] many interesting papers that discuss different aspects of curvature in the noncommutative setting have followed [4–14]. Note that these are only examples of recent progress in the area; several authors have previously considered curvature in this context (see e.g. [15–20]). Although connections on projective modules and their corresponding curvatures are natural objects in noncommutative geometry, classical objects that are built from the curvature tensor, like Ricci and scalar curvature, do not always have straight-forward analogues. Therefore, it is interesting to study as to what extent such concepts are relevant for noncommutative geometry.

For Riemannian manifolds, the Chern–Gauss–Bonnet theorem provides an important link between geometry and topology. It states that the integral of the Pfaffian of the curvature form (of a closed orientable even dimensional manifold) is proportional to the Euler characteristic, which is a topological invariant. For a two dimensional manifold, the Pfaffian is simply the scalar curvature, which reduces the Chern–Gauss–Bonnet theorem to the Gauss–Bonnet theorem. Therefore, to understand similar theorems for two dimensional noncommutative manifolds, one needs to find a proper definition of the scalar curvature. For a Riemannian manifold, the asymptotic expansion of the heat kernel contains information about the scalar curvature in one of the coefficients. The expansion of the heat kernel makes sense even for a noncommutative manifold, and the very same coefficient serves as a definition of noncommutative scalar curvature. For the noncommutative torus, the scalar curvature corresponding to certain perturbations of the flat metric has been computed, and it is possible to

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show that a Gauss–Bonnet type theorem holds; i.e., the trace of the scalar curvature is independent of the metric perturbation [1,2]. However, for higher dimensional manifolds, it is not clear how to define the analogue of the Pfaffian of the curvature form in order to formulate the Chern–Gauss–Bonnet theorem.

In this paper we construct a differential calculus over the noncommutative 4-sphere, in the framework of pseudo-Riemannian calculi [13], and introduce a projective module in close analogy with the space of vector fields on the classical 4-sphere. Moreover, via a suitable localization of the algebra, we find a local trivialization of the projective module and prove the existence of (unique) metric and torsion-free connections for a class of perturbations of the round metric. Finally, we show that in this particular case, there exists a naive analogue of the Pfaffian of the curvature form, which allows us to prove a Chern–Gauss–Bonnet type theorem for the noncommutative 4-sphere.

This paper is organized as follows: In Section 2.1 we briefly recall the concept of a pseudo-Riemannian calculus, and Section 2.2 introduces a particular parametrization of the classical 4-sphere. Sections 3.1 and 3.2 are devoted to the construction of a real metric calculus over the noncommutative 4-sphere, and Section 3.3 discusses certain aspects of localization. These results are then used in Section 3.4 to construct a pseudo-Riemannian calculus, giving metric and torsion-free connections for a class of perturbed metrics. Finally, Section 4 introduces a trace for the noncommutative 4-sphere, and formulates and proves a version of the Chern–Gauss–Bonnet theorem.

2. Preliminaries

2.1. Pseudo-Riemannian calculi

Let us briefly recall the terminology from [13] concerning pseudo-Riemannian calculi, as this is the context in which we shall construct a differential calculus over the noncommutative 4-sphere.

To define a pseudo-Riemannian calculus over an algebra $\mathcal{A}$, we proceed in two steps. First, we define a *real metric calculus* over an algebra $\mathcal{A}$ by choosing a (right) $\mathcal{A}$-module M, together with a non-degenerate bilinear form (the metric), as well as a Lie algebra of derivations and a map φ that associates an element of M to each derivation. Next, a pseudo-Riemannian calculus is defined to be a real metric calculus for which there exists a metric and torsion-free connection on M.

To fix our notation and terminology, let us recall the following definitions:

Definition 2.1. Let M be a right $\mathcal{A}$-module. A map $h : M\times M \to \mathcal{A}$ is called a hermitian form on M if

$$\begin{aligned} &h(U, V + W) = h(U, V) + h(U, W)\\ &h(U, Va) = h(U, V)a\\ &h(U, V)^* = h(V, U). \end{aligned}$$

A hermitian form is *non-degenerate* if $h(U, V) = 0$ for all $V \in M$ implies that $U = 0$. For brevity, we simply refer to a non-degenerate hermitian form as a *metric* on M. The pair (M, h), where M is a right $\mathcal{A}$-module and h is a hermitian form on M, is called a *(right) hermitian $\mathcal{A}$-module*. If h is a metric, we say that (M, h) is a *(right) metric $\mathcal{A}$-module*.

Definition 2.2 (*[13]*). Let (M, h) be a (right) metric $\mathcal{A}$-module, let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a (real) Lie algebra of hermitian derivations and let $\varphi : \mathfrak{g} \to M$ be a $\mathbb{R}$-linear map. If we denote the pair $(\mathfrak{g}, \varphi)$ by $\mathfrak{g}_\varphi$, the triple $(M, h, \mathfrak{g}_\varphi)$ is called a *real metric calculus* if

(1) the image $M_\varphi = \varphi(\mathfrak{g})$ generates M as an $\mathcal{A}$-module,
(2) $h(E, E')^* = h(E, E')$ for all $E, E' \in M_\varphi$.

Definition 2.3 (*[13]*). Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus and let ∇ denote an affine connection on $(M, \mathfrak{g})$. If

$$h(\nabla_d E, E') = h(\nabla_d E, E')^*$$

for all $E, E' \in M_\varphi$ and $d \in \mathfrak{g}$ then $(M, h, \mathfrak{g}_\varphi, \nabla)$ is called a *real connection calculus*.

Definition 2.4 (*[13]*). Let $(M, h, \mathfrak{g}_\varphi, \nabla)$ be a real connection calculus. The calculus is *metric* if

$$d\big(h(U, V)\big) = h\big(\nabla_d U, V\big) + h\big(U, \nabla_d V\big)$$

for all $d \in \mathfrak{g}$, $U, V \in M$, and *torsion-free* if

$$\nabla_{d_1}\varphi(d_2) - \nabla_{d_2}\varphi(d_1) - \varphi\big([d_1, d_2]\big) = 0$$

for all $d_1, d_2 \in \mathfrak{g}$. A metric and torsion-free real connection calculus over M is called a *pseudo-Riemannian calculus over M*.

Given a real metric calculus $(M, h, \mathfrak{g}_\varphi)$, it is natural to ask if it is possible to find an affine connection such that $(M, h, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus. In general, this is not possible, but if such a connection exists, it is unique.

Theorem 2.5 (*[13]*). *Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus over M. Then there exists at most one affine connection ∇ on $(M, \mathfrak{g})$, such that $(M, h, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus.*

2.2. Embedding of S^4 in $\mathbb{R}^5$

The geometric constructions for the noncommutative 4-sphere will closely follow that of classical geometry. Therefore, let us review an explicit parametrization of S^4, giving a chart that covers almost all of the manifold. Furthermore, we present a particular basis for vector fields over that chart.

As a subset of $\mathbb{R}^5$, the 4-dimensional sphere is defined as

$$S^4 = \{(x^1, x^2, x^3, x^4, x^5) \in \mathbb{R}^5 : (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 = 1\},$$

and we let $U_0 \subseteq S^4$ denote the chart of S^4 given by

$$\begin{aligned}
&x^1 = \cos(\xi_1)\cos(\varphi)\cos(\psi) && x^2 = \sin(\xi_1)\cos(\varphi)\cos(\psi)\\
&x^3 = \cos(\xi_2)\sin(\varphi)\cos(\psi) && x^4 = \sin(\xi_2)\sin(\varphi)\cos(\psi)\\
&x^5 = \sin(\psi),
\end{aligned}$$

where $0 < \xi_1, \xi_2 < 2\pi$, $0 < \varphi < \pi/2$ and $-\pi/2 < \psi < \pi/2$. Equivalently, one may consider $z = x^1 + ix^2$, $w = x^3 + ix^4$ and $t = x^5$ with

$$\begin{aligned}
z &= e^{i\xi_1}\cos(\varphi)\cos(\psi)\\
w &= e^{i\xi_2}\sin(\varphi)\cos(\psi)\\
t &= \sin(\psi).
\end{aligned}$$

At each point $p \in U_0$, the tangent space T_pS^4 is spanned by the vectors

$$\begin{aligned}
\partial_{\xi_1}\vec{x} &= (-\sin\xi_1\cos\varphi\cos\psi, \cos\xi_1\cos\varphi, \cos\psi, 0, 0) = (-x^2, x^1, 0, 0, 0)\\
\partial_{\xi_2}\vec{x} &= (0, 0, -\sin\xi_2\sin\varphi\cos\psi, \cos\xi_2\sin\varphi\cos\psi, 0) = (0, 0, -x^4, x^3, 0)\\
\partial_{\varphi}\vec{x} &= (-\cos\xi_1\sin\varphi\cos\psi, -\sin\xi_1\sin\varphi\cos\psi, \cos\xi_2\cos\varphi\cos\psi, \sin\xi_2\cos\varphi\cos\psi, 0)\\
\partial_{\psi}\vec{x} &= (-\cos\xi_1\cos\varphi\sin\psi, -\sin\xi_1\cos\varphi\sin\psi,\\
&\qquad -\cos\xi_2\sin\varphi\sin\psi, -\sin\xi_2\sin\varphi\sin\psi, \cos\psi).
\end{aligned}$$

These vector fields are defined in the local chart U_0 and we would like to extend them to global vector fields on S^4 (however, *not* providing a basis at each point of S^4). As written above, $\partial_{\xi_1}\vec{x}$ and $\partial_{\xi_2}\vec{x}$ may be extended to all of S^4, since all components can be expressed in terms of $x^1, \ldots, x^5$. By rescaling $\partial_\varphi\vec{x}$ and $\partial_\psi\vec{x}$ one obtains

$$\begin{aligned}
-|z||w|\partial_\varphi\vec{x} &= (x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2, 0)\\
-\cos\psi\,\partial_\psi\vec{x} &= (x^1t, x^2t, x^3t, x^4t, -|z|^2 - |w|^2),
\end{aligned}$$

which are well defined as vector fields on S^4. Thus, the globally defined vector fields given by

$$\begin{aligned}
&e_1 = (-x^2, x^1, 0, 0, 0) && e_2 = (0, 0, -x^4, x^3, 0)\\
&e_3 = (x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2, 0) && e_4 = (x^1t, x^2t, x^3t, x^4t, -|z|^2 - |w|^2),
\end{aligned}$$

span the space of vector fields over U_0. For later comparison, let us write down the action of the derivations corresponding to the above vector fields:

$$\begin{aligned}
&\partial_1 z = iz && \partial_1 w = 0 && \partial_1 t = 0\\
&\partial_2 z = 0 && \partial_2 w = iw && \partial_2 t = 0\\
&\partial_3 z = z|w|^2 && \partial_3 w = -w|z|^2 && \partial_3 t = 0\\
&\partial_4 z = zt && \partial_4 w = wt && \partial_4 t = t^2 - 1.
\end{aligned} \tag{2.1}$$

3. The noncommutative 4-sphere

3.1. Basic properties of S^4_θ

For $\theta \in [0, 1)$, we let S^4_θ denote the unital $*$-algebra (over $\mathbb{C}$) generated by Z, W and T, satisfying the relations [21,22]

$$\begin{aligned}
&WZ = qZW \qquad W^*Z = \bar{q}ZW^*\\
&ZZ^* + WW^* + T^2 = \mathbb{1}\\
&T^* = T \qquad [T, Z] = [T, W] = [W, W^*] = [Z, Z^*] = 0,
\end{aligned} \tag{3.1}$$

where $q = e^{i2\pi\theta}$. Furthermore, $ZZ^* \in Z(S^4_\theta)$ and $WW^* \in Z(S^4_\theta)$ where $Z(S^4_\theta)$ denotes the center of S^4_θ. It follows from (3.1) that a linear basis for S^4_θ is given by the elements

$$Z^j(Z^*)^kW^l(W^*)^mT^\epsilon$$

for $j, k, l, m \in \{0, 1, 2, \dots\}$ and $\epsilon \in \{0, 1\}$ (where, e.g., higher powers of T are eliminated by using the relation $T^2 = \mathbb{1} - ZZ^* - WW^*$). For convenience, let us introduce the multi-index notation $I = (j, k, l, m, \epsilon)$ and

$$e^I = Z^j(Z^*)^k W^l (W^*)^m T^\epsilon$$

such that, in this notation, every element $a \in S^4_\theta$ can uniquely be written as

$$a = \sum_I a_I e^I$$

with $a_I \in \mathbb{C}$. It is useful to develop the multi-index notation a bit further. Namely, for $I = (j, k, l, m, \epsilon)$ we write $I = (\hat{I}, \epsilon)$ with $\hat{I} = (j, k, l, m)$. Furthermore, we introduce

$$1_Z = (1, 1, 0, 0, 0) = (\hat{1}_Z, 0) \quad \text{and} \quad 1_W = (0, 0, 1, 1, 0) = (\hat{1}_W, 0),$$

and we write $I+J$ for component-wise addition of multi-indices. Let us now state the result of multiplying two basis elements in the following lemma:

Lemma 3.1. *If $I_1 = (j_1, k_1, l_1, m_1, \epsilon_1)$ and $I_2 = (j_2, k_2, l_2, m_2, \epsilon_2)$ then*

$$e^{I_1}e^{I_2} = \begin{cases} q^{(l_1-m_1)(j_2-k_2)} e^{I_1+I_2} & \text{if } \epsilon_1 + \epsilon_2 \leq 1 \\ q^{(l_1-m_1)(j_2-k_2)} \big(e^{(\hat{I}_1+\hat{I}_2,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_Z,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_W,0)}\big) & \text{if } \epsilon_1 + \epsilon_2 = 2. \end{cases}$$

Proof. Using (3.1) one obtains

$$\begin{aligned} e^I e^J &= Z^{j_1}(Z^*)^{k_1} W^{l_1}(W^*)^{m_1} T^{\epsilon_1} Z^{j_2}(Z^*)^{k_2} W^{l_2}(W^*)^{m_2} T^{\epsilon_2} \\ &= q^{j_2(l_1-m_1)} Z^{j_1+j_2}(Z^*)^{k_1} W^{l_1}(W^*)^{m_1}(Z^*)^{k_2} W^{l_2}(W^*)^{m_2} T^{\epsilon_1+\epsilon_2} \\ &= q^{j_2(l_1-m_1)} q^{k_2(m_1-l_1)} Z^{j_1+j_2}(Z^*)^{k_1+k_2} W^{l_1}(W^*)^{m_1} W^{l_2}(W^*)^{m_2} T^{\epsilon_1+\epsilon_2} \\ &= q^{(l_1-m_1)(j_2-k_2)} Z^{j_1+j_2}(Z^*)^{k_1+k_2} W^{l_1+l_2}(W^*)^{m_1+m_2} T^{\epsilon_1+\epsilon_2}. \end{aligned}$$

Now, if $\epsilon_1 + \epsilon_2 \leq 1$ then the statement in the lemma is proved. If $\epsilon_1 + \epsilon_2 = 2$, then the statement follows after using that $T^2 = \mathbb{1} - ZZ^* - WW^*$, and the fact that both ZZ^* and WW^* are central. □

Let us now proceed to state a few properties of S^4_θ that we shall need in the following.

Proposition 3.2. *The elements ZZ^*, WW^* and $\mathbb{1} - T^2$ are regular (i.e. none of them is a zero divisor).*

Proof. Let us first prove that ZZ^* is not a zero divisor. Thus, let a be an element of S^4_θ, given as

$$a = \sum_I a_I e^I$$

and compute (by using Lemma 3.1)

$$ZZ^* a = \sum_I a_I e^{1_Z} e^I = \sum_I q^{(0-0)(j-k)} a_I e^{I+1_Z} = \sum_I a_I e^{I+1_Z}.$$

Clearly, setting $ZZ^*a = 0$ gives $a_I = 0$ for all I since $\{e^I\}$ is a basis for S^4_θ. Similarly, we consider

$$WW^* a = \sum_I a_I e^{1_W} e^I = \sum_I q^{(1-1)(j-k)} e^{I+1_W} = \sum_I a_I e^{I+1_W}$$

and conclude that $WW^*a = 0$ gives $a = 0$. Finally, we compute

$$\begin{aligned} (\mathbb{1} - T^2)a = (|Z|^2 + |W|^2)a &= \sum_I a_I \big(e^{1_Z} + e^{1_W}\big) e^I = \sum_I a_I e^{I+1_Z} + \sum_I a_I e^{I+1_W} \\ &= \sum_{j=0,\, l,m\geq 1} a_{I-1_W} e^I + \sum_{k=0,\, j,l,m\geq 1} a_{I-1_W} e^I + \sum_{l=0,\, j,k\geq 1} a_{I-1_Z} e^I \\ &\quad + \sum_{m=0,\, j,k,l\geq 1} a_{I-1_Z} e^I + \sum_{j,k,l,m\geq 1} \big(a_{I-1_Z} + a_{I-1_W}\big) e^I. \end{aligned}$$

Note that in the above expression, every basis element appears at most once. Therefore, setting $(\mathbb{1} - T^2)a = 0$ immediately gives $a_{j,k,l,m,\epsilon} = 0$ if at least one of j, k, l, m is zero. If $j, k, l, m \geq 1$ one gets

$$a_{I-(0,0,1,1,0)} = -a_{I-(1,1,0,0,0)} \quad \Rightarrow \quad a_I = -a_{I+(1,1,-1,-1)},$$

which, by iteration, gives

$$a_I = (-1)^n a_{I+(n,n,-n,-n)} \quad \text{for } 0 \le n \le \min(l, m).$$

Hence, since $a_{j,k,l,m,\epsilon} = 0$ if at least one of j, k, l, m is zero, one concludes that

$$a_{(j,k,l,m,\epsilon)} = \begin{cases} (-1)^l a_{j+l,k+l,0,m-l,\epsilon} = 0 \text{ if } l \le m \\ (-1)^m a_{j+m,k+m,l-m,0,\epsilon} = 0 \text{ if } l \ge m \end{cases}$$

which, together with the previous observation, shows that $a = 0$. □

We have already noted that ZZ^*, WW^* and T are central elements. The next results show that if θ is an irrational number, then these elements generate the center of S^4_θ.

Proposition 3.3. *If θ is irrational then $Z(S^4_\theta)$ is generated by ZZ^*, WW^* and T. That is, every $a \in Z(S^4_\theta)$ can be uniquely written as*

$$a = \sum_{j,k,\epsilon} a_{jk\epsilon} (ZZ^*)^j (WW^*)^k T^\epsilon$$

where $a_{jk\epsilon} \in \mathbb{C}$, $j, k \in \{0, 1, 2, \ldots\}$ and $\epsilon \in \{0, 1\}$.

Proof. Let a be an arbitrary (nonzero) central element of S^4_θ and write

$$a = \sum_I a_I e^I.$$

In particular, a has to commute with Z, and one computes

$$[a, Z] = \sum_I a_I \left(e^I e^{(1,0,0,0,0)} - e^{(1,0,0,0,0)} e^I \right) = \sum_I a_I (q^{l-m} - 1) e^{I+(1,0,0,0,0)}.$$

Demanding that $[a, Z] = 0$ gives $(q^{l-m} - 1)a_I = 0$. If $a \neq 0$, there exists an I such that $a_I \neq 0$, which implies that $q^{l-m} = 1$. Since θ is assumed to be irrational it follows that $l = m$. Similarly, if a commutes with W then

$$0 = [a, W] = \sum_I a_I \left(e^I e^{(0,0,1,0,0)} - e^{(0,0,1,0,0)} e^I \right) = \sum_I a_I \left(1 - q^{j-k}\right) e^{I+(0,0,1,0,0)}$$

giving $j = k$ in analogy with the previous case. Thus, an element $a \in Z(S^4_\theta)$ must be of the following form

$$a = \sum_{j,k,\epsilon} a_{j,k,\epsilon} \left(ZZ^*\right)^j \left(WW^*\right)^k T^\epsilon,$$

and it is clear that any element of the above form is in $Z(S^4_\theta)$ since ZZ^*, WW^* and T are central. □

Remark 3.4. Note that Proposition 3.3 does not hold if θ is rational. For instance, if $q^N = 1$ then both Z^N and W^N are central elements.

Let us introduce

$$\begin{aligned} X^1 &= \frac{1}{2}(Z + Z^*) & X^2 &= \frac{1}{2i}(Z - Z^*) \\ X^3 &= \frac{1}{2}(W + W^*) & X^4 &= \frac{1}{2i}(W - W^*) \\ |Z|^2 &= ZZ^* \quad |W|^2 = WW^* & X^5 &= T, \end{aligned}$$

and note that $|Z|^2 = (X^1)^2 + (X^2)^2$ and $|W|^2 = (X^3)^2 + (X^4)^2$, as well as

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 = |Z|^2 + |W|^2 + T^2 = \mathbb{1}.$$

Moreover, the normality of Z and W is equivalent to $[X^1, X^2] = [X^3, X^4] = 0$. Next, let us show that there exist noncommutative analogues of the four derivations appearing in (2.1).

Proposition 3.5. *There exist hermitian derivations $\tilde{\partial}_1$, $\tilde{\partial}_2$, $\tilde{\partial}_3$, $\tilde{\partial}_4$ such that*

$$\begin{aligned} &\tilde{\partial}_1 Z = iZ & &\tilde{\partial}_1 W = 0 & &\tilde{\partial}_1 T = 0 \\ &\tilde{\partial}_2 Z = 0 & &\tilde{\partial}_2 W = iW & &\tilde{\partial}_2 T = 0 \\ &\tilde{\partial}_3 Z = Z|W|^2 & &\tilde{\partial}_3 W = -W|Z|^2 & &\tilde{\partial}_3 T = 0 \\ &\tilde{\partial}_4 Z = ZT & &\tilde{\partial}_4 W = WT & &\tilde{\partial}_4 T = T^2 - \mathbb{1}, \end{aligned}$$

and it follows that

$$
\begin{aligned}
&[\tilde\partial_1,\tilde\partial_2]=[\tilde\partial_1,\tilde\partial_3]=[\tilde\partial_1,\tilde\partial_4]=0\\
&[\tilde\partial_2,\tilde\partial_3]=[\tilde\partial_2,\tilde\partial_4]=0\\
&[\tilde\partial_3,\tilde\partial_4]=-2T\tilde\partial_3.
\end{aligned}
$$

Proof. If the derivations exist, the relations given above (together with the fact that they are hermitian derivations), completely determine their actions via Leibniz' rule. However, for these derivations to be well-defined, one has to check that they respect the defining relations (3.1) of S^4_θ. For instance

$$
\begin{aligned}
\tilde\partial_1(WZ-qZW)&=(\tilde\partial_1W)Z+W(\tilde\partial_1Z)-q(\tilde\partial_1Z)W-qZ(\tilde\partial_1W)\\
&=iWZ-iqZW=i(WZ-qZW)=0,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde\partial_3(WZ-qZW)&=(\tilde\partial_3W)Z+W(\tilde\partial_3Z)-q(\tilde\partial_3Z)W-qZ(\tilde\partial_3W)\\
&=-W|Z|^2Z+WZ|W|^2-qZ|W|^2W+qZW|Z|^2\\
&=(WZ-qZW)|W|^2-(WZ-qZW)|Z|^2=0
\end{aligned}
$$

(using that $|Z|^2$ and $|W|^2$ are central). In this way, relations (3.1) can be checked for the derivations $\tilde\partial_1,\tilde\partial_2,\tilde\partial_3,\tilde\partial_4$. □

3.2. A real metric calculus over S^4_θ

In this section, we shall introduce a differential calculus over S^4_θ in close analogy with the classical parametrization in Section 2.2. The calculus will be constructed in the framework of pseudo-Riemannian calculi, as developed in [13], and briefly reviewed in Section 2.1.

To this end, we introduce four elements of the free (right) module $(S^4_\theta)^5$ that correspond to the classical vector fields e_1,e_2,e_3,e_4 in Section 2.2. However, in order to properly define a connection, one needs to slightly rescale e_1 and e_2. Thus, we consider the following elements of $(S^4_\theta)^5$:

$$
\begin{aligned}
E_1&=(-X^2(\mathbb{1}-T^2),X^1(\mathbb{1}-T^2),0,0,0)\\
E_2&=(0,0,-X^4(\mathbb{1}-T^2),X^3(\mathbb{1}-T^2),0)\\
E_3&=(X^1|W|^2,X^2|W|^2,-X^3|Z|^2,-X^4|Z|^2,0)\\
E_4&=(X^1T,X^2T,X^3T,X^4T,T^2-\mathbb{1}),
\end{aligned}
$$

and let M be the submodule of $(S^4_\theta)^5$ generated by $\{E_1,E_2,E_3,E_4\}$. Note that there are no ordering ambiguities when defining these elements, since $|Z|^2$, $|W|^2$ and T are central. This module is the analogue of the local vector fields over the chart U_0, and the corresponding local triviality is reflected in the following result.

Proposition 3.6. *The module $M=\{E_1a+E_2b+E_3c+E_4d : a,b,c,d\in S^4_\theta\}$ is a free (right) S^4_θ-module of rank 4, and $\{E_1,E_2,E_3,E_4\}$ is a basis for M.*

Proof. By definition, $\{E_1,E_2,E_3,E_4\}$ generates M. To prove that $\{E_1,E_2,E_3,E_4\}$ is a basis, we assume that

$$E_1a+E_2b+E_3c+E_4d=0 \tag{3.2}$$

and show that this implies that $a=b=c=d=0$. Relation (3.2) is equivalent to the equations

$$
\begin{aligned}
-X^2(\mathbb{1}-T^2)a+X^1|W|^2c+X^1Td&=0\\
X^1(\mathbb{1}-T^2)a+X^2|W|^2c+X^2Td&=0\\
-X^4(\mathbb{1}-T^2)b-X^3|Z|^2c+X^3Td&=0\\
X^3(\mathbb{1}-T^2)b-X^4|Z|^2c+X^4Td&=0\\
(\mathbb{1}-T^2)d&=0,
\end{aligned}
$$

which immediately implies that $d=0$ (since $\mathbb{1}-T^2$ is not a zero divisor by Proposition 3.2), and the remaining equations may be written as

$$-X^2(\mathbb{1}-T^2)a+X^1|W|^2c=0 \tag{3.3}$$
$$X^1(\mathbb{1}-T^2)a+X^2|W|^2c=0 \tag{3.4}$$
$$-X^4(\mathbb{1}-T^2)b-X^3|Z|^2c=0 \tag{3.5}$$
$$X^3(\mathbb{1}-T^2)b-X^4|Z|^2c=0. \tag{3.6}$$

The sum of (3.3), multiplied from the left with X^1, and (3.4), multiplied from the left by X^2 gives

$$\big((X^1)^2+(X^2)^2\big)|W|^2c=|Z|^2|W|^2c=0,$$

(using that $[X^1,X^2]=0$) which implies that $c=0$ since neither $|Z|^2$ nor $|W|^2$ is a zero divisor (by Proposition 3.2). Hence, one is left with the equations

$$\begin{aligned}&X^2(\mathbb{1}-T^2)a=0 \qquad X^1(\mathbb{1}-T^2)a=0\\&X^4(\mathbb{1}-T^2)b=0 \qquad X^3(\mathbb{1}-T^2)b=0,\end{aligned}$$

and since $\mathbb{1}-T^2$ is not a zero divisor one obtains

$$\begin{aligned}&X^2a=0 \qquad X^1a=0\\&X^4b=0 \qquad X^3b=0,\end{aligned}$$

giving

$$\begin{aligned}&\big((X^1)^2+(X^2)^2\big)a=|Z|^2a=0\\&\big((X^3)^2+(X^4)^2\big)b=|W|^2b=0,\end{aligned}$$

which implies that $a=b=0$. Thus, we have shown that $E_1a+E_2b+E_3c+E_4d=0$ necessarily gives $a=b=c=d=0$, which proves that $\{E_1,E_2,E_3,E_4\}$ is indeed a basis for M. $\square$

In the module M, we introduce the restriction of the canonical metric on $(S^4_\theta)^5$:

$$h(U,V)=\sum_{a,b=1}^{4}(U^a)^*h_{ab}V^b$$

for $U=E_aU^a$ and $V=E_bV^b$, where

$$h_{ab}=\sum_{i=1}^{5}(E_a^i)^*(E_b^i),$$

giving

$$(h_{ab})=\begin{pmatrix}|Z|^2(\mathbb{1}-T^2)^2 & 0 & 0 & 0\\ 0 & |W|^2(\mathbb{1}-T^2)^2 & 0 & 0\\ 0 & 0 & |Z|^2|W|^2(\mathbb{1}-T^2) & 0\\ 0 & 0 & 0 & \mathbb{1}-T^2\end{pmatrix}.$$

As we shall be interested in perturbations of the standard metric, we introduce

$$h^\delta=\delta h$$

where $\delta\in S^4_\theta$ is assumed to be a hermitian, central and regular element. Since h^δ is diagonal, and each diagonal element is regular, it follows immediately that h^δ is non-degenerate on M; i.e.

$$h(U,V)=0 \text{ for all } V\in M \quad\Rightarrow\quad U=0.$$

Thus, the pair (M,h^δ) is a metric module (cf. Definition 2.1). To construct a real metric calculus over (M,h^δ) (cf. Definition 2.2), we need to associate derivations to E_1,E_2,E_3,E_4. In analogy with the classical situation, we consider the following derivations

$$\begin{aligned}&\partial_1=(\mathbb{1}-T^2)\tilde\partial_1 \quad \partial_2=(\mathbb{1}-T^2)\tilde\partial_2\\&\partial_3=\tilde\partial_3 \qquad\qquad\quad \partial_4=\tilde\partial_4,\end{aligned}$$

with $\tilde\partial_1,\tilde\partial_2,\tilde\partial_3,\tilde\partial_4$ given as in Proposition 3.5. (Note that ∂_1 and ∂_2 are derivations since $\mathbb{1}-T^2$ is central.) These derivations generate an infinite-dimensional Lie algebra.

Proposition 3.7. *For $n\in\mathbb{N}_0$, the hermitian derivations*

$$\partial_1^{(n)}=T^n(\mathbb{1}-T^2)\tilde\partial_1,\quad \partial_2^{(n)}=T^n(\mathbb{1}-T^2)\tilde\partial_2,\quad \partial_3^{(n)}=T^n\tilde\partial_3,\quad \partial_4=\tilde\partial_4$$

span an infinite-dimensional Lie algebra, where

$$\begin{aligned}&[\partial_1^{(n)},\partial_2^{(n)}]=[\partial_1^{(n)},\partial_3^{(n)}]=[\partial_2^{(n)},\partial_3^{(n)}]=0\\&[\partial_4,\partial_i^{(n)}]=(n+2)\partial_i^{(n+1)}-n\partial_i^{(n-1)},\end{aligned}$$

for $i = 1, 2, 3$ *(with the convention that* $n\partial_i^{(n-1)} = 0$ *if* $n = 0$*). Moreover, it follows that*

$$\begin{array}{lll}
\partial_1|Z|^2 = 0 & \partial_1|W|^2 = 0 & \partial_1(\mathbb{1} - T^2) = 0 \\
\partial_2|Z|^2 = 0 & \partial_2|W|^2 = 0 & \partial_2(\mathbb{1} - T^2) = 0 \\
\partial_3|Z|^2 = 2|Z|^2|W|^2 & \partial_3|W|^2 = -2|Z|^2|W|^2 & \partial_3(\mathbb{1} - T^2) = 0 \\
\partial_4|Z|^2 = 2|Z|^2T & \partial_4|W|^2 = 2|W|^2T & \partial_4(\mathbb{1} - T^2) = 2T(\mathbb{1} - T^2),
\end{array}$$

where $\partial_i \equiv \partial_i^{(0)}$ *for* $i = 1, 2, 3$.

Proof. The proof consists of straight-forward computations using the definition of $\tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3, \tilde{\partial}_4$ in Proposition 3.5. □

We let $\mathfrak{g}$ denote the (real) Lie algebra spanned by $\partial_1^{(n)}, \partial_2^{(n)}, \partial_3^{(n)}, \partial_4$, and let $\varphi : \mathfrak{g} \to M$ be the $\mathbb{R}$-linear map defined by

$$\varphi(\partial_i^{(n)}) = E_iT^n \quad \text{for } i = 1, 2, 3,$$
$$\varphi(\partial_4) = E_4.$$

The pair $(\mathfrak{g}, \varphi)$ is denoted by $\mathfrak{g}_\varphi$.

Proposition 3.8. *The triple* $(M, h^\delta, \mathfrak{g}_\varphi)$ *is a real metric calculus over* S^4_θ.

Proof. As already noted, the metric h^δ is non-degenerate on M and, by definition, $\{E_1, E_2, E_3, E_4\}$ generates M, which implies that the image of φ generates M. Finally, since every component of h^δ is hermitian, it follows that $h^\delta(E, E')$ is hermitian for all E, E' in the image of φ. This shows that the triple $(M, h^\delta, \mathfrak{g}_\varphi)$ satisfies all the requirements of a real metric calculus. □

Given a real metric calculus $(M, h^\delta, \mathfrak{g}_\varphi)$, there exists at most one metric and torsion-free connection on the module M (cf. Theorem 2.5). In Section 3.4 we proceed to show that such a connection exists, but let us first discuss certain aspects of localization on S^4_θ.

3.3. The local algebra $S^4_{\theta,\mathrm{loc}}$

For the classical 4-sphere, the vector fields corresponding to E_1, E_2, E_3, E_4 are linearly independent in the chart given in Section 2.2. Thus, as already mentioned, the module M does not correspond to the module of vector fields of S^4, but rather to a local trivialization in the chart U_0. In this chart, the functions $|w|^2$, $|z|^2$ and $1 - t^2$ are invertible, and in analogy with this situation we shall introduce a localization of the algebra S^4_θ in order to be able to perform computations in a "noncommutative chart". Moreover, let us also consider the inverse of $\mathbb{1} + T^2$ (which is globally invertible in the classical setting) as it is an algebraic prototype of the kind of perturbations of the metric that we will consider. To this end, we let S be the multiplicative subset of S^4_θ generated by $\mathbb{1}$, $|Z|^2$, $|W|^2$, $\mathbb{1} - T^2$ and $\mathbb{1} + T^2$. Since every element of S is central, S trivially fulfills the left (and right) Ore condition [23]. Hence, the localization of S^4_θ at S exists, and we denote it by $S^4_{\theta,\mathrm{loc}}$. In other words, $S^4_{\theta,\mathrm{loc}}$ is constructed from S^4_θ by adding the formal inverses of $|Z|^2$, $|W|^2$, $\mathbb{1} - T^2$ and $\mathbb{1} + T^2$. Clearly, $(M, h^\delta, \mathfrak{g}_\varphi)$, as constructed above, is also a real metric calculus over $S^4_{\theta,\mathrm{loc}}$. In what follows, we shall discuss the two algebras in parallel.

Let us take a closer look at the structure of the noncommutative localization we have introduced. The algebra S^4_θ has been localized to include elements, which are classically not globally defined, and the corresponding free module M has been defined, which we claim to be the local trivialization of the module of vector fields. Now, is there a global module of vector fields, for which M is a localization? For the noncommutative 4-sphere, a particular projective module presents itself as a natural candidate. Let $\{e_i\}_{i=1}^5$ denote the standard basis of the free (right) S^4_θ-module $(S^4_\theta)^5$. The endomorphism algebra of $(S^4_\theta)^5$ is $M_5(S^4_\theta)$, which implies that an endomorphism can be given in terms of matrix elements with respect to the standard basis. By defining $\mathcal{P}$ to be the endomorphism given by the matrix entries $\mathcal{P}^{ij} = \delta^{ij}\mathbb{1} - X^iX^j$, one may easily check that $\mathcal{P}$ is a projection; i.e.

$$\begin{aligned}
(\mathcal{P}^2)^{ij} &= \sum_{k=1}^{5} \mathcal{P}^{ik}\mathcal{P}^{kj} = \sum_{k=1}^{5} (\delta^{ik}\mathbb{1} - X^iX^k)(\delta^{kj}\mathbb{1} - X^kX^j) \\
&= \delta^{ij}\mathbb{1} - X^iX^j - X^iX^j + \sum_{k=1}^{5} X^i(X^k)^2X^j = \delta^{ij}\mathbb{1} - X^iX^j = \mathcal{P}^{ij}
\end{aligned}$$

due to the fact that

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 = \mathbb{1}.$$

Let us denote the image of $\mathcal{P}$ by TS^4_θ, which is, by definition, a finitely generated projective module. In classical geometry, $\mathcal{P}$ is the projector that defines the module of vector fields on S^4. Let us now show that, over the local algebra $S^4_{\theta,\mathrm{loc}}$, this module is isomorphic to the module of the real metric calculus we have previously constructed. As defined above, let S be the multiplicative set generated by $\mathbb{1}$, $|Z|^2$, $|W|^2$, $\mathbb{1} - T^2$, $\mathbb{1} + T^2$.

Proposition 3.9. *The localizations, with respect to the multiplicative set S, of TS^4_θ and M are isomorphic as right $S^4_{\theta,\mathrm{loc}}$-modules.*

Proof. First of all, it is easy to check that $E_1, E_2, E_3, E_4 \in TS^4_\theta$; for instance,

$$\sum_{i=1}^{5} X^i E_1^i = X^1(-X^2) + X^2X^1 = 0,$$

since $[X^1, X^2] = 0$, which implies that $\mathcal{P}(E_1) = E_1$ and $E_1 \in TS^4_\theta$. Thus, it follows that $M \subseteq TS^4_\theta$. Next, we will show that $TS^4_\theta \subseteq M$, by explicitly writing $\mathcal{P}(e_i)$ (for $i = 1, 2, 3, 4, 5$) as linear combinations of E_1, E_2, E_3, E_4. Since $\{\mathcal{P}(e_i)\}_{i=1}^5$ generates TS^4_θ, this shows that every element of TS^4_θ can be written in terms of E_1, E_2, E_3, E_4. We claim that

$$\begin{aligned}
\mathcal{P}(e_1) &= -E_1X^2|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_3X^1|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_4X^1T(\mathbb{1} - T^2)^{-1}\\
\mathcal{P}(e_2) &= E_1X^1|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_3X^2|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_4X^2T(\mathbb{1} - T^2)^{-1}\\
\mathcal{P}(e_3) &= -E_2X^4|W|^{-2}(\mathbb{1} - T^2)^{-1} - E_3X^3|W|^{-2}(\mathbb{1} - T^2)^{-1} + E_4X^3T(\mathbb{1} - T^2)^{-1}\\
\mathcal{P}(e_4) &= E_2X^3|W|^{-2}(\mathbb{1} - T^2)^{-1} - E_3X^4|W|^{-2}(\mathbb{1} - T^2)^{-1} + E_4X^4T(\mathbb{1} - T^2)^{-1}\\
\mathcal{P}(e_5) &= -E_4.
\end{aligned}$$

Let us show that $\mathcal{P}(e_1)$ can be written as the linear combination given above. The proof of the other four identities is analogous. First, one checks that

$$\mathcal{P}(e_1) = \big(\mathbb{1} - (X^1)^2, -X^2X^1, -X^3X^1, -X^4X^1, -X^5X^1\big).$$

Next, write

$$\begin{aligned}
U &= -E_1X^2|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_3X^1|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_4X^1T(\mathbb{1} - T^2)^{-1}\\
&= (U^1, U^2, U^3, U^4, U^5),
\end{aligned}$$

and compute the components one by one

$$\begin{aligned}
U^1 &= (X^2)^2|Z|^{-2} + (X^1)^2|W|^2|Z|^{-2}(\mathbb{1} - T^2)^{-1} + (X^1)^2T^2(\mathbb{1} - T^2)^{-1}\\
&= (X^2)^2|Z|^{-2} + (X^1)^2|W|^2|Z|^{-2}(\mathbb{1} - T^2)^{-1}\\
&\qquad - (X^1)^2(\mathbb{1} - T^2)(\mathbb{1} - T^2)^{-1} + (X^1)^2(\mathbb{1} - T^2)^{-1}\\
&= -(X^1)^2 + |Z|^{-2}(\mathbb{1} - T^2)^{-1}\Big((X^2)^2(\mathbb{1} - T^2) + (X^1)^2(|Z|^2 + |W|^2)\Big)\\
&\quad (\text{using } |Z|^2 + |W|^2 + T^2 = \mathbb{1})\\
&= -(X^1)^2 + |Z|^{-2}(\mathbb{1} - T^2)^{-1}\Big((X^2)^2(\mathbb{1} - T^2) + (X^1)^2(\mathbb{1} - T^2)\Big)\\
&= -(X^1)^2 + |Z|^{-2}\big((X^1)^2 + (X^2)^2\big) = \mathbb{1} - (X^1)^2,\\
U^2 &= -X^1X^2|Z|^{-2} + X^2X^1|W|^2|Z|^{-2}(\mathbb{1} - T^2)^{-1} + X^2X^1T^2(\mathbb{1} - T^2)^{-1}\\
&\quad (\text{using } [X^1, X^2] = 0)\\
&= -X^2X^1|Z|^{-2}(\mathbb{1} - T^2)^{-1}\big(\mathbb{1} - T^2 - |W|^2\big) + X^2X^1T^2(\mathbb{1} - T^2)^{-1}\\
&= -X^2X^1|Z|^{-2}(\mathbb{1} - T^2)^{-1}|Z|^2 + X^2X^1T^2(\mathbb{1} - T^2)^{-1}\\
&= -X^2X^1(\mathbb{1} - T^2)^{-1}\big(\mathbb{1} - T^2\big) = -X^2X^1,\\
U^3 &= -X^3X^1(\mathbb{1} - T^2)^{-1} + X^3X^1T^2(\mathbb{1} - T^2)^{-1}\\
&= -X^3X^1(\mathbb{1} - T^2)^{-1}(\mathbb{1} - T^2) = -X^3X^1,\\
U^4 &= -X^4X^1(\mathbb{1} - T^2)^{-1} + X^4X^1T^2(\mathbb{1} - T^2)^{-1}\\
&= -X^4X^1(\mathbb{1} - T^2)^{-1}(\mathbb{1} - T^2) = -X^4X^1,\\
U^5 &= (T^2 - \mathbb{1})X^1T(\mathbb{1} - T^2)^{-1} = -X^1T = -X^1X^5.
\end{aligned}$$

Thus, we have shown that

$$\mathcal{P}(e_1) = -E_1X^2|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_3X^1|Z|^{-2}(\mathbb{1} - T^2)^{-1} + E_4X^1T(\mathbb{1} - T^2)^{-1},$$

which, together with the other four analogous computations, shows that TS^4_θ is contained in M. Combined with the fact that $M \subseteq TS^4_\theta$ one can conclude that $TS^4_\theta = M$ as right $S^4_{\theta,\mathrm{loc}}$-modules. □

3.4. Pseudo-Riemannian calculus

To construct a connection ∇ on M, such that $(M, h^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus, we consider the following class of perturbations. Let us assume that

$$\partial_a\delta = 2\alpha_a\delta,$$

where $\alpha_a \in S^4_{\theta,\mathrm{loc}}$ is hermitian, for $a = 1, 2, 3, 4$. The connection will be constructed over $S^4_{\theta,\mathrm{loc}}$, but we shall see that perturbations in certain directions give connections over S^4_θ.

Proposition 3.10. *Let $\delta \in S^4_{\theta,\mathrm{loc}}$ be a hermitian, regular and central element, such that $\partial_a\delta = 2\alpha_a\delta$, for $a = 1, 2, 3, 4$, where $\alpha_a \in S^4_{\theta,\mathrm{loc}}$ and $\alpha_a^* = \alpha_a$. Then there exists a unique connection ∇, such that $(M, h^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus over $S^4_{\theta,\mathrm{loc}}$, and ∇ is given by*

$$\begin{aligned}
\nabla_1E_1 &= E_1\alpha_1 - E_2\alpha_2|Z|^2|W|^{-2} - E_3\big(\alpha_3|W|^{-2} + \mathbb{1}\big)(\mathbb{1} - T^2) - E_4(\alpha_4 + T)|Z|^2(\mathbb{1} - T^2)\\
\nabla_1E_2 &= \nabla_2E_1 = E_1\alpha_2 + E_2\alpha_1\\
\nabla_1E_3 &= \nabla_3E_1 = E_1(\alpha_3 + |W|^2) + E_3\alpha_1\\
\nabla_1E_4 &= E_1(\alpha_4 + T) + E_4\alpha_1\\
\nabla_4E_1 &= E_1(\alpha_4 + 3T) + E_4\alpha_1\\
\nabla_2E_2 &= -E_1\alpha_1|W|^2|Z|^{-2} + E_2\alpha_2 - E_3\big(\alpha_3|Z|^{-2} - \mathbb{1}\big)(\mathbb{1} - T^2) - E_4(\alpha_4 + T)|W|^2(\mathbb{1} - T^2)\\
\nabla_2E_3 &= \nabla_3E_2 = E_2(\alpha_3 - |Z|^2) + E_3\alpha_2\\
\nabla_2E_4 &= E_2(\alpha_4 + T) + E_4\alpha_2\\
\nabla_4E_2 &= E_2(\alpha_4 + 3T) + E_4\alpha_2\\
\nabla_3E_3 &= -E_1\alpha_1|W|^2(\mathbb{1} - T^2)^{-1} - E_2\alpha_2|Z|^2(\mathbb{1} - T^2)^{-1} + E_3(\alpha_3 + |W|^2 - |Z|^2) - E_4(\alpha_4 + T)|Z|^2|W|^2\\
\nabla_3E_4 &= E_3(\alpha_4 + T) + E_4\alpha_3\\
\nabla_4E_3 &= E_3(\alpha_4 + 3T) + E_4\alpha_3\\
\nabla_4E_4 &= -E_1\alpha_1|Z|^{-2}(\mathbb{1} - T^2)^{-1} - E_2\alpha_2|W|^{-2}(\mathbb{1} - T^2)^{-1} - E_3\alpha_3|Z|^{-2}|W|^{-2} + E_4(\alpha_4 + T),
\end{aligned}$$

and

$$\nabla_{\partial_i^{(n)}}E_a = \big(\nabla_iE_a\big)T^n$$

for $i = 1, 2, 3$, $a = 1, 2, 3, 4$, where $\nabla_a \equiv \nabla_{\partial_a}$.

Proof. Let us recall (cf. [13]) that Kozul's formula

$$\begin{aligned}
2h(\nabla_{d_1}E_2, E_3) &= d_1h(E_2, E_3) + d_2h(E_3, E_1) - d_3h(E_1, E_2)\\
&\quad - h\big(E_1, \varphi([d_2, d_3])\big) + h\big(E_2, \varphi([d_3, d_1])\big) + h\big(E_3, \varphi([d_1, d_2])\big),
\end{aligned} \tag{3.7}$$

where $E_1, E_2, E_3 \in M_\varphi$ and $d_1, d_2, d_3 \in \mathfrak{g}$, gives a straight-forward way of finding a connection on M such that $(M, h^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus. Namely, if one finds $U_{ab} \in M$ such that

$$\begin{aligned}
2h(U_{ab}, E_c) &= \partial_ah(E_b, E_c) + \partial_bh(E_a, E_c) - \partial_ch(E_a, E_b)\\
&\quad - h\big(E_a, \varphi([\partial_b, \partial_c])\big) + h\big(E_b, \varphi([\partial_c, \partial_a])\big) + h\big(E_c, \varphi([\partial_a, \partial_b])\big)
\end{aligned} \tag{3.8}$$

for all $a, b, c \in \{1, 2, 3, 4\}$ then (since the module M is free) one may set $\nabla_aE_b = U_{ab}$, and it follows that $(M, h^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus (see Corollary 3.8 in [13]). It is straight-forward to check that the expressions given in Proposition 3.10 fulfill (3.8). For instance, to check Kozul's formula for ∇_1E_1 one sets

$$K_a = h^\delta(\nabla_1E_1, E_a) - \partial_1h^\delta(E_1, E_a) + \frac{1}{2}\partial_ah^\delta(E_1, E_1) + h^\delta\big(E_1, \varphi([\partial_1, \partial_a])\big)$$

which gives

$$\begin{aligned}
K_1 &= h^\delta(\nabla_1E_1, E_1) - \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2\\
&= \alpha_1h^\delta(E_1, E_1) - \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2\\
&= \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2 - \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2 = 0,\\
K_2 &= h^\delta(\nabla_1E_1, E_2) + \alpha_2\delta|Z|^2(\mathbb{1} - T^2)^2\\
&= -\alpha_2|Z|^2|W|^{-2}h^\delta(E_2, E_2) + \alpha_2\delta|Z|^2(\mathbb{1} - T^2)^2\\
&= -\alpha_2|Z|^2|W|^{-2}|W|^2(\mathbb{1} - T^2)^2 + \alpha_2\delta|Z|^2(\mathbb{1} - T^2)^2 = 0,
\end{aligned}$$

$$
\begin{aligned}
K_3 &= h^\delta(\nabla_1 E_1, E_3) + \frac{1}{2}\partial_3\big(\delta|Z|^2(\mathbb{1}-T^2)^2\big)\\
&= -(\mathbb{1}+\alpha_3|W|^{-2})(\mathbb{1}-T^2)h^\delta(E_3,E_3) + \big(\alpha_3\delta|Z|^2 + \delta|Z|^2|W|^2\big)(\mathbb{1}-T^2)^2\\
&= -(\mathbb{1}+\alpha_3|W|^{-2})\delta|Z|^2|W|^2(\mathbb{1}-T^2)^2 + \big(\alpha_3\delta|Z|^2 + \delta|Z|^2|W|^2\big)(\mathbb{1}-T^2)^2 = 0,\\
K_4 &= h^\delta(\nabla_1 E_1, E_4) + \frac{1}{2}\partial_4\big(\delta|Z|^2(\mathbb{1}-T^2)^2\big) - h^\delta(E_1,E_1)2T\\
&= -(\alpha_4+T)\delta|Z|^2(\mathbb{1}-T^2)^2 + \big(\alpha_4+3T\big)\delta|Z|^2(\mathbb{1}-T^2)^2 - 2\delta|Z|^2T(\mathbb{1}-T^2)^2\\
&= 0.
\end{aligned}
$$

This shows that $\nabla_1 E_1$ satisfies Kozul's formula (3.8). The other connection components can be checked in an analogous way.

Let us now consider the claim that

$$\nabla_{\partial_i^{(n)}} E_a = \big(\nabla_i E_a\big)T^n.$$

This fact is easily derived from Kozul's formula. Namely, one notes that

$$\varphi\big([\partial_a, \partial_i^{(n)}]\big) = \varphi\big([\partial_a, \partial_i]\big)T^n + E_i(\partial_a T^n)$$

and computes using Kozul's formula:

$$
\begin{aligned}
2h^\delta\big(\nabla_{\partial_i^{(n)}} E_b, E_c\big) &= \big(\partial_i h^\delta(E_b,E_c)\big)T^n + \partial_b\big(h^\delta(E_c, E_iT^n)\big) - \partial_c\big(h^\delta(E_iT^n, E_b)\big)\\
&\quad - h^\delta\big(E_i, \varphi([\partial_b,\partial_c])\big)T^n + h^\delta\big(E_b, \varphi([\partial_c, \partial_i^{(n)}])\big) + h^\delta\big(E_c, \varphi([\partial_i^{(n)}, \partial_b])\big)\\
&= \big(\partial_i h^\delta_{bc} + \partial_b h^\delta_{ci} - \partial_c h^\delta_{ib}\big)T^n + h^\delta_{ci}(\partial_b T^n) - h^\delta_{ib}(\partial_c T^n) - h^\delta\big(E_i, \varphi([\partial_b,\partial_c])\big)T^n\\
&\quad + h^\delta\big(E_b, \varphi([\partial_c,\partial_i])\big)T^n + h^\delta_{bi}(\partial_c T^n) + h^\delta\big(E_c, \varphi([\partial_i,\partial_b])\big)T^n - h^\delta_{ci}(\partial_b T^n)\\
&= 2h^\delta\big(\nabla_{\partial_i}E_b, E_c\big)T^n = 2h^\delta\big((\nabla_{\partial_i}E_b)T^n, E_c\big),
\end{aligned}
$$

using that $h^\delta_{ab} = h^\delta_{ba}$ and the fact that T is hermitian and central. Since the metric is non-degenerate, it follows that

$$\nabla_{\partial_i^{(n)}} E_b = (\nabla_{\partial_i} E_b)T^n. \quad \square$$

Note that if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the connection in Proposition 3.10 only involves elements of S^4_θ and is therefore a valid connection for $(M, h^\delta, \mathfrak{g}_\varphi, \nabla)$ over S^4_θ. In particular, this is true for the unperturbed metric; i.e. for $\delta = \mathbb{1}$.

In Section 3.3 we constructed the projective module TS^4_θ and showed that it is isomorphic to M (as a right $S^4_{\theta,\text{loc}}$-module) in Proposition 3.9. As is well known, a projective module defined by a projector $\mathcal{P}$, admits a connection of the form

$$\bar{\nabla}_\partial U = \mathcal{P}\big(e_i \partial(U^i)\big)$$

which is compatible with the canonical metric on the free module. Thus, having argued that one may regard the module M as a localization of the (global) module TS^4_θ, it is natural to ask if the connection on TS^4_θ, defined in the above manner, coincides with the connection found in Proposition 3.10 for the unperturbed metric.

Proposition 3.11. *Let $U = e_i U^i$ be an element of TS^4_θ and set*

$$\bar{\nabla}_a U = \mathcal{P}\big(e_i \partial_a(U^i)\big),$$

for $a = 1, 2, 3, 4$. Then $\bar{\nabla}_a E_b = \nabla_a E_b$ for $a, b = 1, 2, 3, 4$ and $\delta = \mathbb{1}$.

Proof. Let us prove the statement by computing $\bar{\nabla}_a E_b$ for $a, b = 1, 2, 3, 4$ (i.e. 16 components in total) and compare it with Proposition 3.10 for $\delta = \mathbb{1}$. Since the calculations are straight-forward we shall only present one of them here to illustrate how they are performed. Thus,

$$
\begin{aligned}
\bar{\nabla}_1 E_1 &= \mathcal{P}\big(\partial_1(-X^2(\mathbb{1}-T^2), X^1(\mathbb{1}-T^2), 0, 0, 0)\big)\\
&= \mathcal{P}\big((-X^1, -X^2, 0, 0, 0)\big)(\mathbb{1}-T^2)^2\\
&= (-X^1, -X^2, 0, 0, 0)(\mathbb{1}-T^2)^2 - e_iX^i\big(-(X^1)^2 - (X^2)^2\big)(\mathbb{1}-T^2)^2\\
&= (-X^1, -X^2, 0, 0, 0)(\mathbb{1}-T^2)^2 + (X^1, X^2, X^3, X^4, T)|Z|^2(\mathbb{1}-T^2)^2\\
&= \big(X^1(|Z|^2-\mathbb{1}), X^2(|Z|^2-\mathbb{1}), X^3|Z|^2, X^4|Z|^2, T|Z|^2\big)(\mathbb{1}-T^2)^2.
\end{aligned}
$$

Now, for comparison, we find $\nabla_1 E_1$ from Proposition 3.10 when $\delta = \mathbb{1}$:

$$
\begin{aligned}
\nabla_1 E_1 &= -E_3(\mathbb{1}-T^2) - E_4 T|Z|^2(\mathbb{1}-T^2)\\
&= -(X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2, 0)(\mathbb{1}-T^2)\\
&\qquad - (X^1T, X^2T, X^3T, X^4T, T^2-\mathbb{1})T|Z|^2(\mathbb{1}-T^2),
\end{aligned}
$$

and, by using $|W|^2 + T^2|Z|^2 = \mathbb{1} - |Z|^2 - T^2 + T^2|Z|^2 = (\mathbb{1} - T^2)(\mathbb{1} - |Z|^2)$, one obtains

$$\nabla_1 E_1 = -\big(X^1(\mathbb{1} - |Z|^2), X^2(\mathbb{1} - |Z|^2), -X^3|Z|^2, -X^4|Z|^2, -T|Z|^2\big)(\mathbb{1} - T^2)^2$$

which equals $\bar{\nabla}_1 E_1$. The remaining computations are done in an analogous way. □

4. The Chern–Gauss–Bonnet theorem

4.1. The trace

Just as for the noncommutative torus, one may introduce a linear functional on S^4_θ corresponding to integration on the classical manifold. Namely, for a given basis element e^I with $I = (j, k, l, m, \epsilon)$ (in the notation of Section 3.1) one defines a linear map $\phi : S^4_\theta \to C^\infty(S^4)$ via

$$\phi(e^I) = e^{i(j-k)\xi_1}\big(\cos\varphi\cos\psi\big)^{j+k} e^{i(l-m)\xi_2}\big(\sin\varphi\cos\psi\big)^{l+m}(\sin\psi)^\epsilon$$

and

$$\tau(e^I) = \int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_{-\pi/2}^{\pi/2} d\psi \int_0^{\pi/2} d\varphi\, \phi(e^I)\sin\varphi\cos\varphi\cos^3\psi,$$

which are extended to S^4_θ as linear maps (cp. [24] for a similar approach in the unperturbed case). The volume element of the round metric g_0 on S^4 is given by $\sin\varphi\cos\varphi\cos^3\psi\, d\xi_1 d\xi_2 d\psi d\varphi$ and for the perturbed metric δg_0 one obtains

$$dV = \delta^2 \sin\varphi\cos\varphi\cos^3\psi\, d\xi_1 d\xi_2 d\psi d\varphi.$$

In order to reflect the fact that one would like to integrate with respect to the perturbed metric, we introduce

$$\tau_\delta(a) = \tau\big(\delta a \delta\big).$$

Let us note a few properties of the linear functional τ_δ. We start with the following lemma:

Lemma 4.1. *Assume that $\theta \notin \mathbb{Q}$ and $\delta \in Z(S^4_\theta)$. If $e^I \notin Z(S^4_\theta)$ then $\tau_\delta(e^I) = 0$.*

Proof. Let us start by considering $\tau_\delta(e^I)$ when $I = (j, k, l, m, 0)$. Assuming that $\delta \in Z(S^4_\theta)$ and $\theta \notin \mathbb{Q}$, one may write

$$\delta^2 = \sum_{i_1 i_2 \epsilon} a_{i_1 i_2 \epsilon}(|Z|^2)^{i_1}(|W|^2)^{i_2}T^\epsilon$$

by Proposition 3.3, and

$$\begin{aligned}\tau_\delta(e^I) &= \sum_{i_1 i_2 \epsilon} a_{i_1 i_2 \epsilon}\tau\big(e^{(j,k,l,m,0)}(|Z|^2)^{i_1}(|W|^2)^{i_2}T^\epsilon\big)\\ &= \sum_{i_1 i_2 \epsilon} a_{i_1 i_2 \epsilon}\tau\big(e^{(j+i_1,k+i_1,l+i_2,m+i_2,\epsilon)}\big).\end{aligned}$$

Since

$$\int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 e^{ik_1\xi_1}e^{ik_2\xi_2} = \begin{cases}4\pi^2 \text{ if } k_1 = k_2 = 0,\\ 0 \text{ otherwise,}\end{cases}$$

we conclude that $\tau_\delta(e^{(j,k,l,m,0)}) = 0$ if $j \neq k$ or $l \neq m$, which is equivalent to $e^{(j,k,l,m,0)} \notin Z(S^4_\theta)$. Similarly, for $I = (j, k, l, m, 1)$, terms proportional to $a_{i_1 i_2 1}$ are of the form

$$a_{i_1 i_2 1}\tau\Big(e^{(j+i_1,k+i_1,l+i_2,m+i_2,0)} - e^{(j+i_1+1,k+i_1+1,l+i_2,m+i_2,0)} - e^{(j+i_1,k+i_1,l+i_2+1,m+i_2+1,0)}\Big)$$

which, by using the same argument as above, implies that $\tau_\delta(e^{(j,k,l,m,1)}) = 0$ if $j \neq k$ or $l \neq m$. □

Proposition 4.2. *If $\delta \in Z(S^4_\theta)$ and $\theta \notin \mathbb{Q}$, then τ_δ satisfies*

(1) $\tau_\delta([a, b]) = 0$,
(2) $\tau_\delta(a^*) = \overline{\tau_\delta(a)}$,

for all $a, b \in S^4_\theta$.

Proof. To prove (1), we show that $\tau_\delta([e^{I_1}, e^{I_2}]) = 0$. By using Lemma 3.1 one obtains

$$\tau_\delta\big([e^{I_1}, e^{I_2}]\big) = \Big(q^{(l_1-m_1)(j_2-k_2)} - q^{(l_2-m_2)(j_1-k_1)}\Big)\tau_\delta\big(e^{I_1+I_2}\big)$$

if $\epsilon_1 + \epsilon_2 \leq 1$, and

$$\tau_\delta([e^{I_1}, e^{I_2}]) = \left(q^{(l_1-m_1)(j_2-k_2)} - q^{(l_2-m_2)(j_1-k_1)}\right) \times \left(e^{(\hat{I}_1+\hat{I}_2,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_Z,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_W,0)}\right) \tag{4.1}$$

if $\epsilon_1 + \epsilon_2 = 2$. From Lemma 4.1 it follows that if $j_1 + j_2 \neq k_1 + k_2$ or $l_1 + l_2 \neq m_1 + m_2$ then $\tau_\delta([e^{I_1}, e^{I_2}]) = 0$. On the other hand, if $j_1 + j_2 = k_1 + k_2$ and $l_1 + l_2 = m_1 + m_2$ then

$$(l_1 - m_1)(j_2 - k_2) = (l_2 - m_2)(j_1 - k_1)$$

which gives $\tau_\delta([e^{I_1}, e^{I_2}]) = 0$ from (4.1).

For (2), we again consider $a = \sum_I a_I e^I$ and find

$$\tau_\delta(a^*) = \sum_I \overline{a_I} \tau_\delta\left((e^I)^*\right) = \sum_I q^{(j-k)(l-m)} \overline{a_I} \tau_\delta(e^I).$$

Since $\tau_\delta(e^I) = 0$ if $j \neq k$ or $l \neq m$ (by Lemma 4.1), the above sum equals

$$\tau_\delta(a^*) = \sum_I \overline{a_I} \tau_\delta(e^I) = \overline{\tau_\delta(a)}$$

using that $\tau_\delta(e^I) \in \mathbb{R}$ when $j = k$ and $l = m$. $\square$

For the forthcoming discussion of the Chern–Gauss–Bonnet theorem, we extend τ_δ to the commutative subalgebra $Z_{\text{loc}} \subseteq S^4_{\theta,\text{loc}}$ given by

$$Z_{\text{loc}} = \mathbb{C}\left\langle \mathbb{1}, |Z|^2, |Z|^{-2}, |W|^2, |W|^{-2}, T, (\mathbb{1} - T^2)^{-1}, (\mathbb{1} + T^2)^{-1} \right\rangle,$$

by defining a homomorphism (of commutative $*$-algebras) $\phi_0 : Z_{\text{loc}} \to C^\infty(U_0)$ as

$$\begin{aligned} &\phi_0(|Z|^2) = \cos^2(\varphi)\cos^2(\psi) && \phi_0(|W|^2) = \sin^2(\varphi)\cos^2(\psi) \\ &\phi_0(\mathbb{1}) = 1 && \phi_0(T) = \sin(\psi) \end{aligned}$$

as well as

$$\begin{aligned} \phi_0\left((\mathbb{1} - T^2)^{-1}\right) &= \frac{1}{\cos^2(\psi)} = \frac{1}{\phi_0(\mathbb{1} - T^2)} \\ \phi_0\left((\mathbb{1} + T^2)^{-1}\right) &= \frac{1}{1 + \sin^2(\psi)} = \frac{1}{\phi_0(\mathbb{1} + T^2)} \\ \phi_0\left(|Z|^{-2}\right) &= \frac{1}{\cos^2(\varphi)\cos^2(\psi)} = \frac{1}{\phi_0(|Z|^2)} \\ \phi_0\left(|W|^{-2}\right) &= \frac{1}{\sin^2(\varphi)\cos^2(\psi)} = \frac{1}{\phi_0(|W|^2)}. \end{aligned}$$

For ϕ_0 to be well-defined, one needs to check that the above definition is compatible with the relations in Z_{loc}. The only nontrivial relation to check is

$$\begin{aligned} \phi_0(|Z|^2 + |W|^2 + T^2 - \mathbb{1}) &= \cos^2(\varphi)\cos^2(\psi) + \sin^2(\varphi)\cos^2(\psi) + \sin^2(\psi) - 1 \\ &= \cos^2(\psi) + \sin^2(\psi) - 1 = 0, \end{aligned}$$

which shows that ϕ_0 is indeed well-defined. Note that ϕ_0 coincides with ϕ on $Z(S^4_\theta)$. Finally, for $\delta \in Z_{\text{loc}}$, we define

$$\tau_{\delta,\text{loc}}(a) = \int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_{-\pi/2}^{\pi/2} d\psi \int_0^{\pi/2} d\varphi\, \phi_0(a)\phi_0(\delta^2) \cos^3\psi \sin\varphi \cos\varphi,$$

for $a \in Z_{\text{loc}}$, whenever the above integral is convergent. (For instance, the integral does not exists when $a = (\mathbb{1} - T^2)^{-2}$.)

4.2. The Chern–Gauss–Bonnet theorem

For a closed surface Σ, the Gauss–Bonnet theorem states that the integral of the Gaussian curvature over Σ is proportional to the Euler characteristic of Σ. This provides an important link between topology and Riemannian geometry. In particular, since the Euler characteristic is independent of any metric tensor, the integral gives the same value if we perturb the metric. This theorem has been generalized to closed even dimensional Riemannian manifolds, where the scalar curvature is replaced by the Pfaffian of the curvature form. In case of a closed four dimensional manifold M, the Chern–Gauss–Bonnet theorem states that

$$\chi(M) = \frac{1}{32\pi^2} \int_M \left(R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2\right) d\mu \tag{4.2}$$

where R_{abcd} is the Riemann curvature tensor, Ric_{ab} is the Ricci curvature, S denotes the scalar curvature and $\chi(M)$ is the Euler characteristic of M. (Recall that $\chi(S^4) = 2$.) In this section, we will show that there exists an analogue of the Chern–Gauss–Bonnet theorem for the pseudo-Riemannian calculus of S^4_θ we have developed. Our approach is based on the fact that all coefficients of the curvature tensor lie in the commutative subalgebra Z_{loc}, which allows us to compute directly the Pfaffian of the curvature form.

Let us consider a metric perturbation $\delta \in Z_{\mathrm{loc}}$ that is a polynomial in T, and such that δ is invertible in Z_{loc}. It follows that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (in the notation of Section 3.4), since $\partial_1 T = \partial_2 T = \partial_3 T = 0$. Moreover,

$$\partial_4\delta = \delta'(T)(\partial_4 T)\delta^{-1}\delta = -\delta'(T)(\mathbb{1} - T^2)\delta^{-1}\delta$$

where $\delta'(T)$ denotes the (formal) derivative of the polynomial $\delta(T)$ with respect to T, which implies that

$$\alpha \equiv \alpha_4 = -\frac{1}{2}(\mathbb{1} - T^2)\delta'\delta^{-1}.$$

An example of such a perturbation is given by $\delta = (\mathbb{1} + T^2)^N$ which gives

$$\alpha = -NT(\mathbb{1} - T^2)(\mathbb{1} + T^2)^{-1}.$$

Moreover, by α' we shall denote the (formal) derivative of $\alpha(T)$ with respect to T. For easy reference, let us recall the formulas from Proposition 3.10 in the situation where $\alpha_1 = \alpha_2 = \alpha_3 = 0$:

$$\begin{aligned}
&\nabla_1 E_1 = -E_3(\mathbb{1} - T^2) - E_4(\alpha + T)|Z|^2(\mathbb{1} - T^2)\\
&\nabla_2 E_2 = E_3(\mathbb{1} - T^2) - E_4(\alpha + T)|W|^2(\mathbb{1} - T^2)\\
&\nabla_3 E_3 = E_3(|W|^2 - |Z|^2) - E_4(\alpha + T)|Z|^2|W|^2\\
&\nabla_4 E_4 = E_4(\alpha + T)\\
&\nabla_1 E_2 = \nabla_2 E_1 = 0 \qquad \nabla_1 E_3 = \nabla_3 E_1 = E_1|W|^2\\
&\nabla_1 E_4 = E_1(\alpha + T) \qquad \nabla_4 E_1 = E_1(\alpha + 3T)\\
&\nabla_2 E_3 = -E_2|Z|^2 \qquad \nabla_3 E_2 = -E_2|Z|^2\\
&\nabla_2 E_4 = E_2(\alpha + T) \qquad \nabla_4 E_2 = E_2(\alpha + 3T)\\
&\nabla_3 E_4 = E_3(\alpha + T) \qquad \nabla_4 E_3 = E_3(\alpha + 3T).
\end{aligned}$$

It is now straight-forward to compute the curvature:

$$\begin{aligned}
&R(\partial_1, \partial_2)E_1 = -E_2\big(\mathbb{1} - (\alpha + T)^2\big)|Z|^2(\mathbb{1} - T^2)\\
&R(\partial_1, \partial_2)E_2 = E_1\big(\mathbb{1} - (\alpha + T)^2\big)|W|^2(\mathbb{1} - T^2)\\
&R(\partial_1, \partial_2)E_3 = 0 \qquad R(\partial_1, \partial_2)E_4 = 0\\
&R(\partial_1, \partial_3)E_1 = -E_3\big(\mathbb{1} - (\alpha + T)^2\big)|Z|^2(\mathbb{1} - T^2)\\
&R(\partial_1, \partial_3)E_3 = E_1\big(\mathbb{1} - (\alpha + T)^2\big)|Z|^2|W|^2\\
&R(\partial_1, \partial_3)E_2 = 0 \qquad R(\partial_1, \partial_3)E_4 = 0\\
&R(\partial_1, \partial_4)E_1 = -E_4\big(\mathbb{1} + \alpha'\big)|Z|^2(\mathbb{1} - T^2)^2\\
&R(\partial_1, \partial_4)E_4 = E_1(\mathbb{1} + \alpha')(\mathbb{1} - T^2)\\
&R(\partial_1, \partial_4)E_2 = 0 \qquad R(\partial_1, \partial_4)E_3 = 0\\
&R(\partial_2, \partial_3)E_2 = -E_3\big(\mathbb{1} - (\alpha + T)^2\big)|W|^2(\mathbb{1} - T^2)\\
&R(\partial_2, \partial_3)E_3 = E_2\big(\mathbb{1} - (\alpha + T)^2\big)|Z|^2|W|^2\\
&R(\partial_2, \partial_3)E_1 = 0 \qquad R(\partial_2, \partial_3)E_4 = 0\\
&R(\partial_2, \partial_4)E_2 = -E_4(\mathbb{1} + \alpha')|W|^2(\mathbb{1} - T^2)^2\\
&R(\partial_2, \partial_4)E_4 = E_2(\mathbb{1} + \alpha')(\mathbb{1} - T^2)\\
&R(\partial_2, \partial_4)E_1 = 0 \qquad R(\partial_2, \partial_4)E_3 = 0\\
&R(\partial_3, \partial_4)E_3 = -E_4(\mathbb{1} + \alpha')|Z|^2|W|^2(\mathbb{1} - T^2)\\
&R(\partial_3, \partial_4)E_4 = E_3(\mathbb{1} + \alpha')(\mathbb{1} - T^2)\\
&R(\partial_3, \partial_4)E_1 = 0 \qquad R(\partial_3, \partial_4)E_2 = 0
\end{aligned}$$

and the only non-zero curvature components $R_{abpq} = h^{\delta}(E_a, R(\partial_p, \partial_q)E_b)$ turn out to be

$$
\begin{aligned}
R_{1212} &= \delta\big(\mathbb{1} - (\alpha+T)^2\big)|Z|^2|W|^2(\mathbb{1}-T^2)^3\\
R_{1313} &= \delta\big(\mathbb{1} - (\alpha+T)^2\big)|Z|^4|W|^2(\mathbb{1}-T^2)^2\\
R_{1414} &= \delta(\mathbb{1}+\alpha')|Z|^2(\mathbb{1}-T^2)^3\\
R_{2323} &= \delta\big(\mathbb{1} - (\alpha+T)^2\big)|Z|^2|W|^4(\mathbb{1}-T^2)^2\\
R_{2424} &= \delta(\mathbb{1}+\alpha')|W|^2(\mathbb{1}-T^2)^3\\
R_{3434} &= \delta(\mathbb{1}+\alpha')|Z|^2|W|^2(\mathbb{1}-T^2)^2.
\end{aligned}
$$

In the local algebra Z_{loc}, the metric h^{δ} is invertible since δ is invertible. Moreover, every component of the metric, as well as of the curvature, is central, which implies that there exists a naive analogue of the integrand in (4.2). Setting

$$
\begin{aligned}
R^{abcd} &= (h^{\delta})^{ap}(h^{\delta})^{bq}(h^{\delta})^{cr}(h^{\delta})^{ds}R_{pqrs}\\
Ric_{ab} &= (h^{\delta})^{pq}R_{apbq}\\
Ric^{ab} &= (h^{\delta})^{ap}(h^{\delta})^{bq}Ric_{pq}\\
S &= (h^{\delta})^{ab}Ric_{ab}
\end{aligned}
$$

one finds that

$$
R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2 = 24\big(\mathbb{1} - (\alpha+T)^2\big)(\mathbb{1}+\alpha')(\mathbb{1}-T^2)^{-1}\delta^{-2}. \tag{4.3}
$$

Theorem 4.3. *Let $\delta(T)$ be an invertible polynomial in Z_{loc} and define α via the relation $\partial_4\delta = 2\alpha\delta$. If*

$$
\phi_0(\alpha)\big|_{\psi=\frac{\pi}{2}} = \phi_0(\alpha)\big|_{\psi=-\frac{\pi}{2}} = 0,
$$

then

$$
\chi(S^4_\theta) = \frac{1}{32\pi^2}\tau_{\delta,\mathrm{loc}}\big(R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2\big) = 2.
$$

Proof. Since δ is a polynomial in T and $\partial_4 T = T^2 - \mathbb{1}$, one can express α in terms of T and, by a slight abuse of notation, we let $\alpha(t)$ be such that $\phi_0(\alpha) = \alpha(\sin\psi)$. In this notation, the assumption on $\phi_0(\alpha)$ may be stated as $\alpha(1) = \alpha(-1) = 0$.

From the definition of $\tau_{\delta,\mathrm{loc}}$ it follows that

$$
\begin{aligned}
\chi &= \frac{1}{32\pi^2}\tau_{\delta,\mathrm{loc}}\big(R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2\big)\\
&= I_\psi\int_0^{2\pi}d\xi_1\int_0^{2\pi}d\xi_2\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\, d\varphi,
\end{aligned}
$$

where

$$
I_\psi = \frac{24}{32\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\big(1 - (\alpha(\sin\psi) + \sin\psi)^2\big)(1+\alpha'(\sin\psi))\cos\psi\, d\psi.
$$

Substituting $t = \sin\psi$ gives

$$
I_\psi = \frac{24}{32\pi^2}\int_{-1}^{1}\big(1 - (\alpha(t)+t)^2\big)(1+\alpha'(t))dt,
$$

which can easily be integrated to

$$
I_\psi = \frac{24}{32\pi^2}\Big[\alpha(t) + t - \frac{1}{3}\big(\alpha(t)+t\big)^3\Big]_{-1}^{1} = \frac{24}{32\pi^2}\Big(1 - \frac{1}{3} + 1 - \frac{1}{3}\Big) = \frac{1}{\pi^2},
$$

since $\alpha(1) = \alpha(-1) = 0$. Finally, one obtains

$$
\begin{aligned}
\chi &= I_\psi\int_0^{2\pi}d\xi_1\int_0^{2\pi}d\xi_2\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\, d\varphi\\
&= \frac{1}{\pi^2}\int_0^{2\pi}d\xi_1\int_0^{2\pi}d\xi_2\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\, d\varphi = \frac{1}{\pi^2}\cdot 4\pi^2\cdot\frac{1}{2} = 2,
\end{aligned}
$$

which proves the statement. □

Remark 4.4. The condition that $\phi_0(\alpha) = 0$ at $\psi = \pm\pi/2$, in Theorem 4.3, can be understood as a reminiscence of the fact that δ has to be the restriction (to U_0) of a non-zero function $f \in C^\infty(S^4)$. Namely, if δ approaches zero at $\psi = \pm\pi/2$ then

f has to be zero at these boundary points. The purpose of the condition in Theorem 4.3 is to avoid this situation. Let us be a little more precise in the case when $\delta(t)$ (with $t = \sin\psi$) is a polynomial. One finds that

$$\alpha = \frac{\partial_4\delta(\sin\psi)}{2\delta(\sin\psi)} = -\cos\psi\frac{\partial_\psi\delta(\sin\psi)}{2\delta(\sin\psi)} = -\frac{1}{2}\cos^2\psi\frac{\delta'(\sin\psi)}{\delta(\sin\psi)} = -\frac{1}{2}(1-t^2)\frac{\delta'(t)}{\delta(t)}.$$

Clearly, if $\delta(\pm 1) \neq 0$ then $\alpha(\pm 1) = 0$. Conversely, assume that $\delta(1) = 0$. Then, one may write $\delta(t) = (1-t)^n p(t)$ with $p(1) \neq 0$ (and $n > 0$); it follows that

$$\alpha = -\frac{1}{2}(1-t^2)\frac{p'(t)}{p(t)} + \frac{1}{2}n(1+t)$$

which implies $\alpha(1) = n > 0$. The case when $\delta(-1) = 0$ is treated analogously.

In this paper, we have preferred to stay in the purely algebraic regime, and have thus not considered any smooth completion of S^4_θ, in order to stress the point that our results do not depend on the analytic structure. However, we expect that Theorem 4.3 holds true even for more general perturbations in a potentially larger algebra. For instance, if $\delta = e^{\lambda T}$ exists for all $\lambda \in \mathbb{R}$, one obtains $\alpha = \frac{\lambda}{2}(T^2 - \mathbb{1})$ which clearly fulfills the conditions of Theorem 4.3. Moreover, one may consider perturbations given, not only as functions of T, but as more general elements of Z_{loc}. Although our approach to the Chern–Gauss–Bonnet theorem may be too naive to have any impact on the general problem, we hope that our investigations will contribute to the growing understanding of Riemannian curvature in noncommutative geometry.

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