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Almost Einstein and Poincaré–Einstein manifolds in Riemannian signature

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ABSTRACT

An almost Einstein manifold satisfies equations which are a slight weakening of the Einstein equations; Einstein metrics, Poincaré–Einstein metrics, and compactifications of certain Ricci-flat asymptotically locally Euclidean structures are special cases. The governing equation is a conformally invariant overdetermined PDE on a function. Away from the zeros of this function the almost Einstein structure is Einstein, while the zero set gives a scale singularity set which may be viewed as a conformal infinity for the Einstein metric. In this article there are two main results: we give a simple classification of the possible scale singularity spaces of almost Einstein manifolds; we derive geometric results which explicitly relate the intrinsic (conformal) geometry of the conformal infinity to the conformal structure of the ambient almost Einstein manifold. The latter includes new results for Poincaré–Einstein manifolds. Classes of examples are constructed. A compatible generalisation of the constant scalar curvature condition is also developed. This includes almost Einstein as a special case, and when its curvature is suitably negative, is closely linked to the notion of an asymptotically hyperbolic structure.

1. Introduction

A metric is said to be Einstein if its Ricci curvature is proportional to the metric [1]. Despite a long history of intense interest in the Einstein equations many mysteries remain. In high dimensions it is not known if there are any obstructions to the existence of an Einstein metric. There are 3-manifolds and 4-manifolds which do not admit Einstein metrics and the situation is especially delicate in the latter case; see [2] for an overview of some recent progress. Here we consider a specific weakening of the Einstein condition. By its nature this provides an alternative route to studying Einstein metrics but, beyond this, there are several points which indicate that it may be a useful structure in its own right. On the one hand the weakening is very slight, in a sense that will soon be clear. On the other hand, it allows in some interesting cases: at least some manifolds satisfying these equations do not admit Einstein metrics, which suggests a role as a uniformisation type condition; it includes in a natural way Poincaré–Einstein structures and conformally compact Ricci-flat asymptotically locally Euclidean (ALE) spaces, and so Einstein metrics, Poincaré–Einstein structures and these ALE manifolds are special cases of a uniform generalising structure.

Throughout the paper, we consider only metrics g of *Riemannian* signature (meaning that g is positive definite) and the conformal structures these induce; all manifolds shall be assumed to be of dimension $d \geq 3$. On a Riemannian manifold (M^d, g) the Schouten tensor P (or P^g) is a trace adjustment of the Ricci tensor given by

$$\mathrm{Ric}^g = (d-2)P^g + J^g g$$

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where J^g is the metric trace of P^g. Thus a metric is Einstein if and only if the trace-free part of P^g is zero. We will say that (M, g, s) is a *directed almost Einstein* structure if $s \in C^\infty(M)$ is a non-trivial solution to the equation

$$A(g, s) = 0 \quad \text{where } A(g, s) := \text{trace-free}(\nabla^g \nabla^g s + sP^g). \tag{1.1}$$

Here ∇^g is the Levi-Civita connection for g, and the "trace-free" means the trace-free part with respect to taking a metric trace. This is a generalisation of the Einstein condition; we will see shortly that, on the open set where s is non-vanishing, $g^o := s^{-2}g$ is Einstein. On the other hand if g is Einstein then (1.1) holds with $s = 1$. Any attempt to understand the nature and extent of this generalisation should include a description of the possible local structures of the *scale singularity set*, that is the set Σ where s is zero (and where $g^o = s^{-2}g$ is undefined). The main results in this article are some answers to this question and the development of a conformal theory to relate, quite directly, the intrinsic geometric structure of the singularity space Σ to the ambient structure. The classification results for the scale singularity set are new, although simple and elementary. On the other hand the approach to relating the (conformal) geometry of the conformal infinity to the geometry of the ambient structure is more subtle and leads to a number of new results. If s solves (1.1) then so does $-s$, and where s is non-vanishing these solutions determine the same Einstein metric. We shall say that a manifold (M, g) is *almost Einstein* if it admits a covering such that on each open set U of the cover we have that (U, g, s_U) is directed almost Einstein and on overlaps $U \cap V$ we have either $s_U = s_V$ or $s_U = -s_V$. Although there exist almost Einstein spaces which are not directed [3], to simplify the exposition we shall assume here that almost Einstein (AE) manifolds are directed. (So usually we omit the term "directed".) In any case the results apply locally on almost Einstein manifolds which are not directed.

On an Einstein manifold (M, g) the Bianchi identity implies that the scalar curvature Sc^g (i.e. the metric trace of Ric) is constant. Thus simply requiring a metric to be scalar constant is another weakening of the Einstein condition. On compact, connected oriented smooth Riemannian manifolds this may be achieved conformally: this is the outcome of the solution to the "Yamabe problem" due to Yamabe, Trudinger, Aubin and Schoen [4–7]. Just as almost Einstein generalises the Einstein condition, there is a corresponding weakening of the constant scalar curvature condition as follows. We will say that (M, g, s) is a *directed almost scalar constant* structure if $s \in C^\infty(M)$ is a non-trivial solution to the equation $S(g, s) = constant$ where

$$S(g, s) = \frac{2}{d} s(J^g - \Delta^g)s - |ds|_g^2. \tag{1.2}$$

Away from the zero set (which again we denote by Σ) of s we have $S(g, s) = \mathrm{Sc}^{g^o}/d(d-1)$ where $g^o := s^{-2}g$. In particular, off Σ, $S(g, s)$ is constant if and only if Sc^{g^o} is constant. The normalisation is so that if g^o is the metric of a space form then $S(g, s)$ is exactly the sectional curvature. We shall say that a manifold (M, g) is *almost Scalar constant* (ASC) if it is equipped with a covering such that on each open set U of the cover we have that (U, g, s_U) is directed ASC, and on overlaps $U \cap V$ we have either $s_U = s_V$ or $s_U = -s_V$. In fact, in line with our assumptions above and unless otherwise mentioned explicitly, we shall assume below that any ASC structure is directed.

As suggested above, closely related to these notions are certain classes of the so-called conformally compact manifolds that have recently been of considerable interest. We recall how these manifolds are usually described. Let M^d be a compact smooth manifold with boundary $\Sigma = \partial M$. A metric g^o on the interior M^+ of M is said to be conformally compact if it extends (with some specified regularity) to Σ by $g = s^2 g^o$ where g is non-degenerate up to the boundary, and s is a non-negative defining function for the boundary (i.e. Σ is the zero set for s, and ds is non-vanishing along Σ). In this situation the metric g^o is complete and the restriction of g to $T\Sigma$ in $TM|_\Sigma$ determines a conformal structure that is independent of the choice of defining function s; then Σ with this conformal structure is termed the conformal infinity of M^+. (This notion had its origins in the work of Newman and Penrose; see the introduction of [8] for a brief review.) If the defining function is chosen so that $|ds|_g^2 = 1$ along M then the sectional curvatures tend to -1 at infinity and the structure is said to be asymptotically hyperbolic (AH) (see [9] where there is a detailed treatment of the Hodge cohomology of these structures and related spectral theory). The model is the Poincaré hyperbolic ball and thus the corresponding metrics are sometimes called Poincaré metrics. Generalising the hyperbolic ball in another way, one may suppose that the interior conformally compact metric g^o is Einstein with the normalisation $\mathrm{Ric}(g^o) = -ng^o$, where $n = d - 1$, and in this case the structure is said to be Poincaré–Einstein (PE); in fact PE manifolds are necessarily asymptotically hyperbolic. Such structures have been studied intensively recently in relation to the proposed AdS/CFT correspondence of Maldacena [10,11], related fundamental geometric questions [12–20], and through connections to the ambient metric of Fefferman–Graham [21,22].

For simplicity of exposition we shall restrict our attention to smooth AE and ASC structures (M^d, g, s); that is (M, g) is a smooth Riemannian manifold of dimension $d \geq 3$ and $s \in C^\infty(M)$ satisfies either (1.1) (the AE case) or (1.2) (for ASC). Let us write $M^\pm$ for the open subset of M on which s is positive or, respectively, negative and, as above, Σ for the scale singularity set. The first main results (proved in Section 2) are the following classifications for the possible submanifold structures of Σ.

Theorem 1.1. *Let (M^d, g, s) be a directed almost scalar constant structure with M connected. If $S(g, s) > 0$ then s is nowhere vanishing and (M, g^o) has constant scalar curvature $d(d-1)S(g, s)$. If $S(g, s) < 0$ then s is non-vanishing on an open dense set and Σ is either empty or else is a smooth hypersurface; On $M \setminus \Sigma$, Sc^{g^o} is constant and equals $d(d-1)S(g, s)$. Suppose M is closed (i.e. compact without boundary) with $S(g, s) < 0$ and $\Sigma \neq \emptyset$. A constant rescaling of s normalises $S(g, s)$ to -1, and then $(M \setminus M^-)$ is a finite union of connected AH manifolds. Similar for $(M \setminus M^+)$.*

By hypersurface we mean a submanifold of codimension 1 which may include boundary components. In the following we will say that an ASC structure is scalar positive, scalar flat, or scalar negative if, respectively, $S(g, s)$ is positive, zero, or negative.

It seems that almost Einstein manifolds, in the generality we describe here, were introduced in [23] and it was observed there that PE manifolds are a special case; this was explained in detail in [24]. Here, among other things, we see that PE manifolds arise automatically in the scalar negative (i.e. $S(g, s) < 0$) case.

Theorem 1.2. *Let (M, g, s) be a directed almost Einstein structure with M connected. Then s is non-vanishing on an open dense set and (M, g, s) is almost scalar constant. Writing Σ for the scale singularity set, on $M \setminus \Sigma$, g^o is Einstein with scalar curvature $d(d-1)S(g, s)$. There are three cases:*

- *If $S(g, s) > 0$ then the scale singularity set Σ is empty.*
- *If $S(g, s) = 0$ then Σ is either empty or otherwise consists of isolated points and these points are critical points of the function s; in this case for each $p \in M$ with $s(p) = 0$, the metric g^o is asymptotically locally Euclidean (ALE) near p and the Weyl, Cotton, and Bach curvatures vanish at p.*
- *If $S(g, s) < 0$ then Σ is either empty or else is a totally umbilic smooth hypersurface. In particular on a closed $S(g, s) = -1$ almost Einstein manifold $(M \setminus M^-)$ is a finite union of connected Poincaré–Einstein manifolds. Similar for $(M \setminus M^+)$.*

The Cotton and Bach curvatures are defined in, respectively, (4.6) and (4.10) below. Using compactness, the last statement is an easy consequence of Proposition 3.7. That AE implies ASC is part of Theorem 2.3. Given this several parts of the Theorem are immediate from Theorem 1.1 above. The remaining parts of the Theorem summarise Theorem 3.1, Proposition 3.3, Proposition 3.6, and parts of Proposition 4.3 and Corollary 4.4. We shall say that the ALE structures arising as here are conformally *conformally compact* because of the obvious link the term as used above.

Eq. (1.2) is conformally covariant in the sense that for any $\omega \in C^\infty(M)$ we have $S(g, s) = S(e^{2\omega}g, e^{\omega}s)$. Similarly for (1.1) we have $e^{\omega}A(g, s) = A(e^{2\omega}g, e^{\omega}s)$ and so if (M, g, s) is almost Einstein then so is $(M, e^{2\omega}g, e^{w}s)$. Evidently the notions of ASC and AE structure pass to conformal geometry by descending to the equivalence classes, in the space of all such structures, as determined by the equivalence relation $(M, g, s) \sim (M, e^{2\omega}g, e^{w}s)$. This is the point of view we wish to take, throughout g is to be viewed as simply a representative of its conformal class. (We should really view the function s as corresponding to a conformal density σ of weight 1 on the conformal manifold $(M, [g])$, and A as a 2-tensor taking values in this density bundle. We shall postpone this move until Section 2.) The conformal equivalence class $[g, s]$ of (g, s) (under $(g, s) \sim (e^{2\omega}g, e^{\omega}s)$) is a structure which generalises the notion of a metric. This suggests a definition which is convenient for our discussions. A manifold M equipped with the conformal equivalence class $[g, s]$ (in this sense) of (g, s), and where s is nowhere vanishing on an open dense set, is a well-defined structure that we shall term an *almost Riemannian* manifold. Of course the zero set of s is conformally invariant and so is a preferred set $\Sigma \subset M$. An almost Riemannian structure with $\Sigma = \emptyset$ is simply a Riemannian manifold. Note that, in the cases $\Sigma \neq \emptyset$, $S(g, s)$ smoothly extends, to all of M, the natural scalar $Sc^{g^o}/n(n+1)$ which is only defined on $M \setminus \Sigma$. Similarly $A(g, s)$ smoothly extends $sP_0^{g^o}$, where $P_0^{g^o}$ is the trace-free part of P^{g^o}. Thus even though the metric $g^o = s^{-2}g$ is not defined along Σ, nevertheless $A(g, s)$ and $S(g, s)$ are defined globally (at least if we view $A(g, s)$ as representing a density valued tensor) and it is natural to think of these as curvature quantities on almost Riemannian structures. It turns out that AE manifolds, and also the cases of ASC manifolds covered in Theorem 1.1, are necessarily almost Riemannian.

The structures we consider here have an elegant and calculationally effective formulation in terms of conformal tractor calculus. On Riemannian manifolds the metric canonically determines a connection on the tangent bundle, the Levi-Civita connection. On conformal structures we lose this but there is a canonical conformally invariant connection $\nabla^{\mathcal{T}}$ on the (standard conformal) *tractor bundle* $\mathcal{T}$, as described in the next section. On $(M^d, [g])$ this is a rank $(d+2)$ bundle that contains a conformal density twisting of the tangent bundle as a subquotient. The bundle $\mathcal{T}$ also has a (conformally invariant) tractor metric h, of signature $(d+1, 1)$, that is preserved by $\nabla^{\mathcal{T}}$. On a given conformal structure we may ask if there is parallel section of $\mathcal{T}$; that is a section I of $\mathcal{T}$ satisfying $\nabla^{\mathcal{T}}I = 0$. In fact, as we see below (following [25]), this equation is simply a prolongation of (1.1). In particular, on any open set, solving $\nabla^{\mathcal{T}}I = 0$ is equivalent to solving (1.1) and there is an explicit 1–1 relationship between solutions. (We shall write s_I for the solution of (1.1) given by a parallel tractor I.) Thus an almost Einstein structure is a triple $(M, [g], I)$ where I is parallel for the standard tractor connection determined by the conformal structure $[g]$. Since the tractor connection preserves the metric h, the squared length of I, which we denote by the shorthand $|I|^2 := h(I, I)$, is constant on connected AE manifolds (and we henceforth assume M is connected). In fact $S(g, s_I) = -|I|^2$. There is a generalising result for ASC manifolds; see Proposition 2.2.

The geometric study of PE manifolds has been driven by a desire to relate the conformal geometry of the conformal infinity to the metric geometry on the interior. We may obviously extend this programme to the scalar negative (i.e. $S(g, s) < 0$) almost Einstein structures. As indicated above this is a core aim here, and in our treatment (Sections 4 and 6) the tractor structures play a key role. The first main result is Theorem 4.5 which shows, for example, that Σ satisfies a conformal analogue of the Riemannian totally geodesic condition: the intrinsic tractor connection of $(\Sigma, [g_\Sigma])$ exactly agrees with a restriction of the ambient tractor connection. In fact the results are stronger. Summarising part of Theorem 4.5 with Corollary 6.4, along the scale singularity set Σ of a scalar negative AE structure we also have results as in the following theorem. In this Ω is the curvature of the tractor connection for $(M, [g])$ while Ω^Σ is the curvature of the tractor connection for the intrinsic conformal structure of Σ. $\mathbb{W}$ is a natural conformally invariant tractor field equivalent (in dimensions $d \neq 4$) to the curvature of the Fefferman–Graham (ambient) metric over $(M, [g])$, while $\mathbb{W}^{\Sigma}$ is the same for $(\Sigma, [g_{\Sigma}])$.

Theorem 1.3.

$$\Omega(u, v) = \Omega^{\Sigma}(u, v) \quad along\ \Sigma$$

where $u, v \in \Gamma(T\Sigma)$. In dimensions $d \neq 4$ we have the stronger result

$$\Omega(\cdot, \cdot) = \Omega^{\Sigma}(\cdot, \cdot) \quad along\ \Sigma,$$

where here, by trivial extension, we view Ω^{Σ} as a section of $\Lambda^2 T^*M \otimes \mathrm{End}\,\mathcal{T}$. While in dimensions $d \geq 6$ we also have

$$(d-5)\mathbb{W}|_{\Sigma} = (d-4)\mathbb{W}^{\Sigma},$$

where $\mathbb{W}$ is the prolonged conformal curvature quantity (4.9) and again a trivial extension is involved.

In Section 6, Theorem 6.1 we also show that the Fefferman–Graham (obstruction) tensor must vanish on the scale singularity hypersurface of a smooth almost Einstein structure. An alternative direct proof that Σ is Bach-flat, when $n = 4$, is given in Corollary 4.8. A key tool derived in Section 6 is Theorem 6.3 which constructs a Fefferman–Graham ambient metric, formally to all orders, for the even-dimensional conformal structure of a scale singularity set; this construction was heavily influenced by the model in Section 5.1. An important and central aspect of the works [21] and [22] is the direct relationship between the Fefferman–Graham (ambient) metric for conformal manifolds $(\Sigma, [g_{\Sigma}])$ and suitably *even* smooth formal Poincaré–Einstein metrics, with $(\Sigma, [g_{\Sigma}])$ as the conformal infinity (see especially [22, Section 4]); in Section 5.2 there is some discussion of the meaning of even in this context. Here, in contrast, we work in one higher dimension and exploit the use of the Fefferman–Graham metric for the Poincaré–Einstein (or AE) space M itself. In this case we may work with not necessarily even PE (or AE) metrics and exploit the Fefferman–Graham metric globally over M.

In Section 4.4 we describe equations controlling (at least partially) the conformal curvature of almost Einstein structures. Importantly these are given in a form that is suitable for studying boundary problems along Σ that are based directly around the conformal curvature quantities. For example in Proposition 4.6 we observe that in this sense the Yang–Mills equations, applied to the tractor curvature, give the natural conformal equations for four-dimensional almost Einstein structures. The analogue for higher even dimensions is given in Proposition 4.10. In all dimensions we have the following result.

Theorem 1.4. *Let $(M^d, [g], I)$ be an almost Einstein manifold then*

$$I^A \not{D}_A \mathbb{W} = 0.$$

The operator $I^A \not{D}_A$ has the form $\sigma\Delta + lower\ order\ terms$. The statement here is mainly interesting in dimensions $d \geq 5$ and is a part of Theorem 4.7. Since for $d \geq 6$ we have $(d-5)\mathbb{W}|_{\Sigma} = (d-4)\mathbb{W}^{\Sigma}$, for Poincaré–Einstein manifolds (and more generally scalar negative AE structures) the Theorem suggests a Dirichlet type problem with the conformal curvature $\mathbb{W}^{\Sigma}$ of Σ as the boundary (hypersurface) data. The operator $I^A \not{D}_A$ is well defined on almost Einstein manifolds and is linked to the scattering picture of [18] as outlined in Corollary 4.9.

As mentioned, almost Einstein structures provide a generalisation of the notions of Einstein, Poincaré–Einstein and certain conformally compact ALE metrics. Aside from providing a new and uniform perspective on these specialisations, the AE structures provide a natural uniformisation type problem. We may ask for example which smooth manifolds admit an almost Einstein structure. While it is by now a classical result [1] that the sphere products $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{S}^3$ do not admit Einstein metrics it is shown in [3] that these both admit almost Einstein structures; in fact we construct these explicitly as part of a general construction of closed manifolds with almost Einstein structures. In this article we make just a small discussion of examples in Section 5. This includes the conformal sphere as the key model. It admits all scalar types of almost Einstein structure and has a central role in the construction of other examples in [3]. (In fact the standard conformal structure on the sphere admits a continuous curve of AE structures which includes the standard sphere metric, the Euclidean metric pulled back by stereographic projection as well as negative $S(g, s)$ AE structures with g^o hyperbolic off the singularity set. See Corollary 2.4 and the final comments in Section 5.1.) We conclude in Section 5.2 with a discussion of examples found by a doubling construction. Non-Einstein almost Einstein metrics turn up in the constructions and classifications by Derdzinski and Maschler of Kähler metrics which are "almost everywhere" conformal to Einstein by a non-constant rescaling factor; see e.g. [26,27] and the references therein. Some of their examples were inspired by constructions known for some time, such as [28,29]. Examples of non-Einstein $S([g], I) = 0$ AE structures are discussed in [30].

It should also be pointed out that many of the techniques and results we develop apply in other signatures. However there are also fundamental differences in the case of non-Riemannian signature and so here we confine the study to the positive definite setting.

Conversations with Michael Eastwood, Robin Graham, Felipe Leitner, and Paul-Andi Nagy have been much appreciated. It should pointed out that the existence of AE structures which are not directed was observed in the joint work [3] with Leitner and this influenced the presentation here.

2. Almost Einstein structures and conformal tractor calculus

As above let M be a smooth manifold, of dimension $d \geq 3$, equipped with a Riemannian metric g_{ab}. Here and throughout we employ Penrose's abstract index notation. We write $\mathcal{E}^a$ to denote the space of smooth sections of the tangent bundle TM on M, and $\mathcal{E}_a$ for the space of smooth sections of the cotangent bundle T^*M. (In fact we will often use the same symbols for the bundles themselves. Occasionally, to avoid any confusion, we write $\Gamma(\mathcal{B})$ to mean the space of sections of a bundle $\mathcal{B}$.) We write $\mathcal{E}$ for the space of smooth functions and all tensors considered will be assumed smooth without further comment. An index which appears twice, once raised and once lowered, indicates a contraction. The metric g_{ab} and its inverse g^{ab} enable the identification of $\mathcal{E}^a$ and $\mathcal{E}_a$ and we indicate this by raising and lowering indices in the usual way.

With ∇_a denoting the Levi-Civita connection for g_{ab}, and using that this is torsion-free, the Riemann curvature tensor $R_{ab}{}^c{}_d$ is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)V^c = R_{ab}{}^c{}_d V^d \quad \text{where } V^c \in \mathcal{E}^c.$$

This can be decomposed into the totally trace-free *Weyl curvature* C_{abcd} and the symmetric *Schouten tensor* P_{ab} according to

$$R_{abcd} = C_{abcd} + 2g_{c[a}P_{b]d} + 2g_{d[b}P_{a]c}, \tag{2.1}$$

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. Thus P_{ab} is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^c{}_b$:

$$\mathrm{Ric}_{ab} = (n-2)P_{ab} + Jg_{ab}, \qquad J := P^a{}_a.$$

In denoting such curvature quantities we may write e.g. Ric^g or simply Ric depending on whether there is a need to emphasise the metric involved. Also abstract indices will be displayed or suppressed as required for clarity.

Under a conformal rescaling of the metric

$$g \mapsto g^o = s^{-2}g,$$

with $s \in \mathcal{E}$ non-vanishing, the Weyl tensor $C_{ab}{}^c{}_d$ is unchanged (and so we say the Weyl tensor is conformally invariant) whereas the Schouten tensor transforms according to

$$P^{g^o}_{ab} = P^g_{ab} + s^{-1}\nabla_a\nabla_b s - \frac{1}{2}g^{cd}s^{-2}(\nabla_c s)(\nabla_d s)g_{ab}. \tag{2.2}$$

Taking, via g^o, a trace of this we obtain

$$J^{g^o} = s^2 J^g - s\Delta s - \frac{d}{2}|ds|^2_g,$$

where the Δ is the "positive energy" Laplacian. Note that the right-hand side of the last display is $\frac{d}{2}S(g,s)$, with $S(g,s)$ as defined in (1.2). Clearly this is well defined for smooth s even if s may be zero at some points. On the other hand the right-hand side above (and hence $S(g,s)$) is clearly invariant under the conformal transformation $(g,s) \mapsto (e^{2\omega}g, e^{\omega}s)$: this is true away from the zeros of s since there J^{g^o} depends only on the 2-jet of $g^o = s^{-2}g$, but the explicit conformal transformation of the right-hand side is evidently polynomial in e^{ω} and its 2-jet.

Let us digress to prove Theorem 1.1 since it illustrates how an almost Riemannian structure may arise immediately from a formula polynomial in the jets of s.

Proof of Theorem 1.1. Under a dilation $g \mapsto \mu g$ ($\mu \in \mathbb{R}_+$, and fixing s) we have $S(g,s) \mapsto \mu^{-1}S(g,s)$, so to prove the Theorem we may consider just the cases $S(g,s) = 1$ and $S(g,s) = -1$. Suppose that $S(g,s) = 1$ then if $p \in M$ were to be a point where $s_p = 0$ then at p we would have $1 = -|ds|^2_g$ which would be a contradiction. Suppose that $S(g,s) = -1$. Then at any point $p \in M$ where $s_p = 0$ we have $|ds|^2_g = 1$. For the last statements of the Theorem assume that M is closed and the scale singularity set Σ is not empty. Then Σ is a hypersurface (not necessarily connected) which separates M according to the sign of s. The restriction of g^o to the interior of $M \setminus M^-$ (i.e. to M^+) is conformally compact since the restriction of g to $M \setminus M^-$ extends s^2g^o smoothly to the boundary. Finally $(M \setminus M^-, g, s)$ is AH since $|ds|^2_g = 1$ along Σ. By compactness this consists of a finite union of connected AH components. The same analysis applies to $M \setminus M^+$. $\square$

Note that although setting $S(g,s)$ constant is a weakening of the constant scalar curvature condition, Eq. (1.2) is still quite restrictive. For example, it is evident that on closed manifolds with negative Yamabe constant there are no non-trivial solutions with $S(g,s)$ a non-negative constant.

The tensor $A(g,s)$ defined in the Introduction should be compared to the trace-free part of the right-hand side of (2.2) above. Arguing as for $S(g,s)$ above, or by direct calculation, one finds that under $(g,s) \mapsto (e^{2\omega}g, e^{\omega}s)$ we have $A(g,s) \mapsto e^{\omega}A(g,s)$ as mentioned earlier. So both the AE condition and the more general ASC condition are best treated as structures on a conformal manifold. To obtain a clean treatment it is most efficient to draw in some standard objects from conformal geometry; the further details and background may be found in [31,32]. Clearly we may view a conformal structure on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related metrics at the point x. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$

induces a natural line bundle on $(M, [g])$ that we term the conformal density bundle $\mathcal{E}[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. Note $\mathcal{E}[w]$ is trivialised by a choice of metric g from the conformal class, and we write ∇ for the connection corresponding to this trivialisation. It follows immediately that (the coupled) ∇_a preserves the conformal metric. (Note on a fixed conformal structure the conformal densities bundle $\mathcal{E}[-n]$ may be identified in an obvious way with appropriate powers of the 1-density bundle associated to the frame bundle through the representation $|\det(\ )|^{-1}$. See e.g. [31]. Via this identification, the connection we defined on $\mathcal{E}[w]$ agrees with the Levi-Civita connection.)

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes \mathcal{E}[2]$ determined by the conformal structure. This will be henceforth used to identify TM with $T^*M[2]$ even when we have fixed a metric from the conformal class. (For example, with these conventions the Laplacian Δ is given by $\Delta = -\boldsymbol{g}^{ab}\nabla_a\nabla_b = -\nabla^b\nabla_b$.) Although this is conceptually valuable and significantly simplifies many calculations, it is, however, a point where there is potential for confusion. For example in the below, when we write J or J^g we mean $\boldsymbol{g}^{ab}P_{ab}$ where P is the Schouten tensor for some metric g. Thus J is a section of $\mathcal{E}[-2]$ (which depends on g).

In this picture to study the ASC condition we replace $s \in \mathcal{E}$ with a section $\sigma \in \mathcal{E}[1]$ in (1.2) to obtain

$$S([g], \sigma) = \frac{2}{d}\sigma(J^g - \Delta^g)\sigma - |\nabla\sigma|^2_{\boldsymbol{g}}, \tag{2.3}$$

where we have written $|\nabla\sigma|^2_{\boldsymbol{g}}$ as a brief notation for $\boldsymbol{g}^{-1}(\nabla\sigma, \nabla\sigma)$. When the conformal structure is fixed we shall often denote the quantity displayed by simply $S(\sigma)$. Similarly the conformally invariant version of A is the 2-tensor of conformal weight 1 given by

$$A([g], \sigma) := \text{trace-free}(\nabla_a\nabla_b\sigma + P_{ab}\sigma),$$

again we may write simply $A(\sigma)$.

The $A([g], \sigma) = 0$ equation (i.e. (1.1)) becomes

$$\nabla_a\nabla_b\sigma + P_{ab}\sigma + \rho\boldsymbol{g}_{ab} = 0 \tag{2.4}$$

where ρ is a density (in $\mathcal{E}[-1]$) to accommodate the trace part. Here ∇ and P are given with respect to some metric g in the conformal class, but the equation is invariant under conformal rescaling.

We may replace (2.4) with the equivalent first order system

$$\nabla_a\sigma - \mu_a = 0, \quad \text{and} \quad \nabla_a\mu_b + P_{ab} + \boldsymbol{g}_{ab}\rho = 0,$$

where $\mu_a \in \mathcal{E}_a[1] := \mathcal{E}_a \otimes \mathcal{E}[1]$. Differentiating the second of these and considering two possible contractions yields

$$\nabla_a\rho - P_{ab}\mu^b = 0,$$

whence we see that the system has closed up linearly. Eq. (2.4) is equivalent to a connection and a parallel section for this: on any open set in M, a solution of (2.4) is equivalent to $I := (\sigma, \mu_a, \rho) \in \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ satisfying $\nabla^{\mathcal{T}} I = 0$ where

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \boldsymbol{g}_{ab}\rho + P_{ab}\sigma \\ \nabla_a\rho - P_{ab}\mu^b \end{pmatrix}. \tag{2.5}$$

The connection $\nabla^{\mathcal{T}}$ constructed here (following [25]) is the normal conformal tractor connection. We will often write simply ∇ for this when the meaning is clear by context. This is convenient since we will couple the tractor connection to the Levi-Civita connection.

Let us write $J^k\mathcal{E}[1]$ for the bundle of k-jets of germs of sections of $\mathcal{E}[1]$. Considering, at each point of the manifold, sections which vanish to first order at the given point reveals a canonical sequence,

$$0 \to S^2T^*M \otimes \mathcal{E}[1] \to J^2\mathcal{E}[1] \to J^1\mathcal{E}[1] \to 0.$$

This is the jet exact sequence at 2-jets. Via the conformal metric $\boldsymbol{g}$, the bundle of symmetric covariant 2-tensors S^2T^*M decomposes directly into the trace-free part, which we will denote $S^2_0T^*M$, and a pure trace part isomorphic to $\mathcal{E}[-2]$, hence $S^2T^*M \otimes \mathcal{E}[1] = (S^2_0T^*M \otimes \mathcal{E}[1]) \oplus \mathcal{E}[-1]$. The *standard tractor bundle* $\mathcal{T}$ may defined as the quotient of $J^2\mathcal{E}[1]$ by the image of $S^2_0T^*M \otimes \mathcal{E}[1]$ in $J^2\mathcal{E}[1]$. By construction this is invariant, it depends only on the conformal structure. Also by construction, it is an extension of the 1-jet bundle

$$0 \to \mathcal{E}[-1] \xrightarrow{X} \mathcal{T} \to J^1\mathcal{E}[1] \to 0. \tag{2.6}$$

The canonical homomorphism X here will be viewed as a section of $\mathcal{T}[1] = \mathcal{T} \otimes \mathcal{E}[1]$ and, with the jet exact sequence at 1-jets, controls the filtration structure of $\mathcal{T}$.

Next note that there is a tautological operator $D : \mathcal{E}[1] \to \mathcal{T}$ which is simply the composition of the universal 2-jet differential operator $j^2 : \mathcal{E}[1] \to \Gamma(J^2\mathcal{E}[1])$ followed by the canonical projection $J^2\mathcal{E}[1] \to \mathcal{T}$, from the definition of $\mathcal{T}$.

On the other hand, via a choice of metric g, and the Levi-Civita connection it determines, we obtain a differential operator $\mathcal{E}[1] \to \mathcal{E}[1] \oplus \mathcal{E}^1[1] \oplus \mathcal{E}[-1]$ by $\sigma \mapsto (\sigma, \nabla_a\sigma, \frac{1}{d}(\Delta - \mathrm{J})\sigma)$ and this obviously determines an isomorphism

$$\mathcal{T} \stackrel{g}{\cong} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]. \tag{2.7}$$

In the following we shall frequently use (2.7). Sometimes this will be without any explicit comment but also we may write for example $t \stackrel{g}{=} (\sigma, \mu_a, \rho)$, or alternatively $[t]_g = (\sigma, \mu_a, \rho)$, to mean t is an invariant section of $\mathcal{T}$ and (σ, μ_a, ρ) is its image under the isomorphism (2.7). Changing to a conformally related metric $\widehat{g} = e^{2\omega}g$ (ω a smooth function) gives a different isomorphism, which is related to the previous by the transformation formula

$$\widehat{(\sigma, \mu_b, \rho)} = (\sigma, \mu_b + \sigma\Upsilon_b, \rho - \boldsymbol{g}^{bc}\Upsilon_b\mu_c - \frac{1}{2}\sigma\boldsymbol{g}^{bc}\Upsilon_b\Upsilon_c), \tag{2.8}$$

where $\Upsilon := d\omega$. It is straightforward to verify that the right-hand side of (2.5) also transforms in this way and hence $\nabla^{\mathcal{T}}$ gives a conformally invariant connection on $\mathcal{T}$ which we shall also denote by $\nabla^{\mathcal{T}}$. This is the tractor connection. There is also a conformally invariant *tractor metric* h on $\mathcal{T}$ given (as a quadratic form) by

$$(\sigma, \mu, \rho) \mapsto \boldsymbol{g}^{-1}(\mu, \mu) + 2\sigma\rho. \tag{2.9}$$

This is preserved by the connection and clearly has signature $(d + 1, 1)$.

Let us return to our study of Eqs. (2.4) and (1.2). First observe that, given a metric g, via (2.7) the tautological invariant operator D from above is given by the explicit formula

$$D : \mathcal{E}[1] \to \mathcal{T} \qquad \sigma \mapsto \left(\sigma, \nabla_a\sigma, \frac{1}{d}(\Delta\sigma - \mathrm{J}\sigma)\right). \tag{2.10}$$

This is a differential splitting operator, since it is inverted by the canonical tractor X: $h(X, D\sigma) = \sigma$. (To see this one may use that in terms of the splitting (2.7) $X = (0, 0, 1)$.) If a standard tractor I satisfies $I = D\sigma$ for some $\sigma \in \mathcal{E}[1]$ then $\sigma = h(X, I)$ and we shall term I a *scale tractor*. For the study of scale tractors the following result is useful.

Lemma 2.1. *For σ a section of $\mathcal{E}[1]$ we have*

$$|D\sigma|^2 := h(D\sigma, D\sigma) = \frac{2}{d}\sigma(\Delta^g - J^g)\sigma + |\nabla^g\sigma|^2_{\boldsymbol{g}}, \tag{2.11}$$

where $|\nabla\sigma|^2_{\boldsymbol{g}}$ means $\boldsymbol{g}^{ab}(\nabla_a\sigma)\nabla_b\sigma$. In particular, if $\sigma(p) = 0$, at $p \in M$, then

$$|D\sigma|^2(p) = |\nabla\sigma|^2_{\boldsymbol{g}}(p).$$

Proof. This follows easily from formulae (2.9) and (2.10). □

Using Lemma 2.1, we have the following.

Proposition 2.2. *If I is a scale tractor then*

$$|I|^2 = -S(\sigma),$$

where $\sigma = h(X, \sigma)$. In particular off the zero set of σ we have

$$|I|^2 = -\frac{2}{d}\underline{J}^{g^o}$$

where $g^o = \sigma^{-2}\boldsymbol{g}$ and $\underline{J}^{g^o}$ is the g^o trace of P^{g^o}. An ASC structure is a conformal manifold $(M, [g])$ equipped with a scale tractor of constant length.

Proof. Everything is clear except the point made in the second display. Recall that now, in contrast to the Introduction, J^{g^o} denotes $\boldsymbol{g}^{ab}P^{g^o}_{ab}$. So, writing g_o for the inverse to g^o, we have

$$\sigma^2 J^{g^o} = \sigma^2\boldsymbol{g}^{ab}P^{g^o}_{ab} = g_o^{ab}P_{ab} =: \underline{J}^{g^o},$$

this is the metric g^o trace of the Schouten tensor P^{g^o}. On the other hand, away from the zero set of σ, we may calculate in the scale σ and we have $\nabla^{g^o}\sigma = 0$, whence $-2\sigma^2 J^{g^o}/d$ is exactly the right-hand side of (2.11). □

Now collecting our observations we obtain the basic elements of the tractor picture for AE structures, as follows.

Theorem 2.3. *A directed almost Einstein structure is a conformal manifold $(M^{n+1}, [g])$ equipped with a parallel (standard) tractor $I \neq 0$. The mapping from non-trivial solutions of (2.4) to parallel tractors is by $\sigma \mapsto D\sigma$ with inverse $I \mapsto \sigma := h(I, X)$. If $I \neq 0$ is parallel and $\sigma := h(I, X)$ then the structure $(M, [g], \sigma)$ is ASC with $S([g], \sigma) = -|I|^2$. On the open set where σ is nowhere vanishing $g^o := \sigma^{-2}\mathbf{g}$ is Einstein with $\mathrm{Ric}^{g^o} = -n|I|^2 g^o$.*

Proof. The first observation is immediate from the construction in (2.5) of the tractor connection as a prolongation of Eq. (2.4) for an almost Einstein structure.

Next observe that if $I \stackrel{g}{=} (\sigma,\ \mu_a,\ \rho)$ is a parallel section for $\nabla^{\mathcal{T}}$ then it follows immediately from formula (2.5) that necessarily

$$\big(\sigma, \mu_a, \rho\big) = \left(\sigma, \nabla_a \sigma, \frac{1}{d}(\Delta\sigma - J\sigma)\right), \tag{2.12}$$

that is I is a scale tractor, $I = D\sigma$. From the formula for the tractor metric it follows that $\sigma = h(X, I)$.

Since the tractor connection preserves the tractor metric it follows that if I is a parallel tractor then $|I|^2 := h(I, I)$ is constant. Thus an almost Einstein structure is ASC as claimed.

For the final statement we use that I parallel implies that σ satisfies (2.4). On the set where σ is nowhere vanishing we may use the metric $g^o = \sigma^{-2}\mathbf{g}$. The corresponding Levi-Civita connection annihilates σ and then (2.4) asserts that P^{g^o} is trace-free. □

In view of the Theorem we shall often use the notation $(M, [g], I)$ to denote a directed almost Einstein manifold. In this context I should be taken as parallel and non-zero.

There is a useful immediate consequence of the Theorem, as follows.

Corollary 2.4. *On a fixed conformal structure $(M, [g])$ the set of directed AE structures is naturally a vector space with the origin removed. In particular if I_1 and I_2 are two linearly independent directed AE structures then for each $t \in \mathbb{R}$*

$$I_t := (\sin t) I_1 + (\cos t) I_2$$

is a directed AE structure. In this case given $p \in M$ there is $t \in \mathbb{R}$ so that $\sigma_t(p) := h(X, I_t)_p = 0$.

One might suspect that generically non-scalar positive AE manifolds will have non-empty scale singularity sets. The Corollary shows that this certainly is the case on a fixed conformal structure with two linearly independent AE structures.

3. Classification of the scale singularity set

Given a standard tractor I and $\sigma := h(X, I)$ let us write $S(I)$ as an alternate notation for $S(\sigma)$. As before we write

$$\Sigma := \{p \in M \mid \sigma(p) = 0\}$$

and term this the *scale singularity set* of I; this is the set where $g^o = \sigma^{-2}\mathbf{g}$ is undefined. In this section we shall establish the following Theorem before extending our results to a proof of Theorem 1.2.

Theorem 3.1. *Let $(M, [g], I)$ be an almost Einstein structure of Riemannian signature. There are three cases:*

- $|I|^2 < 0$, *which is equivalent to $S(I) > 0$, then Σ is empty and $(M, \sigma^{-2}\mathbf{g})$ is Einstein with positive scalar curvature;*
- $|I|^2 = 0$, *which is equivalent to $S(I) = 0$, then Σ is either empty or consists of isolated points, and $(M \setminus \Sigma, \sigma^{-2}\mathbf{g})$ is Ricci-flat;*
- $|I|^2 > 0$, *which is equivalent to $S(I) < 0$, then the scale singularity set Σ is either empty or else is a totally umbilic smooth hypersurface, and $(M \setminus \Sigma, \sigma^{-2}\mathbf{g})$ is Einstein of negative scalar curvature.*

The curvature statements follow from Theorem 2.3. Also from there we have that an AE manifold is ASC. Thus from Theorem 1.1 we have at once both the first result and also that if, alternatively, $|I|^2 > 0$ then the singularity set is either empty or is a hypersurface. The proof is completed via Propositions 3.3 and 3.6 below.

We shall make a general observation which sheds light on the scalar flat case. From Theorem 2.3, I parallel implies $I = D\sigma$, for some density σ in $\mathcal{E}[1]$. An obvious question is whether, at any point $p \in M$, we may have $j^1_p\sigma = 0$, i.e. whether the 1-jet of σ may vanish at p. Evidently this is impossible if $|I|^2 \neq 0$. We observe here (cf. [23]) that, in any case, if $I = D\sigma \neq 0$ is parallel then the zeros of $j^1\sigma$ are isolated. In fact we have a slightly stronger result. As usual here we write $\sigma = h(X, I)$.

Lemma 3.2. *Suppose that $I \neq 0$ is parallel and $j^1_p\sigma = 0$. Then there is a neighbourhood of p such that, in this neighbourhood, σ is non-vanishing away from p.*

Proof. Suppose that $I \neq 0$ is parallel and $j^1_p\sigma = 0$. Since I is parallel $I = D\sigma$. This with $j^1_p\sigma = 0$ implies that, at p, and in the scale g, we have $I \stackrel{g}{=} (0\ , 0\ , \rho)$ for some density ρ with $\rho(p) \neq 0$. Thus from (2.5) (or equivalently (2.4)) we have $(\nabla_a\nabla_b\sigma)(p) = -\rho(p)\mathbf{g}_{ab}(p)$. Trivialising the density bundles using the metric g the latter is equivalent to $(\nabla_a\nabla_b s)(p) = -r(p)g_{ab}(p)$ where the smooth function r satisfies $r(p) \neq 0$. (Here we use that $g = \tau^{-2}\mathbf{g}$ for some non-vanishing τ in $\mathcal{E}[1]$ and $s = \tau^{-1}\sigma$ while $r = \tau\rho$. Then since ∇ is the Levi-Civita for g we have $\nabla\tau = 0$.) So, in terms of coordinates based at p, the first non-vanishing term in the Taylor series for s (based at p) is $-rg_{ij}x^ix^j$. □

Note that an ASC structure is scalar flat if and only if g^o is Ricci-flat on $M \setminus \Sigma$. In the following $\sigma := h(X, I)$.

Proposition 3.3. *If $(M, [g], I)$ is an ASC structure with $j^1_p\sigma = 0$, at some point p, then $(M, [g], I)$ is scalar flat. Conversely if $(M, [g], I)$ is a scalar flat ASC structure then, at any $p \in M$ with $\sigma(p) = 0$ we have $j^1_p\sigma = 0$.*

If $(M, [g], I)$ is a scalar flat AE structure then, at any $p \in M$ with $\sigma(p) = 0$ we have $j^1_p\sigma = 0$ and $j^2_p\sigma \neq 0$. For any scalar flat AE manifold the scale singularity set consists of isolated points.

Proof. Since by definition $I = D\sigma$, from Lemma 2.1 it is immediate that, at any point p with $\sigma(p) = 0$, we have $S(\sigma)(p) = 0$ if and only if $j^1_p\sigma = 0$. (Alternatively this is visible directly from formula (2.3).) The first two statements follow immediately, as by definition $S(\sigma)$ is constant on an ASC manifold.

Now we consider AE manifolds. These are ASC and so we have the first results. Since I is parallel, we have $I = D\sigma$. If an AE manifold is scalar flat then, at a point p where $\sigma(p) = 0$, we have $j^1_p\sigma = 0$ and so from (2.12) the tractor I is of the form $I \stackrel{g}{=} (0,\ 0,\ \rho)$ at p. On the other hand, since $I \neq 0$ is parallel, it follows that $D\sigma = I$ is nowhere zero on M. Hence (since D is a second order differential operator) $j^2\sigma$ is non-vanishing. In fact, from (2.10), at any point p where $j^1_p\sigma$ vanishes we have $\rho(p) = \frac{1}{d}(\Delta\sigma)(p) \neq 0$. The last statement is now an immediate consequence of Lemma 3.2. $\square$

Remark. Note that $j^1_p\sigma = 0$ means that when we work in terms of a background metric g we have $j^1_p s = 0$ for the function s corresponding to σ and so p is a critical point of s. In fact it is already clear from (1.2) that, even for ASC structures, if $S(g, s) = 0$ then $s_p = 0$ implies p is a critical point. ■

3.1. Conformal hypersurfaces and the scale singularity set

Let us first recall some facts concerning general hypersurfaces in a conformal manifold $(M^d, [g])$, $d \geq 3$. If Σ is a boundary component of a Riemannian (or conformal) manifold then, without further comment, we will assume that the conformal structure extends smoothly to a collar of the boundary. Our results will not depend on the choice of extension. So in the following we suppose that Σ is an embedded codimension 1 submanifold of M.

Let n_a be a section of $\mathcal{E}_a[1]$ such that, along Σ, n_a is a conormal that satisfies $|n|^2_{\mathbf{g}} := \mathbf{g}^{ab}n_a n_b = 1$. Note that the latter is a conformally invariant condition since $\mathbf{g}^{-1}$ has conformal weight -2. Now in the scale g, the mean curvature of Σ is given by

$$H^g = \frac{1}{d-1}\big(\nabla_a n^a - n^a n^b \nabla_a n_b\big),$$

as a conformal -1-density. This is independent of how n_a is extended off Σ. Now under a conformal rescaling, $g \mapsto \widehat{g} = e^{2\omega}g$, H transforms to $H^{\widehat{g}} = H^g + n^a\Upsilon_a$ where $\Upsilon = d\omega$ (and we use this notation below without further mention). Thus we obtain a conformally invariant section N of $\mathcal{T}|_\Sigma$

$$N \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix},$$

and from (2.9) $h(N, N) = 1$ along Σ. Obviously N is independent of any choices in the extension of n_a off Σ. This is the *normal tractor* of [25] and may be viewed as a tractor bundle analogue of the unit conormal field from the theory of Riemannian hypersurfaces.

Recall that a point p in a hypersurface is an umbilic point if, at that point, the trace-free part of the second fundamental form is zero. This is a conformally invariant condition. A hypersurface is totally umbilic if this holds at all points. Differentiating N tangentially along Σ using $\nabla^{\mathcal{T}}$, directly from (2.5) we obtain the following result.

Lemma 3.4. *If the normal tractor N is parallel, with respect to $\nabla^{\mathcal{T}}$, along a hypersurface Σ then the hypersurface Σ is totally umbilic.*

In fact constancy of N along a hypersurface is equivalent to total umbilicity. This is (Proposition 2.9) from [25].

We return now to the study of ASC structures, on which I denotes the scale tractor. First we see that the normal tractor is linked, in an essential way, to the ambient geometry off the hypersurface.

Proposition 3.5. *Let $(M^d, [g], I)$ be a scalar negative ASC structure with scale singularity set $\Sigma \neq \emptyset$ and $|I|^2 = 1$. Then, with N denoting the normal tractor for Σ, we have $N = I|_\Sigma$.*

Proof. As usual let us write $\sigma := h(X, I)$. By definition

$$I = D\sigma \stackrel{g}{=} \begin{pmatrix} \sigma \\ \nabla_a\sigma \\ \frac{1}{d}(\Delta\sigma - J\sigma) \end{pmatrix}.$$

Let us write $n_a := \nabla_a \sigma$. Along Σ we have $\sigma = 0$, therefore

$$I|_\Sigma \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ \frac{1}{d}\Delta\sigma \end{pmatrix},$$

and from Lemma 2.1 $|n|^2_{\mathbf{g}} = 1$, since $|I|^2 = 1$. So $n_a|_\Sigma$ is a conformal weight 1 conormal field for Σ.

Next we calculate the mean curvature H in terms of σ. Recall $(d-1)H = \nabla^a n_a - n^a n^b \nabla_b n_a$, on Σ. We calculate the right-hand side in a neighbourhood of Σ. Since $n_a = \nabla_a \sigma$, we have $\nabla^a n_a = -\Delta\sigma$. On the other hand

$$n^a n^b \nabla_b n_a = \frac{1}{2} n^b \nabla_b (n^a n_a) = \frac{1}{2} n^b \nabla_b \left(1 - \frac{2}{d}\sigma\Delta\sigma + \frac{2}{d} J\sigma^2\right),$$

where we used that $|D\sigma|^2 = 1$. Now along Σ we have $1 = n^a n_a = n^a \nabla_a \sigma$, and so there this simplifies to

$$n^a n^b \nabla_b n_a = -\frac{1}{d}\Delta\sigma.$$

Putting these results together, we have

$$(d-1)H = \frac{1}{d}(1-d)\Delta\sigma \Rightarrow H = -\frac{1}{d}\Delta\sigma.$$

Thus

$$I|_\Sigma \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H \end{pmatrix},$$

as claimed. □

A consequence for AE structures follows easily.

Proposition 3.6. *Let $(M^d, [g], I)$ be a scalar negative almost Einstein structure with scale singularity set $\Sigma \neq \emptyset$ and $|I|^2 = 1$. Then Σ is a totally umbilic hypersurface with $I|_\Sigma = N$, the normal tractor for Σ.*

Proof. Since an AE structure $(M, [g], I)$ is ASC it follows from Proposition 3.5 above that along the singularity hypersurface I agrees with the normal tractor N. On the other hand, since I is parallel everywhere, it follows that N is parallel along Σ and so, from Lemma 3.4, Σ is totally umbilic. □

Proposition 2.8 of LeBrun's [8] also gives a proof that the conformal infinity of a PE metric is totally umbilic.

Proof of Theorem 3.1. The remaining point is to show that if $(M, [g], I)$ is AE with $|I|^2 > 0$ and a singularity hypersurface Σ, then this is totally umbilic. This is immediate from the previous Proposition as multiplying I with a positive constant yields a yields a parallel tractor with the same singularity set. □

Much of Theorem 1.2 repackages the tractor based statements in Theorem 3.1 above, other parts follow from Theorem 1.1. To complete the proof we simply need to describe PE manifolds in the same language, and this is our final aim for this section.

Proposition 3.7. *Suppose that M is a compact manifold with boundary Σ, and $(M, [g], I)$ is an almost Einstein structure with $|I|^2 = 1$, and such that the scale singularity set is Σ. Then $(M, [g], I)$ is a Poincaré–Einstein manifold with the interior metric $g^o = \sigma^{-2}\mathbf{g}$, where $\sigma := h(X, I)$. Conversely Poincaré–Einstein manifolds are scalar negative almost Einstein structures.*

Proof. Suppose that $(M, [g], I)$ is an AE structure as described. Since AE manifolds are ASC, with the parallel tractor I giving the scale tractor of the ASC structure, it follows from Theorem 1.1 that $(M, [g], \sigma)$ is AH. But I parallel means that $g^o = \sigma^{-2}\mathbf{g}$ is Einstein on $M \setminus \Sigma$, and there $|I|^2 = 1$ is equivalent to $\mathrm{Ric}(g^o) = -ng^o$. The converse direction is also straightforward, or see [24]. □

4. Conformal geometry of Σ versus conformal geometry of M

Here for almost Einstein manifolds we shall derive basic equations satisfied by the conformal curvatures. In particular for Poincaré–Einstein manifolds, and more generally for scalar negative almost Einstein manifolds, we shall study the relationship between the conformal geometry of M and the intrinsic conformal geometry of the scale singularity set Σ. Since Σ is a hypersurface, a first step is to understand the conformal structure induced on an arbitrary hypersurface in a conformal manifold and in particular the relationship between the intrinsic conformal tractor bundle of Σ and the ambient tractor bundle of M. This is the subject of Section 4.1. On the other hand we have already observed that on scalar negative AE manifolds the singularity set is umbilic. So the main aim of this section is to deepen this picture. We shall see that the along the singularity hypersurface the intrinsic tractor connection necessarily agrees with an obvious restriction of the ambient tractor connection. This has immediate consequences for the relationship between the intrinsic and ambient conformal curvature quantities, but we are able to also show that there is an even stronger compatibility between the conformal curvatures of $(\Sigma, [g_\Sigma])$ and those of $(M, [g])$. Finally we shall derive equations on the latter that partly establish a Dirichlet type problem based directly on the conformal curvature quantities.

4.1. Conformal hypersurfaces

Here we revisit (cf. Section 3.1) the study of a general hypersurface Σ in a conformal manifold $(M^d, [g])$, $d \geq 3$. This time our aim is to see, in this general setting, how the conformal structure of the hypersurface is linked that of the ambient space.

With respect to the embedding map, each metric g from the conformal class on M pulls back to a metric g_Σ on Σ. Thus the ambient conformal structure of M induces a conformal structure $[g_\Sigma]$ on Σ^n $(n+1=d)$; we shall refer to this as the *intrinsic* conformal structure of Σ. Given the relationship of the intrinsic and ambient conformal structures it follows easily that the intrinsic conformal density bundle of weight w, $\mathcal{E}^\Sigma[w]$ is canonically isomorphic to $\mathcal{E}[w]|_\Sigma$ and we shall no longer distinguish these. It is also clear that, since g_Σ is determined by g, the trivialisations g_Σ and g induce on, respectively, $\mathcal{E}^\Sigma[w]$ and $\mathcal{E}[w]$ are consistent. In particular the Levi-Civita connection on $\mathcal{E}^\Sigma[w]$ agrees with the restriction of the Levi-Civita connection on $\mathcal{E}[w]$ (and see Section 2 for a discussion of the latter).

If $n \geq 3$ then $(\Sigma, [g_\Sigma])$ has an intrinsic tractor bundle $\mathcal{T}_\Sigma$. We want to relate this to $\mathcal{T}$ along Σ. Note that $\mathcal{T}|_\Sigma$ has a canonical rank $n+2$ subbundle, viz. $N^\perp$ the subbundle orthogonal (with respect to h) to the normal tractor N. As noted in [33], there is a canonical (conformally invariant) isomorphism

$$N^\perp \xrightarrow{\cong} \mathcal{T}_\Sigma. \tag{4.1}$$

To see this let n_a denote a weight 1 conormal field along Σ. There is a canonical inclusion of $T\Sigma$ in $TM|_\Sigma$ and we identify $T^*\Sigma$ with the annihilator subbundle in $T^*M|_\Sigma$ of n^a. These identifications do not require choosing a metric from the conformal class. Now calculating in a scale g on M the tractor bundle $\mathcal{T}$, and hence also $N^\perp$, decomposes into a triple via (2.7). Then the mapping of the isomorphism is (cf. [34])

$$[N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b - Hn_b\sigma \\ \rho + \dfrac{1}{2}H^2\sigma \end{pmatrix} \in [\mathcal{T}_\Sigma]_{g_\Sigma} \tag{4.2}$$

where, as usual, H denotes the mean curvature of Σ in the scale g and g_Σ is the pull-back of g to Σ. Since (σ, μ_b, ρ) is a section of $[N^\perp]_g$ we have $n^a\mu_a = H\sigma$. Using this one easily verifies that the mapping is conformally invariant: If we transform to $\widehat{g} = e^{2\omega}g$, $\omega \in \mathcal{E}$, then (σ, μ_b, ρ) transforms according to (2.8). Using that $\widehat{H} = H + n^a\Upsilon_a$ one calculates that the image of (σ, μ_b, ρ) (under the map displayed) transforms by the intrinsic version of (2.8), that is by (2.8) except where Υ_a is replaced by $\Upsilon_a^\Sigma = \Upsilon_a - n_a n^b\Upsilon_b$ (which on Σ agrees with $d^\Sigma\omega$, the intrinsic exterior derivative of ω). This signals that the explicit map displayed in (4.2) descends to a conformally invariant map (4.1). We henceforth use this to identify $N^\perp$ with $\mathcal{T}_\Sigma$, and write $\mathrm{Proj}_\Sigma : \mathcal{T}|_\Sigma \to \mathcal{T}_\Sigma$ for the orthogonal projection afforded by N.

So far we understand the tractor bundle on Σ for $n \geq 3$. In the case of $n = 2$, Σ does not in general have a preferred intrinsic conformal tractor connection. There is much to be said in this case but for our current purposes it will be most economical to proceed as follows. In the case of a dimension $n = 2$ surface Σ in $(M^3, [g])$, we shall *define* $\mathcal{T}_\Sigma$ to be the subbundle orthogonal to N in $\mathcal{T}|_\Sigma$. As in other dimensions we write $\mathrm{Proj}_\Sigma : \mathcal{T}|_\Sigma \to \mathcal{T}_\Sigma$ for the corresponding orthogonal projection. Then for $d = 3$, equivalently $n = 2$, we define the tractor connection on Σ to be the orthogonal projection of the ambient tractor connection. That is, working locally, for $v \in \Gamma(T\Sigma)$ and $T \in \mathcal{T}_\Sigma = N^\perp$ we extend these smoothly to $v \in \Gamma(TM)$ and $T \in \mathcal{T}$. Then we *define* $\nabla_v^{\mathcal{T}_\Sigma} T := \mathrm{Proj}_\Sigma(\nabla_v^{\mathcal{T}} T)$ along Σ. It is verified by standard arguments that this is independent of the extension choices and defines a connection on $\mathcal{T}_\Sigma$.

Finally we observe a useful alternative approach to the arguments above via a result that, for other purposes, we will call on later.

Proposition 4.1. *Let Σ be an orientable hypersurface in an orientable conformal manifold $(M, [g])$. In a neighbourhood of Σ there is a metric $\widehat{g}$ in the conformal class so that Σ is minimal, i.e. $H^{\widehat{g}} = 0$.*

Proof. For simplicity let us calculate in the metric g and write H^g to be the mean curvature of Σ as a *function* along Σ. Take any smooth extension of this to a function on M. By a standard argument one can show that in a neighbourhood of Σ there is a *normal* defining function s for Σ, that is Σ is the zero set of s, and along Σ the 1-form ds satisfies $|ds|_g^2 = 1$. Then $n^a := g^{ab}\nabla_b s$ is a unit normal vector field along Σ. Recall the conformal transformation of the mean curvature: If $\widehat{g} = e^{2\omega}g$, for some $\omega \in \mathcal{E}$ then $e^\omega H^{\widehat{g}} = H^g + n^a\Upsilon_a = H^g + n^a\nabla_a\omega$. Thus if we take $\omega := -sH^g$ then $H^{\widehat{g}} = 0$. $\square$

Dropping the 'hat' on $\widehat{g}$, we see that with such g (satisfying $H^g = 0$) the map (4.2) simplifies significantly in this normalisation; the splittings of $N^\perp$ and $\mathcal{T}_\Sigma$ then agree in the "obvious way". This is consistent with conformal transformation: The condition $H = 0$ does not fix the representative metric g, even along Σ. For example at the 1-jet level the remaining freedom along M is to conformally rescale by $g \mapsto e^{2\omega}g$ where $n^a\nabla_a\omega = 0$. This is exactly as required to preserve the agreement of the splittings of $N^\perp$ and $\mathcal{T}_\Sigma$. In fact this was the point of view taken in [33]. From there one easily recovers formula (4.2).

Finally we note here that the rescaling involved in the proof of the proposition above is global, and especially natural in the case of directed ASC and AE structures.

Corollary 4.2. *Let $(M, [g], I)$ be a directed scalar negative ASC manifold with a scale singularity set. Then there is a metric $\widehat{g} \in [g]$ with respect to which Σ is a minimal hypersurface. In particular if $(M, [g], I)$ is a directed AE manifold then Σ is totally geodesic with respect to $\widehat{g}$.*

Proof. Suppose that σ is the conformal weight 1 density defining a $S(\sigma) = -1$ ASC structure with a non-trivial scale singularity hypersurface Σ. Write H^g (now as a -1 density) for the mean curvature of Σ with respect to an arbitrary background metric g and extend this smoothly to M. Then Σ has mean curvature zero with respect to the metric $\widehat{g} = e^{2\omega}g$ where $\omega := -H^g\sigma$. For the last statement we recall that if σ satisfies (2.4) then Σ is totally umbilic and this is a conformally invariant condition. $\square$

4.2. Tractor curvature

We digress briefly to recall some further background. In this section we work on an arbitrary conformal manifold $(M^d, [g])$. It will be convenient to introduce the alternative notation $\mathcal{E}^A$ for the tractor bundle $\mathcal{T}$ and its space of smooth sections. Here the index indicates an abstract index in the sense of Penrose and so we may write, for example, $V^A \in \mathcal{E}^A$ to indicate a section of the standard tractor bundle. Using the abstract index notation the tractor metric is denoted h_{AB} with inverse h^{BC}. These will be used to lower and raise indices in the usual way.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we immediately see that

$$Y_AX^A = 1, \qquad Z_{Ab}Z^A{}_c = \boldsymbol{g}_{bc}$$

and that all other quadratic combinations that contract the tractor index vanish.

Given a choice of conformal scale we have the corresponding Levi-Civita connection on tensor and density bundles and we can use the coupled Levi-Civita tractor connection to act on sections of the tensor product of a tensor bundle with a tractor bundle and so forth. This operation is defined via the Leibniz rule in the usual way. In particular we have

$$\nabla_a X_A = Z_{Aa}, \qquad \nabla_a Z_{Ab} = -P_{ab}X_A - Y_A\boldsymbol{g}_{ab}, \qquad \nabla_a Y_A = P_{ab}Z_A{}^b. \tag{4.3}$$

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E \tag{4.4}$$

for $V^C \in \mathcal{E}^C$. Using (4.3) and the usual formulae for the curvature of the Levi-Civita connection we calculate (cf. [32])

$$\Omega_{abCE} = Z_C{}^cZ_E{}^eC_{abce} - X_CZ_E{}^eA_{eab} + X_EZ_C{}^eA_{eab} \tag{4.5}$$

where

$$A_{abc} := 2\nabla_{[b}P_{c]a} \tag{4.6}$$

is the *Cotton tensor*.

Next we note that there is a conformally invariant differential operator between weighted tractor bundles

$$\mathbb{D}_A \colon \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1],$$

given a choice of conformal scale g by the formula

$$\mathbb{D}_AV := (d + 2w - 2)wY_AV + (d + 2w - 2)Z_{Aa}\nabla^aV + X_A(\Delta - wJ)V. \tag{4.7}$$

This is the (Thomas) tractor-D operator as recovered in [25]; see [35,36] for an invariant derivation. The conformal operator D from Section 2 is simply $\frac{1}{d}$ times $\mathbb{D}$ applied to $\mathcal{E}[1]$. (It is convenient to retain the two notations, rather than carry the factor $1/d$ into many calculations.) Using $\mathbb{D}$ we obtain (following [35,36]) a conformally invariant curvature quantity as follows

$$W_{BC}{}^E{}_F := \frac{3}{d-2}\mathbb{D}^AX_{[A}\Omega_{BC]}{}^E{}_F, \tag{4.8}$$

where $\Omega_{BC}{}^{E}{}_{F} := Z_B{}^b Z_C{}^c \Omega_{bc}{}^{E}{}_{F}$. In a choice of conformal scale, W_{ABCE} is given by

$$(d-4)\left(Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e C_{abce} - 2Z_A{}^a Z_B{}^b X_{[C} Z_{E]}{}^e A_{eab} - 2X_{[A} Z_{B]}{}^b Z_C{}^c Z_E{}^e A_{bce}\right) + 4X_{[A} Z_{B]}{}^b X_{[C} Z_{E]}{}^e B_{eb}, \tag{4.9}$$

where

$$B_{ab} := \nabla^c A_{acb} + P^{dc} C_{dacb} \tag{4.10}$$

is known as the *Bach tensor* or the Bach curvature. From formula (4.9) it is clear that W_{ABCD} has Weyl tensor type symmetries. It is shown in [31] and [32] that the tractor field W_{ABCD} has an important relationship to the ambient metric of Fefferman and Graham. See also Section 4.4 below.

For later use we recall here some standard identities which arise from the Bianchi identity $\nabla_{a_1} R_{a_2 a_3 de} = 0$, where sequentially labelled indices are skewed over:

$$\nabla_{a_1} C_{a_2 a_3 cd} = g_{ca_1} A_{da_2 a_3} - g_{da_1} A_{ca_2 a_3}; \tag{4.11}$$

$$(n-3) A_{abc} = \nabla^d C_{dabc}; \tag{4.12}$$

$$\nabla^a P_{ab} = \nabla_b \mathrm{J}; \tag{4.13}$$

$$\nabla^a A_{abc} = 0. \tag{4.14}$$

4.3. Further geometry of the singularity set

We are now set to return to the almost Einstein setting. Via the projectors, a general tractor $I^A \in \mathcal{E}^A$ expands to

$$I^E = Y^E \sigma + Z^{Ed} \mu_d + X^E \rho,$$

where, for example, $\sigma = X_A I^A$. Hence

$$\Omega_{abCE} I^E = \sigma Z_C{}^c A_{cab} + Z_C{}^c \mu^d C_{abcd} - X_C \mu^d A_{dab}.$$

Now assume that $I^A \neq 0$ is parallel (of any length). As a point of notation: in this case we shall write $I^E = Y^E \sigma + Z^{Ed} n_d + X^E \rho$. That is $n_d = \nabla_d \sigma$. Then the left-hand side of the last display vanishes, whence the coefficients of $Z_C{}^c$ and X_C must vanish, i.e.,

$$\sigma A_{cab} + n^d C_{abcd} = 0 \quad \text{and} \quad n^d A_{dab} = 0.$$

Away from the zero set of σ, we have that $\sigma^{-1} n^d = \sigma^{-1} \nabla^d \sigma$ is a gradient and the first equation of the display is the condition that the metric is conformal to a Cotton metric (cf. e.g. [1,37,38]). On the other hand at a point p where $\sigma(p) = 0$ the same equation shows that

$$C_{abcd} \nabla^d \sigma = C_{abcd} n^d = 0 \quad \text{at } p. \tag{4.15}$$

Once again using formulae (4.3) for the tractor connection we obtain

$$\nabla^a \Omega_{acDE} = (d-4) Z_D{}^d Z_E{}^e A_{cde} - X_D Z_E{}^e B_{ec} + X_E Z_D{}^e B_{ec}. \tag{4.16}$$

This too is annihilated by contraction with the parallel tractor I^E and so we obtain

$$(d-4) Z_D{}^d n^e A_{cde} - X_D n^e B_{ec} + \sigma Z_D{}^d B_{dc} = 0.$$

From the coefficient of $Z_D{}^d$ we have

$$\sigma B_{dc} + (d-4) n^e A_{cde} = 0.$$

In dimension four B_{dc} is conformally invariant and this recovers the well-known result that, in this dimension, it vanishes on the conformally Einstein part of M. But then by continuity it follows that the Bach tensor vanishes everywhere on M. In other dimensions the last display shows that $n^e A_{cde} = 0$ at any zeros of σ. This with (4.15) gives the first part of the following.

Proposition 4.3. *Consider an almost Einstein manifold $(M, [g], I)$ and write $\sigma := I_A X^A$ and $n_a := \nabla_a \sigma$. We have*

$$\sigma A_{cab} + n^d C_{abcd} = 0, \Rightarrow n^c A_{cab} = 0, \quad \textit{and}$$
$$\sigma B_{ac} + (d-4) n^e A_{cae} = 0 \Rightarrow n^a B_{ab} = 0,$$

everywhere on M. Hence for any point p with $\sigma(p) = 0$ we have

$$n^a C_{abcd} = 0 \textit{ at } p.$$

In dimension $d = 4$ we have $C_{abcd}(p) = 0$, while $B_{ab} = 0$ on M. In dimensions $d \neq 4$ we have:

$$n^a \Omega_{abCD} = 0 \quad \textit{at } p.$$

In any dimension, if $j^1_p \sigma = 0$ then

$$C_{abcd} = 0 = A_{bcd} \Leftrightarrow \Omega_{abCD} = 0 \quad \textit{at} \quad p, \quad \textit{and} \quad W_{ABCD}(p) = 0.$$

Proof. The displayed implications follow by contracting n^a into the equations and using the symmetries of A and C. In dimension 4 $n^aC_{abcd} = 0$ at p implies $C_{abcd}(p) = 0$, when I is not null (and so $n^a(p) \neq 0$). It remains to establish the final claims. If at some point p we have $j^1_p\sigma = 0$, then, at p we have $I^A = \rho X^A$ with $\rho(p) \neq 0$. So from $I^E\nabla_a\Omega_{bcDE} = 0$ it follows that $X^E\nabla_a\Omega_{bcDE} = 0$ at p. But, $\nabla_a X^E = Z^E{}_a$ and from (4.5) we have $X^E\Omega_{bcDE} = 0$ everywhere. So $Z_D{}^dC_{bcda} - X_DA_{abc} = Z^E{}_a\Omega_{bcDE} = 0$ at p. This immediately yields the result of the last display. But we also have that X^AW_{ABCD} and I^AW_{ABCD} are zero everywhere, so an easy variation of the last argument also shows that W_{ABCD} vanishes at p. $\square$

In the case that I is null $g^o = \sigma^{-2}\mathbf{g}$ is Ricci-flat on $M \setminus \Sigma$. So, from the last part of the Proposition, it follows that g^o is asymptotically flat (locally) as we approach any points of Σ. Following [37] let us say a conformal manifold of dimension $d \geq 4$ is *weakly generic* at $p \in M$ if the only solution at p to $C_{abcd}v^d = 0$ is $v^d_p = 0$; then say that $(M, [g])$ is *weakly generic* if this holds at all points of M. From the Proposition above and Corollary 2.4 we see that the rank of the Weyl tensor obstructs certain AE structures. Summarising we have the following.

Corollary 4.4. *Let $(M^d, [g], I)$ be an almost Einstein structure with $S(I) = -|I|^2 = 0$. Then (M, g^o) is asymptotically locally Euclidean as we approach any point p with $\sigma(p) = 0$. If $(M^{d\geq 4}, [g], I)$ is an AE structure with scale singularity set $\Sigma \neq \emptyset$ then $(M, [g])$ is not weakly generic. If $(M, [g])$ admits any two linearly independent AE structures then it is nowhere weakly generic.*

$S(I) = 0$ AE structures were studied via a different approach in [30]; as well as some of the results mentioned here, they show that if M is closed, or $(M \setminus \Sigma, g^o)$ is complete, then $\Sigma \neq \emptyset$ implies that (M, g) is conformally diffeomorphic to the standard sphere. They also discuss the asymptotic flatness in preferred coordinates based at p.

Now we specialise to the case of a scalar negative almost Einstein manifold $(M, [g], I)$, with a non-empty scale singularity set Σ. We may suppose, without loss of generality, that $|I|^2 = 1$. From Corollary 4.2 we may also assume that g is a metric in the conformal class so that $H^g = 0$, where H^g is the mean curvature of the hypersurface Σ.

As usual we identify $T\Sigma$ with its image in $TM|_\Sigma$ under the obvious inclusion and $T^*\Sigma$ with the annihilator (in T^*M) of n^a. In our calculations here we will reserve the abstract indices i, j, k, l for $T\Sigma \subset TM|_\Sigma$ and its dual. For example R_{ijcd} means the restriction of the Riemannian curvature $R_{abcd} = R^g_{abcd}$ to tangential (to Σ) directions in the first two slots. Now, calculating in the metric g, recall that the Riemannian curvature R_{abcd} decomposes into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the Schouten tensor P_{ab}, according to (2.1). It follows that along Σ

$$R_{ijkl} = C_{ijkl} + 2\mathbf{g}^\Sigma_{k[i}P_{j]l} + 2\mathbf{g}^\Sigma_{l[j}P_{i]k},$$

where we have used that the intrinsic conformal metric on Σ is just the restriction of the ambient conformal metric. The Levi-Civita connection ∇ on (M, g) induces a connection on $T\Sigma$; this is by differentiating tangentially followed by orthogonal projection into $\Gamma(T\Sigma)$. By the well-known Gauss formula this recovers the Levi-Civita connection for g_Σ. On the other hand since Σ is totally geodesic for g the section giving the orthogonal projection is itself parallel, so we find the standard result that for totally geodesic hypersurfaces the intrinsic parallel transport along Σ agrees with the ambient parallel transport (as applied to tangent vectors). It follows immediately that $R_{ijkl} = R^\Sigma_{ijkl}$, where by R^Σ we mean the intrinsic Riemannian curvature of $(\Sigma, g|_\Sigma)$. But since $n^aC_{abcd} = 0$ we have that $C_{ijkl}|_\Sigma$ is completely trace-free with respect to $\mathbf{g}^\Sigma$ and so has Weyl tensor type symmetries, as a tensor on Σ. It follows easily that, for $d \geq 4$, the right-hand side of the last display necessarily gives the canonical decomposition of R^Σ_{ijkl} into its Weyl and Schouten parts. On the other hand since n_a is parallel along Σ we have $R_{ij}{}^c{}_dn_c = 0$ and using again (2.1), but now applied to R_{ijcd}, we see that $P_{ib}n^b = 0$. That is, along Σ

$$C^\Sigma_{ijkl} = C_{ijkl}, \qquad P^\Sigma_{ij} = P_{ij} \quad \text{and} \quad P_{ib}n^b = 0. \tag{4.17}$$

Note that since the Weyl curvature of any 3 manifold is identically zero, in the case of dimension $d = 4$ we have $C^\Sigma \equiv 0$. Thus in this dimension the display is consistent with Proposition 4.3 where we observed that $C|_\Sigma = 0$.

As discussed in Section 4.1, $\mathcal{T}_\Sigma$ may be identified with $N^\perp$ (i.e. the annihilator subbundle of the normal tractor) in $\mathcal{T}|_\Sigma$ and we shall continue to make this identification. Since N is parallel along Σ, $\nabla^\mathcal{T}$ preserves this subbundle. Now recall that the conformal density bundles on Σ are just the restrictions of their ambient counterparts: $\mathcal{E}^\Sigma[w] = \mathcal{E}[w]|_\Sigma$. When we work with the metric g, which has $H^g = 0$, then the splittings of the tractor bundles $N^\perp$ and $\mathcal{T}_\Sigma$ also coincide in the obvious way (see the Remark concluding Section 4.1), and in particular (via the intrinsic version of (2.7)) $\mathcal{T}_\Sigma$ decomposes to $\mathcal{E}^\Sigma[1] \oplus \mathcal{E}^\Sigma_i[1] \oplus \mathcal{E}^\Sigma[-1]$ where the weight one 1-forms on Σ, $\mathcal{E}^\Sigma_i[1]$ may be identified with $n^\perp$ in $\mathcal{E}_a[1]|_\Sigma$. It follows from these observations, the explicit formula (2.5) expressed with respect to the metric g, and the second result in the display (4.17), that the tractor parallel transport on $\Sigma^{n\geq 3}$ is just the restriction of the ambient. Although we used special scales for the argument it suffices to use any metric from the conformal class to verify the agreement since the connections are conformally invariant. Let us summarise the consequences.

Theorem 4.5. *Let $(M^{d\geq 3}, [g], I)$ be a scalar negative almost Einstein structure with a non-empty scale singularity hypersurface Σ. The tractor connection of $(M, [g])$ preserves the intrinsic tractor bundle of Σ, where the latter is viewed as a subbundle of the ambient tractors: $\mathcal{T}_\Sigma \subset \mathcal{T}$. Furthermore the intrinsic tractor parallel transport of $\nabla^{\mathcal{T}_\Sigma}$ coincides with the restriction of the parallel transport of $\nabla^\mathcal{T}$.*

We have

$$\Omega(u, v) = \Omega^\Sigma(u, v) \quad \textit{along } \Sigma$$

where $u, v \in \Gamma(T\Sigma)$. In dimensions $d \neq 4$ we have the stronger result

$$\Omega(\cdot,\cdot) = \Omega^{\Sigma}(\cdot,\cdot) \quad \text{along } \Sigma,$$

where here, by trivial extension, we view Ω^{Σ} as a section of $\Lambda^2 T^*M \otimes \operatorname{End} \mathcal{T}$.

Proof. In the case of $d = 3$ the agreement of the parallel transport is immediate from the definition of the tractor connection $\nabla^{\mathcal{T}_\Sigma}$ and that the normal tractor N^A is parallel along Σ. In the remaining dimensions this was established immediately above. From this, and the fact that on Σ we have $\Omega(u, v)N = 0$, it follows at once that $\Omega(u, v) = \Omega^{\Sigma}(u, v)$ along Σ, as claimed. For dimensions $d \neq 4$ we have from Proposition 4.3 above that $\Omega(n, \cdot) = 0$, whence the final claim. □

Remark. To obtain the result that the intrinsic tractor parallel transport of $\nabla^{\mathcal{T}_\Sigma}$ coincides with the restriction of the ambient parallel transport of $\nabla^{\mathcal{T}}$ to sections of $\mathcal{T}_\Sigma$ uses that Σ is totally umbilic and that $n^a C_{abcd} = 0$ along Σ. These conditions are sufficient for the agreement of the connections. ■

4.4. Extending off Σ

Given a conformal manifold $(\Sigma, [g_\Sigma])$ we may ask if this can arise as the scale singularity set of a scalar negative almost Einstein manifold. Narrowing the problem, we may begin with a fixed smooth (or with specified regularity) codimension 1 embedding of Σ in a manifold M and consider the Dirichlet type problem of finding a directed AE structure $(M, [g], I)$ with $(\Sigma, [g_\Sigma])$ as the scale singularity set; issues include whether or not there is any solution and, if there is, then whether $(\Sigma, [g_\Sigma])$ determines $(M, [g], I)$ uniquely. This is exactly the problem of finding on M a conformal structure $[g]$ and on this a solution σ to the conformally invariant equation $\nabla_a\nabla_b\sigma + P_{ab}\sigma + \rho\boldsymbol{g}_{ab} = 0$ (i.e. (2.4)) such that Σ is the zero set of σ (and then there is the question of whether the pair $([g], \sigma)$ is unique). We want to derive consequences of this equation that make the nature of this problem more transparent. We have seen already that this may be viewed as finding on M a conformal structure admitting a parallel tractor parallel tractor I with $I|_\Sigma$ agreeing with the normal tractor N along Σ.

The data on Σ is a conformal structure, and, for any solution $[g_\Sigma]$ is simply the pull-back of the ambient conformal structure $[g]$ on M. By Theorem 4.5 we know (at least to some order along Σ) how the ambient conformal curvature is related to the intrinsic conformal curvature of $(\Sigma, [g_\Sigma])$. Thus it seems natural to derive the equations which control how this extends off Σ. With less ambition we shall not attempt here to study the full boundary problem. Rather we seek to find equations which control the conformal curvature quantities off Σ and which are also well defined along Σ.

First note that it follows from the Bianchi identity (4.13) that Einstein manifolds (M^d, g^o) are Cotton, i.e. $A^{g^o}_{abc} = 0$. In dimension $d = 3$ the Weyl tensor vanishes identically and the Cotton tensor is conformally invariant. Thus almost Einstein 3-manifolds are Cotton and hence conformally flat. So if $(M^3, [g], I)$ is scalar positive then it is a positive sectional curvature space form. If $(M^3, [g], I)$ has $S(I) \leq 0$ then I may have a scale singularity set Σ, but off this the structure (M, g^o) is either hyperbolic (if $S(I) < 0$) or locally Euclidean (if $S(I) = 0$).

From (4.16) one easily concludes that on an Einstein manifold (M^d, g^o) the tractor curvature satisfies the (full) Yang–Mills equations, that is $\nabla^a \Omega_{ab}{}^C{}_D = 0$ (see also [39]) (where the connection ∇ is in the scale g^o). In dimension $d = 4$ this equation is conformally invariant. Thus almost Einstein 4-manifolds are globally Yang–Mills. Combining with relevant results from Proposition 4.3 and Theorem 4.5, let us summarise.

Proposition 4.6. *Let $(M^4, [g], I)$ be an almost Einstein manifold. Then the tractor curvature satisfies the conformally invariant Yang–Mills equations,*

$$\nabla^a \Omega_{ab}{}^C{}_D = 0.$$

If I is scalar negative then along any singularity hypersurface Σ of I we have

$$C_{abcd} = 0 \quad \text{and} \quad \Omega(u, v) = \Omega^{\Sigma}(u, v) \quad \text{along } \Sigma$$

where $u, v \in \Gamma(T\Sigma)$.

Note that in dimension 4 the tractor curvature is Yang Mills if and only if the conformal structure is Bach-flat. However the Proposition suggests that it is useful to view the Bach-flat condition as a Yang–Mills equation in order to formulate an extension problem (or boundary problem in the PE case). We note that in [8] LeBrun established the existence and uniqueness of a real analytic self-dual Poincaré–Einstein metric in dimension 4 defined near the boundary with prescribed real analytic conformal infinity. If a four-dimensional metric is self-dual then so is its tractor curvature and hence the tractor connection is Yang–Mills.

Before we continue we need some further notation. Let us write $\sharp$ (*hash*) for the natural tensorial action of sections A of $\operatorname{End}(\mathcal{T})$ on tractor sections. For example, on a covariant 2-tractor T_{AB}, we have

$$A\sharp T_{AB} = -A^C{}_A T_{CB} - A^C{}_B T_{AC}.$$

If A is skew for $\boldsymbol{h}$, then at each point, A is $\mathfrak{so}(h)$-valued. The hash action then commutes with the raising and lowering of indices and preserves the $SO(h)$-decomposition of tractor bundles.

As a section of the tensor square of the $\boldsymbol{h}$-skew bundle endomorphisms of $\mathcal{T}$, the curvature quantity W has a double hash action on tractors T; we write $W\sharp\sharp T$ for this. Now for dimensions $d \neq 4$ we use this to construct a Laplacian operator on (possibly conformally weighted) tractor sections. For T a section of $(\otimes^k \mathcal{T})[w]$ and $d \neq 4$ we make the definition

$$\not{\Delta} T := (\Delta - wJ)T - \frac{1}{2(d-4)} W\sharp\sharp T.$$

Then from this we obtain a variant of the usual tractor-D operator as follows:

$$\not{D}_A T := (d+2w-2)wY_A T + (d+2w-2)Z_{Aa}\nabla^a T + X_A \not{\Delta} T.$$

In terms of this operator we have,

Theorem 4.7. *Let $(M^d, [g], I)$ be an almost Einstein manifold. Then if $d = 4$ we have $W_{BCDE} = 0$. In dimension 6 the conformally invariant equation*

$$\not{\Delta} W_{A_1A_2B_1B_2} = 0$$

holds. In all dimensions we have

$$I^A \not{D}_A W_{BCEF} = 0. \tag{4.18}$$

Also

$$W_{BCEF} I^F = 0. \tag{4.19}$$

In particular if I is scalar negative and Σ denotes the singularity hypersurface for I then $W_{BCEF}N^F = 0$ along Σ.

Proof. From (4.9) it follows that, in dimension 4, $W = 0$ is equivalent setting the Bach tensor to zero and, as noted earlier, this is equivalent to the conformal tractor connection being Yang Mills. We have this, in particular, on almost Einstein manifolds. Since I is parallel it annihilates the tractor curvature, i.e. $\Omega_{bc}{}^E{}_F I^E = 0$. But since it is parallel and has conformal weight 0, I commutes with the tractor-D operator $\mathbb{D}$. Thus (4.19) follows from (4.8). We also note here that since I commutes with $\mathbb{D}$, and, on the other hand, any contraction of I with W is zero, it follows by an elementary argument that I commutes with $\not{D}$.

In dimension 6 we have from [40] that

$$\not{\Delta} W_{A_1A_2B_1B_2} = K X_{A_1} Z_{A_2}{}^a X_{B_1} Z_{B_2}{}^b \mathcal{B}_{ab},$$

where K is a non-zero constant and $\mathcal{B}_{ab}$ is the Fefferman–Graham (obstruction) tensor (see also [23]). The sequentially labelled indices here are implicitly skewed over. But the conformal invariant $\mathcal{B}_{ab}$ is zero on Einstein manifolds [21,41,40] and hence also (by continuity) on almost Einstein manifolds.

It remains to establish (4.18). Since W has conformal weight -2, it follows that when $d = 6$ we have $I^A \not{D}_A W_{BCEF} = \sigma \not{\Delta} W_{A_1A_2B_1B_2}$, where as usual σ denotes the conformal density $X^A I_A$. Thus (4.18) holds in dimension 6. Let us suppose now that $d \neq 4, 6$. Here we will use the link between the standard tractor bundle on $(M, [g])$ and the Fefferman–Graham (FG) metric of [21,22]. This link was developed in [31,32,42] but here we use especially the notation and results from [40]. (It should be noted however that here we use the opposite sign for the Laplacian.) The arguments we use below are a minor variation of similar developments from those sources.

For a Riemannian conformal manifold $(M^d, [g])$ the Fefferman–Graham ambient manifold [21] is a signature $(d+1, 1)$ pseudo-Riemannian manifold with $\mathcal{Q}$ as an embedded submanifold. There is some further background on the FG metric in Section 6. Suitably homogeneous tensor fields on the ambient manifold, upon restriction to $\mathcal{Q}$, determine tractor fields on the underlying conformal manifold. In particular, in dimensions other than 4, W_{ABCD} is the tractor field equivalent to $(d-4)\boldsymbol{R}_{ABCD}|_{\mathcal{Q}}$ where $\boldsymbol{R}$ is the curvature of the FG ambient metric. Under this correspondence the FG ambient metric applied to homogeneous tensors along $\mathcal{Q}$, descends to the tractor metric. Ambient differential operators that are suitably tangential and homogeneous (see e.g. [42,40]) also descend to operators between tractor bundles or subquotients thereof. For example the tractor connection arises from ambient parallel transport along $\mathcal{Q}$.

On the FG ambient manifold let us define a Laplacian operator $\not{\boldsymbol{\Delta}}$ by the formula

$$\not{\boldsymbol{\Delta}} := \boldsymbol{\Delta} - \frac{1}{2}\boldsymbol{R}\#\#.$$

Then in all dimensions $d \neq 4, 6$, $\not{\boldsymbol{\Delta}}\boldsymbol{R}|_{\mathcal{Q}} = 0$, [40, Section 3.2]. On the other hand $\not{D}_A$ corresponds to the ambient operator

$$(d+2w-2)\boldsymbol{\nabla} + \boldsymbol{X}\not{\boldsymbol{\Delta}} =: \not{\boldsymbol{D}} : \boldsymbol{\mathcal{T}}^{\Phi}(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^{\Phi}(w-1)$$

where $\boldsymbol{\mathcal{T}}^{\Phi}(w)$ indicates the space of sections, homogeneous of weight w, of some ambient tensor bundle. (NB: An ambient tensor T is homogeneous of weight w if and only if $\boldsymbol{\nabla}_{\boldsymbol{X}} T = wT$.) From the Bianchi identity on the FG ambient manifold, and the fact that $\not{\boldsymbol{\Delta}}\boldsymbol{R}|_{\mathcal{Q}} = 0$, it follows that on the ambient manifold we have

$$\not{\boldsymbol{D}}_{[A}\boldsymbol{R}_{BC]DE} = 0,$$

along $\mathcal{Q}$. This descends to

$$\not{D}_{[A}W_{BC]DE} = 0.$$

So we have

$$I^A\not{D}_{[A}W_{BC]DE} = 0.$$

But since $I^AW_{ABDE} = 0$ and I commutes with $\not{D}$ (4.18) follows. □

From the Theorem we may conclude some restrictions on the intrinsic conformal structure. For example we have the following.

Corollary 4.8. *Let $(M^5, [g], I)$ be a scalar negative almost Einstein manifold with scale singularity set $\Sigma \neq \emptyset$. Then the induced conformal structure $(\Sigma^4, [g_\Sigma])$ is Bach-flat.*

Proof. In dimensions $d \neq 6$ Eq. (4.18), i.e. $I^A\not{D}_AW = 0$, on M implies that along any scale singularity subspace we Σ have

$$\delta W = 0$$

where δ is the (conformally invariant) tractor twisted conformal Robin operator [33,24] applied to W; in terms of g we have $\delta W = n^a\nabla^g_aW + 2H^gW$ where H^g is the mean curvature of Σ and ∇ is the usual (density coupled) the tractor connection.

To simplify the presentation let us temporarily display the first two abstract indices of the tractor W, but suppress the last pair; we shall write W_{BC} rather than W_{BCDE}. From the defining formula (4.8) for W_{BC} it follows easily that

$$W_{BC} = (d-4)Z_B{}^bZ_C{}^c\Omega_{bc} - X_BZ_C{}^b\nabla^a\Omega_{ab} + X_CZ_B{}^b\nabla^a\Omega_{ab}. \tag{4.20}$$

This is expression (13) from [32]. Exploiting Corollary 4.2, let us calculate in a metric g with respect to which Σ is totally geodesic. Setting $d = 5$, applying $\delta = n^a\nabla_a$ to (4.20), and using the tractor connection formulae (4.3) we see that the coefficient of $Z_B{}^bZ_C{}^c$ is

$$n^a\nabla_a\Omega_{bc} - n_b\nabla^a\Omega_{ac} + n_c\nabla^a\Omega_{ab},$$

where Ω_{bc} is the tractor curvature of the ambient conformal structure $(M, [g])$ (where we have suppressed the tractor indices). Evidently a part of the condition $\delta W|_\Sigma = 0$ is that the last display is zero along Σ. Thus, in particular, n^b contracted into this must vanish, that is

$$n^bn^a\nabla_a\Omega_{bc} - \nabla^a\Omega_{ac} + n_cn^b\nabla^a\Omega_{ab} = 0 \quad \text{along } \Sigma.$$

But using that Σ is totally geodesic and, from Theorem 4.5, that $\Omega(u, v) = \Omega^\Sigma(u, v)$ along Σ where $u, v \in \Gamma(T\Sigma)$, this exactly states that

$$g^{ij}_\Sigma\nabla^\Sigma_i\Omega^\Sigma_{jk} = 0,$$

where g_Σ is the intrinsic metric on Σ induced by g and Ω^Σ_{jk} is the tractor curvature of its conformal class. Thus the conformally invariant intrinsic tractor curvature of the (Σ, g_Σ) satisfies the Yang–Mills equations. As mention earlier, in dimension 4 these are conformally invariant and are equivalent to the structure $(\Sigma, [g_\Sigma])$ being Bach-flat. □

There is an analogue of this result for higher odd d; see Theorem 6.1 below. It is likely that there is a proof of Theorem 6.1 using only Eq. (4.18), but certainly approaching this directly (as in the proof for $d = 5$ above) would rapidly become technical for increasing dimension. Section 6 gives a simple and conceptual treatment, using the Fefferman–Graham metric.

Remarks: From Eq. (4.18) it follows that the conformal aspects of the asymptotics of Poincaré–Einstein metrics are controlled by the operator $I^A\not{D}_A$.

In dimension 6 the main equation (4.18) (or equivalently $\not{\Box}W_{A_1A_2B_1B_2} = 0$) is equivalent to requiring $(M, [g])$ to have vanishing Fefferman–Graham tensor.

In dimensions other than 3, 4 and 6, and *off* Σ, a key part of (4.18) is the harmonic equation $\Delta C - \frac{1}{2}R\sharp\sharp C = 0$ on the Weyl curvature which holds on Cotton (and hence Einstein) manifolds, as follows easily from the Bianchi identities (4.11) and (4.12). However in dimensions other than 3 we cannot conclude that there is a scale for which an AE manifold is Cotton (everywhere). On the other hand Eq. (4.18) holds globally on an AE manifold ($d \neq 4$).

The following sheds some light on the meaning of Eq. (4.18) and its relation to possible boundary problems. This follows easily from the Theorem and the definitions of the operators involved, except we have also called on Corollary 6.4.

Corollary 4.9. *On an Einstein manifold $(M^{(n+1)\geq 3}, g^o)$ we have*

$$\left(\Delta^{g^o} + 4\frac{n-2}{n+1}J^{g^o}\right)W - \frac{1}{2(n-3)}W\sharp\sharp W = 0.$$

In particular this holds on an almost manifold $(M^{(n+1)\geq 3}, [g], I)$ off the zero set Σ of $\sigma = h(X, I)$. If $|I|^2 = 1$ and Σ is non-empty, then on $M \setminus \Sigma$ we have

$$(\Delta^{g^o} - 2(n-2))W - \frac{1}{2(n-3)} W \sharp\sharp W = 0,$$

while along the hypersurface Σ we have

$$N \lrcorner W = 0, \quad \text{and, if } n \geq 5, \quad (n-4)W|_\Sigma = (n-3)W^\Sigma,$$

while, if $n \neq 5$,

$$\delta W = 0 \quad \text{along } \Sigma,$$

where δ is the conformal Robin operator applied to W; in terms of g we have $\delta W = n^a \nabla^g_a + 2H^g W$ is the where H^g is the mean curvature of Σ.

It is shown in [24] that on densities $I^A D_A$ agrees with the Laplacians arising in the scattering problems treated in [18]. The operator $(\Delta^{g^o} - 2(n-2))$ here is a tractor twisted version of such. We have used n rather than d in the formulae here to simplify comparisons with [24] and [18].

We have seen in dimension 3, 4 and 6 that there are conformally invariant equations controlling the conformal curvature of an AE manifold. This is achieved trivially in dimension 3. As a final note for this section we point out that there is an analogue of the results for dimensions 4 and 6 to higher even dimensions.

Proposition 4.10. *Almost Einstein manifolds $(M^{d\text{ even}}, [g], I)$, $d \geq 4$, satisfy the conformally invariant equation that the Fefferman–Graham tensor vanishes. This may be expressed in the form*

$$0 = \not\Delta_{d/2-2} W = \Delta^{d/2-2} W + \text{lower order terms}, \tag{4.21}$$

where by $\not\Delta_0$ and Δ^0 we mean the operator given by multiplication by 1.

The linear operator $\not\Delta_{d/2-2}$ is constructed in [40], and the result here is an easy consequence of the results there for Einstein manifolds. Once again on a scalar negative AE manifold with a scale singularity set Σ, (4.21) expresses the vanishing Fefferman–Graham tensor condition in a form suitable to link with the conformal curvature data on Σ (using Corollary 6.4, or for $n = 3, 4$ Theorem 4.5). It should be interesting to construct compatible conformal boundary operators for W along embedded submanifolds Σ so that these yield a well posed and conformal elliptic problem for the conformal curvature Ω. Close analogues of the conformal boundary operators developed in [33] should play a role.

Remark. Note that conformal equations, such as (4.21), offer the chance to split the problem of finding almost Einstein structures (or Poincaré–Einstein metrics) into a conformal problem, say controlled by (4.21) with further boundary operators along Σ, and a second part where one would find a compatible "scale" σ. We should expect that a solution to the conformal problem is necessary but in general not sufficient. However one may ask if (in Riemannian signature and say on closed even manifolds) (4.21) plus the (clearly necessary) vanishing of the conformal invariant

$$\Omega_{ab}{}^{C}{}_{F_1} \Omega_{cd}{}^{D}{}_{F_2} \cdots \Omega_{ef}{}^{E}{}_{F_{d+1}},$$

where the sequentially labelled indices are skewed over, is sufficient for a conformal manifold to necessarily admit an almost Einstein structure locally. A corresponding global question is whether a smooth section K of $\mathcal{T}$ satisfying $\Omega_{ab}{}^{C}{}_{D} K^D = 0$ plus (4.21) is sufficient to conclude that the conformal structure on a closed even manifold admits a directed almost Einstein structure. In dimension 4 there is a positive answer to this if we restrict to K such that $h(X, K)$ is non-vanishing [43]; in this case the structure must be Einstein. ■

5. Examples and the model

5.1. The model—Almost Einstein structures on the sphere

Proposition 5.1. *The d-sphere, with its standard conformal structure, admits a $(d+2)$-dimensional space of compatible directed almost Einstein structures. For each $S \in \mathbb{R}$ there is an almost Einstein structure I on $\mathbb{S}^d$ with $S(I) = S$.*

The AE structures on the sphere also may be viewed as examples of ASC structures on the sphere. In any case we shall see that, in a sense, "most" of these are scalar negative (which might at first seem counterintuitive).

Before we prove this let us recall a construction of the standard conformal structure on the sphere. Consider a $(d+2)$-dimensional real vector space $\mathbb{V}$ equipped with a non-degenerate bilinear form $\mathcal{H}$ of signature $(d+1, 1)$. The *null cone* $\mathcal{N}$ of zero-length vectors form a quadratic variety in $\mathbb{V}$. Choosing a time orientation, let us write $\mathcal{N}_+$ for the forward part of $\mathcal{N} \setminus \{0\}$. Under the ray projectivisation of $\mathbb{V}$ the forward cone $\mathcal{N}_+$ is mapped to a quadric in $\mathbb{P}_+(\mathbb{V}) \cong \mathbb{S}^{d+1}$. This image is topologically a sphere $\mathbb{S}^d$ and we will write π for the submersion $\mathcal{N}_+ \to \mathbb{S}^d$. Each point $p \in \mathcal{N}_+$ determines a positive definite inner product on $T_{x=\pi p}\mathbb{S}^d$ by $g_x(u, v) = \mathcal{H}_p(u', v')$ where $u', v' \in T_p\mathcal{N}_+$ are lifts of $u, v \in T_x\mathbb{S}^d$. For a given vector $u \in T_x\mathbb{S}^d$ two lifts to $p \in \mathcal{N}_+$ differ by a vertical vector field. Since any vertical vector is normal (with respect to $\mathcal{H}$) to the cone it follows that g_x is independent of the choices of lifts. Clearly then, each section of π determines a metric on $\mathbb{S}$ and by construction this is smooth if the section is. (Evidently the metric agrees with the pull-back of $\mathcal{H}$ via the section concerned.) Now, viewed as a metric on $T\mathbb{R}^{d+2}$, $\mathcal{H}$ is homogeneous of degree 2 with respect to the standard Euler vector field E on $\mathbb{V}$, that is $\mathcal{L}_E\mathcal{H} = 2\mathcal{H}$, where $\mathcal{L}$ denotes the Lie derivative. In particular this holds on the cone, which we note is generated by E.

Write $\boldsymbol{g}$ for the restriction of $\mathcal{H}$ to vector fields in $T\mathcal{N}_+$ which are the lifts of vector fields on $\mathbb{S}^d$. Then for any pair $u, v \in \Gamma(T\mathbb{S}^d)$, with lifts to vector fields u', v' on $\mathcal{N}_+$, $\boldsymbol{g}(u', v')$ is a function on $\mathcal{N}_+$ homogeneous of degree 2, and which is independent of how the vector fields were lifted. Evidently $\mathcal{N}_+$ may be identified with the total space of the bundle of conformally related metrics (i.e. $\mathcal{Q}$ in Section 2) for the standard sphere. Thus $\boldsymbol{g}(u', v')$ may be identified with a conformal density of weight 2 on $\mathbb{S}^d$. That is, this construction determines a section of $S^2T^*\mathbb{S}^d \otimes E[2]$ that we shall also denote by $\boldsymbol{g}$. By construction this is a conformal metric (see Section 2) on $\mathbb{S}^d$. Fix a future pointing vector I in $\mathbb{V}$ with $|I|^2 := \mathcal{H}(I, I) = -1$. Regarding $\mathbb{V}$ as an affine space, view I as a constant section of $T\mathbb{V}$. Write X^A for standard coordinates on $\mathbb{V}$ (i.e. via an isomorphism $\mathbb{V} \cong \mathbb{R}^{d+2}$). It is straightforward to verify that the $\mathcal{H}(I, X) = 1$ hyperplane meets $\mathcal{N}_+$ in a copy of $\mathbb{S}^d$ and the metric induced by this section of π is a standard metric on $\mathbb{S}^d$. Thus $\boldsymbol{g}$ is a standard conformal structure on the sphere. We are ready to prove the Proposition.

Proof of Proposition 5.1. It is easily verified that $G := SO(\mathcal{H}) \cong SO_0(d + 1, 1)$ (the identity connected component of the Lorentz group) acts transitively on the sphere. Thus the conformal sphere may be identified with G/P where P is the parabolic subgroup of G which stabilises a nominated ray in $\mathcal{N}_+$. Now $G \to G/P$ may be viewed as a flat Cartan bundle over $G/P = \mathbb{S}^d$ and the standard tractor bundle $\mathcal{T}$ is $G \times_P \mathbb{V}$ where $\mathbb{V}$ is viewed as a P-module, by restriction. Here $G \times_P \mathbb{V} = G \times \mathbb{V}/\sim$ where the equivalence relation is $(gp, v) \sim (g, p \cdot v)$ with $g \in G, p \in P$ and where "$\cdot$" indicates the standard representation of G on $\mathbb{V}$. The bundle $G \times_P \mathbb{V}$ is trivialised canonically by the map $(g, v) \mapsto (gP, g \cdot v)$ and so we have a connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$ induced from the trivial connection on $(G/P) \times \mathbb{V}$. It is straightforward to verify that this is the normal tractor connection. (In fact this is essentially a tautology; one view the idea of a normal conformal connection tractor as modelled on this homogeneous case.) Thus in this case the tractor connection is globally flat, and the bundle $\mathcal{T}$ admits $(d + 2)$ linearly independent parallel sections. □

Using the embedding of $\mathcal{N}_+$ in $\mathbb{V}$ we can explicitly describe the almost Einstein structures of the Proposition. For example we may construct a scalar negative AE structure on $\mathbb{S}^d$ as follows. Take a vector $I \in \mathbb{V}$ of squared length 1 (i.e. $|I|^2 = 1$). We shall use the same notation for the covector $\mathcal{H}(I, \cdot)$. By the standard parallel transport (of $\mathbb{V}$ viewed as an affine structure) view this as a constant section of $T^*\mathbb{V}$. Then as above, writing X^A for standard coordinates on $\mathbb{V}$, the intersection of the hyperplane $I_AX^A = 1$ with $\mathcal{N}_+$, which we shall denote S_+, is a section of π over an open cap C_+ of the sphere. Similarly the intersection of the hyperplane $I_AX^A = -1$ with $\mathcal{N}_+$, which we shall denote S_-, is a section of π over another open cap C_- of the sphere. On the other hand the hyperplane $I_AX^A = 0$ (parallel to the previous) intersects $\mathcal{N}_+$ in a cone of one lower dimension. The image Σ of this under π is a copy of $\mathbb{S}^n$ embedded in $\mathbb{S}^d$ (where as usual $d = n + 1$). It is easily deduced that $\mathbb{S}^d$ is the union of the three submanifolds C_-, Σ, and C_+ and that, for example, with respect to (a restriction of) the smooth structure on $\mathbb{S}^d$, the embedded Σ is a boundary for its union with C_+. This follows because any forward null ray though the origin and parallel to the $I_AX^A = 1$ hyperplane lies in the hyperplane $I_AX^A = 0$, whereas every other forward null ray through the original meets either the $I_AX^A = 1$ hyperplane or the $I_AX^A = -1$ hyperplane. Let us write g^o for the metric that the sections $S_\pm$ give on $C_\pm$. Note that the hypersurface $\mathbb{S}^n$ canonically has no more than a conformal structure. This may obviously be viewed as arising as a restriction of the conformal structure on $\mathbb{S}^d$. Equivalently we may view its conformal structure as arising in the same way as the conformal structure on $\mathbb{S}^{d+1}$, except in this case by the restriction of π to the sub-cone $I_AX^A = 0$ in $\mathcal{N}_+$, and from (the restriction of) $\mathcal{H}$ along this sub-cone. In the following we write g to denote any metric from the standard conformal class on $\mathbb{S}^d$. Note that on $C_\pm$ this is conformally related to g^o.

Now let us henceforth identify, without further mention, each function on $\mathcal{N}_+$ which is homogeneous of degree $w \in \mathbf{R}$ with the corresponding conformal density of weight w. With $\sigma := I_AX^A$, as above, note that $\sigma^{-2}\boldsymbol{g}$ is homogeneous of degree 0 on $\mathcal{N}_+$ and agrees with the restriction of $\mathcal{H}$ along $S_\pm$. Thus on $C_\pm$ we have $\sigma^{-2}\boldsymbol{g} = g^o$; $\sigma^{-2}\boldsymbol{g}$ recovers the metric determined by $S_\pm$. Similarly on $\mathbb{S}^d$ we have $g = \tau^{-2}\boldsymbol{g}$, where τ is a non-vanishing conformal density of weight 1. So on $C_+ \cup C_-$, $g^o = s^{-2}g$ where s is the *function* σ/τ. We see that g^o is conformally compact on $\mathbb{S}^d \setminus C_+$, and also on $\mathbb{S}^d \setminus C_-$.

We may now understand this structure via the tractor bundle on $\mathbb{S}^d$. Let us write ρ^t for the natural action of $\mathbf{R}_+$ on $\mathcal{N}_+$ and then ρ^t_* for the derivative of this. Now modify the latter action on $T\mathbb{V}$ by rescaling: we write $t^{-1}\rho^t_*$ for the action of $\mathbf{R}_+$ on $T\mathbb{V}$ which takes $u \in T_p\mathbb{V}$ to $t^{-1}(\rho^t_* u) \in T_{\rho^t(p)}\mathbb{V}$. Note that u and $t^{-1}(\rho^t_* u)$ are mutually parallel, according to the affine structure on $\mathbb{V}$. It is easily verified that the quotient of $T\mathbb{V}|_{\mathcal{N}_+}$ by the $\mathbf{R}_+$ action just defined is a rank $d + 2$ vector bundle $\mathcal{T}$ on M. The parallel transport of $\mathbb{V}$ determines a parallel transport on $\mathcal{T}$, that is a connection $\nabla^{\mathcal{T}}$. Since $\mathbb{V}$ is totally parallel this connection is flat. The twisting of ρ^t_* to $t^{-1}\rho^t_*$ is designed so that the metric $\mathcal{H}$ on $\mathbf{R}^{d+2}$ also descends to give a (signature $(d + 1, 1)$) metric h on $\mathcal{T}$ and clearly this is preserved by the connection. In fact $(\mathcal{T}, h, \nabla^{\mathcal{T}})$ is the usual normal standard tractor bundle. This is proved under far more general circumstances in [31] (see also [32]); it is shown there that the tractor bundle may be recovered from the Fefferman–Graham ambient metric by an argument generalising that above. In this picture the Euler vector field $E = X^A\partial/\partial X^A$ (using the summation convention), which generates the fibres of π, descends to the canonical tractor field $X \in \mathcal{T}[1]$.

It follows from these observations that, since the vector field I is parallel on $\mathbb{V}$, its restriction to $\mathcal{N}_+$ is equivalent to a parallel section of $\mathcal{T}$; we shall also denote this by I. So this is an almost Einstein structure on $\mathbb{S}^d$; $|I|^2 = 1$ means that the almost Einstein structure we recover has $S(\sigma) = -1$, whence has $\mathrm{Ric}(g^o) = -ng^o$ on $C_\pm$. The zero set for $\sigma = h(X, I)$ is exactly Σ. So we see that $(\mathbb{S}^d, [g], I)$ is an almost Einstein manifold. Since it is conformally flat with $S(\sigma) = -1$ it is what may be termed an *almost hyperbolic* structure on the sphere. The fact that along Σ the parallel tractor I gives the normal tractor N is especially natural in this picture since Σ is determined by a hyperplane orthogonal to I. Finally we observe that it follows from Proposition 3.7 that the spaces $(\mathbb{S}^d \setminus C_\pm, [g], I)$ are Poincaré–Einstein manifolds, in fact each equivalent to the conformal compactification of the hyperbolic ball.

Since the group G acts transitively on length 1 spacelike vectors, from the picture above we see that any scalar negative AE structure on the sphere is related to the one constructed by a conformal transformation after an $\mathbb{R}_+$ action on the parallel tractor I.

The scalar flat almost Einstein structures are obtained by a similar construction to the scalar negative case above. Note that if I is a non-zero null vector in $\mathbb{V}$ then the hyperplane $\mathcal{H}(I, X) = 1$ meets all future null rays in $\mathcal{N}_+$ except the one parallel to I. So the almost Einstein structure determined by I has a single isolated point of scale singularity. The Einstein metric g^o is conformally related to the round metric, and $|I|^2 = 0$ means that $S(I) = 0$ and so g^o is flat; this is the usual Euclidean structure on the sphere minus a point. It is straightforward to conclude that the map, along null generators, relating this Euclidean almost Einstein structure and the standard sphere embedded in the cone (as described earlier) is the usual stereographic projection.

In a partial summary then, if I_1 and I_2 are constant vectors in $\mathbb{V}$ with $|I_1|^2 = -1$ and $|I_2|^2 = 1$ then, as parallel tractors on $\mathbb{S}^d$ these determine, respectively the standard sphere metric and almost hyperbolic structures. We can interpolate between these via Corollary 2.4 and we note that for some $t \in \mathbb{R}$ the parallel tractor $I_t := (\sin t)I_1 + (\cos t)I_2$ is null and so determines a Ricci-flat structure in the conformal class, that is a Euclidean metric on the sphere minus a point. For each $t \in \mathbb{R}$ the isotropy subgroup G_{I_t} of $G = SO_0(\mathcal{H})$ fixing the vector I_t clearly acts transitively and by isometries on the connected components of $\mathbb{S}^d \setminus \Sigma_t$, where Σ_t is the scale singularity set of I_t.

5.2. Doubling and almost hyperbolic constructions

One route to constructing further compact almost Einstein manifolds is via the doubling of compact Poincaré–Einstein manifolds. So suppose that M is a compact Poincaré–Einstein with conformal infinity Σ. The double we seek is a gluing along Σ,

$$M_{(2)} := (M \sqcup M)/\Sigma$$

where the identification of the two copies of Σ is the obvious one. As pointed out in [44], for example, this may be equipped with a smooth structure compatible with the smooth structure on M and so that the natural involution exchanging the factors is also smooth. Now extend the PE metric g^o of M to a metric on $M_{(2)}$ by symmetry. This will be smooth if g^o is *even* in the sense of [22, Section 4] (following [17]): Locally along the collar the metric may be put in normal form, relative to some g^Σ from the conformal class on Σ,

$$g^o = s^{-2}(ds^2 + g^\Sigma_s) \tag{5.1}$$

where s satisfies $|ds|^2_g = 1$ and g_s is a 1-parameter family of metrics on Σ such that $g^\Sigma_0 = g^\Sigma$. The metric is *even* if for each point of Σ, and with the metric g^o in this form, we have that $ds^2 + g_s$ is the restriction to $M \times [0, \infty)$ of a smooth metric g on a neighbourhood $\mathcal{U} \subset \Sigma \times (-\infty, \infty)$ such that $\mathcal{U}$ and g are invariant under the map $s \mapsto -s$.

Infinite volume hyperbolic manifolds provide a source of even PE manifolds. From Theorem 7.4 in [22] (building on [45,46]) we have that if (M, g^o) is a hyperbolic PE manifold then locally along the conformal infinity Σ it may be put in the normal form (5.1) where (in terms of local coordinates (s, x^i), with x^i the coordinates on Σ) we have

$$(g_s)_{ij} = g^\Sigma_{ij} - P^\Sigma_{ij}s^2 + \frac{1}{4}g^{kl}_\Sigma P^\Sigma_{ik}P^\Sigma_{lj}s^4;$$

here if $d \geq 4$ then P^Σ_{ij} is the intrinsic Schouten tensor of g^Σ, while if $d = 3$ then P^Σ_{ij} is a symmetric 2-tensor on Σ satisfying $2g^{ij}_\Sigma P^\Sigma_{ij} = \mathrm{Sc}^{g^\Sigma}$ and $2g^{ij}_\Sigma \nabla_i P^\Sigma_{jk} = \nabla^\Sigma_k \mathrm{Sc}^{g^\Sigma}$. In this case g^o is manifestly even.

Let Γ be a convex co-compact, torsion-free, discrete group of orientation preserving isometries of $\mathbb{H}^d$. Then the orbit space $M_+ := \Gamma \setminus \mathbb{H}^d$ is a hyperbolic manifold of infinite volume. Such M_+ may be conformally compactified [9,44] to yield a (hyperbolic) PE manifold. Thus, by the doubling construction, to each group Γ as above we may associate a closed almost Einstein structure.

Rather than the usual model of the hyperbolic ball we may realise $\mathbb{H}^d$ as a hyperbolic cap of the sphere as described in Section 5.1; it is not difficult to see that we may arrange that the cap is the right-hemisphere of a standard round sphere, where the latter is given also as a section of the cone as Section 5.1. Then the convex co-compact Γ arises as a discrete subgroup of $G_I \subset G = SO(\mathcal{H})$, where G_I is the isotropy subgroup of G which fixes the length 1 parallel tractor I defining the hyperbolic manifold. Now Γ also acts on the hyperbolic left-hemisphere and, by symmetry, in both cases the conformal infinity Σ_Γ may be identified with the orbit space $\Gamma \setminus \Omega_\Gamma(\Sigma)$ where $\Omega_\Gamma(\Sigma)$ is the open subset of the sphere $\Sigma \cong \mathbb{S}^n$ where Γ acts properly discontinuously. The smooth structure on the doubling of M is induced by the usual smooth structure on the sphere and the action of Γ on $\mathbb{S}^d$ evidently preserves the solution of (2.4) giving I. In this sense we may view the doubling of M as arising from the orbit space of Γ on $\mathbb{S}^d$ equipped with a standard almost hyperbolic structure (i.e. hyperbolic almost Einstein structure as in Section 5.1), but where we first remove the limit of this action in Σ. The AE structure on this is a solution to (2.4) descended from a solution on $\mathbb{S}^d$.

6. The Fefferman–Graham metric for an AE manifold and obstructions

The Fefferman–Graham tensor (also called "the obstruction tensor") is a natural conformally invariant symmetric trace-free 2-tensor $\mathcal{B}_{ab}$ on manifolds of even dimension n that has the form $\Delta^{n/2-2}\nabla^c\nabla^d C_{acbd} + \textit{lower order terms}$. In the case $n = 4$ it agrees with the Bach tensor while in higher even dimensions it is due to Fefferman and Graham [21]. In Corollary 4.8 we found that the Bach tensor necessarily vanishes on the scale singularity set of AE 5-manifolds. Here we prove the analogue of that result for higher odd dimensions. In the process of proving this we obtain an extension to Theorem 4.5.

For $\pi : \mathcal{Q} \to M^d$ a Riemannian conformal structure, let us use ρ to denote the $\mathbb{R}_+$ action on $\mathcal{Q}$ given by $\rho(s)(x, g_x) = (x, s^2 g_x)$. An *ambient manifold* is a smooth $(d + 2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$–action ρ and an $\mathbb{R}_+$–equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We write $\boldsymbol{X} \in \Gamma(T\tilde{M})$ for the fundamental field generating the $\mathbb{R}_+$–action. That is, for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$, we have $\boldsymbol{X}f(u) = (\mathrm{d}/\mathrm{d}t)f(\rho(\mathrm{e}^t)u)|_{t=0}$. For an ambient manifold $\tilde{M}$, an *ambient metric* is a pseudo–Riemannian metric $\boldsymbol{h}$ of signature $(d + 1, 1)$ on $\tilde{M}$ satisfying the conditions: (i) $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{h} = 2\boldsymbol{h}$, where $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$; (ii) for $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$. In [21] (and see [22]) Fefferman and Graham considered formally the Gursat problem of obtaining $\mathrm{Ric}(\boldsymbol{h}) = 0$. They proved that for the case of $d = 2$ and $d \geq 3$ odd this may be achieved to all orders, while for $d \geq 4$ even the problem is obstructed at finite order by the tensor $\mathcal{B}_{ab}$; for d even one may obtain $\mathrm{Ric}(\boldsymbol{h}) = 0$ up to the addition of terms vanishing to order $d/2 - 1$. (See [22] for the statements concerning uniqueness. For extracting results via tractors we do not need this, as discussed in e.g. [31,32].) We shall henceforth call any (approximately or otherwise) Ricci-flat ambient metric a *Fefferman–Graham metric*.

Since an AE manifold $(M^d, [g], I)$ has, by definition, a conformal structure we may construct the Fefferman–Graham metric, as for any conformal manifold. We have already exploited this in the proof of Theorem 4.7. On the other hand if $S(I) < 0$ and the scale singularity set Σ is non-empty then, as discussed in Section 4, this embedded n-manifold ($n = d-1$) has induced on it a conformal structure $(\Sigma, [g_\Sigma])$. We may ask how the Fefferman–Graham metric for $(\Sigma, [g_\Sigma])$ is related to the Fefferman–Graham metric for $(M^d, [g])$. In Theorem 6.3 below for n even (so $d = n+1$ odd) we show that $(\Sigma, [g_\Sigma])$ admits a Fefferman–Graham which is Ricci-flat to all orders. Thus we obtain the following.

Theorem 6.1. *Suppose that $(\Sigma^n, [g_\Sigma])$ is the scale singularity space of a scalar negative almost Einstein manifold, then the Fefferman–Graham tensor of $(\Sigma^n, [g_\Sigma])$ is zero.*

Using Theorem 1.2, this result also follows from [22, Theorem 4.8] or [41, Theorem 2.1]. The proof here follows a rather different tack.

In the subsequent discussion of ambient metrics all results can be assumed to hold formally to all orders unless stated otherwise. We typically use bold symbols or tilde symbols for the objects on $\tilde{M}$. For example $\boldsymbol{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. It is assumed the reader is somewhat familiar with treatments of Fefferman–Graham metrics. In particular, as used in the proof of Theorem 4.7, we use that suitably homogeneous tensor fields of $\tilde{M}|_{\mathcal{Q}}$ correspond to tractor fields. The notation and approach here follows that in [42,31,40].

Lemma 6.2. *Let $(M^d, [g], I)$ be an AE manifold with $d \geq 3$ odd. There is a parallel 1-form field $\mathbf{I}$ on $\tilde{M}$ such that $\mathbf{I}|_{\mathcal{Q}}$ is the homogeneous (of weight 0) section of $T^*\tilde{M}|_{\mathcal{Q}}$ corresponding to I.*

Proof. Let $\sigma := h(X, I)$, as usual. This corresponds to a function on $\mathcal{Q}$ homogeneous of degree 1. Since d is odd, this may be extended "harmonically" to all orders (e.g. [47]). That is there is a smooth homogeneous degree 1 function $\boldsymbol{\sigma}$ on $\tilde{M}$ such that $\boldsymbol{\Delta\sigma} = 0$ and $\boldsymbol{\sigma}|_{\mathcal{Q}}$ is the homogeneous function corresponding to the conformal density σ.

The operator $\tilde{\mathbb{D}}_A = (d + 2w - 2)\boldsymbol{\nabla}_A + \boldsymbol{X}_A\boldsymbol{\Delta}$, on $\tilde{M}$, corresponds to the tractor-D operator $\mathbb{D}$ [31,32]. This acts tangentially along $\mathcal{Q}$ in the sense of [42] and [40]. Define $\boldsymbol{I}_A := \boldsymbol{\nabla}_A\boldsymbol{\sigma} = \frac{1}{d}\tilde{\mathbb{D}}_A\boldsymbol{\sigma}$. This has the required properties. Obviously $\mathbf{I}|_{\mathcal{Q}}$ corresponds to $I_A = \frac{1}{d}\mathbb{D}_A\sigma$. (Recall on a density σ of weight 1, $\frac{1}{d}\mathbb{D}_A\sigma = D\sigma$.) Now note that $\boldsymbol{\Delta I}_B = \boldsymbol{\Delta\nabla}_B\boldsymbol{\sigma} = \boldsymbol{\nabla}_B\boldsymbol{\Delta\sigma} = 0$, to all orders, as the FG metric is Ricci-flat to all orders. Now $\mathbb{D}_A I_B = 0$ on M, and so $\tilde{\mathbb{D}}_A\boldsymbol{I}_B|_{\mathcal{Q}} = 0$. Using the previous result we conclude $((d-2)\boldsymbol{\nabla}_A\boldsymbol{I}_B)|_{\mathcal{Q}} = 0$. Now by induction we get that $\boldsymbol{I}_A$ is parallel to all orders: Suppose that $\boldsymbol{\nabla}_{A_2}\cdots\boldsymbol{\nabla}_{A_{i+1}}\boldsymbol{I}_B|_{\mathcal{Q}} = 0$ for $i = 1, \ldots, k$ then, since $\tilde{\mathbb{D}}$ acts tangentially, we get

$$\tilde{\mathbb{D}}_{A_1}\boldsymbol{\nabla}_{A_2}\cdots\boldsymbol{\nabla}_{A_{k+1}}\boldsymbol{I}_B|_{\mathcal{Q}} = 0.$$

Thus, along $\mathcal{Q}$,

$$(d-2k-2)\boldsymbol{\nabla}_{A_1}\boldsymbol{\nabla}_{A_2}\cdots\boldsymbol{\nabla}_{A_{k+1}}\boldsymbol{I}_B + \boldsymbol{X}_{A_1}\boldsymbol{\Delta\nabla}_{A_2}\cdots\boldsymbol{\nabla}_{A_{k+1}}\boldsymbol{I}_B = 0.$$

To study the second term we may commute the Laplacian $\boldsymbol{\Delta}$ to the right of the ∇'s with free indices. We see then that this entire term drops out as $\boldsymbol{\Delta I}$ vanishes to all orders while the other terms pick up curvature and hence involve at most k derivatives of $\boldsymbol{I}$ (and so exit by the inductive hypothesis). On the other hand, since d is odd, $(d-2k-2) \neq 0$. □

Using this we obtain the key result.

Theorem 6.3. *Let $(M^d, [g], I)$ be a scalar negative AE manifold with $d \geq 3$ odd and $\Sigma \neq \emptyset$. Write $\boldsymbol{I}$ for the parallel 1-form field on $\tilde{M}$ corresponding (as in the Lemma above) to I. Write $\boldsymbol{\Sigma}$ for the hypersurface given as the zero set of $\boldsymbol{\sigma} := \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{I})$. This has a metric $\boldsymbol{h}_{\boldsymbol{\Sigma}}$ induced from $\boldsymbol{h}$, it is totally geodesic, and $(\boldsymbol{\Sigma}, \boldsymbol{h}_{\boldsymbol{\Sigma}})$ is a Fefferman–Graham metric for $(\Sigma, [g_\Sigma])$, which is formally smooth and Ricci-flat to all orders.*

Proof. First some observations. Since $\boldsymbol{d\sigma} \neq 0$, (and in particular this holds along $\boldsymbol{\Sigma}$) it is clear that $\boldsymbol{\Sigma}$ is a smooth hypersurface, and its intersection with $\mathcal{Q}$ is the inverse image of Σ with respect to the standard map $\mathcal{Q} \to M$. Since the conformal structure $[g_\Sigma]$ of Σ is induced from the conformal structure of the ambient space $(M, [g])$ it follows easily that, when restricted to the tangents of this intersection, $\boldsymbol{h}_{\boldsymbol{\Sigma}}$ agrees with the tautological 2-form (which we have since the intersection of $\boldsymbol{\Sigma}$ with $\mathcal{Q}$ is naturally identified with the bundle of metrics in the conformal class over $(\Sigma, [g_\Sigma])$).

Since $\boldsymbol{\sigma}$ is homogeneous of degree 1 we have $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{\sigma} = \boldsymbol{\sigma}$ and so along $\boldsymbol{\Sigma}$ the field $\boldsymbol{X}$ is everywhere tangent to $\boldsymbol{\Sigma}$. Clearly $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{h}^{\boldsymbol{\Sigma}} = 2\boldsymbol{h}^{\boldsymbol{\Sigma}}$ from the analogous property for $\boldsymbol{h}$.

Since $\boldsymbol{d\sigma}$ is parallel $\boldsymbol{h}^{-1}(\boldsymbol{d\sigma}, \boldsymbol{d\sigma})$ is constant and agrees with $|I|^2 > 0$. In particular, along $\boldsymbol{\Sigma}$, $\boldsymbol{N} := \boldsymbol{d\sigma}$ gives a parallel conormal field for $\boldsymbol{\Sigma}$, of non-zero pointwise length. So $\boldsymbol{\Sigma}$ is totally geodesic.

Once again using that $\boldsymbol{d\sigma}$ is parallel we have that $\boldsymbol{R}_{AB}{}^C{}_D\boldsymbol{N}_C = 0$ along $\boldsymbol{\Sigma}$. It follows that the intrinsic Ricci curvature $\mathbf{Ric}^{\boldsymbol{\Sigma}}$ agrees with the tangential restriction of the ambient Ricci curvature. But the latter is everywhere zero to all orders, and therefore so is $\mathbf{Ric}^{\boldsymbol{\Sigma}}$. □

Corollary 6.4. *If $(M^d, [g], I)$ is a scalar negative AE structure with $d \geq 6$ and a non-empty scale singularity space Σ, then $(d-5)W|_\Sigma = (d-4)W^\Sigma$.*

In the Corollary we view, by trivial extension, W^Σ as a section of $\otimes^4 \mathcal{T}$.

Proof. If $d \geq 7$ is odd then this is immediate from the proof above. Since $\boldsymbol{\Sigma}$ is totally geodesic and $\boldsymbol{N}$ annihilates the curvature $\boldsymbol{R}$ of the Fefferman–Graham metric $\boldsymbol{h}$, it follows that along $\boldsymbol{\Sigma}$ we have $\boldsymbol{R} = \boldsymbol{R}^{\boldsymbol{\Sigma}}$ (using a trivial extension to view $\boldsymbol{R}^{\boldsymbol{\Sigma}}$ as a section of $\otimes T^*\tilde{M}$). But, as used in Section 4.4, $(d-4)\boldsymbol{R}|_{\mathcal{Q}}$ is the ambient tensor field equivalent to the tractor W; $(n-4)\boldsymbol{R}^{\boldsymbol{\Sigma}}$ similarly corresponds to W^Σ.

If $d \geq 6$ is even it is straightforward to verify that the results in Lemma 6.2 and in Theorem 6.3 hold to sufficient order to obtain the result here. □

In some sense the Corollary applies to all dimensions $d \geq 3$ except for $d = 4$ as follows. When $d = 5$ since $[g_\Sigma]$ is Bach-flat we have $W^\Sigma = 0$. However $W|_\Sigma$ gives a tractor field equivalent to $\boldsymbol{R}^\Sigma$. For $d = 3$ AE manifolds both $\boldsymbol{R}$ and $\boldsymbol{R}^{\boldsymbol{\Sigma}}$ are zero.

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