



# Stacked central configurations for the spatial six-body problem

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## ABSTRACT

In this paper we show the existence of three new families of stacked spatial central configurations for the six-body problem with the following properties: four bodies are at the vertices of a regular tetrahedron and the other two bodies are on a line connecting one vertex of the tetrahedron with the center of the opposite face.

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## 1. Introduction

Given  $n$  punctual masses  $m_1, m_2, \dots, m_n$  with position vectors  $r_1, r_2, \dots, r_n$  the *Newtonian  $n$ -body problem* in celestial mechanics consists in studying the motion of these masses interacting amongst themselves through no other forces than their mutual gravitational attraction according to Newton's gravitational law [1].

Usually  $r_j \in \mathbb{R}^d$  for  $d = 2, 3$ . The Euclidean distance between the bodies of masses  $m_i$  and  $m_j$  is denoted by  $r_{ij} = |r_i - r_j|$  and we take the *inertial barycentric* system, that is the center of mass of the system, given by  $\sum_{j=1}^n m_j r_j / M$ , where  $M = m_1 + \dots + m_n$  is the total mass, is considered at the origin of the inertial system.

Two configurations  $(r_1, r_2, \dots, r_n)$  and  $(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_n)$  of the  $n$  bodies are *similar* if we can pass from one to the other doing a dilation and a rotation of  $\mathbb{R}^d$  (with respect to the inertial barycentric system). A *homographic* solution of the  $n$ -body problem is a solution such that the configuration of the  $n$  bodies at the instant  $t$  remains similar to itself as  $t$  varies.

The first three homographic solutions were found in 1767 by Euler [2] in the three-body problem, for which three bodies are *collinear* at any time. In 1772 Lagrange [3] found two additional homographic solutions in the three-body problem, where the three bodies are at any time in the vertices of an *equilateral triangle*.

At a given instant  $t = t_0$  the configuration of the  $n$  bodies is *central* if the gravitational acceleration  $\ddot{r}_j$  acting on every mass point  $m_j$  is proportional to its position  $r_j$  (referred to the inertial barycentric system), that is  $\ddot{r}_j = \lambda r_j$  with  $\lambda \neq 0$  for

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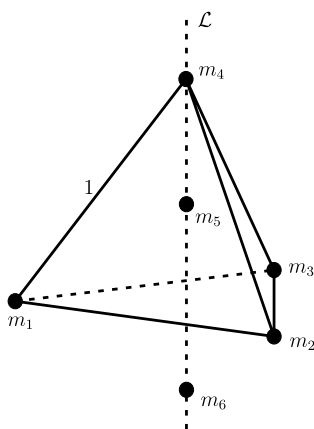


Fig. 1. Four bodies at the vertices of a regular tetrahedron and two bodies on a line  $\mathcal{L}$  connecting one vertex with the center of the opposite face.

all  $j = 1, \dots, n$ . Two central configurations are *related* if we can pass from one to another through a dilation and a rotation (centered at the center of mass). So we can study the classes of central configurations defined by the above equivalence relation.

The knowledge of central configurations allows us to compute homographic solutions (see [4]); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see [5]); if the  $n$  bodies are going to a simultaneous collision then the bodies tend to a central configuration (see [6]). See also the references [7–9].

In this paper we are interested in spatial central configurations, that is  $d = 3$ . More precisely, we are interested in *stacked spatial central configurations*, that is, spatial central configurations for the  $n$ -body problem where a proper subset of the  $n$  bodies are already on a central configuration.

Natural examples of stacked central configurations are given by the *nested* central configurations. The existence of double-nested planar central configurations for  $2n$  bodies is known for two nested regular  $n$ -gons with common center [10, 11]. A partial extension of this result is given in [12] where Llibre and Mello prove the existence of triple-nested triangular central configurations for the planar nine-body problem.

Double-nested spatial central configurations for  $2n$  bodies were studied for two nested regular polyhedra in [13]. More recently the same authors studied central configurations of three regular polyhedra for the spatial  $3n$ -body problem in [14]. See also [15] where nested regular tetrahedra are studied.

In 2005 Hampton [16] provides a new family of stacked planar central configurations for the five-body problem with three bodies at the vertices of an equilateral triangle and the other two bodies located symmetrically with respect to a perpendicular bisector.

In 2008 Llibre and Mello [17] give new examples of stacked planar central configurations for the five-body problem with three bodies at the vertices of an equilateral triangle and the two other bodies on a perpendicular bisector.

Recently Hampton and Santoprete [18] provided new examples of stacked spatial central configurations for the seven-body problem where the bodies are arranged as concentric three- and two-dimensional simplexes.

In this paper we find new classes of stacked spatial central configurations for the six-body problem which have four bodies at the vertices of a regular tetrahedron and the other two bodies are on a line connecting one vertex of the tetrahedron with the center of the opposite face; see Fig. 1. These new central configurations are generalizations of the ones studied by Llibre and Mello in [17].

Consider a regular tetrahedron whose sides have length 1 and let  $A, B, C, D, E$  and  $F$  be the following points on the line  $\mathcal{L}$  connecting the vertex at  $r_4$  with the center of the opposite face (see Fig. 2):  $A$  is the center of the tetrahedron,  $B$  is the vertex at  $r_4$ ,  $C$  is the point symmetric with the point  $B$  relative to the plane that contains  $r_1, r_2$  and  $r_3$ ,  $E$  is the barycenter of the equilateral triangle with vertices at  $r_1, r_2$  and  $r_3$ ,  $D$  and  $F$  are the points where the sphere with center at  $B$  and radius 1 meet the line  $\mathcal{L}$ . Let  $AB \subset \mathcal{L}$  be the segment whose endpoints are  $A$  and  $B$ , and in a similar way we define the segments  $BF, CD, DE$  and  $EA$ .

**Theorem 1.** Assume that we have four bodies with masses  $m_1, m_2, m_3$  and  $m_4$  at the vertices of a regular tetrahedron whose sides have length 1 and two bodies with masses  $m_5$  and  $m_6$  on the line  $\mathcal{L}$  connecting the vertex at  $r_4$  with the center of the opposite face according to Fig. 1. In order that the six bodies can be in a central configuration the following statements hold:

- (a) The three masses  $m_1, m_2$  and  $m_3$  are equal.
- (b) Only one of the position vectors  $r_5$  or  $r_6$  must be in the segment  $AB$ . See Fig. 2.

Without loss of generality we can assume that  $r_5 \in AB$ . Then there exist a position  $G \in AB$ , non-empty segments  $I^1(G) \subset CD$ ,  $I^2(G) \subset EA$ ,  $I^3(G) \subset BF$  and positive masses  $m_1 = m_2 = m_3, m_4, m_5$  and  $m_6(i), i = 1, 2, 3$ , such that  $r_1, r_2, r_3$  and  $r_4$  are at the

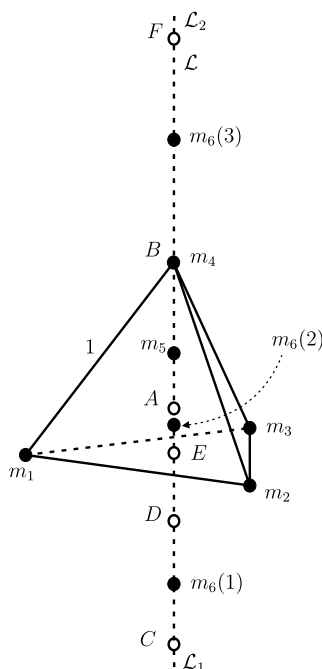


Fig. 2. Six bodies in stacked spatial central configurations.

vertices of the tetrahedron,  $r_5$  is at  $G$  and  $r_6(1) \in I^1(G)$ , or  $r_6(2) \in I^2(G)$ , or  $r_6(3) \in I^3(G)$ , form three central configurations. See Fig. 2.

Numerical results presented at the end of the paper provide numerical evidence of the following improvement of Theorem 1.

Without loss of generality we can assume that  $r_5 \in AB$ . Then for all positions  $G \in AB$ , there exist non-empty segments  $I^1(G) \subset CD$ ,  $I^2(G) \subset EA$ ,  $I^3(G) \subset BF$  and positive masses  $m_1 = m_2 = m_3, m_4, m_5$  and  $m_6(i), i = 1, 2, 3$ , such that  $r_1, r_2, r_3$  and  $r_4$  are at the vertices of the tetrahedron,  $r_5$  is at  $G$  and  $r_6(1) \in I^1(G)$ , or  $r_6(2) \in I^2(G)$ , or  $r_6(3) \in I^3(G)$ , form three different central configurations.

## 2. Proof of Theorem 1

The equations of motion of the  $n$ -body problem are given by

$$\ddot{r}_i = - \sum_{\substack{j=1 \\ j \neq i}}^n m_j \frac{r_i - r_j}{r_{ij}^3}, \quad (1)$$

for  $i = 1, 2, \dots, n$ . Here the gravitational constant is taken equal to 1 and  $r_j \in \mathbb{R}^3$ .

From the definition of the central configuration, Eq. (1) can be written as

$$\lambda r_i = - \sum_{\substack{j=1 \\ j \neq i}}^n m_j \frac{r_i - r_j}{r_{ij}^3}, \quad (2)$$

for  $i = 1, 2, \dots, n$ .

For the spatial central configurations, instead of working with Eq. (2) we shall work with the Dziobek equations (see equation (6), p. 295 of [18] and the references therein)

$$f_{ijh} = \sum_{k \neq i, j, h} m_k (R_{ik} - R_{jk}) \Delta_{ijkh} = 0, \quad (3)$$

where  $R_{ij} = 1/r_{ij}^3$  and  $\Delta_{ijkh} = (r_i - r_j) \wedge (r_j - r_h) \cdot (r_h - r_k)$ . Thus  $\Delta_{ijkh}$  gives six times the signed volume of the tetrahedron formed by  $m_i, m_j, m_h$  and  $m_k$ . Moreover  $f_{ijh} = -f_{jih}$ . Thus Eq. (3) is a set of  $n(n-1)(n-2)/2$  equations.

Assume  $\Delta_{1234} > 0$ . Define the following subsets of  $\mathcal{L}$  (see Fig. 2):

$$\mathcal{L}_1 = \{r_i \in \mathcal{L} : r_{1i} > 1 = r_{12}, \Delta_{123i} < 0, i = 5, 6\},$$

$$\mathcal{L}_2 = \{r_i \in \mathcal{L} : r_{4i} > 1 = r_{12}, \Delta_{123i} > 0, i = 5, 6\}.$$

Thus one has

$$\mathcal{L} = \mathcal{L}_1 \cup \{C\} \cup CD \cup \{D\} \cup DE \cup \{E\} \cup EA \cup \{A\} \cup AB \cup \{B\} \cup BF \cup \{F\} \cup \mathcal{L}_2.$$

For the six-body problem, Eq. (3) is a set of 60 equations. Our class of configurations with six bodies as in Fig. 1 without collisions must satisfy, among other conditions,

$$\begin{aligned} r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34} = 1, \quad r_{15} = r_{25} = r_{35}, \quad r_{16} = r_{26} = r_{36}, \\ \Delta_{1425} = -\Delta_{1435} = -\Delta_{2415} = \Delta_{2435} = \Delta_{3415} = -\Delta_{3425}, \\ \Delta_{1426} = -\Delta_{1436} = -\Delta_{2416} = \Delta_{2436} = \Delta_{3416} = -\Delta_{3426}, \\ \Delta_{1523} = -\Delta_{1532} = -\Delta_{2513} = \Delta_{2531} = \Delta_{3512} = -\Delta_{3521}, \\ \Delta_{1524} = -\Delta_{1534} = -\Delta_{2514} = \Delta_{2534} = \Delta_{3514} = -\Delta_{3524}, \\ \Delta_{1526} = -\Delta_{1536} = -\Delta_{2516} = \Delta_{2536} = \Delta_{3516} = -\Delta_{3526}, \\ \Delta_{2613} = -\Delta_{2631} = -\Delta_{1623} = \Delta_{1632} = \Delta_{3612} = -\Delta_{3621}, \\ \Delta_{2614} = -\Delta_{2634} = -\Delta_{1624} = \Delta_{1634} = \Delta_{3614} = -\Delta_{3624}, \\ \Delta_{2615} = -\Delta_{2635} = -\Delta_{1625} = \Delta_{1635} = \Delta_{3615} = -\Delta_{3625}, \end{aligned}$$

and  $\Delta_{456i} = 0$ , for  $i = 1, 2, 3$  and any permutation of the indices  $i, 4, 5$  and  $6$ .

Taking into account the above symmetries it follows that  $f_{123} = 0, f_{124} = 0, f_{125} = 0, f_{126} = 0, f_{132} = 0, f_{134} = 0, f_{135} = 0, f_{136} = 0, f_{145} = 0, f_{146} = 0, f_{231} = 0, f_{234} = 0, f_{235} = 0, f_{236} = 0, f_{245} = 0, f_{246} = 0, f_{345} = 0, f_{346} = 0, f_{456} = 0, f_{465} = 0$  and  $f_{564} = 0$  of (3) are trivially satisfied.

**Lemma 2.** *If the position of the body 5 is symmetric with the position of the body 4, that is  $r_5$  is at  $C$ , then there are no positions for the body 6 on  $\mathcal{L}$  and positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.*

**Proof.** The assumption of the lemma forces that  $r_{12} = r_{15}, r_{16} \neq r_{12}$  and  $r_{15} \neq r_{16}$ . From the equations  $f_{142} = 0, f_{143} = 0, f_{241} = 0, f_{243} = 0, f_{341} = 0$  and  $f_{342} = 0$  of (3) one has

$$(R_{12} - R_{45})\Delta_{1425}m_5 + (R_{16} - R_{46})\Delta_{1426}m_6 = 0. \quad (4)$$

From the equations  $f_{152} = 0, f_{153} = 0, f_{251} = 0, f_{253} = 0, f_{351} = 0$  and  $f_{352} = 0$  of (3) one has

$$(R_{12} - R_{45})\Delta_{1524}m_4 + (R_{16} - R_{56})\Delta_{1526}m_6 = 0. \quad (5)$$

Note that the coefficient of the mass  $m_5$ ,  $(R_{12} - R_{45})\Delta_{1425}$ , in Eq. (4) is positive. So in order to have positive solutions of this equation the coefficient of the mass  $m_6$ ,  $(R_{16} - R_{46})\Delta_{1426}$ , must be negative. This happens only when  $r_{16} > r_{46}$  and  $\Delta_{1426} > 0$ . This implies that the coefficient of the mass  $m_6$ ,  $(R_{16} - R_{56})\Delta_{1526}$ , in Eq. (5) is negative. But the coefficient of the mass  $m_4$  in this equation is already negative. Thus the two masses  $m_4$  and  $m_6$  have opposite signs. ■

**Remark 3.** An analogous result can be obtained by changing the bodies 5 and 6 in Lemma 2. The same happens in Lemmas 4 and 5.

**Lemma 4.** *If the position of the body 5 is at  $A$  then there are no positions for the body 6 on  $\mathcal{L}$  and positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.*

**Proof.** By assumption  $r_{15} = r_{45}$  and from Lemma 2 one has  $r_{16} \neq r_{12}$ . From the equations  $f_{142} = 0, f_{143} = 0, f_{241} = 0, f_{243} = 0, f_{341} = 0$  and  $f_{342} = 0$  of (3) one has

$$(R_{16} - R_{46})\Delta_{1426}m_6 = 0.$$

As  $m_6 > 0$  and  $\Delta_{1426} \neq 0$  one has  $R_{16} = R_{46}$  or equivalently  $r_{16} = r_{46}$ . This implies that  $r_6 = r_5$ , that is a collision of the bodies 5 and 6. ■

**Lemma 5.** *If the position of the body 5 is at  $E$  then there are no positions for the body 6 on  $\mathcal{L}$  and positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.*

**Proof.** By assumption,  $\Delta_{1523} = 0$ . From the equations  $f_{152} = 0, f_{153} = 0, f_{251} = 0, f_{253} = 0, f_{351} = 0$  and  $f_{352} = 0$  of (3) one has

$$(R_{12} - R_{45})\Delta_{1524}m_4 + (R_{16} - R_{56})\Delta_{1526}m_6 = 0. \quad (6)$$

From the equations  $f_{142} = 0, f_{143} = 0, f_{241} = 0, f_{243} = 0, f_{341} = 0$  and  $f_{342} = 0$  of (3) one has

$$(R_{15} - R_{45})\Delta_{1425}m_5 + (R_{16} - R_{46})\Delta_{1426}m_6 = 0. \quad (7)$$

Note that the coefficient of the mass  $m_4$ ,  $(R_{12} - R_{45})\Delta_{1524}$ , in Eq. (6) is positive. So in order to have positive solutions of this equation the coefficient of the mass  $m_6$ ,  $(R_{16} - R_{56})\Delta_{1526}$ , must be negative. This happens only when  $r_{16} > r_{56}$  and  $\Delta_{1526} > 0$ . This implies that the coefficient of the mass  $m_5$ ,  $(R_{15} - R_{45})\Delta_{1425}$ , in Eq. (7) is positive. But the coefficient of the mass  $m_6$  in this equation is already positive. Thus the two masses  $m_5$  and  $m_6$  have opposite signs. ■

**Lemma 6.** Assume that we have four bodies with masses  $m_1, m_2, m_3$  and  $m_4$  at the vertices of a regular tetrahedron whose sides have length 1 and two bodies with masses  $m_5$  and  $m_6$  on the line  $\mathcal{L}$  according to Fig. 2. In order that the six bodies can be in a central configuration the following statements hold:

- (a) The three masses  $m_1, m_2$  and  $m_3$  are equal.  
 (b) Only one of the position vectors  $r_5$  or  $r_6$  must be in the segment  $AB$ .

**Proof.** From Lemmas 2, 4 and 5 we can assume that the  $r_5 \neq A, r_5 \neq C, r_5 \neq E, r_6 \neq A, r_6 \neq C$  and  $r_6 \neq E$ . From the equations  $f_{254} = 0, f_{256} = 0, f_{264} = 0, f_{265} = 0, f_{452} = 0, f_{462} = 0$  and  $f_{562} = 0$  of (3) one has

$$(R_{12} - R_{16})\Delta_{2641}(m_1 - m_3) = 0.$$

Thus  $m_1 = m_3$  since  $r_{12} \neq r_{16}$  and  $\Delta_{2641} \neq 0$ . By the same arguments, from the equations  $f_{154} = 0, f_{156} = 0, f_{164} = 0, f_{165} = 0, f_{451} = 0, f_{461} = 0$  and  $f_{561} = 0$  of (3) one has  $m_2 = m_3$  while from the equations  $f_{354} = 0, f_{356} = 0, f_{364} = 0, f_{365} = 0, f_{453} = 0, f_{463} = 0$  and  $f_{563} = 0$  of (3) one has  $m_1 = m_2$ . Therefore  $m_1 = m_2 = m_3$ . This proves item (a) of the lemma.

Taking into account that  $m_1 = m_2 = m_3$  the other 18 equations of (3) reduce to 3 equations:

- (i) From the equations  $f_{142} = 0, f_{143} = 0, f_{241} = 0, f_{243} = 0, f_{341} = 0$  and  $f_{342} = 0$  of (3) one has

$$(R_{15} - R_{45})\Delta_{1425}m_5 + (R_{16} - R_{46})\Delta_{1426}m_6 = 0. \quad (8)$$

- (ii) From the equations  $f_{152} = 0, f_{153} = 0, f_{251} = 0, f_{253} = 0, f_{351} = 0$  and  $f_{352} = 0$  of (3) one has

$$(R_{12} - R_{15})\Delta_{1523}m_1 + (R_{12} - R_{45})\Delta_{1524}m_4 + (R_{16} - R_{56})\Delta_{1526}m_6 = 0. \quad (9)$$

- (iii) From the equations  $f_{162} = 0, f_{163} = 0, f_{261} = 0, f_{263} = 0, f_{361} = 0$  and  $f_{362} = 0$  of (3) one has

$$(R_{12} - R_{16})\Delta_{1623}m_1 + (R_{12} - R_{46})\Delta_{1624}m_4 + (R_{15} - R_{56})\Delta_{1625}m_5 = 0. \quad (10)$$

In order to have positive solutions the coefficients of Eq. (8) must have opposite signs. There are two possibilities: either  $(R_{15} - R_{45})\Delta_{1425} < 0$  and  $(R_{16} - R_{46})\Delta_{1426} > 0$  or  $(R_{15} - R_{45})\Delta_{1425} > 0$  and  $(R_{16} - R_{46})\Delta_{1426} < 0$ .

(1)  $(R_{15} - R_{45})\Delta_{1425} < 0$  and  $(R_{16} - R_{46})\Delta_{1426} > 0$ . The inequality  $(R_{15} - R_{45})\Delta_{1425} < 0$  is satisfied only when  $r_{15} > r_{45}$  and  $\Delta_{1425} > 0$ , that is only when  $r_5 \in AB$ . Now the inequality  $(R_{16} - R_{46})\Delta_{1426} > 0$  is satisfied when either  $r_{16} < r_{46}$ , that is  $r_6 \in \mathcal{L}_1 \cup CD \cup DE \cup EA$ , or  $\Delta_{1426} < 0$ , that is  $r_6 \in BF \cup \mathcal{L}_2$ .

(2)  $(R_{15} - R_{45})\Delta_{1425} > 0$  and  $(R_{16} - R_{46})\Delta_{1426} < 0$ . The inequality  $(R_{15} - R_{45})\Delta_{1425} > 0$  is satisfied when either  $r_{15} < r_{45}$ , that is  $r_5 \in \mathcal{L}_1 \cup CD \cup DE \cup EA$ , or  $\Delta_{1425} < 0$ , that is  $r_5 \in BF \cup \mathcal{L}_2$ . Now the inequality  $(R_{16} - R_{46})\Delta_{1426} < 0$  is satisfied only when  $r_{16} > r_{46}$  and  $\Delta_{1426} > 0$ , that is only when  $r_6 \in AB$ .

From (1) and (2) above, item (b) of the lemma is proved. ■

The first part of Theorem 1 is proved.

**Remark 7.** From now on we can assume that  $r_5 \in AB$ . See Lemma 6.

**Lemma 8.** If  $r_6 \in \mathcal{L}_1$  then there are no positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.

**Proof.** From the assumption one has

$$R_{12} - R_{16} > 0, \quad R_{12} - R_{46} > 0, \quad R_{15} - R_{56} > 0, \\ \Delta_{1623} < 0, \quad \Delta_{1624} < 0, \quad \Delta_{1625} < 0.$$

Therefore the three coefficients of Eq. (10) are negative. This implies that two of the masses  $m_1, m_4, m_5$  have opposite signs. ■

**Lemma 9.** If  $r_6 \in \mathcal{L}_2$  then there are no positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.

**Proof.** From the assumption one has

$$R_{12} - R_{16} > 0, \quad R_{12} - R_{46} > 0, \quad R_{15} - R_{56} > 0, \\ \Delta_{1623} > 0, \quad \Delta_{1624} > 0, \quad \Delta_{1625} > 0.$$

Therefore the three coefficients of Eq. (10) are positive. This implies that two of the masses  $m_1, m_4, m_5$  have opposite signs. ■

**Lemma 10.** If  $r_6 \in DE$  then there are no positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.

**Proof.** From the assumption one has

$$R_{12} - R_{16} < 0, \quad R_{12} - R_{46} < 0, \quad R_{15} - R_{56} < 0, \\ \Delta_{1623} < 0, \quad \Delta_{1624} < 0, \quad \Delta_{1625} < 0.$$

Therefore the three coefficients of Eq. (10) are positive. This implies that two of the masses  $m_1, m_4, m_5$  have opposite signs. ■

Without loss of generality we can take a coordinate system such that the bodies with masses  $m_1, m_2, m_3, m_4, m_5$  and  $m_6$  are at  $(\sqrt{3}/3, 0, 0), (-\sqrt{3}/6, -1/2, 0), (-\sqrt{3}/6, 1/2, 0), (0, 0, \sqrt{6}/3), (0, 0, x)$  and  $(0, 0, y)$ , respectively. Thus one has  $A = (0, 0, \sqrt{6}/12), B = (0, 0, \sqrt{6}/3), C = (0, 0, -\sqrt{6}/3), D = (0, 0, -1 + \sqrt{6}/3), E = (0, 0, 0)$  and  $F = (0, 0, 1 + \sqrt{6}/3)$ . Therefore  $r_5 \in AB$  if and only if  $\sqrt{6}/12 < x < \sqrt{6}/3$ .

From the coefficients of Eqs. (8)–(10), respectively, define the functions

$$n_{13} = (R_{15} - R_{45})\Delta_{1425}, \quad n_{14} = (R_{16} - R_{46})\Delta_{1426}, \\ n_{21} = (R_{12} - R_{15})\Delta_{1523}, \quad n_{22} = (R_{12} - R_{45})\Delta_{1524}, \quad n_{24} = (R_{16} - R_{56})\Delta_{1526}, \\ n_{31} = (R_{12} - R_{16})\Delta_{1623}, \quad n_{32} = (R_{12} - R_{46})\Delta_{1624}, \quad n_{33} = (R_{15} - R_{56})\Delta_{1625},$$

which can be written in terms of the above coordinates as

$$n_{13}(x, y) = \left( \frac{\sqrt{2} - \sqrt{3}x}{6} \right) \left( \frac{1}{(x^2 + \frac{1}{3})^{3/2}} - \frac{27}{((\sqrt{6} - 3x)^2)^{3/2}} \right), \\ n_{14}(x, y) = \left( \frac{\sqrt{2} - \sqrt{3}y}{6} \right) \left( \frac{1}{(y^2 + \frac{1}{3})^{3/2}} - \frac{27}{((\sqrt{6} - 3y)^2)^{3/2}} \right), \\ n_{21}(x, y) = \frac{\sqrt{3}x}{2} \left( 1 - \frac{1}{(x^2 + \frac{1}{3})^{3/2}} \right), \\ n_{22}(x, y) = \left( \frac{\sqrt{3}x - \sqrt{2}}{6} \right) \left( 1 - \frac{27}{((\sqrt{6} - 3x)^2)^{3/2}} \right), \\ n_{24}(x, y) = \left( \frac{x - y}{2\sqrt{3}} \right) \left( \frac{1}{(y^2 + \frac{1}{3})^{3/2}} - \frac{1}{((x - y)^2)^{3/2}} \right), \\ n_{31}(x, y) = -\frac{\sqrt{3}y}{2} \left( 1 - \frac{1}{(y^2 + \frac{1}{3})^{3/2}} \right), \\ n_{32}(x, y) = \left( \frac{\sqrt{2} - \sqrt{3}y}{6} \right) \left( 1 - \frac{27}{((\sqrt{6} - 3y)^2)^{3/2}} \right), \\ n_{33}(x, y) = \left( \frac{x - y}{2\sqrt{3}} \right) \left( \frac{1}{(x^2 + \frac{1}{3})^{3/2}} - \frac{1}{((x - y)^2)^{3/2}} \right).$$

Eqs. (8)–(10) define three hyperplanes through the origin in the space  $(m_1, m_4, m_5, m_6)$ . The mass vector is parallel to the straight line defined by the intersection of these three hyperplanes which can be written as  $T = (T_1, -T_2, T_3, -T_4)$ , where  $T_1 = n_{13}n_{24}n_{32} + n_{14}n_{22}n_{33}$ ,  $T_2 = n_{13}n_{24}n_{31} + n_{14}n_{21}n_{33}$ ,  $T_3 = n_{14}(n_{21}n_{32} - n_{22}n_{31})$  and  $T_4 = n_{13}(n_{21}n_{32} - n_{22}n_{31})$ . Thus there will be positive masses  $m_1, m_4, m_5$  and  $m_6$  solutions of Eqs. (8)–(10) if and only if the components of the vector  $T$  have the same sign.

**Lemma 11.** Consider  $x = 3/10$  and  $y = -1/2$ , that is  $r_5 \in AB$  and  $r_6 \in CD$ . Then there are positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.

**Proof.** Substituting  $x = 3/10$  and  $y = -1/2$  into  $n_{13}, n_{14}, n_{21}, n_{22}, n_{24}, n_{31}, n_{32}$  and  $n_{33}$  and then calculating  $T_1, T_2, T_3$  and  $T_4$  it follows that

$$T_1 \left( \frac{3}{10}, -\frac{1}{2} \right) < 0, \quad T_2 \left( \frac{3}{10}, -\frac{1}{2} \right) > 0, \quad T_3 \left( \frac{3}{10}, -\frac{1}{2} \right) < 0, \quad T_4 \left( \frac{3}{10}, -\frac{1}{2} \right) > 0.$$

Therefore the components of the mass vector  $T$  have the same sign. The lemma is proved.

In order to give some information about the values of the masses consider  $m_1 = m_2 = m_3 = 10$ ,

$$m_4 = -\frac{90\theta_1}{889\theta_2},$$

where

$$\begin{aligned} \theta_1 &= -169428788777 \sqrt{3}\sqrt{127} + 68197707648 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{7} \\ &\quad + 96005855232 \sqrt{127}\sqrt{7} - 1733111767086 \sqrt{7}\sqrt{2} \\ &\quad - 1398415595283 \sqrt{3}\sqrt{7} - 55810140602 \sqrt{127}\sqrt{2} \\ &\quad + 7165469970760 + 277475849024 \sqrt{3}\sqrt{2}, \\ \theta_2 &= -6885857196 \sqrt{7}\sqrt{2} - 3517815545 \sqrt{3}\sqrt{7} + 1654729020 \sqrt{127}\sqrt{2} \\ &\quad - 2168765627 \sqrt{3}\sqrt{127} + 12461781528 + 1743814656 \sqrt{127}\sqrt{7} \\ &\quad + 159252480 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{7} + 1138062240 \sqrt{3}\sqrt{2}, \\ m_5 &= -\frac{146304\theta_3}{35 \left( 3 + 2\sqrt{3}\sqrt{2} \right)^2 \left( 1536 \sqrt{3}\sqrt{127} - 16129 \right) \theta_4}, \end{aligned}$$

where

$$\begin{aligned} \theta_3 &= -9279529102449 \sqrt{3}\sqrt{127} + 910848433824 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{7} \\ &\quad + 2443137682041 \sqrt{127}\sqrt{7} - 3512003097036 \sqrt{7}\sqrt{2} \\ &\quad - 3064630331483 \sqrt{3}\sqrt{7} - 8621348150970 \sqrt{127}\sqrt{2} \\ &\quad + 21107073194110 \sqrt{3}\sqrt{2} + 110325167369661, \\ \theta_4 &= -6885857196 \sqrt{7}\sqrt{2} - 3517815545 \sqrt{3}\sqrt{7} + 1654729020 \sqrt{127}\sqrt{2} \\ &\quad - 2168765627 \sqrt{3}\sqrt{127} + 12461781528 + 1743814656 \sqrt{127}\sqrt{7} \\ &\quad + 159252480 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{7} + 1138062240 \sqrt{3}\sqrt{2}, \\ m_6 &= -\frac{40320 \left( 1881 \sqrt{3}\sqrt{127} - 2810 \sqrt{127}\sqrt{2} + 16129 \right) \theta_5}{127 \left( -9 + 10\sqrt{3}\sqrt{2} \right)^2 \left( 27\sqrt{3}\sqrt{7} + 34\sqrt{7}\sqrt{2} - 49 \right) \left( 1536 \sqrt{3}\sqrt{127} - 16129 \right) \theta_6}, \end{aligned}$$

where

$$\begin{aligned} \theta_5 &= -9279529102449 \sqrt{3}\sqrt{127} + 910848433824 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{7} \\ &\quad + 2443137682041 \sqrt{127}\sqrt{7} - 3512003097036 \sqrt{7}\sqrt{2} \\ &\quad - 3064630331483 \sqrt{3}\sqrt{7} - 8621348150970 \sqrt{127}\sqrt{2} \\ &\quad + 21107073194110 \sqrt{3}\sqrt{2} + 110325167369661, \\ \theta_6 &= -6885857196 \sqrt{7}\sqrt{2} - 3517815545 \sqrt{3}\sqrt{7} + 1654729020 \sqrt{127}\sqrt{2} \\ &\quad - 2168765627 \sqrt{3}\sqrt{127} + 12461781528 + 1743814656 \sqrt{127}\sqrt{7} \\ &\quad + 159252480 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{7} + 1138062240 \sqrt{3}\sqrt{2}. \end{aligned}$$

By a long but elementary calculation Eqs. (8)–(10) are satisfied. The extensive calculations involved in this lemma have been corroborated with the software Maple 9. The values of the above masses with an accuracy of ten decimal round-off coordinates are

$$m_4 = 6.7458781851, \quad m_5 = 10.1934123989, \quad m_6 = 8.0305718049. \quad \blacksquare$$

**Lemma 12.** Consider  $x = 3/10$  and  $y = 1/10$ , that is  $r_5 \in AB$  and  $r_6 \in EA$ . Then there are positive masses  $m_i, i = 1, \dots, 6$ , such that these bodies form a central configuration.

**Proof.** Substituting  $x = 3/10$  and  $y = 1/10$  into  $n_{13}, n_{14}, n_{21}, n_{22}, n_{24}, n_{31}, n_{32}$  and  $n_{33}$  and then calculating  $T_1, T_2, T_3$  and  $T_4$  it follows that

$$T_1\left(\frac{3}{10}, \frac{1}{10}\right) > 0, \quad T_2\left(\frac{3}{10}, \frac{1}{10}\right) < 0, \quad T_3\left(\frac{3}{10}, \frac{1}{10}\right) > 0, \quad T_4\left(\frac{3}{10}, \frac{1}{10}\right) < 0.$$

Therefore the components of the mass vector  $T$  have the same sign. The lemma is proved.

In order to give some information about the values of the masses consider  $m_1 = m_2 = m_3 = 10$ ,

$$m_4 = \frac{90\psi_1}{13081\psi_2},$$

where

$$\begin{aligned} \psi_1 = & 400775035943835 \sqrt{3}\sqrt{103} - 367935503618123 \sqrt{3}\sqrt{127} \\ & - 79367052441600 \sqrt{127}\sqrt{103} + 340184683326250 \sqrt{127}\sqrt{2} \\ & - 446181650457150 \sqrt{2}\sqrt{103} - 391767956870560 \sqrt{3}\sqrt{2} \\ & + 42152670665280 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{103} + 1140836863198798, \end{aligned}$$

$$\begin{aligned} \psi_2 = & -828144000 \sqrt{127}\sqrt{103} + 155520000 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{103} \\ & - 336046827905 \sqrt{3}\sqrt{103} + 123201043920 \sqrt{3}\sqrt{2} \\ & + 344038847249 \sqrt{3}\sqrt{127} + 578025295560 \sqrt{2}\sqrt{103} \\ & - 449497813320 \sqrt{127}\sqrt{2} - 656045558874, \end{aligned}$$

$$m_5 = \frac{4572\psi_3}{515 \left(24\sqrt{3}\sqrt{127} - 16129\right) \left(-3 + 10\sqrt{3}\sqrt{2}\right)^2 \psi_4},$$

where

$$\begin{aligned} \psi_3 = & 277476547044902110 \sqrt{3}\sqrt{103} - 208952726731151670 \sqrt{3}\sqrt{127} \\ & + 223267858814565828 \sqrt{127}\sqrt{2} - 298005698143418421 \sqrt{2}\sqrt{103} \\ & - 2925421091076103921 \sqrt{3}\sqrt{2} - 118173007491268230 \sqrt{127}\sqrt{103} \\ & + 51801039529144776 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{103} + 5985425952439496070, \end{aligned}$$

$$\begin{aligned} \psi_4 = & -828144000 \sqrt{127}\sqrt{103} + 155520000 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{103} \\ & - 336046827905 \sqrt{3}\sqrt{103} + 123201043920 \sqrt{3}\sqrt{2} \\ & + 344038847249 \sqrt{3}\sqrt{127} - 656045558874 \\ & + 578025295560 \sqrt{2}\sqrt{103} - 449497813320 \sqrt{127}\sqrt{2}, \end{aligned}$$

$$m_6 = -\frac{3708 \left(-1881 \sqrt{3}\sqrt{127} + 2810 \sqrt{127}\sqrt{2} - 16129\right) \psi_5}{635 \left(24\sqrt{3}\sqrt{127} - 16129\right) \left(-603 \sqrt{3}\sqrt{103} + 2090 \sqrt{2}\sqrt{103} - 10609\right) \psi_6},$$

where

$$\begin{aligned} \psi_5 = & 277476547044902110 \sqrt{3}\sqrt{103} - 208952726731151670 \sqrt{3}\sqrt{127} \\ & + 223267858814565828 \sqrt{127}\sqrt{2} - 298005698143418421 \sqrt{2}\sqrt{103} \\ & - 2925421091076103921 \sqrt{3}\sqrt{2} - 118173007491268230 \sqrt{127}\sqrt{103} \\ & + 51801039529144776 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{103} + 5985425952439496070, \end{aligned}$$

$$\begin{aligned} \psi_6 = & \left(-9 + 10\sqrt{3}\sqrt{2}\right)^2 \left(-828144000 \sqrt{127}\sqrt{103} - 656045558874\right. \\ & + 155520000 \sqrt{3}\sqrt{127}\sqrt{2}\sqrt{103} - 336046827905 \sqrt{3}\sqrt{103} \\ & + 123201043920 \sqrt{3}\sqrt{2} + 344038847249 \sqrt{3}\sqrt{127} \\ & \left.+ 578025295560 \sqrt{2}\sqrt{103} - 449497813320 \sqrt{127}\sqrt{2}\right). \end{aligned}$$

By a long but elementary calculation Eqs. (8)–(10) are satisfied. The extensive calculations involved in this lemma have been corroborated with the software Maple 9. The values of the above masses with an accuracy of ten decimal round-off coordinates are

$$m_4 = 8.0398208730, \quad m_5 = 0.0828851657, \quad m_6 = 0.0962271225. \quad \blacksquare$$

**Lemma 13.** Consider  $x = 3/10$  and  $y = 1$ , that is  $r_5 \in AB$  and  $r_6 \in BF$ . Then there are positive masses  $m_i$ ,  $i = 1, \dots, 6$ , such that these bodies form a central configuration.

**Proof.** Substituting  $x = 3/10$  and  $y = 1$  into  $n_{13}$ ,  $n_{14}$ ,  $n_{21}$ ,  $n_{22}$ ,  $n_{24}$ ,  $n_{31}$ ,  $n_{32}$  and  $n_{33}$  and then calculating  $T_1$ ,  $T_2$ ,  $T_3$  and  $T_4$  it follows that

$$T_1\left(\frac{3}{10}, 1\right) > 0, \quad T_2\left(\frac{3}{10}, 1\right) < 0, \quad T_3\left(\frac{3}{10}, 1\right) > 0, \quad T_4\left(\frac{3}{10}, 1\right) < 0.$$

Therefore the components of the mass vector  $T$  have the same sign. The lemma is proved.

In order to give some information about the values of the masses consider  $m_1 = m_2 = m_3 = 10$ ,

$$m_4 = \frac{45\chi_1}{508\chi_2},$$

where

$$\begin{aligned} \chi_1 &= 53153630040 \sqrt{3}\sqrt{127}\sqrt{2} - 218046103912 \sqrt{127}\sqrt{2} + 1243241083487 \\ &\quad + 169782922943 \sqrt{3}\sqrt{127} - 129631817952 \sqrt{127} - 1242920324976 \sqrt{3} \\ &\quad - 430417266086 \sqrt{3}\sqrt{2} + 1513598400600 \sqrt{2}, \\ \chi_2 &= -302306823 \sqrt{127} + 127894410 \sqrt{3}\sqrt{127}\sqrt{2} + 831828240 \sqrt{127}\sqrt{2} \\ &\quad - 644882824 \sqrt{3}\sqrt{127} - 4369346100 \sqrt{2} + 3421428641 \sqrt{3} \\ &\quad - 4211991576 + 1781931920 \sqrt{3}\sqrt{2}, \\ m_5 &= -\frac{56007\chi_3}{80(1029\sqrt{3}\sqrt{127} - 16129)(-3 + \sqrt{3}\sqrt{2})^2\chi_4}, \end{aligned}$$

where

$$\begin{aligned} \chi_3 &= 587871472464 \sqrt{3}\sqrt{127}\sqrt{2} - 2792763123759 \sqrt{127}\sqrt{2} - 31866467162793 \\ &\quad + 2357179926093 \sqrt{3}\sqrt{127} - 1447895560896 \sqrt{127} - 4363013925568 \sqrt{3} \\ &\quad + 12332808521953 \sqrt{3}\sqrt{2} + 5395481085936 \sqrt{2}, \\ \chi_4 &= -302306823 \sqrt{127} + 127894410 \sqrt{3}\sqrt{127}\sqrt{2} + 831828240 \sqrt{127}\sqrt{2} \\ &\quad - 644882824 \sqrt{3}\sqrt{127} - 4369346100 \sqrt{2} + 3421428641 \sqrt{3} \\ &\quad - 4211991576 + 1781931920 \sqrt{3}\sqrt{2}, \\ m_6 &= \frac{4410(-1881\sqrt{3}\sqrt{127} + 2810\sqrt{127}\sqrt{2} - 16129)\chi_5}{127(1029\sqrt{3}\sqrt{127} - 16129)(-9\sqrt{3} + 11\sqrt{2} + 8)(-9 + 10\sqrt{3}\sqrt{2})^2\chi_6}, \end{aligned}$$

where

$$\begin{aligned} \chi_5 &= 587871472464 \sqrt{3}\sqrt{127}\sqrt{2} - 2792763123759 \sqrt{127}\sqrt{2} - 31866467162793 \\ &\quad + 2357179926093 \sqrt{3}\sqrt{127} - 1447895560896 \sqrt{127} - 4363013925568 \sqrt{3} \\ &\quad + 12332808521953 \sqrt{3}\sqrt{2} + 5395481085936 \sqrt{2}, \\ \chi_6 &= -302306823 \sqrt{127} + 127894410 \sqrt{3}\sqrt{127}\sqrt{2} + 831828240 \sqrt{127}\sqrt{2} \\ &\quad - 644882824 \sqrt{3}\sqrt{127} - 4369346100 \sqrt{2} + 3421428641 \sqrt{3} \\ &\quad - 4211991576 + 1781931920 \sqrt{3}\sqrt{2}. \end{aligned}$$

By a long but elementary calculation Eqs. (8)–(10) are satisfied. The extensive calculations involved in this lemma have been corroborated with the software Maple 9. The values of the above masses with an accuracy of ten decimal round-off coordinates are

$$m_4 = 2.8165087541, \quad m_5 = 145.0492841643, \quad m_6 = 9.1870655797. \quad \blacksquare$$

Define  $G = (0, 0, 3/10) \in AB$ ,  $G_1 = (0, 0, -1/2) \in I^1(G) \subset CD$ ,  $G_2 = (0, 0, 1/10) \in I^2(G) \subset EA$ , and  $G_3 = (0, 0, 1) \in I^3(G) \subset BF$ . The proof of the last part of Theorem 1 follows from Lemmas 11–13 and the open conditions on the mass vector  $T$ . See Figs. 3–5.

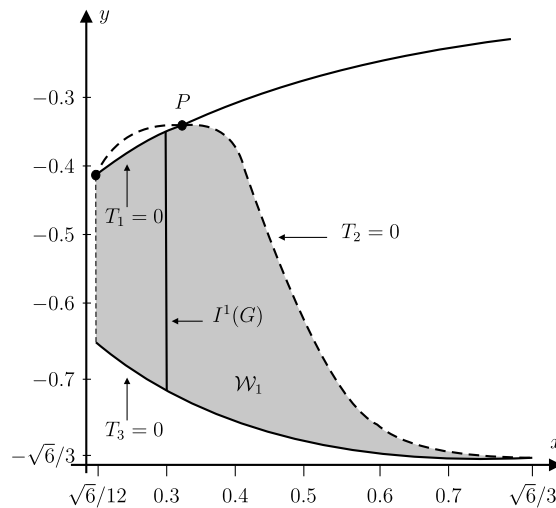


Fig. 3. The region  $\mathcal{W}_1$  and the segment  $I^1(G)$  from Theorem 1.

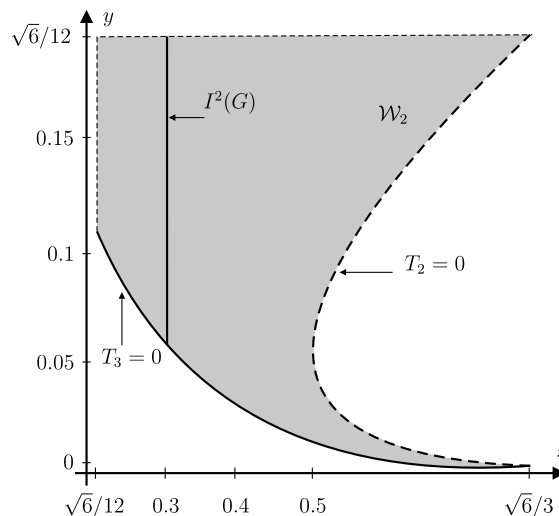


Fig. 4. The region  $\mathcal{W}_2$  and the segment  $I^2(G)$  from Theorem 1.

Consider  $r_5 \in AB$  and  $r_6 \in CD$ . It follows that  $n_{13} < 0$ ,  $n_{14} > 0$ ,  $n_{21} < 0$ ,  $n_{22} > 0$ ,  $n_{24} > 0$ ,  $n_{31} > 0$ ,  $n_{32} < 0$ . In particular, as  $n_{13} \neq 0$  and  $n_{14} \neq 0$  it follows that  $T_3 = 0$  if and only if  $T_4 = 0$ . Define the region

$$\mathcal{W}_1 = \left\{ (x, y) : \frac{\sqrt{6}}{12} < x < \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3} < y < \frac{\sqrt{6}}{3} - 1, T_1 < 0, T_2 > 0, T_3 < 0, T_4 > 0 \right\}.$$

In Fig. 3 the region  $\mathcal{W}_1$  is illustrated. Note that  $(3/10, -1/2) \in I^1(G) \subset \mathcal{W}_1$  according to Theorem 1. Note also that the projection of the region  $\mathcal{W}_1$  on the  $x$ -axis is all of the segment  $AB$ . The point  $P$  is defined by the intersection of the curves  $T_1 = 0$  and  $T_2 = 0$  and has coordinates  $x = 1/3$  and  $y = -1/3$ .

Consider  $r_5 \in AB$  and  $r_6 \in EA$ . It follows that  $n_{13} < 0$ ,  $n_{14} > 0$ ,  $n_{21} < 0$ ,  $n_{22} > 0$ ,  $n_{31} < 0$ ,  $n_{32} > 0$ . In particular, as  $n_{13} \neq 0$  and  $n_{14} \neq 0$  it follows that  $T_3 = 0$  if and only if  $T_4 = 0$ . Furthermore the function  $T_1$  is positive. Define the region

$$\mathcal{W}_2 = \left\{ (x, y) : \frac{\sqrt{6}}{12} < x < \frac{\sqrt{6}}{3}, 0 < y < \frac{\sqrt{6}}{12}, T_1 > 0, T_2 < 0, T_3 > 0, T_4 < 0 \right\}.$$

The region  $\mathcal{W}_2$  is depicted in Fig. 4. Note that  $(3/10, 1/10) \in I^2(G) \subset \mathcal{W}_2$  according to Theorem 1. Note also that the projection of the region  $\mathcal{W}_2$  on the  $x$ -axis is all of the segment  $AB$ .

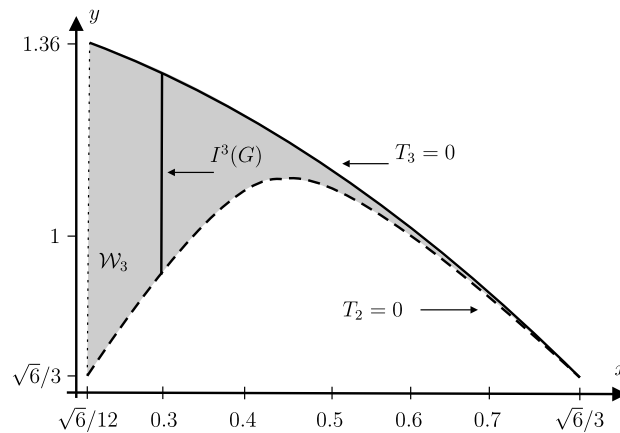


Fig. 5. The region  $\mathcal{W}_3$  and the segment  $I^3(G)$  from Theorem 1.

Consider  $r_5 \in AB$  and  $r_6 \in BF$ . It follows that  $n_{13} < 0$ ,  $n_{14} > 0$ ,  $n_{21} < 0$ ,  $n_{22} > 0$ ,  $n_{24} > 0$ ,  $n_{31} > 0$ ,  $n_{32} < 0$ . In particular, as  $n_{13} \neq 0$  and  $n_{14} \neq 0$  it follows that  $T_3 = 0$  if and only if  $T_4 = 0$ . Furthermore the function  $T_1$  is positive. Define the region

$$\mathcal{W}_3 = \left\{ (x, y) : \frac{\sqrt{6}}{12} < x < \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3} < y < 1 + \frac{\sqrt{6}}{3}, T_1 > 0, T_2 < 0, T_3 > 0, T_4 < 0 \right\}.$$

In Fig. 5 the region  $\mathcal{W}_3$  is illustrated. Note that  $(3/10, 1) \in I^3(G) \subset \mathcal{W}_3$  according to Theorem 1. Note also that the projection of the region  $\mathcal{W}_3$  on the  $x$ -axis is all of the segment  $AB$ .

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