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Noncommutative Bloch analysis of Bochner Laplacians with nonvanishing gauge fields

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ABSTRACT

Given an invariant gauge potential and a periodic scalar potential $\tilde{V}$ on a Riemannian manifold $\tilde{M}$ with a discrete symmetry group Γ, consider a Γ-periodic quantum Hamiltonian $\tilde{H} = -\tilde{\Delta}_{\mathrm{B}} + \tilde{V}$ where $\tilde{\Delta}_{\mathrm{B}}$ is the Bochner Laplacian. Both the gauge group and the symmetry group Γ can be noncommutative, and the gauge field need not vanish. On the other hand, Γ is supposed to be of type I. With any unitary representation Λ of Γ one associates a Hamiltonian $H^{\Lambda} = -\Delta_{\mathrm{B}}^{\Lambda} + V$ on $M = \tilde{M}/\Gamma$ where V is the projection of $\tilde{V}$ onto M. We describe a construction of the Bloch decomposition of $\tilde{H}$ into a direct integral whose components are H^{Λ}, with Λ running over the dual space $\hat{\Gamma}$. The evolution operator and the resolvent decompose correspondingly. Conversely, given $\Lambda \in \hat{\Gamma}$, one can express the propagator $\mathcal{K}_t^{\Lambda}(y_1, y_2)$ (the kernel of $\exp(-\mathrm{i}tH^{\Lambda})$) in terms of the propagator $\tilde{\mathcal{K}}_t(y_1, y_2)$ (the kernel of $\exp(-\mathrm{i}t\tilde{H})$) as a weighted sum over Γ. Such a formula is known in theoretical physics for the case when the gauge field vanishes and $\tilde{M}$ is a universal covering space of a multiply connected manifold M. We show that these constructions are mutually inverse. Analogous formulas exist for resolvents and their kernels (Green functions) as well.

1. Introduction

Suppose there is given a connected Riemannian manifold $\tilde{M}$ with a discrete symmetry group Γ, and an invariant Hermitian vector fiber bundle with connection $(\tilde{\mathfrak{V}}, \tilde{\mathfrak{h}}, \tilde{\nabla})$ over $\tilde{M}$. This means that $(\tilde{\mathfrak{V}}, \tilde{\mathfrak{h}}, \tilde{\nabla}) = \pi^*(\mathfrak{V}, \mathfrak{h}, \nabla)$ where $(\mathfrak{V}, \mathfrak{h}, \nabla)$ is a Hermitian vector fiber bundle with connection over $M = \tilde{M}/\Gamma$ and $\pi : \tilde{M} \to M$ is the projection. Let us consider a Γ-periodic Hamilton operator in $L^2(\tilde{\mathfrak{V}})$ of the form $\tilde{H} = -\tilde{\Delta}_{\mathrm{B}} + \tilde{V}$ where $\tilde{\Delta}_{\mathrm{B}}$ is the Bochner Laplacian and $\tilde{V}$ is a Γ-invariant semibounded real function on $\tilde{M}$ ($\tilde{V}$ is the pull-back of a function V on M). For any finite-dimensional unitary representation Λ of Γ in a vector space $\mathscr{L}_{\Lambda}$, one constructs a Hermitian vector fiber bundle $\mathfrak{V}^{\Lambda}$ over M with a connection ∇^{Λ}, and consequently a Hamiltonian $H^{\Lambda} = -\Delta_{\mathrm{B}}^{\Lambda} + V$ in $L^2(\mathfrak{V}^{\Lambda})$. An important feature of the construction is that the operator $\tilde{H}$ decomposes into a direct integral with components H^{Λ} where Λ runs over all equivalence classes of irreducible unitary representations of Γ. The evolution operator and the resolvent decompose correspondingly.

This type of construction is well known in the cases when either the connection ∇ is flat (and so $\Delta_{\mathrm{B}} = \Delta_{\mathrm{LB}}$ is the Laplace–Beltrami operator) [1–4], or the group Γ is commutative [5,6]. The general case when the connection is not flat and the symmetry group Γ need not be commutative is treated in [7] in the framework of noncommutative geometry using the theory of C^* algebras. The subject of the current paper is, too, an extension of the construction of the Bloch decomposition to such a general case. In contrast to [7], however, our construction relies on standard techniques of differential geometry

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and, in particular, the theory of C^* algebras is not employed at all. We follow closely the presentation given in [8] for the particular case of a flat connection and a noncommutative symmetry group Γ.

The action of Γ on $\tilde{M}$ is usually supposed to be co-compact [1–3,9,5,6,4]. Following [8], we relax this assumption while introducing the Hamiltonian $-\tilde{\Delta}_B + \tilde{V}$ as the Friedrichs extension of the corresponding symmetric operator defined on smooth sections with compact supports.

In the framework of Feynman path integrals, there was derived a remarkable formula relating the propagators $\mathcal{K}_t^\Lambda(x, x_0)$ and $\tilde{\mathcal{K}}_t(x, x_0)$ associated respectively with the Hamiltonians H^Λ and $\tilde{H}$ [10,11]. A similar relation is known to hold for heat kernels [12]. Another important application in mathematics of this type of formula is in the derivation of the Selberg trace formula [13,14]. We show that such a formula makes sense also in the more general case with a nonvanishing gauge field. Moreover, an analogous formula can be derived for Green functions.

2. Basic notions and notation

All geometric objects are supposed to be smooth. Let $\tilde{M}$ be a connected Riemannian manifold (Hausdorff and second countable) with a discrete and at most countable symmetry group Γ. The action of Γ on $\tilde{M}$ is assumed to be smooth (every $\gamma \in \Gamma$ acts as a diffeomorphism on $\tilde{M}$), free, isometric and properly discontinuous. Denote by $\tilde{\mu}$, the measure on $\tilde{M}$ induced by the Riemannian metric. The quotient $M = \tilde{M}/\Gamma$ is a connected Riemannian manifold with an induced measure μ. The factorization defines a principal fiber bundle $\pi : \tilde{M} \to M$ with the structure group Γ. All L^2 spaces based on manifolds M and $\tilde{M}$ are everywhere understood with the measures μ and $\tilde{\mu}$, respectively.

To be specific, let us recall that the assumption on the properly discontinuous action implies that for any compact set $K \subset \tilde{M}$, the intersection $K \cap \gamma \cdot K$ is nonempty for at most finitely many elements $\gamma \in \Gamma$. Moreover, any point $y \in \tilde{M}$ has a neighborhood U such that the sets $\gamma \cdot U$, $\gamma \in \Gamma$, are mutually disjoint (see, for instance, [15]).

Furthermore, assume that on $\tilde{M}$, there is given a Γ-invariant gauge potential. Geometrically this means that over $\tilde{M}$ there is given an invariant Hermitian vector fiber bundle with connection (covariant derivative) $(\tilde{\mathfrak{V}}, \tilde{\mathfrak{h}}, \tilde{\nabla})$. That is, the action of Γ on $\tilde{M}$ lifts to an action Ψ on $\tilde{\mathfrak{V}}$ which is fiber-wise linear, and both the Hermitian product on fibers, $\tilde{\mathfrak{h}}$, and the covariant derivative $\tilde{\nabla}$ are invariant with respect to the action Ψ.

Let us denote by L_γ the left action of $\gamma \in \Gamma$ on $\tilde{M}$, i.e. $L_\gamma(y) = \gamma \cdot y$ for $y \in \tilde{M}$, and by $C^\infty(\tilde{\mathfrak{V}})$ the vector space of smooth sections of $\tilde{\mathfrak{V}}$. Notice that the invariance of $\tilde{\nabla}$ means that for any $\gamma \in \Gamma$ fixed and at any point $y \in \tilde{M}$,

$$\forall \varphi \in C^\infty(\tilde{\mathfrak{V}}),\ \forall X \in T_y\tilde{M}, \quad \tilde{\nabla}_X \left(\Psi_{\gamma^{-1}}\varphi(\gamma \cdot y)\right) = \Psi_{\gamma^{-1}} \left(\tilde{\nabla}_{\gamma \cdot X}\varphi(\gamma \cdot y)\right) \in \tilde{\mathfrak{V}}_y, \tag{1}$$

where $\gamma \cdot X \equiv (\mathrm{d}L_\gamma)_y X \in T_{\gamma\cdot y}\tilde{M}$. This property can be reformulated as follows. Let W_γ, $\gamma \in \Gamma$, be the one-parameter family of linear operators on $C^\infty(\tilde{\mathfrak{V}})$ defined by

$$\forall \varphi \in C^\infty(\tilde{\mathfrak{V}}), \quad (W_\gamma \varphi)(y) = \Psi_\gamma \varphi(\gamma^{-1} \cdot y). \tag{2}$$

One clearly has $W_\gamma W_{\gamma'} = W_{\gamma\gamma'}$, $\forall \gamma, \gamma' \in \Gamma$. Relation (1) means that for all smooth vector fields $\xi \in C^\infty(T\tilde{M})$,

$$\forall \gamma \in \Gamma, \quad \tilde{\nabla}_\xi W_\gamma = W_\gamma \tilde{\nabla}_{\gamma^{-1}\cdot\xi}.$$

The assumptions mean that $(\tilde{\mathfrak{V}}, \tilde{\mathfrak{h}}, \tilde{\nabla})$ admits a factorization with respect to the action of Γ, and thus over M there exists a Hermitian vector fiber bundle with connection, $(\mathfrak{V}, \mathfrak{h}, \nabla)$, such that $(\tilde{\mathfrak{V}}, \tilde{\mathfrak{h}}, \tilde{\nabla}) = \pi^*(\mathfrak{V}, \mathfrak{h}, \nabla)$ (the usual pull-back by the projection $\pi : \tilde{M} \to M$). Conversely, any such a pull-back is naturally Γ-invariant. The vector space $C^\infty(\mathfrak{V})$ is identified with the subspace in $C^\infty(\tilde{\mathfrak{V}})$ formed by those smooth sections φ which satisfy $W_\gamma\varphi = \varphi$, $\forall \gamma \in \Gamma$; in more detail,

$$\forall \gamma \in \Gamma,\ \forall y \in \tilde{M}, \quad \varphi(\gamma \cdot y) = \Psi_\gamma \varphi(y). \tag{3}$$

Let $\xi \in C^\infty(TM)$ be a vector field on M and $\tilde{\xi} \in C^\infty(T\tilde{M})$ be the unique vector field on $\tilde{M}$ such that $\mathrm{d}\pi(\tilde{\xi}) = \xi$ (hence $\tilde{\xi}$ is Γ-invariant). If $\varphi \in C^\infty(\tilde{\mathfrak{V}})$ fulfills (3), then $\tilde{\nabla}_{\tilde{\xi}}\varphi$ fulfills (3) as well. This defines the covariant derivative ∇_ξ in $\mathfrak{V}$.

The Bochner Laplacian $\tilde{\Delta}_B$ is a second-order differential operator acting on smooth sections of $\tilde{\mathfrak{V}}$ whose construction depends on the covariant derivative $\tilde{\nabla}$ and on the Riemannian metric $\tilde{\mathfrak{g}}$ defined on cotangent spaces on $\tilde{M}$. If $\varphi \in C^\infty(\tilde{\mathfrak{V}})$, then $\tilde{\nabla}\varphi$ belongs to $C^\infty(T^*\tilde{M} \otimes \tilde{\mathfrak{V}})$. The differential operator $\tilde{\Delta}_B$ is unambiguously determined by the equality

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(\tilde{\mathfrak{V}}), \quad \int_{\tilde{M}} \tilde{\mathfrak{h}}(\varphi_1, -\tilde{\Delta}_B\varphi_2)\,\mathrm{d}\tilde{\mu} = \int_{\tilde{M}} \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{h}}(\tilde{\nabla}\varphi_1, \tilde{\nabla}\varphi_2)\,\mathrm{d}\tilde{\mu}. \tag{4}$$

The Bochner Laplacian $\tilde{\Delta}_B$ is Γ-invariant in the sense that, using the defining relation (2),

$$\forall \gamma \in \Gamma, \quad \tilde{\Delta}_B W_\gamma = W_\gamma \tilde{\Delta}_B.$$

Analogously, one introduces the Bochner Laplacian Δ_{B} on M which is associated with the covariant derivative ∇ and with the Riemannian metric $\mathfrak{g}$. If $\sigma \in C^\infty(\mathfrak{V})$ is represented by $\varphi \in C^\infty(\tilde{\mathfrak{V}})$ fulfilling (3), then $\Delta_{\mathrm{B}}\sigma$ is represented by $\tilde{\Delta}_{\mathrm{B}}\varphi$.

Finally, we summarize several basic facts concerning harmonic analysis on Γ. In general, the harmonic analysis is well established for locally compact groups of type I [16], and this is why we assume in the sequel that Γ belongs to this class. In addition, all irreducible representations of type I groups are finite-dimensional, and even the dimension is uniformly bounded [17, Korollar I]. This fact facilitates various algebraic constructions throughout the paper. On the other hand, it is known that a countable discrete group is of type I if and only if it has an Abelian normal subgroup of finite index [17, Satz 6]. This means, unfortunately, that there exist covering spaces of interest whose structure group Γ is not of type I, and here we do not treat such cases.

Let $\hat{\Gamma}$ be the dual space to Γ (the quotient space of the space of irreducible unitary representations of Γ). In the case in question the Haar measure on Γ is nothing but the counting measure. Let $\hat{m}$ be the Plancherel measure on $\hat{\Gamma}$. For $\Lambda \in \hat{\Gamma}$ denote by $\mathscr{I}(\mathscr{L}_\Lambda)$ the space of linear maps on $\mathscr{I}_\Lambda$. Note that one can naturally identify $\mathscr{I}(\mathscr{L}_\Lambda) \equiv \mathscr{L}_\Lambda \otimes \mathscr{L}_\Lambda^*$ ($\mathscr{L}_\Lambda^*$ is the dual space to $\mathscr{L}_\Lambda$) and thus $\mathscr{I}(\mathscr{L}_\Lambda)$ becomes equipped with the scalar product $\langle A_1, A_2\rangle = \mathrm{Tr}(A_1^* A_2)$, $\forall A_1, A_2 \in \mathscr{I}(\mathscr{L}_\Lambda)$.

The Fourier transform is defined as a unitary mapping

$$\mathscr{F} : L^2(\Gamma) \to \int_{\hat{\Gamma}}^{\oplus} \mathscr{I}(\mathscr{L}_\Lambda)\,\mathrm{d}\hat{m}(\Lambda).$$

For $f \in L^1(\Gamma) \subset L^2(\Gamma)$, one has

$$\mathscr{F}[f](\Lambda) = \sum_{\gamma\in\Gamma} f(\gamma)\Lambda(\gamma).$$

Conversely, if f is of the form $f = g * h$ (the convolution) where $g, h \in L^1(\Gamma)$, and $\hat{f} = \mathscr{F}[f]$, then

$$f(s) = \int_{\hat{\Gamma}} \mathrm{Tr}[\Lambda(s)^* \hat{f}(\Lambda)]\,\mathrm{d}\hat{m}(\Lambda).$$

Using the fact that Γ is a countable discrete group as well as the unitarity of the Fourier transform, one finds that

$$\hat{m}(\hat{\Gamma}) \le \int_{\hat{\Gamma}} \dim \mathscr{L}_\Lambda\,\mathrm{d}\hat{m}(\Lambda) = 1.$$

The following rules satisfied by the Fourier transformation are also of importance:

$$\forall r \in \Gamma,\ \forall f \in L^2(\Gamma), \quad \mathscr{F}[f(r\cdot\gamma)](\Lambda) = \Lambda(r^{-1})\mathscr{F}[f(\gamma)](\Lambda) \tag{5}$$

(here $\mathscr{F}$ acts in the variable $\gamma \in \Gamma$), and, conversely,

$$\forall r \in \Gamma,\ \forall \hat{f} \in \int_{\hat{\Gamma}}^{\oplus} \mathscr{I}(\mathscr{L}_\Lambda)\,\mathrm{d}\hat{m}(\Lambda), \quad \mathscr{F}^{-1}[\Lambda(r)\hat{f}(\Lambda)](\gamma) = \mathscr{F}^{-1}[\hat{f}(\Lambda)](r^{-1}\gamma) \tag{6}$$

(here $\mathscr{F}^{-1}$ acts in the variable $\Lambda \in \hat{\Gamma}$).

3. A construction of the noncommutative Bloch decomposition

3.1. Associated vector fiber bundles over M

Since $\dim \tilde{M} = \dim M$, in the principal fiber bundle $\tilde{M}$ over M with the structure group Γ, there exists a unique connection which is necessarily flat. Given a finite-dimensional unitary representation Λ of Γ in $\mathscr{L}_\Lambda$, one can associate with the principal fiber bundle a vector fiber bundle $E(\Lambda)$ over M with a typical fiber $\mathscr{L}_\Lambda$ [18]. Provided that the representation is unitary, $E(\Lambda)$ naturally acquires a Hermitian structure. The flat connection in $\tilde{M}$ carries over to the vector fiber bundle $E(\Lambda)$ as a Hermitian covariant derivative which is again flat.

Suppose $(\mathfrak{V}_j, \mathfrak{h}_j, \nabla_j)$, $j = 1, 2$, are two Hermitian vector fiber bundles with connection over M. The tensor product $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ is a vector fiber bundle over M with fibers $(\mathfrak{V}_1 \otimes \mathfrak{V}_2)_x = (\mathfrak{V}_1)_x \otimes (\mathfrak{V}_2)_x$, $x \in M$, and it is again equipped with a Hermitian structure in a canonical way. Moreover, a Hermitian connection ∇_{12} is naturally defined in $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ by the rule: for any vector field $\xi \in C^\infty(TM)$,

$$\forall \varphi_1 \in C^\infty(\mathfrak{V}_1),\ \varphi_2 \in C^\infty(\mathfrak{V}_2), \quad \nabla_{12}(\xi)\varphi_1 \otimes \varphi_2 = (\nabla_1(\xi)\varphi_1) \otimes \varphi_2 + \varphi_1 \otimes (\nabla_2(\xi)\varphi_2).$$

Definition 1. Let $(\mathfrak{V}^\Lambda, \mathfrak{h}^\Lambda, \nabla^\Lambda)$ be the Hermitian vector fiber bundle with connection over M obtained as the tensor product of $(\mathfrak{V}, \mathfrak{h}, \nabla)$ and the associated vector fiber bundle $E(\Lambda)$ (which is supposed to be equipped with the Hermitian structure and the Hermitian covariant derivative, as recalled above).

Though the construction of associated fiber bundles is standard, let us indicate some intermediate objects occurring in the construction for the sake of future reference. Denote by $(\tilde{\mathfrak{V}}^\Lambda, \tilde{\mathfrak{h}}^\Lambda, \tilde{\nabla}^\Lambda)$ the Hermitian vector fiber bundle with connection over $\tilde{M}$ with fibers $\tilde{\mathfrak{V}}_y^\Lambda = \tilde{\mathfrak{V}}_y \otimes \mathscr{L}_\Lambda, y \in \tilde{M}$. The Hermitian product on $\tilde{\mathfrak{V}}_y^\Lambda$ is defined in the usual way. The covariant derivative $\tilde{\nabla}^\Lambda$ is defined so that for all $\varphi \in C^\infty(\tilde{\mathfrak{V}})$ and $v \in \mathscr{L}_\Lambda$, one has

$$\tilde{\nabla}^\Lambda \varphi \otimes v = (\tilde{\nabla}\varphi) \otimes v \in C^\infty(T^*\tilde{M} \otimes \tilde{\mathfrak{V}} \otimes \mathscr{L}_\Lambda).$$

Let Ψ^Λ be the action of Γ on $\tilde{\mathfrak{V}}^\Lambda = \tilde{\mathfrak{V}} \otimes \mathscr{L}_\Lambda$ defined by

$$\Psi_\gamma^\Lambda = \Psi_\gamma \otimes \Lambda(\gamma), \quad \forall \gamma \in \Gamma.$$

Moreover, analogously to (2), one introduces a one-parameter family of linear operators on $C^\infty(\tilde{\mathfrak{V}}^\Lambda)$ called W_γ^Λ, $\gamma \in \Gamma$, i.e. one puts

$$W_\gamma^\Lambda = W_\gamma \otimes \Lambda(\gamma), \quad \gamma \in \Gamma.$$

Observe that, with this definition, $(\tilde{\mathfrak{V}}^\Lambda, \tilde{\mathfrak{h}}^\Lambda, \tilde{\nabla}^\Lambda)$ is again Γ-invariant. Furthermore, very similarly to (4), one introduces the Bochner Laplacian $\tilde{\Delta}_B^\Lambda$ as a differential operator acting on smooth sections of $\tilde{\mathfrak{V}}^\Lambda$, and one readily finds that

$$\forall \varphi \in C^\infty(\tilde{\mathfrak{V}}),\ \forall v \in \mathscr{L}_\Lambda, \quad \tilde{\Delta}_B^\Lambda \varphi \otimes v = (\tilde{\Delta}_B \varphi) \otimes v.$$

Note that the Bochner Laplacian $\tilde{\Delta}_B^\Lambda$ commutes with all W_γ^Λ, $\gamma \in \Gamma$.

$(\mathfrak{V}^\Lambda, \mathfrak{h}^\Lambda, \nabla^\Lambda)$ is in fact nothing but the factorization of $(\tilde{\mathfrak{V}}^\Lambda, \tilde{\mathfrak{h}}^\Lambda, \tilde{\nabla}^\Lambda)$ with respect to the action Ψ^Λ of Γ. Again one has $(\tilde{\mathfrak{V}}^\Lambda, \tilde{\mathfrak{h}}^\Lambda, \tilde{\nabla}^\Lambda) = \pi^*(\mathfrak{V}^\Lambda, \mathfrak{h}^\Lambda, \nabla^\Lambda)$. It is convenient to identify smooth (or measurable) sections ψ of $\mathfrak{V}^\Lambda$ with smooth (measurable) sections φ of $\tilde{\mathfrak{V}}^\Lambda$ fulfilling

$$\forall \gamma \in \Gamma, \quad \varphi(\gamma \cdot y) = \Psi_\gamma^\Lambda \varphi(y) \tag{7}$$

everywhere (or almost everywhere) on $\tilde{M}$. In that case we say that φ is an equivariant section.

The Bochner Laplacian Δ_B^Λ associated with ∇^Λ and $\mathfrak{g}$ is introduced on M in the standard way, similarly to (4).

We conclude this subsection with an auxiliary construction.

Definition 2. Let us define $\Phi^\Lambda : C_0^\infty(\tilde{\mathfrak{V}}^\Lambda) \to C_0^\infty(\mathfrak{V}^\Lambda)$ so that $\forall \sigma \in C_0^\infty(\tilde{\mathfrak{V}}^\Lambda)$, $\Phi^\Lambda\sigma$ is represented by the series

$$\varphi = \sum_{\gamma \in \Gamma} W_\gamma^\Lambda \sigma \in C_0^\infty(\tilde{\mathfrak{V}}^\Lambda), \quad \text{i.e. } \varphi(y) = \sum_{\gamma \in \Gamma} \Psi_\gamma^\Lambda \sigma(\gamma^{-1} \cdot y). \tag{8}$$

Remark. Note that φ is in fact a smooth section of $\tilde{\mathfrak{V}}^\Lambda$ for the action of Γ is properly discontinuous. Moreover, φ fulfills (7) and thus it represents a smooth section of $\mathfrak{V}^\Lambda$ whose support is contained in $\pi(\operatorname{supp} \sigma)$ and so is compact.

Lemma 3. *The range of Φ^Λ is equal to the whole space $C_0^\infty(\mathfrak{V}^\Lambda)$.*

Proof. Given that $\psi \in C_0^\infty(\mathfrak{V}^\Lambda)$, one can assume, without loss of generality, that there exists an open neighborhood $U \supset \operatorname{supp} \psi$ such that the principal fiber bundle $\pi : \tilde{M} \to M$ is trivial over U. Let $\eta : U \to \eta(U) \subset \tilde{M}$ be a smooth section, and $\varphi \in C^\infty(\tilde{\mathfrak{V}}^\Lambda)$ be the equivariant section (i.e. fulfilling (7)) representing ψ. Then, the restriction $\varphi|_{\eta(U)}$ is a smooth section of $\tilde{\mathfrak{V}}^\Lambda$ over the open set $\eta(U)$ with a compact support, and it extends naturally to a global section $\sigma \in C_0^\infty(\tilde{\mathfrak{V}}^\Lambda)$ (vanishing outside of $\eta(U)$). Both φ and the series $\sum_{\gamma \in \Gamma} W_\gamma^\Lambda \sigma$ fulfill (7). Moreover, the two sections coincide on $\eta(U)$ and so they coincide everywhere. Hence $\Phi^\Lambda \sigma = \psi$. □

Remark 4. Using once more the fact that the action of Γ is properly discontinuous and that $\tilde{\Delta}_B^\Lambda$ commutes with all operators W_s^Λ, one observes that

$$\forall \sigma \in C_0^\infty(\tilde{\mathfrak{V}}^\Lambda), \quad \Delta_B^\Lambda \Phi^\Lambda \sigma = \Phi^\Lambda \tilde{\Delta}_B^\Lambda \sigma. \tag{9}$$

3.2. The Bloch decomposition

We remind the reader that Γ is assumed to be of type I. As already recalled in Section 2, this in particular means that all irreducible representations of Γ are finite-dimensional. Thus, all representation Hilbert spaces $\mathscr{L}_\Lambda$ are finite-dimensional, which is why all tensor products in the construction of associated vector bundles, as described in Section 3.1, make sense.

Suppose we are given a real measurable Γ-invariant function $\tilde{V}$ on $\tilde{M}$ bounded from below. Hence, $\tilde{V} = \pi^*V$ for a basically unique real measurable function V on M which is bounded from below as well. The differential operator $-\tilde{\Delta}_B + \tilde{V}$ is well defined on the domain $C_0^\infty(\tilde{\mathfrak{V}}) \subset L^2(\tilde{\mathfrak{V}})$. Moreover, since this is a densely defined operator which is readily seen to be symmetric and semibounded, one can apply a standard procedure resulting in a distinguished self-adjoint extension with the same lower bound, the so-called Friedrichs extension [19, Chp. VI Section 2]. This extension is in some sense minimal (its form domain is the smallest one among all self-adjoint extensions), and the basic steps of its construction are roughly as follows. Using the differential operator $-\tilde{\Delta}_{\mathrm{B}}+\tilde{V}$ one defines, in a standard manner, a quadratic form on the domain $C_0^\infty(\tilde{\mathfrak{V}})$. Since this densely defined quadratic form is semibounded, it is closable. Now it is known (by a result which is sometimes called the first representation theorem) that in turn a unique self-adjoint operator is associated with the closed semibounded quadratic form, and this is exactly the sought minimal self-adjoint extension (see also [20, Chp. 5] or [21, Chp. X Section 3]). Let us denote the Friedrichs extension by $\tilde{H}$.

Furthermore, note that the linear operators W_γ, $\gamma\in\Gamma$, defined in (2) map bijectively the vector space $C_0^\infty(\tilde{\mathfrak{V}})$ onto itself and preserve the L^2 norm, and so they extend unambiguously to unitary operators on $L^2(\tilde{\mathfrak{V}})$. The Hamiltonian $\tilde{H}$ commutes with all unitary operators W_γ, and in this sense it is Γ-periodic.

Let Λ be again a finite-dimensional unitary representation of Γ. Similarly as above, let us denote by H^Λ the Friedrichs extension of the differential operator $-\Delta_{\mathrm{B}}^\Lambda+V$ defined on the domain $C_0^\infty(\mathfrak{V}^\Lambda)\subset L^2(\mathfrak{V}^\Lambda)$. Note that if $\psi\in C_0^\infty(\mathfrak{V}^\Lambda)$ is represented by $\varphi\in C_0^\infty(\tilde{\mathfrak{V}}^\Lambda)$ fulfilling (7), then $H^\Lambda\psi$ is represented by $(-\tilde{\Delta}_{\mathrm{B}}^\Lambda+\tilde{V})\varphi$.

In the first step of the generalized Bloch analysis, one decomposes $\tilde{H}$ into a direct integral over $\hat{\Gamma}$ with components being equal to H^Λ. As a corollary one obtains a similar relationship for the corresponding evolution operators and resolvents. The decomposition is achieved by applying a unitary mapping Φ described below. Its construction is basically a modification of the construction presented in Section IV of [8], and therefore we omit here some details and, first of all, some proofs which resemble those given in [8]. In particular, this is true for the proof of the following lemma.

Lemma 5. *For $f\in L^2(\tilde{\mathfrak{V}})$ and $y\in\tilde{M}$ put*

$$\forall\gamma\in\Gamma,\quad f_y(\gamma)=\Psi_\gamma f(\gamma^{-1}\cdot y)\in\tilde{\mathfrak{V}}_y. \tag{10}$$

Then, f_y is well defined for almost all $x\in M$ and all $y\in\pi^{-1}(\{x\})$ and belongs to $L^2(\Gamma,\tilde{\mathfrak{V}}_y)\equiv\tilde{\mathfrak{V}}_y\otimes L^2(\Gamma)$.

Observe that the tensor product $L^2(\mathfrak{V}^\Lambda)\otimes\mathscr{L}_\Lambda^*$ can be identified with the Hilbert space formed by those measurable sections ψ of $\tilde{\mathfrak{V}}^\Lambda\otimes\mathscr{L}_\Lambda^*\equiv\tilde{\mathfrak{V}}\otimes\mathscr{I}(\mathscr{L}_\Lambda)$ which satisfy

$$\forall\gamma\in\Gamma,\quad \psi(\gamma\cdot y)=\left(\Psi_\gamma\otimes L_{\Lambda(\gamma)}\right)\psi(y)\quad\text{a.e. on }\tilde{M}, \tag{11}$$

with $L_{\Lambda(\gamma)}\in\mathrm{End}(\mathscr{I}(\mathscr{L}_\Lambda))$ being the linear operator on $\mathscr{I}(\mathscr{L}_\Lambda)$ acting by multiplication from the left, $L_{\Lambda(\gamma)}A=\Lambda(\gamma)A$, $\forall A\in\mathscr{I}(\mathscr{L}_\Lambda)$, and which have finite L^2 norms (with integration taken over M). The next lemma readily follows from the unitarity of the Fourier transformation and from property (5).

Lemma 6. *For any $f\in L^2(\tilde{\mathfrak{V}})$ let $f_y\in\tilde{\mathfrak{V}}_y\otimes L^2(\Gamma)$, $y\in\tilde{M}$, be as defined in (10). Then, the measurable section of the vector bundle $\tilde{\mathfrak{V}}\otimes\mathscr{I}(\mathscr{L}_\Lambda)$ given by*

$$\tilde{M}\ni y\mapsto(1\otimes\mathscr{F})[f_y](\Lambda)\in\tilde{\mathfrak{V}}_y\otimes\mathscr{I}(\mathscr{L}_\Lambda)\equiv\left(\tilde{\mathfrak{V}}\otimes\mathscr{I}(\mathscr{L}_\Lambda)\right)_y, \tag{12}$$

is well defined for a.a. $\Lambda\in\hat{\Gamma}$, and for those Λ it satisfies the equivariance condition (11) and so it represents a measurable section of $\mathfrak{V}^\Lambda\otimes\mathscr{L}_\Lambda^*$. The section has a finite L^2 norm and thus it belongs to $L^2(\mathfrak{V}^\Lambda)\otimes\mathscr{L}_\Lambda^*$.

Definition 7. We define

$$\Phi:L^2(\tilde{\mathfrak{V}})\to\int_{\hat{\Gamma}}^{\oplus}L^2(\mathfrak{V}^\Lambda)\otimes\mathscr{L}_\Lambda^*\,\mathrm{d}\hat{m}(\Lambda) \tag{13}$$

so that for all $f\in L^2(\tilde{\mathfrak{V}})$, the components $\Phi[f](\Lambda)$ are given by the prescription:

$$\text{for a.a. }\Lambda\in\hat{\Gamma},\ \text{a.a. }y\in\tilde{M},\quad \Phi[f](\Lambda)(y)=(1\otimes\mathscr{F})[f_y](\Lambda)\in\left(\tilde{\mathfrak{V}}\otimes\mathscr{I}(\mathscr{L}_\Lambda)\right)_y. \tag{14}$$

According to Lemma 6, $\Phi[f](\Lambda)$ fulfills (11) and can be identified with an element from $L^2(\mathfrak{V}^\Lambda)\otimes\mathscr{L}_\Lambda^*$. In particular, if $f\in L^1(\tilde{\mathfrak{V}})\cap L^2(\tilde{\mathfrak{V}})$, then

$$\Phi[f](\Lambda)(y)=\sum_{\gamma\in\Gamma}\Psi_\gamma f(\gamma^{-1}\cdot y)\otimes\Lambda(\gamma). \tag{15}$$

Lemma 8. *Φ is a unitary mapping.*

Proof. Let $p_2:\Gamma\times\tilde{M}\to\tilde{M}$ be the projection onto the second component, and $\check{\mathscr{H}}$ be the Hilbert space formed by measurable sections $\check{\psi}$ of $p_2^*\tilde{\mathfrak{V}}$ which satisfy

$$\forall r,\gamma\in\Gamma,\ \text{for a.a. }y\in\tilde{M},\quad \check{\psi}(r,\gamma\cdot y)=\Psi_\gamma\check{\psi}(\gamma^{-1}r,y). \tag{16}$$

Hence $\check{\psi}(\gamma, y) = \Psi_\gamma \check{\psi}(1, \gamma^{-1} \cdot y)$. It follows that the function

$$y \mapsto \sum_{\gamma\in\Gamma} \|\check{\psi}(\gamma, y)\|^2 = \sum_{\gamma\in\Gamma} \|\check{\psi}(1, \gamma \cdot y)\|^2$$

is Γ-invariant and projects to a function $\psi_*(x)^2$ defined on M. The norm on $\check{\mathscr{H}}$ is given by the integral

$$\|\check{\psi}\|^2 = \int_M \psi_*(x)^2 \,\mathrm{d}\mu(x).$$

Consider the linear mapping

$$\Theta : L^2(\tilde{\mathfrak{V}}) \to \check{\mathscr{H}} : f \mapsto \check{\psi}, \qquad \check{\psi}(\gamma, y) = \Psi_\gamma f(\gamma^{-1} \cdot y). \tag{17}$$

By a simple computation, one can check that, for all $f \in L^2(\tilde{\mathfrak{V}})$, the image $\check{\psi} = \Theta f$ actually fulfills (16) and $\|\check{\psi}\| = \|f\|$. Moreover, Θ is clearly invertible, $(\Theta^{-1}\check{\psi})(y) = \check{\psi}(1, y)$. Hence, Θ is a unitary mapping. Observe that if $\check{\psi} \in \check{\mathscr{H}}$, then

$$\text{for a.a. } y \in \tilde{M}, \quad \check{\psi}(\cdot, y) \in \tilde{\mathfrak{V}}_y \otimes L^2(\Gamma).$$

Further let us set

$$\hat{\mathscr{H}} = \int_{\hat{\Gamma}}^{\oplus} L^2(\mathfrak{V}^\Lambda) \otimes \mathscr{L}_\Lambda^* \,\mathrm{d}\hat{m}(\Lambda).$$

Using an analogous identification as above, if $\hat{\psi} \in \hat{\mathscr{H}}$, then $\hat{\psi}(\Lambda, y) \in \tilde{\mathfrak{V}}_y \otimes \mathscr{I}(\mathscr{L}_\Lambda)$ is defined almost everywhere on $\hat{\Gamma} \times \tilde{M}$ and fulfills

$$\forall \gamma \in \Gamma, \text{ for a.a. } (\Lambda, y) \in \hat{\Gamma} \times \tilde{M}, \quad \hat{\psi}(\Lambda, \gamma \cdot y) = (\Psi_\gamma \otimes L_{\Lambda(\gamma)})\hat{\psi}(\Lambda, y). \tag{18}$$

Observe that if $\hat{\psi} \in \hat{\mathscr{H}}$, then

$$\text{for a.a. } y \in \tilde{M}, \quad \hat{\psi}(\cdot, y) \in \tilde{\mathfrak{V}}_y \otimes \int_{\hat{\Gamma}}^{\oplus} \mathscr{I}(\mathscr{L}_\Lambda) \,\mathrm{d}\hat{m}(\Lambda).$$

Next we introduce two mutually inverse linear mappings, $\Xi : \check{\mathscr{H}} \to \hat{\mathscr{H}} : \check{\psi} \mapsto \hat{\psi}$, and $\Xi^{-1} : \hat{\mathscr{H}} \to \check{\mathscr{H}} : \hat{\psi} \mapsto \check{\psi}$, defined by the equalities

$$\begin{aligned} \hat{\psi}(\Lambda, y) &= (1 \otimes \mathscr{F})[\check{\psi}(\cdot, y)](\Lambda) \in \tilde{\mathfrak{V}}_y \otimes \mathscr{I}(\mathscr{L}_\Lambda), \\ \check{\psi}(\gamma, y) &= (1 \otimes \mathscr{F}^{-1})[\hat{\psi}(\cdot, y)](\gamma) \in \tilde{\mathfrak{V}}_y. \end{aligned} \tag{19}$$

If $\check{\psi}$ satisfies (16), then property (5) of the Fourier transformation implies that the image $\hat{\psi} = \Xi\check{\psi}$ fulfills

$$\hat{\psi}(\Lambda, r \cdot y) = (1 \otimes \mathscr{F})[\Psi_r \check{\psi}(r^{-1}\gamma, y)](\Lambda) = (\Psi_r \otimes L_{\Lambda(r)})\hat{\psi}(\Lambda, y).$$

Conversely, if $\hat{\psi}$ satisfies (18), then property (6) of the Fourier transformation implies that the image $\check{\psi} = \Xi^{-1}\hat{\psi}$ fulfills

$$\check{\psi}(r, \gamma \cdot y) = \Psi_\gamma (1 \otimes \mathscr{F}^{-1})[(1 \otimes \Lambda(\gamma))\, \hat{\psi}(\Lambda, y)](r) = \Psi_\gamma \check{\psi}(\gamma^{-1}r, y).$$

The unitarity of the Fourier transformation implies that the mappings Ξ and Ξ^{-1} are unitary as well.

Comparing the definitions of Φ, Θ and Ξ given in (14), (17) and (19), respectively, one can see that $\Phi = \Xi\Theta$. Hence Φ is unitary. □

Remark 9. Alternatively, one can define Φ as follows. For a given $\Lambda \in \hat{\Gamma}$, let us identify $L^2(\mathfrak{V}^\Lambda) \otimes \mathscr{L}_\Lambda^*$ with $\mathrm{Lin}(\mathscr{L}_\Lambda, L^2(\mathfrak{V}^\Lambda))$. For $\varphi \in C_0^\infty(\tilde{\mathfrak{V}})$, define a linear mapping $\Phi[\varphi](\Lambda) : \mathscr{L}_\Lambda \to L^2(\mathfrak{V}^\Lambda)$ by

$$\forall v \in \mathscr{L}_\Lambda, \quad \Phi[\varphi](\Lambda)v = \Phi^\Lambda \varphi \otimes v, \tag{20}$$

with Φ^Λ being given in (8). This way one gets a linear mapping

$$\Phi : C_0^\infty(\tilde{\mathfrak{V}}) \to \int_{\hat{\Gamma}}^{\oplus} L^2(\mathfrak{V}^\Lambda) \otimes \mathscr{L}_\Lambda^* \,\mathrm{d}\hat{m}(\Lambda)$$

that can be verified to be an isometry and so it unambiguously extends from $C_0^\infty(\tilde{\mathfrak{V}})$ to $L^2(\tilde{\mathfrak{V}})$. It is not difficult to see that definitions (14) and (20) of the mapping Φ in fact coincide.

Now we are ready to describe the Bloch decomposition. Put

$$\Sigma = \overline{\bigcup_{\Lambda\in\hat{\Gamma}} \operatorname{spec}(H^\Lambda)}$$

and, for z running over the corresponding resolvent sets,

$$\tilde{R}(z) = (\tilde{H} - z)^{-1}, \qquad R^\Lambda(z) = (H^\Lambda - z)^{-1}.$$

Furthermore,

$$\tilde{U}(t) = \exp(-\mathrm{i}t\tilde{H}), \qquad U^\Lambda(t) = \exp(-\mathrm{i}tH^\Lambda), \quad t \in \mathbb{R}.$$

Theorem 10. *The unitary mapping Φ decomposes the Hamiltonian $\tilde{H}$, i.e.*

$$\Phi\tilde{H}\Phi^{-1} = \int_{\hat{\Gamma}}^{\oplus} H^\Lambda \otimes 1\, \mathrm{d}\hat{m}(\Lambda). \tag{21}$$

Consequently,

$$\Phi\tilde{U}(t)\Phi^{-1} = \int_{\hat{\Gamma}}^{\oplus} U^\Lambda(t) \otimes 1\, \mathrm{d}\hat{m}(\Lambda), \quad t \in \mathbb{R}, \tag{22}$$

and

$$\Phi\tilde{R}(z)\Phi^{-1} = \int_{\hat{\Gamma}}^{\oplus} R^\Lambda(z) \otimes 1\, \mathrm{d}\hat{m}(\Lambda), \quad z \in \mathbb{C} \setminus \Sigma. \tag{23}$$

Proof. From (20) one deduces that if $\varphi \in C_0^\infty(\tilde{\mathfrak{V}})$ then $\Phi[\varphi](\Lambda) \in C_0^\infty(\mathfrak{V}^\Lambda) \otimes \mathscr{L}_\Lambda^*$, $\forall \Lambda \in \hat{\Gamma}$. Taking into account also (9), one has

$$\left(\Delta_\mathrm{B}^\Lambda \otimes 1\right) \Phi[\varphi](\Lambda) = \Phi[\tilde{\Delta}_\mathrm{B}\varphi](\Lambda).$$

Moreover, if $f \in L^2(\tilde{\mathfrak{V}})$, then $(\tilde{V}f)_y = \tilde{V}(y)f_y$ and so

$$\forall \Lambda \in \hat{\Gamma}, \quad \Phi[\tilde{V}f](\Lambda) = (V \otimes 1)\Phi[f](\Lambda).$$

Altogether this implies that

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(\tilde{\mathfrak{V}}),\ \forall \Lambda \in \hat{\Gamma}, \quad \Phi[(-\tilde{\Delta}_\mathrm{B} + \tilde{V})\varphi_2](\Lambda) = \left((-\Delta_\mathrm{B}^\Lambda + V) \otimes 1\right) \Phi[\varphi_2](\Lambda),$$

and

$$\begin{aligned}\langle \varphi_1, (-\tilde{\Delta}_\mathrm{B} + \tilde{V})\varphi_2\rangle &= \left\langle \Phi[\varphi_1], \Phi[(-\tilde{\Delta}_\mathrm{B} + \tilde{V})\varphi_2]\right\rangle \\ &= \int_{\hat{\Gamma}}^{\oplus} \left\langle \Phi[\varphi_1](\Lambda), \left((-\Delta_\mathrm{B}^\Lambda + V) \otimes 1\right) \Phi[\varphi_2](\Lambda)\right\rangle \mathrm{d}\hat{m}(\Lambda).\end{aligned}$$

Closing the quadratic forms one arrives at equality (21). $\square$

Remark 11. Note that clearly $\operatorname{spec}(\tilde{H}) \subset \Sigma$. Furthermore, it is also obvious that starting from equality (21), one can derive a decomposition of any operator of the form $F(\tilde{H})$ where F is a continuous function on Σ. For example, the choice $F(x) = \exp(-tx)$, with $t > 0$, may be of interest, and this way one gets a decomposition of the Schrödinger semigroup into a direct integral

$$\Phi \exp(-t\tilde{H})\Phi^{-1} = \int_{\hat{\Gamma}}^{\oplus} \exp(-tH^\Lambda) \otimes 1\, \mathrm{d}\hat{m}(\Lambda), \quad t > 0.$$

Remark. The construction described earlier in [8] is a particular case of the construction presented above. In more detail, the settings in [8] are as follows: $\mathfrak{V} = M \times \mathbb{C}$ is the trivial line bundle over M, the Hermitian structure $\mathfrak{h}$ is given by the standard scalar product in $\mathbb{C}$, and $\nabla = \mathrm{d}$ is the trivial flat connection. If $\pi : \tilde{M} \to M$ is a covering map (not necessarily universal) with a structure group Γ, then $\tilde{\mathfrak{V}} = \pi^*\mathfrak{V} = \tilde{M} \times \mathbb{C}$, and the group Γ acts on $\tilde{\mathfrak{V}}$ as an identity on the fibers, $\Psi_\gamma(y, z) = (\gamma \cdot y, z)$ for $(y, z) \in \tilde{M} \times \mathbb{C}$ and $\gamma \in \Gamma$.

4. A formula for propagators and Green functions

In Eq. (22), the evolution operator $\tilde{U}(t)$ is expressed in terms of $U^{\Lambda}(t)$, $\Lambda \in \hat{\Gamma}$. It is possible to invert this relationship and to derive a formula giving an expression for the propagator associated with H^{Λ} in terms of the propagator associated with $\tilde{H}$. The propagators are regarded as distributional kernels of the corresponding evolution operators.

If $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are vector fiber bundles over manifolds N_1 and N_2, respectively, then the symbol $\mathfrak{W}_1 \boxtimes \mathfrak{W}_2$ stands for the vector fiber bundle over $N_1 \times N_2$ with fibers

$$(\mathfrak{W}_1 \boxtimes \mathfrak{W}_2)_{(x_1,x_2)} = (\mathfrak{W}_1)_{x_1} \otimes (\mathfrak{W}_2)_{x_2}, \quad (x_1, x_2) \in N_1 \times N_2.$$

If $\mathfrak{W}$ is a vector fiber bundle over a Riemannian manifold N, then the dual space to $C_0^{\infty}(\mathfrak{W})$ is formed by distributional sections of the dual bundle $\mathfrak{W}^*$ (with $\mathfrak{W}_x^*$ being the dual space to $\mathfrak{W}_x$, $x \in N$).

Suppose that $\mathfrak{W}$ is a Hermitian vector fiber bundle over a Riemannian manifold N. Let

$$C^{\infty}(\mathfrak{W}) \to C^{\infty}(\mathfrak{W}^*) : \sigma \mapsto \overline{\sigma},$$

be the canonical antilinear isomorphism following fiber-wise from the Riesz lemma.

The action Ψ of the group Γ on $\tilde{\mathfrak{V}}$ induces naturally an action Ψ' on the dual vector fiber bundle $\tilde{\mathfrak{V}}^*$. Analogously to (2), one introduces operators W'_{γ} on $C^{\infty}(\tilde{\mathfrak{V}}^*)$,

$$\forall \sigma \in C^{\infty}(\tilde{\mathfrak{V}}^*), \quad (W'_{\gamma}\sigma)(y) = \Psi'_{\gamma}\sigma(\gamma^{-1} \cdot y).$$

Notice that

$$\forall \varphi \in C^{\infty}(\tilde{\mathfrak{V}}), \quad W'_{\gamma}\overline{\varphi} = \overline{W_{\gamma}\varphi}.$$

The operators W_{γ}, $\gamma \in \Gamma$, defined in (2) on smooth sections can be extended to distributional sections. Let α be a distributional section of $\tilde{\mathfrak{V}}$, i.e. a continuous functional on $C_0^{\infty}(\tilde{\mathfrak{V}}^*)$. Then, for $\gamma \in \Gamma$,

$$\forall \sigma \in C_0^{\infty}(\tilde{\mathfrak{V}}^*), \quad W_{\gamma}\alpha(\sigma) = \alpha(W'_{\gamma^{-1}}\sigma). \tag{24}$$

Similarly, let β be a distributional section of $\tilde{\mathfrak{V}}^*$. Then

$$\forall \varphi \in C_0^{\infty}(\tilde{\mathfrak{V}}), \quad W'_{\gamma}\beta(\varphi) = \beta(W_{\gamma^{-1}}\varphi). \tag{25}$$

Of course, in definition (24), (25), one takes into account the invariance of measure $\tilde{\mu}$ with respect to the action of Γ.

For $\varphi_1, \varphi_2 \in C_0^{\infty}(\mathfrak{W})$ denote by $\overline{\varphi_1} \otimes \varphi_2$ the section of $\mathfrak{W}^* \boxtimes \mathfrak{W}$ given by $(\overline{\varphi_1} \otimes \varphi_2)(x_1, x_2) = \overline{\varphi_1}(x_1) \otimes \varphi_2(x_2)$, $(x_1, x_2) \in N_1 \times N_2$. Let B be a bounded operator in $L^2(\mathfrak{W})$. As a corollary of the Schwartz kernel theorem (see, for example, [22, Theorem 5.2.1]), one introduces the kernel β of B as a distributional section of $(\mathfrak{W}^* \boxtimes \mathfrak{W})^* \equiv \mathfrak{W} \boxtimes \mathfrak{W}^*$ that is unambiguously given by the relation

$$\forall \varphi_1, \varphi_2 \in C_0^{\infty}(\mathfrak{W}), \quad \beta(\overline{\varphi_1} \otimes \varphi_2) = \langle \varphi_1, B\varphi_2 \rangle.$$

The map $B \mapsto \beta$ is injective.

If B is a bounded operator in $L^2(\tilde{\mathfrak{V}})$, then B is Γ-periodic, i.e. B commutes with all W_{γ}, $\gamma \in \Gamma$, if and only if the distributional kernel β satisfies the invariance condition

$$\forall \gamma \in \Gamma, \quad (W_{\gamma} \otimes W'_{\gamma})\beta = \beta. \tag{26}$$

Suppose that B is a bounded operator in $L^2(\mathfrak{V}^{\Lambda})$, $\Lambda \in \hat{\Gamma}$. The kernel β of B is a distributional section of $\mathfrak{V}^{\Lambda} \boxtimes (\mathfrak{V}^{\Lambda})^*$. Alternatively, one can characterize B by a distributional section β^{Λ} of

$$\tilde{\mathfrak{V}}^{\Lambda} \boxtimes (\tilde{\mathfrak{V}}^{\Lambda})^* \equiv (\tilde{\mathfrak{V}} \boxtimes \tilde{\mathfrak{V}}^*) \otimes \mathscr{I}(\mathscr{L}_{\Lambda})$$

given by

$$\forall v_1, v_2 \in \mathscr{L}_{\Lambda}, \ \forall \varphi_1, \varphi_2 \in C_0^{\infty}(\tilde{\mathfrak{V}}),$$
$$\langle v_1, \beta^{\Lambda}(\overline{\varphi_1} \otimes \varphi_2)v_2 \rangle_{\mathscr{L}_{\Lambda}} = \beta\left(\overline{(\Phi^{\Lambda}\varphi_1 \otimes v_1)} \otimes (\Phi^{\Lambda}\varphi_2 \otimes v_2)\right) = \langle \Phi^{\Lambda}\varphi_1 \otimes v_1, B\Phi^{\Lambda}\varphi_2 \otimes v_2 \rangle. \tag{27}$$

Here we regard β^{Λ} as a continuous functional on $C_0^{\infty}(\tilde{\mathfrak{V}}^* \boxtimes \tilde{\mathfrak{V}})$ with values in $\mathscr{I}(\mathscr{L}_{\Lambda})$. Moreover, β^{Λ} fulfills

$$\forall \gamma \in \Gamma, \quad (W_{\gamma} \otimes 1)\beta^{\Lambda} = \Lambda(\gamma^{-1})\beta^{\Lambda}, \qquad (1 \otimes W'_{\gamma})\beta^{\Lambda} = \beta^{\Lambda}\Lambda(\gamma). \tag{28}$$

Again, the map $B \mapsto \beta^{\Lambda}$ is injective.

Let t be a real parameter. Denote by $\tilde{\mathcal{K}}_t$ the kernel of $\tilde{U}(t)$, and by $\mathcal{K}_t^{\Lambda}$ the kernel of $U^{\Lambda}(t)$. Thus, $\tilde{\mathcal{K}}_t$ is a distributional section of $\tilde{\mathfrak{V}} \boxtimes \tilde{\mathfrak{V}}^*$, and $\mathcal{K}_t^{\Lambda}$ is a distributional section of $\tilde{\mathfrak{V}} \boxtimes \tilde{\mathfrak{V}}^*$ with values in $\mathscr{I}(\mathscr{L}_{\Lambda})$. Moreover, the kernel $\mathcal{K}_t^{\Lambda}$ is Λ-equivariant in the sense of (28).

Let us rewrite the Bloch decomposition of the propagator (22) in terms of kernels. The following lemma is a straightforward modification of Lemma 12 and Proposition 13 in [8] and so we omit the proof.

Lemma 12. *For all $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{\mathfrak{Y}})$ and $\gamma \in \Gamma$, the function*

$$\Lambda \mapsto \mathrm{Tr}[\Lambda(\gamma)^* \mathcal{K}_t^\Lambda(\overline{\varphi_1} \otimes \varphi_2)]$$

is integrable on $\hat{\Gamma}$ and one has

$$(W_\gamma \otimes 1)\tilde{\mathcal{K}}_t(\overline{\varphi_1} \otimes \varphi_2) = \int_{\hat{\Gamma}} \mathrm{Tr}[\Lambda(\gamma)^* \mathcal{K}_t^\Lambda(\overline{\varphi_1} \otimes \varphi_2)]\, \mathrm{d}\hat{m}(\Lambda). \tag{29}$$

Definition 13. For $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{\mathfrak{Y}})$ arbitrary but fixed we set

$$\forall \gamma \in \Gamma, \quad F_t(\gamma) = (W_\gamma \otimes 1)\tilde{\mathcal{K}}_t\,(\overline{\varphi_1} \otimes \varphi_2) = \tilde{\mathcal{K}}_t\left(\overline{W_{\gamma^{-1}}\varphi_1} \otimes \varphi_2\right),$$

and

$$\forall \Lambda \in \hat{\Gamma}, \quad G_t(\Lambda) = \mathcal{K}_t^\Lambda(\overline{\varphi_1} \otimes \varphi_2) \in \mathscr{I}(\mathscr{L}_\Lambda).$$

Absolutely in the same manner as in the proof of Lemma 14 in [8], one can show the following lemma.

Lemma 14. *$F_t \in L^2(\Gamma)$ and $\Lambda \mapsto \|G_t(\Lambda)\|$ is a bounded function on $\hat{\Gamma}$. Recalling that $\hat{m}(\hat{\Gamma}) \leq 1$, one has $\|G_t(\cdot)\| \in L^1(\hat{\Gamma}) \cap L^2(\hat{\Gamma})$. In particular,*

$$G_t \in \int_{\hat{\Gamma}}^{\oplus} \mathscr{I}(\mathscr{L}_\Lambda)\, \mathrm{d}\hat{m}(\Lambda).$$

In view of Lemma 14, the following proposition is an easy corollary of (29).

Proposition 15. *For all $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{\mathfrak{Y}})$, one has*

$$F_t = \mathscr{F}^{-1}[G_t], \quad G_t = \mathscr{F}[F_t]. \tag{30}$$

Remark 16. Rewriting formally the second equality in (30) gives

$$\mathcal{K}_t^\Lambda(y_1, y_2) = \sum_{\gamma \in \Gamma} \left((\Psi_\gamma \otimes 1) \cdot \tilde{\mathcal{K}}_t(\gamma^{-1} \cdot y_1, y_2)\right) \otimes \Lambda(\gamma). \tag{31}$$

Alternatively, using the fact that the Hamiltonian $\tilde{H}$ is Γ-periodic and that the kernel $\tilde{\mathcal{K}}_t$ obeys the invariance condition (26), one can write

$$\mathcal{K}_t^\Lambda(y_1, y_2) = \sum_{\gamma \in \Gamma} \left((1 \otimes \Psi'_\gamma) \cdot \tilde{\mathcal{K}}_t(y_1, \gamma^{-1} \cdot y_2)\right) \otimes \Lambda(\gamma^{-1}).$$

In the case of a flat connection formula (31) coincides with the formula for propagators on multiply connected spaces as described in [10,11].

A formula analogous to (31) can also be derived for the corresponding Green functions. From the theoretical point of view, it can be even more convenient to work with Green functions instead of propagators for some properties of Green functions are easier to control than those of propagators. Let us sketch how the procedure should be modified for this purpose.

Recall (23). Let $z \in \mathbb{C} \setminus \Sigma$ be a spectral parameter. Denote by $\tilde{\mathcal{G}}_z$ the kernel of $\tilde{R}(z)$, and by $\mathcal{G}_z^\Lambda$ the kernel of $R^\Lambda(z)$. Thus, $\tilde{\mathcal{G}}_z$ is a distributional section of $\tilde{\mathfrak{Y}} \boxtimes \tilde{\mathfrak{Y}}^*$, and $\mathcal{G}_z^\Lambda$ is a distributional section of $\tilde{\mathfrak{Y}} \boxtimes \tilde{\mathfrak{Y}}^*$ with values in $\mathscr{I}(\mathscr{L}_\Lambda)$. Moreover, the kernel $\mathcal{G}_z^\Lambda$ is Λ-equivariant. $\tilde{\mathcal{G}}_z$ and $\mathcal{G}_z^\Lambda$ are the Green functions of $\tilde{H}$ and H^Λ, respectively.

One can again rewrite the Bloch decomposition of the Green function (23) in terms of kernels.

Lemma 17. *For all $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{\mathfrak{Y}})$ and $\gamma \in \Gamma$, the function*

$$\Lambda \mapsto \mathrm{Tr}[\Lambda(\gamma)^* \mathcal{G}_z^\Lambda(\overline{\varphi_1} \otimes \varphi_2)]$$

is integrable on $\hat{\Gamma}$ and one has

$$(W_\gamma \otimes 1)\tilde{\mathcal{G}}_z(\overline{\varphi_1} \otimes \varphi_2) = \int_{\hat{\Gamma}} \mathrm{Tr}[\Lambda(\gamma)^* \mathcal{G}_z^\Lambda(\overline{\varphi_1} \otimes \varphi_2)]\, \mathrm{d}\hat{m}(\Lambda). \tag{32}$$

Proof. Let us sketch the basic steps. First one finds that

$$\langle \Phi[\varphi_1](\Lambda), (R^\Lambda(z) \otimes 1)\Phi[\varphi_2](\Lambda)\rangle = \mathrm{Tr}[\mathcal{G}_z^\Lambda(\overline{\varphi_1} \otimes \varphi_2)].$$

Taking into account the unitarity of Φ, it also follows that

$$\int_{\hat{\Gamma}} \left|\mathrm{Tr}[\mathcal{G}_z^{\Lambda}(\overline{\varphi_1} \otimes \varphi_2)]\right| \, \mathrm{d}\hat{m}(\Lambda) \leq \frac{\|\varphi_1\| \, \|\varphi_2\|}{\mathrm{dist}(z, \Sigma)},$$

and so the function $\Lambda \mapsto \mathrm{Tr}[\mathcal{G}_z^{\Lambda}(\overline{\varphi_1} \otimes \varphi_2)]$ is integrable on $\hat{\Gamma}$. Relation (23) can be rewritten as

$$\tilde{\mathcal{G}}_z(\overline{\varphi_1} \otimes \varphi_2) = \int_{\hat{\Gamma}} \mathrm{Tr}[\mathcal{G}_z^{\Lambda}(\overline{\varphi_1} \otimes \varphi_2)] \, \mathrm{d}\hat{m}(\Lambda),$$

and replacing φ_1 by $W_{\gamma^{-1}}\varphi_1$, $\gamma \in \Gamma$, and taking into account the Λ-equivariance of $\mathcal{G}_z^{\Lambda}$ one gets (32). $\square$

Definition 18. For $\varphi_1, \varphi_2 \in C_0^{\infty}(\tilde{\mathfrak{Y}})$ arbitrary but fixed put

$$\forall \gamma \in \Gamma, \quad F_z(\gamma) = (W_\gamma \otimes 1)\tilde{\mathcal{G}}_z \left(\overline{\varphi_1} \otimes \varphi_2\right) = \tilde{\mathcal{G}}_z \left(\overline{W_{\gamma^{-1}}\varphi_1} \otimes \varphi_2\right),$$

and

$$\forall \Lambda \in \hat{\Gamma}, \quad G_z(\Lambda) = \mathcal{G}_z^{\Lambda}(\overline{\varphi_1} \otimes \varphi_2) \in \mathscr{I}(\mathscr{L}_\Lambda).$$

Lemma 19. *$F_z \in L^2(\Gamma)$ and $\Lambda \mapsto \|G_z(\Lambda)\|$ is a bounded function on $\hat{\Gamma}$. Since $\hat{m}(\hat{\Gamma}) \leq 1$ one has $\|G_z(\cdot)\| \in L^1(\hat{\Gamma}) \cap L^2(\hat{\Gamma})$. In particular,*

$$G_z \in \int_{\hat{\Gamma}}^{\oplus} \mathscr{I}(\mathscr{L}_\Lambda) \, \mathrm{d}\hat{m}(\Lambda).$$

Proof. One can proceed very similarly as in the proof of Lemma 14 in [8]. Let us just indicate a couple of modifications. Assuming (without loss of generality) that the sets $\gamma \cdot \mathrm{supp}(\varphi_1)$, $\gamma \in \Gamma$, are mutually disjoint one derives the estimate

$$\sum_{\gamma \in \Gamma} |F_z(\gamma)|^2 \leq \|\varphi_1\|^2 \|\tilde{R}(z)\varphi_2\|^2 \leq \frac{\|\varphi_1\|^2 \|\varphi_2\|^2}{\mathrm{dist}(z, \Sigma)^2}.$$

Furthermore, still assuming that the sets $\gamma \cdot \mathrm{supp}(\varphi_j)$, $\gamma \in \Gamma$, are mutually disjoint both for $j = 1$ and $j = 2$, one gets

$$\|G_z(\Lambda)\| \leq \max_{\Lambda \in \hat{\Gamma}} (\dim(\mathscr{L}_\Lambda)) \, \frac{\|\varphi_1\| \, \|\varphi_2\|}{\mathrm{dist}(z, \Sigma)}.$$

The lemma follows. $\square$

Finally, equality (32) implies the following proposition.

Proposition 20. *For all $\varphi_1, \varphi_2 \in C_0^{\infty}(\tilde{\mathfrak{Y}})$,*

$$F_z = \mathscr{F}^{-1}[G_z], \quad G_z = \mathscr{F}[F_z]. \tag{33}$$

Remark 21. The second equality in (33) can be formally rewritten as

$$\mathcal{G}_z^{\Lambda}(y_1, y_2) = \sum_{\gamma \in \Gamma} \left((\Psi_\gamma \otimes 1)\tilde{\mathcal{G}}_z(\gamma^{-1} \cdot y_1, y_2)\right) \otimes \Lambda(\gamma). \tag{34}$$

And this is the formula for Green functions. Alternatively, in view of the invariance condition (26), one can also write

$$\mathcal{G}_z^{\Lambda}(y_1, y_2) = \sum_{\gamma \in \Gamma} \left((1 \otimes \Psi'_\gamma)\tilde{\mathcal{G}}_z(y_1, \gamma^{-1} \cdot y_2)\right) \otimes \Lambda(\gamma^{-1}).$$

5. Particular cases and examples

5.1. A reduction to the maximal Abelian covering space

As already mentioned there are multiply connected manifolds (configuration spaces) of interest whose fundamental group is not of type I. A well-known example is a two-dimensional Euclidean space with two or more omitted points whose fundamental group is isomorphic to the free group with one generator per omitted point. If one is interested in the propagator formula (31) or in the Green function formula (34) for magnetic Schrödinger operators, i.e. one considers a situation when the gauge group is $U(1)$, then one may avoid working with the universal covering space $\tilde{M}$ of M whose structure group is $\pi_1(M)$ and instead employ the maximal Abelian covering space $\hat{M}$ whose structure group is Abelian and isomorphic to $H_1(M;\mathbb{Z})$. The notion of the maximal Abelian covering space already proved itself to be useful in the spectral analysis of magnetic Schrödinger operators [23]. The authors are indebted to Takuya Mine for pointing out to them this possibility.

In what follows we restrict ourselves to the formula for Green functions since it is much easier to handle than the propagator formula. Moreover, the covariant derivative is supposed to be flat on M and hence trivial on $\tilde{M}$, and scalar potentials are not considered ($V = 0$).

Let us recall that a maximal Abelian covering space of M is a covering space $\hat{p} : \hat{M} \to M$ such that $\hat{p}_*\pi_1(\hat{M}) = [\pi_1(M), \pi_1(M)]$. It is known to exist and to be unique up to equivalence. This is a normal covering space and the covering group is isomorphic to $\pi_1(M)/[\pi_1(M), \pi_1(M)] = H_1(M;\mathbb{Z})$ (see, for instance, [24]). Still assuming that M is a connected Riemannian manifold, $\tilde{M}$ as well as $\hat{M}$ become Riemannian manifolds in a unique manner so that the corresponding projections are isometric at every point. The action of the covering group on $\tilde{M}$ or $\hat{M}$ is then isometric, free, transitive on the fibers and properly discontinuous [15].

If the gauge group is $U(1)$, then only one-dimensional representations of $\pi_1(M)$ are relevant. Since any one-dimensional representation of $\pi_1(M)$ is trivial on $[\pi_1(M), \pi_1(M)]$, it induces a representation of $\pi_1(M)/[\pi_1(M), \pi_1(M)] = H_1(M;\mathbb{Z})$ – the structure group of the covering space $\hat{M} \to M$. In that case the universal covering space can be reduced to the maximal Abelian covering space. Because the group $H_1(M;\mathbb{Z})$ is Abelian, one may refer to Proposition 20 to justify formula (34). The knowledge of the Green function $\hat{\mathcal{G}}_z$ on $\hat{M}$ is required, however. Suppose one knows the Green function $\tilde{\mathcal{G}}_z$ on $\tilde{M}$ rather than the Green function $\hat{\mathcal{G}}_z$ on $\hat{M}$. Since $\tilde{M} \to \hat{M}$ is a covering space with the structure group $[\pi_1(M), \pi_1(M)]$, one can formally construct $\hat{\mathcal{G}}_z$ as the sum

$$\hat{\mathcal{G}}_z(y, y_0) = \sum_{\gamma\in[\pi_1(M),\pi_1(M)]} \tilde{\mathcal{G}}_z(\gamma \cdot y, y_0).$$

Below we aim to verify its convergence in the sense of distributions.

We are going to consider a bit more general situation. Let X be a connected Riemannian manifold, and Γ be a discrete symmetry group of X. Suppose the action of Γ on X is isometric, free and properly discontinuous. Let H be the free Hamiltonian in $L^2(X)$ introduced as the Friedrichs extension of the symmetric positive operator $-\Delta_{\mathrm{LB}}$ (the Laplace–Beltrami operator) with the domain $C_0^\infty(X)$. The symmetric operator is known to be essentially self-adjoint on complete Riemannian manifolds [25,26]. Otherwise, in the general case, H is sometimes said to be determined by the generalized Dirichlet boundary conditions [27], the form domain of H coincides with the Sobolev space $H_0^1(X)$, and the closed quadratic form associated with H reads

$$H_0^1(X) \ni f \mapsto \int_X \mathfrak{g}(\mathrm{d}f, \mathrm{d}f)\, \mathrm{d}\mu.$$

Denote by $R(z) = (H - z)^{-1}$, $\operatorname{Re} z < 0$, the corresponding resolvent restricted to the left halfplane. The free Green function $\mathcal{G}_z$ is a distribution on $X \times X$ defined by

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(X), \quad \mathcal{G}_z(\overline{\varphi_1} \otimes \varphi_2) = \langle \varphi_1, R(z)\varphi_2 \rangle.$$

Note that

$$\mathcal{G}_z(\gamma \cdot x, \gamma \cdot y) = \mathcal{G}_z(x, y), \quad \forall \gamma \in \Gamma.$$

Put $\hat{X} = X/\Gamma$. Denote by $\hat{H}$ the free Hamiltonian on $\hat{X}$. $\hat{H}$ is again the Friedrichs extension of minus the Laplace–Beltrami operator with the domain $C_0^\infty(\hat{X})$. It is convenient to identify $L^2(\hat{X})$ with the Hilbert space $\mathscr{H}$ formed by Γ-periodic functions on X which are L^2 integrable over a fundamental domain. Then, as a differential operator, $\hat{H}$ coincides with $-\Delta_{\mathrm{LB}}$. Put $\hat{R}(z) = (\hat{H} - z)^{-1}$, $\operatorname{Re} z < 0$.

Again, $C_\Gamma^\infty(X)$ stands for the vector space of smooth Γ-periodic functions on X, and we define

$$\Phi_0 : C_0^\infty(X) \to C_\Gamma^\infty(X) : \varphi \mapsto \sum_{\gamma\in\Gamma} L_\gamma^*\varphi. \tag{35}$$

If $\varphi \in C_0^\infty(X)$, then $\Phi_0\varphi = \hat{p}^*\sigma$ for a unique $\sigma \in C_0^\infty(\hat{X})$. This defines a linear mapping $C_0^\infty(X) \to C_0^\infty(\hat{X})$ which is surjective (see Lemma 3). Particularly, it follows that

$$\Phi_0\left(C_0^\infty(X)\right) \subset \operatorname{Dom}(\hat{H}) \subset \mathscr{H}$$

is a core of $\hat{H}$.

The free Green function $\hat{\mathcal{G}}_z$ on $\hat{X}$ (associated with $\hat{H}$) is identified with a $\Gamma \times \Gamma$-periodic distribution on $X \times X$ unambiguously determined by the relation

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(X), \quad \hat{\mathcal{G}}_z(\overline{\varphi_1} \otimes \varphi_2) = \langle \Phi_0\varphi_1, \hat{R}(z)\Phi_0\varphi_2 \rangle$$

(the scalar product is taken in $\mathscr{H}$).

Consider the semigroup $\exp(-tH)$, $t > 0$. The corresponding distributional kernel $p(t; x, y)$ is the heat kernel on X (for the generalized Dirichlet boundary conditions). The heat kernel $p(t; x, y)$ is known to be a smooth and strictly positive function on $(0, +\infty) \times X \times X$ which is symmetric in the variables x and y. It is unambiguously characterized as the smallest positive fundamental solution of the heat equation on X. Moreover, one has

$$\forall x \in X, \quad \int_X p(t; x, y)\, \mathrm{d}\mu(y) \le 1 \tag{36}$$

[28]. Under certain assumptions it is even true that inequality (36) becomes in fact an equality for any $x \in X$ [29], for example when X is a complete Riemannian manifold of Ricci curvature bounded from below [28]. This is consistent with the probabilistic interpretation—for a fixed x, $p(t; x, y)$ is the probability density of a diffusion process when a particle departs from the point x at time 0 and reaches a variable point y at time $t > 0$ [30]. For our purposes inequality (36) is pretty sufficient, however.

Note that the Green function equals the Laplace transform of the heat kernel,

$$\mathcal{G}_z(x, y) = \int_0^\infty \mathrm{e}^{zt} p(t; x, y)\, \mathrm{d}t, \quad \operatorname{Re} z < 0. \tag{37}$$

This also means that $\mathcal{G}_z(x, y)$ is a regular distribution and $R(z)$ is an integral operator.

Lemma 22. *On a general Riemannian manifold X and for all $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$, the integral operator $R(z)$ is bounded on $L^1(X)$ with the upper bound $1/|\operatorname{Re} z|$. In particular, $\forall \varphi \in C_0^\infty(X)$, $R(z)\varphi \in L^1(X)$.*

Proof. From (36) and (37) it follows that, for all $\varphi \in L^1(X)$,

$$\|R(z)\varphi\|_1 \le \int_X \int_X |\mathcal{G}_z(x, y)|\, |\varphi(y)|\, \mathrm{d}\mu(x)\mathrm{d}\mu(y) \le \frac{\|\varphi\|_1}{|\operatorname{Re} z|}.$$

This proves the lemma. □

Lemma 23. *Suppose $D \subset X$ is a fundamental domain of the action of Γ, $\varphi \in C_0^\infty(X)$. Then*

$$\|\Phi_0 R(z)\varphi|_D\|_1 \le \frac{\|\varphi\|_1}{|\operatorname{Re} z|}, \qquad \|\Phi_0 R(z)\varphi|_D\|_\infty \le C_{\operatorname{supp}(\varphi)} \frac{\|\varphi\|_\infty}{|\operatorname{Re} z|}, \tag{38}$$

where $C_{\operatorname{supp}(\varphi)} \ge 0$ depends only on $\operatorname{supp}(\varphi)$. Consequently,

$$\|\Phi_0 R(z)\varphi|_D\|_2 \le \frac{1}{|\operatorname{Re} z|} \sqrt{C_{\operatorname{supp}(\varphi)} \|\varphi\|_1 \|\varphi\|_\infty} \tag{39}$$

(the norms of the restrictions to the domain D occurring on the left-hand side are taken in $L^p(D)$, and otherwise the norms are taken in $L^p(X)$ for an appropriate p).

Proof. First observe that $f \in L^1(X)$ implies $\Phi_0 f \in L^1(D)$ and $\|\Phi_0 f|_D\|_1 \le \|f\|_1$ where, similarly to (35), $\Phi_0 f(x) = \sum_{\gamma \in \Gamma} f(\gamma \cdot x)$. In fact, one has

$$\|\Phi_0 f|_D\|_1 = \int_D \left| \sum_{\gamma \in \Gamma} f(\gamma \cdot x) \right| \mathrm{d}\mu(x) \le \sum_{\gamma \in \Gamma} \int_{\gamma \cdot D} |f(x)|\, \mathrm{d}\mu(x) = \|f\|_1.$$

Now, to get the first inequality in (38), it suffices to apply Lemma 22. Further, one can assume, without loss of generality, that the sets $\gamma \cdot \operatorname{supp}(\varphi)$, $\gamma \in \Gamma$, are mutually disjoint (in that case $C_{\operatorname{supp}(\varphi)} = 1$). Then

$$|\Phi_0 R(z)\varphi(x)| \le \|\varphi\|_\infty \sum_{\gamma \in \Gamma} \int_{\gamma^{-1} \cdot \operatorname{supp}(\varphi)} \mathcal{G}_{\operatorname{Re}(z)}(x, y)\, \mathrm{d}\mu(y) \le \frac{\|\varphi\|_\infty}{|\operatorname{Re} z|}.$$

Finally, one has $\|f\|_2^2 \le \|f\|_1 \|f\|_\infty$. □

Proposition 24. *The equality*

$$\hat{\mathcal{G}}_z(x_1, x_2) = \sum_{\gamma \in \Gamma} \mathcal{G}_z(\gamma \cdot x_1, x_2) \tag{40}$$

holds on $X \times X$ in the sense of distributions.

Proof. To show that series (40) converges in the sense of distributions, it suffices to verify its convergence on every test function $\varphi \in C_0^\infty(X \times X)$. Such a simplification is possible in view of the definition of the topology in the distribution space and, in particular, owing to the completeness of this space [31, Section 5.3]. Choose $\varphi \in C_0^\infty(X \times X)$. Then $\operatorname{supp}\varphi \subset K_1 \times K_2$ for some compact subsets $K_1, K_2 \subset X$. There exist open sets $U_j \subset X$, $1 \le j \le N$, so that $K_1 \subset \cup U_j$ and the sets $\gamma \cdot U_j$, $\gamma \in \Gamma$, are mutually disjoint for every j. One has, for $\operatorname{Re} z < 0$,

$$\begin{aligned}\sum_{\gamma\in\Gamma} \big|\mathcal{G}_z(\gamma\cdot x_1, x_2)\big(\varphi(x_1,x_2)\big)\big| &\le \|\varphi\|_\infty \sum_{\gamma\in\Gamma}\int_{K_1\times K_2} \mathcal{G}_{\operatorname{Re}(z)}(\gamma\cdot x_1, x_2)\,\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\\ &\le \|\varphi\|_\infty \sum_{j=1}^{N}\sum_{\gamma\in\Gamma}\int_{\gamma\cdot U_j\times K_2} \mathcal{G}_{\operatorname{Re}(z)}(x_1, x_2)\,\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\\ &\le \frac{N\|\varphi\|_\infty}{|\operatorname{Re} z|}\mu(K_2).\end{aligned}$$

Let us denote $\mathcal{S}_z(x_1,x_2) = \sum_{\gamma\in\Gamma}\mathcal{G}_z(\gamma\cdot x_1, x_2) \in \mathscr{D}'(X\times X)$. In particular, one has

$$\forall \varphi_1,\varphi_2 \in C_0^\infty(X), \quad \mathcal{S}_z(\varphi_1\otimes\varphi_2) = \sum_{\gamma\in\Gamma}\mathcal{G}_z(L_\gamma^*\varphi_1\otimes\varphi_2).$$

Showing that $\hat{\mathcal{G}}_z = \mathcal{S}_z$ means proving the equality

$$\forall \varphi_1,\varphi_2 \in C_0^\infty(X), \quad \langle \Phi_0\varphi_1, \hat{R}(z)\Phi_0\varphi_2\rangle = \int_X \overline{\Phi_0\varphi_1}R(z)\varphi_2\,\mathrm{d}\mu$$

(the integral makes sense since $\|\Phi_0\varphi_1\|_\infty < \infty$, $\|R(z)\varphi_2\|_1 < \infty$). This is further equivalent to

$$\forall \varphi \in C_0^\infty(X), \quad \hat{R}(z)\Phi_0\varphi = \Phi_0 R(z)\varphi.$$

To verify this relation, one has to show that

$$\forall \varphi \in C_0^\infty(X), \quad \Phi_0 R(z)\varphi \in \operatorname{Dom}(\hat{H}) \quad \text{and} \quad (\hat{H}-z)\Phi_0R(z)\varphi = \Phi_0\varphi. \tag{41}$$

Let us prove (41). From (39), it follows that $\Phi_0R(z)\varphi$ is a Γ-periodic measurable function on X which is L^2 integrable over a fundamental domain; so it belongs to the Hilbert space $\hat{\mathscr{H}}$. Since the series $\sum_{\gamma\in\Gamma} R(z)\varphi(\gamma\cdot x)$ converges in the L^1 norm over any compact subset of X it converges in the sense of distributions. Hence, in the sense of distributions,

$$(-\Delta_{\mathrm{LB}}-z)\sum_{\gamma\in\Gamma} R(z)\varphi(\gamma\cdot x) = \sum_{\gamma\in\Gamma}\left((-\Delta_{\mathrm{LB}}-z)R(z)\varphi\right)(\gamma\cdot x) = \sum_{\gamma\in\Gamma}\varphi(\gamma\cdot x) = \Phi_0\varphi(x),$$

i.e. $(-\Delta_{\mathrm{LB}}-z)\Phi_0R(z)\varphi = \Phi_0\varphi$. Since $-\Delta_{\mathrm{LB}}-z$ is an elliptic operator with smooth coefficients and $\Phi_0\varphi$ is a smooth function, both of them on X, by the elliptic regularity theorem, $\Phi_0R(z)\varphi$ is smooth as well (see, for example, [32, Appendix 4 Section 5]). By the Green formula,

$$\forall \varphi,\psi \in C_0^\infty(X), \quad \left\langle(\hat{H}-\bar{z})\Phi_0\psi, \Phi_0R(z)\varphi\right\rangle = \langle\Phi_0\psi, \Phi_0\varphi\rangle.$$

This equality implies (41) since $\Phi_0\big(C_0^\infty(X)\big)$ is a core of $\hat{H}$. □

5.2. The case of a trivial line bundle over M

The Green function formula (34) (or the propagator formula (31)) essentially simplifies in the particular case when the gauge group is $U(1)$ and the line bundle over M is trivial. Then the line bundle over $\tilde{M}$ is trivial as well. On the other hand, the connection need not be flat. Let us shortly indicate the basic modifications.

Suppose $\mathfrak{V} = M \times \mathbb{C}$ with a Hermitian structure $\mathfrak{h}$ given by the standard scalar product in $\mathbb{C}$, and with a connection $\nabla = \mathrm{d} + \alpha$ where α is a one-form on M with values in $\mathrm{i}\mathbb{R}$. If $\pi : \tilde{M} \to M$ is a covering space with a structure group Γ, then $\tilde{\mathfrak{V}} = \pi^*\mathfrak{V} = \tilde{M}\times\mathbb{C}$. The group Γ acts on $\tilde{\mathfrak{V}}$ as an identity on the fibers, $\Psi_\gamma(y,z) = (\gamma\cdot y, z)$ for $(y,z) \in \tilde{M}\times\mathbb{C}$ and $\gamma\in\Gamma$. Furthermore, $\tilde{\nabla} = \mathrm{d}+\tilde{\alpha}$, $\tilde{\alpha} = \pi^*\alpha$. Let us denote by L_γ^* the pullback mapping for the left action of $\gamma\in\Gamma$ on $\tilde{M}$. Note that the invariance condition (1) for the connection $\tilde{\nabla}$ means that $L_\gamma^*\tilde{\alpha} = \tilde{\alpha}$, $\forall\gamma\in\Gamma$, and this is clearly satisfied. If Λ is a one-dimensional unitary representation of Γ, then $\tilde{\mathfrak{V}}^\Lambda = \tilde{M}\times\mathbb{C}$, $\tilde{\mathfrak{h}}^\Lambda = \tilde{\mathfrak{h}}$, $\tilde{\nabla}^\Lambda = \tilde{\nabla}$. Note, however, that $\Psi_\gamma^\Lambda(y,z) = (\gamma\cdot y, \Lambda(\gamma)z)$.

Suppose η is a nowhere-vanishing complex function on $\tilde{M}$ such that $\forall\gamma\in\Gamma$, $L_\gamma^*\eta = \Lambda(\gamma)\eta$. Clearly, $|\eta|$ is Γ-periodic. Replacing η by $\eta/|\eta|$ one can assume that $|\eta| \equiv 1$ on $\tilde{M}$. Then η defines a trivialization of $\mathfrak{V}^\Lambda$. In that case $\mathfrak{V}^\Lambda = M\times\mathbb{C}$, $\mathfrak{h}^\Lambda = \mathfrak{h}$ and $\alpha^\Lambda = \alpha + (\mathrm{d}\eta)\eta^{-1}$ (more precisely, $(\mathrm{d}\eta)\eta^{-1}$ is a Γ-invariant one-form on $\tilde{M}$ which projects to a one-form on M).

One also identifies $L^2(\tilde{\mathfrak{V}}) = L^2(\tilde{M})$, $L^2(\mathfrak{V}^\Lambda) = L^2(M)$. Let $\Delta_{\mathrm{B}}^\Lambda$ be the Bochner Laplacian corresponding to the connection $\nabla^\Lambda = \mathrm{d} + \alpha^\Lambda$. Thus, we consider $H^\Lambda = -\Delta_{\mathrm{B}}^\Lambda + V$ as an operator in $L^2(M)$ rather than in the Hilbert space of Λ-equivariant functions on $\tilde{M}$. Let us denote the latter space by $\mathscr{H}^\Lambda$. The two spaces are related by the unitary mapping

$$L^2(M) \to \mathscr{H}^\Lambda : \psi \mapsto (\pi^*\psi)\eta. \tag{42}$$

With this trivialization, it is natural to define the Green function $\mathcal{G}_z^\Lambda$ of H^Λ in the standard manner as a distribution on $M \times M$,

$$\forall \psi_1, \psi_2 \in C_0^\infty(M), \quad \mathcal{G}_z^\Lambda(\overline{\psi_1} \otimes \psi_2) = \langle \psi_1, R^\Lambda(z)\psi_2 \rangle.$$

One has to keep in mind that $\mathcal{G}_z^\Lambda$ depends on the choice of η. This definition differs from that given in (27) and so one has to somewhat modify the Green function formula (34). To this end, it is useful to identify distributions on M with Γ-periodic distributions on $\tilde{M}$ as explained in the following remark.

Remark 25. Denote by $C_\Gamma^\infty(\tilde{M})$ the vector space of Γ-periodic smooth functions on $\tilde{M}$. Let us define

$$\Phi_0 : C_0^\infty(\tilde{M}) \to C_\Gamma^\infty(\tilde{M}) : \varphi \mapsto \sum_{\gamma \in \Gamma} L_\gamma^* \varphi.$$

For $f \in \mathscr{D}'(M)$ one introduces $\pi^* f \in \mathscr{D}'(\tilde{M})$ by the prescription

$$\forall \varphi \in C_0^\infty(\tilde{M}), \quad \pi^* f(\varphi) = f(\psi) \text{ where } \pi^*\psi = \Phi_0\varphi.$$

The functional $\pi^* f$ is well defined on $C_0^\infty(\tilde{M})$, and it is not difficult to verify that it is continuous. Hence $\pi^* f \in \mathscr{D}'(\tilde{M})$. Moreover, $\pi^* f$ is Γ-periodic.

Conversely, suppose $\tilde{g} \in \mathscr{D}'(\tilde{M})$ is Γ-periodic. Note that for every $\psi \in C_0^\infty(M)$, there exists $\varphi \in C_0^\infty(\tilde{M})$ such that $\Phi_0\varphi = \pi^*\psi$; φ is, however, ambiguous. On the other hand, if for some $\varphi \in C_0^\infty(\tilde{M})$, $\Phi_0\varphi = 0$, then $\tilde{g}(\varphi) = 0$. In fact, there exists $\chi \in C_0^\infty(\tilde{M})$ such that $\Phi_0\chi \equiv 1$ on a neighborhood of $\operatorname{supp}\varphi$. Since $\tilde{g}$ is Γ-invariant, one has

$$\tilde{g}(\varphi) = \tilde{g}(\varphi\Phi_0\chi) = \tilde{g}(\chi\Phi_0\varphi) = 0.$$

Let us define a functional g on $C_0^\infty(M)$,

$$\forall \psi \in C_0^\infty(M), \quad g(\psi) = \tilde{g}(\varphi) \quad \text{where } \varphi \in C_0^\infty(\tilde{M}) \text{ is s.t. } \Phi_0\varphi = \pi^*\psi.$$

Then g is well defined and continuous. One observes that g is the only distribution on M satisfying $\pi^* g = \tilde{g}$.

One concludes that the mapping $\mathscr{D}'(M) \to \mathscr{D}'(\tilde{M}) : f \mapsto \pi^* f$ induces an isomorphism of $\mathscr{D}'(M)$ onto the space of Γ-periodic distributions on $\tilde{M}$.

Let us regard $\mathcal{G}_z^\Lambda$ as a $\Gamma \times \Gamma$-periodic distribution on $\tilde{M} \times \tilde{M}$. Using the unitary map (42), it is a matter of straightforward manipulations to show that the Green function formula now reads

$$\mathcal{G}_z^\Lambda(y_1, y_2) = \overline{\eta(y_1)}\eta(y_2) \sum_{\gamma \in \Gamma} \Lambda(\gamma) \tilde{\mathcal{G}}_z(\gamma^{-1} \cdot y_1, y_2). \tag{43}$$

5.3. The case of a trivializable line bundle over $\tilde{M}$

Next we are going to discuss a still rather particular case but more general than in the preceding subsection. The gauge group is again supposed to be $U(1)$. The line bundle over M need not be trivial neither is the connection required to be flat. We assume, however, that after a pull-back one gets a line bundle over the covering space $\tilde{M}$ which is trivializable. As is well known, this surely happens if $H^2(\tilde{M}, \mathbb{Z}) = 0$.

The assumption that the line bundle $\tilde{\mathfrak{V}} = \pi^*\mathfrak{V}$ be trivializable means that there exists a nowhere vanishing smooth section $\eta \in C^\infty(\tilde{\mathfrak{V}})$. Without loss of generality, we assume that $\tilde{\mathfrak{h}}(\eta, \eta) = 1$. Using η one passes from $\tilde{\mathfrak{V}}$ to the trivial line bundle $\tilde{M} \times \mathbb{C}$. In particular, one has the isomorphism

$$C^\infty(\tilde{M}) \to C^\infty(\tilde{\mathfrak{V}}) : \varphi \mapsto \varphi\eta. \tag{44}$$

The Hermitian structure in $\tilde{M} \times \mathbb{C}$ is given by the standard scalar product in $\mathbb{C}$, the covariant derivative $\tilde{\nabla}$ becomes $\mathrm{d} + \tilde{\alpha}$ where $\tilde{\alpha}$ is a one-form on $\tilde{M}$ with values in $\mathrm{i}\mathbb{R}$.

Given $y \in \tilde{M}$ one can compare the values $\eta(y)$ and $\eta(\gamma \cdot y)$ for any $\gamma \in \Gamma$ since $\tilde{\mathfrak{V}}_y = \tilde{\mathfrak{V}}_{\gamma \cdot y} = \mathfrak{V}_{\pi(y)}$. Abusing somewhat the previously used notation let us write

$$\forall \gamma \in \Gamma, \ \forall y \in \tilde{M}, \quad \eta(\gamma \cdot y) = \Psi_\gamma(y)^{-1}\eta(y),$$

where $\Psi_\gamma(y) \in \mathbb{C}$ is unambiguously determined by this equality. Then for any $\gamma \in \Gamma$, Ψ_γ is a smooth complex function on $\tilde{M}$ with values in the unit circle. The fiber-wise linear action of Γ on $\tilde{M} \times \mathbb{C}$ takes the form

$$\tilde{M} \times \mathbb{C} \to \tilde{M} \times \mathbb{C} : (y, z) \mapsto \big(\gamma \cdot y, \Psi_\gamma(y)z\big), \quad \text{with } \gamma \in \Gamma.$$

The composition rule for this action means that

$$\forall \gamma_1, \gamma_2 \in \Gamma, \quad \big(L^*_{\gamma_2}\Psi_{\gamma_1}\big)\Psi_{\gamma_2} = \Psi_{\gamma_1\gamma_2}.$$

Here again, L^*_γ stands for the pull-back mapping of the left action of $\gamma \in \Gamma$ on $\tilde{M}$.

Sections of the line bundle $\mathfrak{V}$ over M can be naturally identified with Γ-invariant sections of the line bundle $\tilde{\mathfrak{V}}$ over $\tilde{M}$. Employing the isomorphism (44) this means that smooth (measurable) sections of $\mathfrak{V}$ are identified with those smooth (measurable) functions φ on $\tilde{M}$ which satisfy everywhere (almost everywhere) the condition

$$\forall \gamma \in \Gamma, \quad \varphi(\gamma \cdot y) = \Psi_\gamma(y)\varphi(y),$$

i.e. $L^*_\gamma\varphi = \Psi_\gamma\varphi$. The operators W_γ defined in (2) for the general case now act either in $C^\infty(\tilde{M})$ or as unitary operators in $L^2(\tilde{M})$ according to the rule

$$\forall \gamma \in \Gamma, \quad W_\gamma\varphi(y) = \Psi_\gamma(\gamma^{-1} \cdot y)\varphi(\gamma^{-1} \cdot y). \tag{45}$$

Thus, sections of $\mathfrak{V}$ correspond exactly to those functions on $\tilde{M}$ which satisfy $W_\gamma\varphi = \varphi, \forall \gamma \in \Gamma$.

From the physical point of view, it is of interest to observe that now the invariance condition (1) for the connection $\tilde{\nabla}$ in general does not mean that $L^*_\gamma\tilde{\alpha} = \tilde{\alpha}$ for $\gamma \in \Gamma$. A straightforward computation gives the correct invariance condition for this case, namely

$$\forall \gamma \in \Gamma, \quad L^*_{\gamma^{-1}}\tilde{\alpha} = \tilde{\alpha} + \Psi_\gamma^{-1}\mathrm{d}\Psi_\gamma.$$

On the other hand, the curvature $\tilde{\Omega} = \mathrm{d}\tilde{\alpha}$ or, in other words, the two form of the magnetic field (up to a constant multiplier) fulfills $L^*_\gamma\tilde{\Omega} = \tilde{\Omega}, \forall \gamma \in \Gamma$. In the analysis of physical systems with an invariant nonzero magnetic flux the so-called magnetic translations turn out to be a useful tool [33,34]. In our formalism the magnetic translations coincide with the operators W_γ defined in (45). Particularly, the symmetry of the Hamiltonian $\tilde{H} = -\Delta_B + \tilde{V}$ is reflected by the fact that it commutes with all $W_\gamma,\ \gamma \in \Gamma$.

Let Λ be a one-dimensional unitary representation of Γ. The line bundle $\tilde{\mathfrak{V}}^\Lambda$ is again identified with $\tilde{M} \times \mathbb{C}$, and the Hermitian structure and the covariant derivative remain unchanged. What is modified, however, is the action of Γ on $\tilde{M} \times \mathbb{C}$. The modified action was called Ψ^Λ_γ in Section 3.1 and now it takes the form

$$\tilde{M} \times \mathbb{C} \to \tilde{M} \times \mathbb{C} : (y, z) \mapsto \big(\gamma \cdot y, \Lambda(\gamma)\Psi_\gamma(y)z\big), \quad \text{with } \gamma \in \Gamma.$$

Sections of $\mathfrak{V}^\Lambda$ are again identified with functions φ on $\tilde{M}$ fulfilling

$$\forall \gamma \in \Gamma, \quad \varphi(\gamma \cdot y) = \Lambda(\gamma)\Psi_\gamma(y)\varphi(y). \tag{46}$$

This identification implies an isomorphism of $L^2(\mathfrak{V}^\Lambda)$ with the Hilbert space of Λ-equivariant functions on $\tilde{M}$, i.e. with the Hilbert space $\mathscr{H}^\Lambda$ formed by measurable functions on $\tilde{M}$ satisfying (46) almost everywhere and being square integrable over a fundamental domain of the action of Γ. Then, the formal differential expression corresponding to the Hamiltonian $\tilde{H}$ acting in $L^2(\tilde{M})$ is the same as that for H^Λ acting in $\mathscr{H}^\Lambda$.

The Green function of $\tilde{H}$ is a distribution $\tilde{\mathcal{G}}_z$ on $\tilde{M} \times \tilde{M}$. According to (26), it has the symmetry property

$$\forall \gamma \in \Gamma, \quad \tilde{\mathcal{G}}_z(\gamma \cdot y_1, \gamma \cdot y_2) = \Psi_\gamma(y_1)\overline{\Psi_\gamma(y_2)}\tilde{\mathcal{G}}_z(y_1, y_2).$$

The Green function $\mathcal{G}^\Lambda_z$ of H^Λ is a distribution on $\tilde{M} \times \tilde{M}$ as well. According to (28) it is Λ-equivariant, and this property now reads

$$\forall \gamma \in \Gamma, \quad \mathcal{G}^\Lambda_z(\gamma \cdot y_1, y_2) = \Lambda(\gamma)\Psi_\gamma(y_1)\mathcal{G}^\Lambda_z(y_1, y_2), \qquad \mathcal{G}^\Lambda_z(y_1, \gamma \cdot y_2) = \Lambda(\gamma^{-1})\overline{\Psi_\gamma(y_2)}\mathcal{G}^\Lambda_z(y_1, y_2).$$

Finally, in this example the Green function formula (34) takes the form

$$\begin{aligned} \mathcal{G}^\Lambda_z(y_1, y_2) &= \sum_{\gamma \in \Gamma} \Lambda(\gamma)\Psi_\gamma(\gamma^{-1} \cdot y_1)\tilde{\mathcal{G}}_z(\gamma^{-1} \cdot y_1, y_2) \\ &= \sum_{\gamma \in \Gamma} \Lambda(\gamma^{-1})\overline{\Psi_\gamma(\gamma^{-1} \cdot y_2)}\tilde{\mathcal{G}}_z(y_1, \gamma^{-1} \cdot y_2). \end{aligned} \tag{47}$$

5.4. A constant magnetic field on a torus

As an illustration of the formalism described in Section 5.3 let us consider an example in which M is the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. This example shares the basic nontrivial features of the approach and, at the same time, it is sufficiently simple to allow for explicit computations. The Riemannian structure on $\mathbb{T}^2$ is induced by the standard scalar product in $\mathbb{R}^2$. Let $\tilde{M}$ be the universal covering space of the torus, i.e. $\tilde{M} = \mathbb{R}^2$. Hence the covering group is $\Gamma = (2\pi\mathbb{Z})^2$. The gauge group is supposed to be $U(1)$ and thus only line bundles are considered. We do not require that line bundles over M be trivial. For convenience we choose a connection in such a way that the magnetic field (curvature) is constant in the natural coordinate system on the torus. Since any line bundle over $\mathbb{R}^2$ is trivializable, one can actually employ the notation and the formalism introduced in Section 5.3.

The magnetic Bloch analysis on a torus has already been discussed in detail in [5] for the case when there is no scalar potential. Here we focus on the inverse procedure, i.e. on the formula for Green functions (34). As a slight modification if compared to [5], we prefer to use the Landau gauge rather than the symmetric one. The former gauge is also frequently used in the physical literature dedicated to quantum systems describing a particle on a torus in a constant magnetic field [35,36].

The standard coordinates on $\mathbb{R}^2$ are denoted (x, y) while the angle coordinates on $\mathbb{T}^2$ are denoted (ϕ, θ). The projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is given by

$$\phi = x \,(\mathrm{mod}\ 2\pi), \qquad \theta = y \,(\mathrm{mod}\ 2\pi).$$

Since $H^2(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}$, equivalence classes of line bundles over $\mathbb{T}^2$ are labeled by integers. To describe a line bundle over $\mathbb{T}^2$ in terms of transition functions, we cover the torus by two cylinders:

$$U_1 = (0, 2\pi) \times \mathbb{T}^1, \qquad U_2 = (-\pi, \pi) \times \mathbb{T}^1.$$

The coordinates on U_1 and U_2 are (ϕ_1, θ) and (ϕ_2, θ), respectively, and so the coordinate θ is the same for both cylinders. The intersection $U_1 \cap U_2$ equals the disjoint union $U_{12A} \cup U_{12B}$ where U_{12A} is $(0, \pi) \times \mathbb{T}^1$ both in the coordinates (ϕ_1, θ) and (ϕ_2, θ) while U_{12B} is identified either with $(\pi, 2\pi) \times \mathbb{T}^1$ in the coordinates (ϕ_1, θ) or with $(-\pi, 0) \times \mathbb{T}^1$ in the coordinates (ϕ_2, θ). The transformation of coordinates on $U_1 \cap U_2$ is given in an obvious way, namely

$$\begin{aligned} &\phi_2 = \phi_1 \quad \text{if } 0 < \phi_1 < \pi,\ 0 < \phi_2 < \pi, \\ &\phi_2 = \phi_1 - 2\pi \quad \text{if } \pi < \phi_1 < 2\pi,\ -\pi < \phi_2 < 0. \end{aligned}$$

Any line bundle over a cylinder is equivalent to a trivial one. We glue together $U_1 \times \mathbb{C}$ and $U_2 \times \mathbb{C}$ by a transition function τ defined on $U_1 \cap U_2$, with values in $U(1)$. We put

$$\tau = 1 \quad \text{on } U_{12A}, \qquad \tau = \mathrm{e}^{-\mathrm{i}N\theta} \quad \text{on } U_{12B},\ \text{where } N \in \mathbb{Z} \text{ is fixed}.$$

Let $\mathfrak{V}_N$ denote the resulting line bundle over $\mathbb{T}^2$. Then a section in $\mathfrak{V}_N$ is determined by a couple of complex functions (ψ_1, ψ_2) defined on U_1 and U_2, respectively, so that $\psi_1 = \tau\psi_2$ on $U_1 \cap U_2$. A connection in $\mathfrak{V}_N$ is determined by a couple of one-forms (α_1, α_2) defined respectively on U_1 and U_2, with values in $\mathrm{i}\mathbb{R}$, and such that $\alpha_2 = \alpha_1 + \tau^{-1}\mathrm{d}\tau$ on $U_1 \cap U_2$. Our choice is

$$\alpha_1 = \frac{\mathrm{i}N}{2\pi}\phi_1\mathrm{d}\theta, \qquad \alpha_2 = \frac{\mathrm{i}N}{2\pi}\phi_2\mathrm{d}\theta.$$

For the curvature (the two-form of the magnetic field) we get $\Omega = \mathrm{d}\alpha = \mathrm{i}N/(2\pi)\,\mathrm{d}\phi \wedge \mathrm{d}\theta$, and one has

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}^2} \Omega = N.$$

Since τ takes its values in $U(1)$, the Hermitian structure in $\mathfrak{V}_N$ is induced by the standard scalar product in $\mathbb{C}$.

A crucial role in the formalism of Section 5.3 is played by the family of functions Ψ_γ, $\gamma \in \Gamma$. To find these functions, we first need to describe the line bundle $\tilde{\mathfrak{V}}_N = \pi^*\mathfrak{V}_N$. To this end, let us cover $\mathbb{R}^2$ by two countable families of open strips, namely $\tilde{U}_{1,k} = (2\pi k, 2\pi(k+1)) \times \mathbb{R}$ and $\tilde{U}_{2,k} = (\pi(2k-1), \pi(2k+1)) \times \mathbb{R}$, $k \in \mathbb{Z}$. Then, $\pi(\tilde{U}_{1,k}) = U_1$ and $\pi(\tilde{U}_{2,k}) = U_2$, $\forall k \in \mathbb{Z}$. A section of $\tilde{\mathfrak{V}}_N$ is determined by two countable families of functions, namely $\varphi_{1,k}$ defined on $\tilde{U}_{1,k}$ and $\varphi_{2,k}$ defined on $\tilde{U}_{2,k}$, $k \in \mathbb{Z}$, such that

$$\varphi_{1,k}(x, y) = \varphi_{2,k}(x, y) \quad \text{on } \big(2\pi k, \pi(2k+1)\big) \times \mathbb{R},$$

and

$$\varphi_{1,k}(x, y) = \mathrm{e}^{-\mathrm{i}Ny}\varphi_{2,k+1}(x, y) \quad \text{on } \big(\pi(2k+1), 2\pi(k+1)\big) \times \mathbb{R}.$$

For a nowhere vanishing smooth section η of $\tilde{\mathfrak{V}}_N$, we choose the families

$$\eta_{1,k}(x, y) = \mathrm{e}^{\mathrm{i}kNy}, \qquad \eta_{2,k}(x, y) = \mathrm{e}^{\mathrm{i}kNy}, \quad k \in \mathbb{Z}.$$

Then η determines a trivialization of $\tilde{\mathfrak{V}}_N$, and one finds that

$$\Psi_\gamma(x, y) = \mathrm{e}^{-\mathrm{i}mNy} \quad \text{for } \gamma = (2\pi m, 2\pi n) \in (2\pi\mathbb{Z})^2.$$

Below we consider the case with vanishing scalar potential so that explicit computations are possible. Furthermore, we leave aside the case $N = 0$ which has particular properties but whose discussion is very elementary. Thus, the Hamiltonian $\tilde{H}$ describes a charged particle in a homogeneous magnetic field on the plane,

$$\tilde{H} = -\partial_x^2 - \left(\partial_y + \frac{\mathrm{i}N}{2\pi}\,x\right)^2 \quad \text{in } L^2(\mathbb{R}^2, \mathrm{d}x\mathrm{d}y). \tag{48}$$

The symmetry of $\tilde{H}$ is demonstrated by the fact that it commutes with all magnetic translations W_γ, $\gamma \in \Gamma$. One can establish the symmetry even under somewhat more general circumstances, for any real N (not necessarily an integer). Defining the operators $W_{a,b}$, with $a, b \in \mathbb{R}$, by the relation

$$W_{a,b}\varphi(x, y) = \mathrm{e}^{-\mathrm{i}Nay/(2\pi)}\varphi(x - a, y - b),$$

one observes that $\tilde{H}W_{a,b} = W_{a,b}\tilde{H}$. Note that $W_{a_1,b_1}W_{a_2,b_2} = \mathrm{e}^{\mathrm{i}Na_2b_1/(2\pi)}W_{a_1+a_2,b_1+b_2}$ but if $\gamma_1, \gamma_2 \in (2\pi\mathbb{Z})^2$ and $N \in \mathbb{Z}$, then $W_{\gamma_1}W_{\gamma_2} = W_{\gamma_1+\gamma_2}$.

As is well known, using the Fourier transformation in the y variable one can decompose $\tilde{H}$ into a direct integral whose components are unitarily equivalent to the Hamiltonian of the harmonic oscillator with parameter values: $\hbar = 1$, the mass equals $1/2$ and the frequency $\omega = 2|N|/(2\pi)$. Consequently, one can express the Green function of $\tilde{H}$ in terms of the Green function $\mathcal{G}_z^{\mathrm{ho}}$ of the harmonic oscillator (with the indicated values of parameters). One has

$$\tilde{\mathcal{G}}_z(x_1, y_1; x_2, y_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{G}_z^{\mathrm{ho}}\left(x_1 + \frac{2\pi k}{N}, x_2 + \frac{2\pi k}{N}\right)\mathrm{e}^{\mathrm{i}k(y_1-y_2)}\,\mathrm{d}k, \tag{49}$$

and

$$\mathcal{G}_z^{\mathrm{ho}}(x_1, x_2) = \frac{1}{\sqrt{2\pi\omega}}\Gamma\left(\frac{1}{2} - \frac{z}{\omega}\right)D_{-\frac{1}{2}+\frac{z}{\omega}}\left(\sqrt{\omega}x_>\right)D_{-\frac{1}{2}+\frac{z}{\omega}}\left(-\sqrt{\omega}\,x_<\right)$$

where $D_\nu(x)$ is the parabolic cylinder function, $x_> = \max\{x_1, x_2\}$, $x_< = \min\{x_1, x_2\}$ (see [37] and references therein).

Let us write a one-dimensional representation Λ of Γ in the form

$$\Lambda(\gamma) = \mathrm{e}^{2\pi\mathrm{i}(\mu m+\nu n)} \quad \text{for } \gamma = (2\pi m, 2\pi n) \in (2\pi\mathbb{Z})^2$$

where $\mu, \nu \in [0, 1)$ (here we change the meaning of the symbol μ). The Hamiltonian H^Λ as a differential operator has the same form as that given on the RHS of (48) but it acts in the Hilbert space $\mathscr{H}^\Lambda$ formed by measurable functions φ on $\mathbb{R}^2$ which are square integrable over the fundamental domain $D = (0, 2\pi)^2$ and satisfy, almost everywhere on $\mathbb{R}^2$,

$$\varphi(x + 2\pi, y) = \mathrm{e}^{2\pi\mathrm{i}\mu}\mathrm{e}^{-\mathrm{i}Ny}\varphi(x, y), \qquad \varphi(x, y + 2\pi) = \mathrm{e}^{2\pi\mathrm{i}\nu}\varphi(x, y).$$

The Green function of H^Λ can be derived with the help of formula (47). Using (49) and the rapid decay of the parabolic cylinder functions,

$$D_\nu(x) = \mathrm{e}^{-x^2/4}x^\nu\left(1 + O\left(x^{-1}\right)\right) \quad \text{as } x \to +\infty,$$

one can apply the Poisson summation in (47) to arrive at the equality

$$\begin{aligned}\mathcal{G}_z^\Lambda(x_1, y_1; x_2, y_2) = \frac{1}{2\pi}\sum_{s=0}^{|N|-1}\sum_{k\in\mathbb{Z}^2}\mathcal{G}_z^{\mathrm{ho}}&\left(x_1 + 2\pi k_1 + \frac{2\pi(s+\nu)}{N}, x_2 + 2\pi k_2 + \frac{2\pi(s+\nu)}{N}\right)\\ &\times \mathrm{e}^{\mathrm{i}k_1(Ny_1-2\pi\mu)-\mathrm{i}k_2(Ny_2-2\pi\mu)+\mathrm{i}(s+\nu)(y_1-y_2)}.\end{aligned} \tag{50}$$

Note that

$$\mathcal{G}_z^\Lambda(x_1, y_1; x_2, y_2) = \mathrm{e}^{\mathrm{i}\nu y_1}\mathcal{G}_z^{\Lambda=1}\left(x_1 + \frac{2\pi\nu}{N}, y_1 - \frac{2\pi\mu}{N}; x_2 + \frac{2\pi\nu}{N}, y_2 - \frac{2\pi\mu}{N}\right)\mathrm{e}^{-\mathrm{i}\nu y_2}. \tag{51}$$

A formula analogous to (50) also holds for the kernel $p^\Lambda(t; x_1, y_1; x_2, y_2)$ of the Schrödinger semigroup $\exp(-tH^\Lambda)$, $t > 0$. Let $p^{\mathrm{ho}}(t; x_1, x_2)$ denote the kernel of the Schrödinger semigroup $\exp(-tH^{\mathrm{ho}})$, $t > 0$, where H^{ho} is the Hamiltonian of the harmonic oscillator (for the proper choice of parameters). Using the formula for the Schrödinger semigroups (replacing $\mathcal{G}_z(\ldots)$ by $p(t; \ldots)$ in (50)), one can compute the traces. After some simple manipulations one finds that

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{e}^{-tH^\Lambda}\right) &= \int_0^{2\pi}\int_0^{2\pi} p^\Lambda(t; x, y; x, y)\,\mathrm{d}x\mathrm{d}y\\ &= |N|\int_{\mathbb{R}} p^{\mathrm{ho}}(t; u, u)\,\mathrm{d}u\\ &= |N|\mathrm{Tr}\left(\mathrm{e}^{-tH^{\mathrm{ho}}}\right)\\ &= \frac{2|N|}{\sinh(t\omega/2)}.\end{aligned}$$

This equality makes it possible to compare the spectra of H^{Λ} and H^{ho}. Thus, one deduces that

$$\operatorname{spec} H^{\Lambda} = \left\{ \frac{|N|}{\pi} \left(\ell + \frac{1}{2} \right);\ \ell = 0, 1, 2, \ldots \right\} \tag{52}$$

where the multiplicity of each eigenvalue of H^{Λ} equals $|N|$. Of course, the spectrum of H^{Λ} can also be derived by solving directly the corresponding eigenvalue equation [38–40,36].

Equality (52) particularly implies that all operators H^{Λ}, $\Lambda \in \hat{\Gamma}$, are mutually unitarily equivalent (this is not true for $N = 0$). The corresponding unitary mapping can be immediately deduced from (51). Consider the linear mapping

$$T^{\Lambda} : \mathscr{H}^{\Lambda=1} \to \mathscr{H}^{\Lambda}, \quad T^{\Lambda}\varphi(x, y) = e^{i\nu y}\varphi\left(x + \frac{2\pi\nu}{N}, y - \frac{2\pi\mu}{N} \right).$$

One readily verifies that T^{Λ} is well defined, unitary and $H^{\Lambda} = T^{\Lambda}H^{\Lambda=1}(T^{\Lambda})^{-1}$.

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