Journal of Geometry and Physics 114 (2017) 109–123

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Gaussian and mean curvatures for discrete asymptotic nets

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ARTICLE INFO

Article history:
Received 14 April 2016
Received in revised form 28 November 2016
Accepted 30 November 2016
Available online 9 December 2016

Keywords:
Discrete differential geometry
Integrable systems

ABSTRACT

We propose discretisations of Gaussian and mean curvatures of surfaces parametrised in terms of asymptotic coordinates and examine their relevance in the context of integrable discretisations of classical classes of surfaces and their underlying integrable systems. We also record discrete analogues of the classical relation between the Gaussian curvature of hyperbolic surfaces and the torsion of their asymptotic lines.

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1. Introduction

It has been established that discrete systems play a fundamental role in the theory of integrable systems and geometric tools to construct and analyse integrable systems have proven to be very useful in this connection. In particular, the intimate relationship between discrete differential geometry and discrete integrable systems has been well documented [1]. Recently, the development of a discrete curvature theory for polyhedral surfaces which respects the integrability properties inherent in its classical continuous counterpart has been a focus of attention. Gaussian and mean curvatures for discrete curvature nets (circular nets) have been proposed in [2] and related to a discrete Steiner formula for parallel nets [3]. A more general curvature theory for discrete surfaces based on mesh parallelity has been put forward in [4]. Moreover, the theory of edge-constraint nets [5] gives rise to the definition of fundamental forms and curvatures for discretisations of surfaces which are parametrised in terms of general coordinates.

Asymptotic coordinates on hyperbolic surfaces are fundamental in classical differential geometry [6] and have been shown to constitute a rich source of integrable connections (see, e.g., [7,8] and references therein). Discrete asymptotic nets, that is, lattices of $\mathbb{Z}^2$ combinatorics with planar stars have proven to be the canonical discrete analogue of surfaces parametrised in terms of asymptotic coordinates [1]. Here, we propose Gaussian and mean curvatures for discrete asymptotic nets and demonstrate their compatibility with a variety of standard discretisations of classical classes of surfaces, namely pseudospherical, minimal and Bianchi surfaces and (generalised) affine spheres, and their underlying integrable systems. Specifically, we show that the Gaussian curvature of discrete pseudospherical surfaces is constant and pairs of conjugate discrete minimal surfaces have vanishing mean curvature in the current sense and coinciding Gaussian curvature. Furthermore, the square root of the negative reciprocal of the vertex Gaussian curvature of discrete Bianchi surfaces is "discrete harmonic" with respect to a weighted discrete d'Alembert operator. In the classical setting, the affine distance of a point on a surface in centro-affine differential geometry may be expressed in terms of the Gaussian curvature of the surface at that point and the Euclidean distance of the corresponding tangent plane to the origin. It is shown that this connection may be transferred to the discrete setting and we demonstrate that the affine distance of discrete affine spheres is constant. We

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http://dx.doi.org/10.1016/j.geomphys.2016.11.031
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also prove that discrete generalised affine spheres are algebraically characterised by the affine distance obeying a "discrete logarithmic" harmonicity condition.

The approach presented here is based on known canonical discrete versions of the classical Lelieuvre formulae and associated Moutard equation [9,10]. These may also be used to determine compact expressions for the discrete torsion of discrete asymptotic lines and its relation to the discrete Gaussian curvature is shown to mirror the well-known connection in the classical setting.

2. Classical Gaussian and mean curvatures

There exist various equivalent definitions of Gaussian and mean curvatures of surfaces Σ in a three-dimensional Euclidean space $\mathbb{R}^3$. For instance, Steiner's formula [11] relates the change of the area of an infinitesimal surface element of a surface Σ to the Gaussian and mean curvatures of Σ as one passes through the associated family of parallel surfaces $\Sigma^{\parallel}$. Thus, if Σ is a surface parametrised by

$$\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3, \quad (x, y) \mapsto \boldsymbol{r}(x, y)$$

and

$$\boldsymbol{N} = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{|\boldsymbol{r}_x \times \boldsymbol{r}_y|}$$

constitutes an associated unit normal then a parallel surface $\Sigma^{\parallel}$ at (constant) distance μ is represented by

$$\boldsymbol{r}^{\parallel} = \boldsymbol{r} + \mu \boldsymbol{N}$$

so that

$$\boldsymbol{r}^{\parallel}_x \times \boldsymbol{r}^{\parallel}_y = \boldsymbol{r}_x \times \boldsymbol{r}_y + \mu(\boldsymbol{r}_x \times \boldsymbol{N}_y + \boldsymbol{N}_x \times \boldsymbol{r}_y) + \mu^2 \boldsymbol{N}_x \times \boldsymbol{N}_y.$$

Since all terms in the above identity are proportional to $\boldsymbol{N}$, we may, in short-hand notation, formulate Steiner's formula as

$$\frac{\boldsymbol{r}^{\parallel}_x \times \boldsymbol{r}^{\parallel}_y}{\boldsymbol{r}_x \times \boldsymbol{r}_y} = 1 - 2\mu\mathcal{H} + \mu^2\mathcal{K},$$

where the Gaussian and mean curvatures $\mathcal{K}$ and $\mathcal{H}$ of the surface Σ are given by

$$\mathcal{K} = \frac{\boldsymbol{N}_x \times \boldsymbol{N}_y}{\boldsymbol{r}_x \times \boldsymbol{r}_y}, \quad \mathcal{H} = -\frac{\boldsymbol{r}_x \times \boldsymbol{N}_y + \boldsymbol{N}_x \times \boldsymbol{r}_y}{2(\boldsymbol{r}_x \times \boldsymbol{r}_y)}. \tag{1}$$

Here, we are concerned with surfaces which may be parametrised in terms of real asymptotic coordinates, that is, surfaces of negative Gaussian curvature $\mathcal{K}$ (hyperbolic surfaces). A surface Σ is parametrised in terms of asymptotic coordinates if the second fundamental form of Σ is purely off-diagonal, that is,

$$\boldsymbol{r}_x \cdot \boldsymbol{N}_x = \boldsymbol{r}_y \cdot \boldsymbol{N}_y = 0.$$

This is equivalent to stating that the second derivatives $\boldsymbol{r}_{xx}$ and $\boldsymbol{r}_{yy}$ are tangent to Σ. Hence, (x, y) constitute asymptotic coordinates on Σ if and only if the position vector $\boldsymbol{r}$ and the normal $\boldsymbol{N}$ are related by

$$\boldsymbol{r}_x \sim \boldsymbol{N}_x \times \boldsymbol{N}, \quad \boldsymbol{r}_y \sim \boldsymbol{N} \times \boldsymbol{N}_y.$$

It turns out that it is possible to introduce a scaled normal

$$\mathcal{V} = \rho \boldsymbol{N}$$

such that (modulo interchanging x and y)

$$\boldsymbol{r}_x = \mathcal{V}_x \times \mathcal{V}, \quad \boldsymbol{r}_y = \mathcal{V} \times \mathcal{V}_y. \tag{2}$$

The latter are known as Lelieuvre formulae [6]. These encode the fact that surfaces parametrised in terms of asymptotic coordinates are encapsulated in the classical Moutard equation [12]

$$\mathcal{V}_{xy} = h\mathcal{V}. \tag{3}$$

Indeed, the pair (2) is compatible if and only if the compatibility condition $\mathcal{V}_{xy} \times \mathcal{V} = \boldsymbol{0}$ holds. Accordingly, there exists a function h such that the Moutard equation (3) results.

The geometric meaning of the magnitude $|\rho| = |\mathcal{V}|$ of the Lelieuvre normal is revealed by calculating the Gaussian curvature of Σ. Indeed, if we formulate the Lelieuvre formulae as

$$\boldsymbol{r}_x = \rho^2 \boldsymbol{N}_x \times \boldsymbol{N}, \quad \boldsymbol{r}_y = \rho^2 \boldsymbol{N} \times \boldsymbol{N}_y$$

then we conclude that

$$\mathcal{K} = -\frac{1}{\rho^4} = -\frac{1}{|\mathcal{V}|^4}. \tag{4}$$

We also observe that

$$\mathcal{H} = \frac{(h\rho - \rho_{xy})\mathcal{V}}{\boldsymbol{r}_x \times \boldsymbol{r}_y} = \frac{\rho\mathcal{V}_{xy} - \rho_{xy}\mathcal{V}}{\boldsymbol{r}_x \times \boldsymbol{r}_y} \tag{5}$$

by virtue of the normal component

$$\rho_{xy} - \rho \boldsymbol{N}_x \cdot \boldsymbol{N}_y = h\rho \tag{6}$$

of the Moutard equation (3). The applicability and usefulness of the Lelieuvre formulae, the Moutard equation and the associated representation of the Gaussian and mean curvatures will now be illustrated in the context of classical integrable classes of surfaces which are naturally parametrised in terms of asymptotic coordinates.

2.1. Pseudospherical surfaces

By definition, pseudospherical surfaces are surfaces of constant negative Gaussian curvature [6]. Without loss of generality, we may assume that $\mathcal{K} = -1$ and $\rho = 1$ by virtue of the representation (4). Accordingly, $\mathcal{V} = \boldsymbol{N}$ and the function h in the Moutard equation (3) may be read off (6). Pseudospherical surfaces are therefore governed by the nonlinear sigma model [8,13,14]

$$\boldsymbol{N}_{xy} = h\boldsymbol{N}, \quad \boldsymbol{N}^2 = 1$$

or, explicitly,

$$\boldsymbol{N}_{xy} + (\boldsymbol{N}_x \cdot \boldsymbol{N}_y)\boldsymbol{N} = \boldsymbol{0}, \quad \boldsymbol{N}^2 = 1.$$

The latter constitutes an avatar of the classical integrable sine–Gordon equation [6]

$$\omega_{xy} = \sin\omega,$$

where ω denotes the angle between the asymptotic lines on pseudospherical surfaces. The sine–Gordon equation is also relevant in a physical context in that it has made appearances in connection with, most notably, Frenkel's and Kontorova's crystal dislocation theory [15] and Josephson junctions in the theory of superconductivity [16].

2.2. Affine spheres

The discovery by Tzitzéica [17] of a class of surfaces, the properties of which are preserved by affine transformations is widely regarded as the beginning of the theory of affine differential geometry. Tzitzéica demonstrated that the ratio of the Gaussian curvature at a point of a surface and the fourth power of the distance from the origin to the tangent plane at that point is invariant under centro-affine transformations. Hence, for a hyperbolic surface, the quantity

$$D^4 = -\frac{d^4}{\mathcal{K}}, \quad d = \boldsymbol{r} \cdot \boldsymbol{N} \tag{7}$$

constitutes an invariant with respect to centro-affine transformations. The quantity D is known as the affine distance of (the points of) the surface to the origin. Comparison with the representation (4) of the Gaussian curvature $\mathcal{K}$ reveals that the affine distance may be formulated as

$$D = \boldsymbol{r} \cdot \mathcal{V}.$$

The latter representation turns out to be convenient in the proof of the centro-affine invariance of D.

Affine spheres are now defined by the requirement that the affine distance be constant so that, without loss of generality,

$$D = \boldsymbol{r} \cdot \mathcal{V} = 1. \tag{8}$$

Accordingly, affine spheres are governed by the Lelieuvre formulae (2) subject to the constraint (8). In this case, one may show that, modulo a suitable reparametrisation of the asymptotic coordinates, the function h in the Moutard equation (3) obeys the integrable Tzitzéica equation [17–19]

$$(\ln h)_{xy} = h - h^{-2}.$$

The latter has also been shown to be another avatar of the "affinsphären equation" [20] which arises naturally in the Lagrangian description of an anisentropic gasdynamic system for certain constitutive laws [21]. Its "tri-linear" form is given by

$$\begin{vmatrix} \tau & \tau_x & \tau_{xx} \\ \tau_y & \tau_{xy} & \tau_{xxy} \\ \tau_{yy} & \tau_{xyy} & \tau_{xxyy} \end{vmatrix} + \frac{1}{4}\tau^3 = 0, \tag{9}$$

where $h = -2(\ln\tau)_{xy}$.

Differentiation of the affine sphere condition (8) shows that

$$\boldsymbol{r}\cdot\mathcal{V}_x=\boldsymbol{r}\cdot\mathcal{V}_y=0$$

so that one may interpret $\boldsymbol{r}$ as a normal of a surface Σ^* with position vector $\mathcal{V}$. If one takes the cross-product of the Lelieuvre formulae (2) with $\boldsymbol{r}$ then one obtains

$$\mathcal{V}_x=\boldsymbol{r}\times\boldsymbol{r}_x,\quad \mathcal{V}_y=\boldsymbol{r}_y\times\boldsymbol{r} \tag{10}$$

which implies that $\boldsymbol{r}$ represents a Lelieuvre normal associated with Σ^*. In fact, since the constraint (8) is symmetric in $\boldsymbol{r}$ and $\mathcal{V}$, the surface Σ^* constitutes another (polar-reciprocal) affine sphere. It is noted that the pair (2), (10) of Lelieuvre formulae characterises the class of affine spheres since it implies that $D=1$.

2.3. Generalised affine spheres

In general, the Lelieuvre normal $\mathcal{V}$ associated with a surface Σ may not be interpreted as the position vector of a surface parametrised in terms of asymptotic coordinates. However, if we introduce a further scaling of the Lelieuvre normal according to

$$\mathcal{V}^*=\frac{\mathcal{V}}{D}$$

so that

$$\boldsymbol{r}\cdot\mathcal{V}^*=1$$

then $\boldsymbol{r}$ and $\mathcal{V}^*$ are related by the pairs

$$\boldsymbol{r}_x=D^2\mathcal{V}^*_x\times\mathcal{V}^*,\qquad \boldsymbol{r}_y=D^2\mathcal{V}^*\times\mathcal{V}^*_y$$
$$\mathcal{V}^*_x=\frac{1}{D^2}\boldsymbol{r}\times\boldsymbol{r}_x,\qquad \mathcal{V}^*_y=\frac{1}{D^2}\boldsymbol{r}_y\times\boldsymbol{r}.$$

Thus, $\mathcal{V}^*$ may be regarded as the position vector of a surface Σ^* which is parametrised in terms of asymptotic coordinates and the corresponding Lelieuvre normal is

$$\boldsymbol{r}^*=\frac{\boldsymbol{r}}{D}.$$

Furthermore, the associated affine distance is given by

$$D^*=\boldsymbol{r}^*\cdot\mathcal{V}^*=\frac{1}{D}.$$

In [22], Tzitzéica generalised affine spheres by demanding that the affine distance be a function of one asymptotic variable only. In [23], it has been pointed out that this class of surfaces constitutes a special case of the class of surfaces for which $\ln D$ is "harmonic" with respect to the second fundamental form, that is,

$$(\ln D)_{xy}=0 \tag{11}$$

and which is governed by an integrable "generalised Tzitzéica system". Furthermore, it is evident that D^* also satisfies the harmonicity condition (11) so that the associated generalised affine spheres come in pairs $(\boldsymbol{r},\mathcal{V}^*)$.

2.4. Minimal surfaces

Classical minimal surfaces are naturally parametrised in terms of both curvature and asymptotic coordinates. In the case of asymptotic coordinates, the minimality condition $\mathcal{H}=0$ is equivalent to

$$\rho_{xy}=h\rho \tag{12}$$

by virtue of the representation (5) of the mean curvature or, alternatively, the orthogonality condition

$$\boldsymbol{N}_x\cdot\boldsymbol{N}_y=0 \tag{13}$$

which may be concluded from $(1)_2$ or directly from (6). Hence, minimal surfaces are governed by the Moutard equations (3) and (12) subject to $\mathcal{V}^2=\rho^2$. In terms of a four-vector $\mathsf{V}=(\mathcal{V},\rho)$ in a Minkowski space $\mathbb{R}^{3,1}$, this is encapsulated in the nonlinear sigma model

$$\mathsf{V}_{xy}=h\mathsf{V},\quad \langle\mathsf{V},\mathsf{V}\rangle=0.$$

According to Bonnet [6], any minimal surface Σ parametrised in terms of asymptotic coordinates gives rise to another (conjugate or adjoint) minimal surface $\bar{\Sigma}$ which is parametrised in terms of curvature coordinates and isometric to Σ and vice versa. Indeed, it is readily seen that the compatibility condition for the pair

$$\bar{\boldsymbol{r}}_x=\boldsymbol{N}\times\boldsymbol{r}_x,\quad \bar{\boldsymbol{r}}_y=\boldsymbol{N}\times\boldsymbol{r}_y \tag{14}$$

is given by

$$\mathcal{H} \sim \boldsymbol{r}_x \times \boldsymbol{N}_y + \boldsymbol{N}_x \times \boldsymbol{r}_y = 0.$$

Now, since $\bar{\boldsymbol{N}} = \boldsymbol{N}$ and the Lelieuvre formulae (2) lead to

$$\bar{\boldsymbol{r}}_x = \rho^2 \boldsymbol{N}_x, \quad \bar{\boldsymbol{r}}_y = -\rho^2 \boldsymbol{N}_y, \tag{15}$$

the orthogonality property (13) yields

$$\bar{\boldsymbol{r}}_x \cdot \bar{\boldsymbol{r}}_y = 0, \quad \bar{\boldsymbol{r}}_{xy} \cdot \bar{\boldsymbol{N}} = 0$$

so that, with respect to the surface $\bar{\Sigma}$, the coordinates (x, y) are both orthogonal and conjugate and therefore constitute curvature coordinates [6]. Moreover,

$$\bar{\mathcal{H}} \sim \bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{N}}_y + \bar{\boldsymbol{N}}_x \times \bar{\boldsymbol{r}}_y = 0.$$

Finally, the defining relations (14) imply that

$$(d\bar{\boldsymbol{r}})^2 = (d\boldsymbol{r})^2$$

which confirms that the surfaces Σ and $\bar{\Sigma}$ are isometric. Hence, in particular, the Gaussian curvatures of the two minimal surfaces Σ and $\bar{\Sigma}$ coincide. The converse of Bonnet's observation may be shown in a similar manner.

3. Discrete Gaussian and mean curvatures

Here, we investigate to which extent the theory and examples presented in the previous section may be translated into the discrete language. To this end, we consider the standard discretisation of discrete asymptotic nets which has been studied and applied extensively in discrete differential geometry [1]. Thus, a discrete asymptotic net (asymptotic lattice) is a map

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3, \quad (n_1, n_2) \mapsto \boldsymbol{r}(n_1, n_2)$$

such that the stars of the lattice are planar, that is, any vertex $\boldsymbol{r}$ and its four neighbours $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_{\bar{1}}$, $\boldsymbol{r}_{\bar{2}}$ are coplanar. Here, and in the following, we suppress arguments of functions and denote unit increments and decrements of the independent variables n_k by subscripts k and $\bar{k}$ respectively so that, for instance,

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}(n_1, n_2), & \boldsymbol{r}_1 &= \boldsymbol{r}(n_1 + 1, n_2) \\ \boldsymbol{r}_{\bar{2}} &= \boldsymbol{r}(n_1, n_2 - 1), & \boldsymbol{r}_{12} &= \boldsymbol{r}(n_1 + 1, n_2 + 1). \end{aligned}$$

If $\boldsymbol{N}$ denotes a unit normal to the "tangent" plane which contains the vertex $\boldsymbol{r}$ and its four neighbours then the asymptotic net condition may be expressed as

$$\Delta_1 \boldsymbol{r} \sim \boldsymbol{N}_1 \times \boldsymbol{N}, \quad \Delta_2 \boldsymbol{r} \sim \boldsymbol{N} \times \boldsymbol{N}_2,$$

where the difference operator Δ_k is defined by $\Delta_k f = f_k - f$. As in the continuous case, it may be shown [24] that there exists a scaled normal

$$\mathcal{V} = \rho \boldsymbol{N}$$

which satisfies the discrete Lelieuvre formulae [9,10]

$$\Delta_1 \boldsymbol{r} = \mathcal{V}_1 \times \mathcal{V}, \quad \Delta_2 \boldsymbol{r} = \mathcal{V} \times \mathcal{V}_2. \tag{16}$$

Furthermore, the compatibility condition $\Delta_2 \Delta_1 \boldsymbol{r} = \Delta_1 \Delta_2 \boldsymbol{r}$ is readily shown to result in $(\mathcal{V}_{12} + \mathcal{V}) \times (\mathcal{V}_1 + \mathcal{V}_2) = \boldsymbol{0}$ which is equivalent to the discrete Moutard equation

$$\mathcal{V}_{12} + \mathcal{V} = H(\mathcal{V}_1 + \mathcal{V}_2) \tag{17}$$

for some function H. Conversely, any solution $(\mathcal{V}, H)$ of the Moutard equation (17) gives rise to a discrete asymptotic net $\boldsymbol{r}$ via the discrete Lelieuvre formulae (16).

Steiner's formula has also proven to be a useful tool in the context of discrete curvature theory [3–5]. Here, our approach is of a different mixed geometric and algebraic nature and its connection with, for instance, the theory of so-called edge-constraint nets [5] is currently being investigated. As pointed out at the beginning of Section 2, the definition (1) of the Gaussian and mean curvatures is meaningful since the numerators and denominators of $\mathcal{K}$ and $\mathcal{H}$ are proportional to the unit normal $\boldsymbol{N}$. If we are concerned with discrete analogues of these curvatures which are associated with the quadrilaterals of a discrete asymptotic net then it is natural to consider a normal $\mathcal{N}$ which is defined on the dual lattice, that is, a map

$$\mathcal{N} : (\mathbb{Z}^2)^* \to \mathbb{R}^3.$$

In the current context, a canonical choice for such a normal is given by the cross-product of the diagonals of the quadrilaterals, that is,

$$\mathcal{N} \sim (\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1).$$

In fact, the discrete Lelieuvre formulae (16) and the discrete Moutard equation (17) reveal that

$$\mathcal{N} \sim \mathcal{V}_1 + \mathcal{V}_2 \sim \mathcal{V}_{12} + \mathcal{V} \tag{18}$$

which provides a first indication of the geometric significance of the magnitude of the discrete Lelieuvre normal $\mathcal{V}$. If we introduce lattice parameters ϵ, δ and variables $x = \epsilon n_1, y = \delta n_2$ then we may employ the Taylor expansions

$$\boldsymbol{r}_1 = \boldsymbol{r} + \epsilon \boldsymbol{r}_x + \cdots, \quad \boldsymbol{r}_2 = \boldsymbol{r} + \delta \boldsymbol{r}_y + \cdots, \quad \boldsymbol{r}_{12} = \boldsymbol{r} + \epsilon \boldsymbol{r}_x + \delta \boldsymbol{r}_y + \epsilon\delta \boldsymbol{r}_{xy} + \cdots$$

in the formal continuum limit $\epsilon, \delta \to 0$. Accordingly,

$$\begin{aligned}(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1) &= (\epsilon \boldsymbol{r}_x + \delta \boldsymbol{r}_y) \times (\delta \boldsymbol{r}_y - \epsilon \boldsymbol{r}_x) + \cdots \\ &= 2\epsilon\delta \boldsymbol{r}_x \times \boldsymbol{r}_y + \cdots.\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}(\boldsymbol{N}_{12} \times \boldsymbol{N}) \times (\boldsymbol{N}_2 \times \boldsymbol{N}_1) &= [(\epsilon \boldsymbol{N}_x + \delta \boldsymbol{N}_y) \times \boldsymbol{N}] \times [(\delta \boldsymbol{N}_y - \epsilon \boldsymbol{N}_x) \times \boldsymbol{N}] + \cdots \\ &= 2\epsilon\delta \boldsymbol{N}_x \times \boldsymbol{N}_y + \cdots.\end{aligned}$$

The left-hand side of the above expression is proportional to $\mathcal{N}$ since the diagonals $\boldsymbol{r}_2 - \boldsymbol{r}_1$ and $\boldsymbol{r}_{12} - \boldsymbol{r}$ may be regarded as the lines of intersection of the tangent planes at $\boldsymbol{r}$ and $\boldsymbol{r}_{12}$ and at $\boldsymbol{r}_2$ and $\boldsymbol{r}_1$ respectively so that

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 \parallel \boldsymbol{N}_{12} \times \boldsymbol{N}, \quad \boldsymbol{r}_{12} - \boldsymbol{r} \parallel \boldsymbol{N}_2 \times \boldsymbol{N}_1$$

as may also be deduced from the discrete Lelieuvre formulae. Accordingly, we adopt the definition

$$\mathcal{K} = \frac{(\boldsymbol{N}_{12} \times \boldsymbol{N}) \times (\boldsymbol{N}_2 \times \boldsymbol{N}_1)}{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)} \tag{19}$$

for the Gaussian curvature of a discrete asymptotic net.

In order to define a discrete analogue of the mean curvature $(1)_2$, we first note that

$$\boldsymbol{r}_x \times \boldsymbol{N}_y + \boldsymbol{N}_x \times \boldsymbol{r}_y = (\boldsymbol{r}_x \times \boldsymbol{N})_y + (\boldsymbol{N} \times \boldsymbol{r}_y)_x.$$

A natural discrete version of the latter is given by

$$\Delta_2(\Delta_1 \boldsymbol{r} \times \boldsymbol{N}^{(1)}) + \Delta_1(\boldsymbol{N}^{(2)} \times \Delta_2 \boldsymbol{r}),$$

where, for combinatorial reasons, the quantities $\boldsymbol{N}^{(k)}$ must be normals which are defined on the edges $\Delta_k \boldsymbol{r}$. In fact, if we make the choice

$$\boldsymbol{N}^{(k)} = \frac{\boldsymbol{N}_k + \boldsymbol{N}}{1 + \boldsymbol{N}_k \cdot \boldsymbol{N}} \tag{20}$$

then $\boldsymbol{N}^{(k)}$ is orthogonal to the edge $\Delta_k \boldsymbol{r}$ and is scaled in such a manner that it becomes the unit normal $\boldsymbol{N}$ in the formal continuum limit. Hence, we define the mean curvature of a discrete asymptotic net to be

$$\mathcal{H} = -\frac{\Delta_2(\Delta_1 \boldsymbol{r} \times \boldsymbol{N}^{(1)}) + \Delta_1(\boldsymbol{N}^{(2)} \times \Delta_2 \boldsymbol{r})}{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}. \tag{21}$$

In order for this to be meaningful, we now demonstrate that, once again, the numerator of $\mathcal{H}$ is proportional to $\mathcal{N}$. Thus, the discrete Lelieuvre formulae (16) may be used to formulate the discrete mean curvature as

$$\mathcal{H} = \frac{\Delta_2(\rho_1 \rho \Delta_1 \boldsymbol{N}) + \Delta_1(\rho_2 \rho \Delta_2 \boldsymbol{N})}{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)} \tag{22}$$

which is the discrete analogue of the representation

$$\mathcal{H} = \frac{(\rho^2 \boldsymbol{N}_x)_y + (\rho^2 \boldsymbol{N}_y)_x}{2(\boldsymbol{r}_x \times \boldsymbol{r}_y)}$$

of the classical mean curvature. In terms of the discrete Lelieuvre normal $\mathcal{V}$, the discrete mean curvature may therefore be cast into the form

$$\begin{aligned}\mathcal{H} &= \frac{(\rho_1 + \rho_2)(\mathcal{V}_{12} + \mathcal{V}) - (\rho_{12} + \rho)(\mathcal{V}_1 + \mathcal{V}_2)}{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)} \\ &= \frac{[H(\rho_1 + \rho_2) - (\rho_{12} + \rho)](\mathcal{V}_1 + \mathcal{V}_2)}{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}\end{aligned} \tag{23}$$

as required. Here, we have made use of the discrete Moutard equation (17). It is observed that the above variants of the discrete mean curvature constitute natural discretisations of the formulae (5).

Evaluation of the discrete Gaussian curvature (19) by means of the discrete Lelieuvre formulae (16) now results in the representation

$$\mathcal{K} = -\frac{1}{\rho\rho_1\rho_2\rho_{12}} = -\frac{\operatorname{sgn}(\rho\rho_1\rho_2\rho_{12})}{|\mathcal{V}||\mathcal{V}_1\|\mathcal{V}_2\|\mathcal{V}_{12}|} \tag{24}$$

which constitutes a natural discretisation of the expression (4) for the classical Gaussian curvature. One may regard the above discrete Gaussian curvature as the geometric mean of a Gaussian curvature defined on the vertices of a discrete asymptotic net, namely

$$\mathcal{K}^v = -\frac{1}{\rho^4} = -\frac{1}{|\mathcal{V}|^4} \tag{25}$$

which coincides with the definition of the Gaussian curvature for semi-discrete asymptotic nets presented in [25]. However, it is emphasised that the "vertex Gaussian curvature" $\mathcal{K}^v$ depends on the choice of the discrete Lelieuvre normal $\mathcal{V}$. Indeed, in the discrete case, the Lelieuvre normal is only defined up to a scaling by a constant α and its reciprocal $1/\alpha$ of the Lelieuvre normals associated with the "even" and "odd" sublattices

$$D_2^e = \{(n_1, n_2) : n_1 + n_2 \text{ even}\}, \quad D_2^o = \{(n_1, n_2) : n_1 + n_2 \text{ odd}\} \tag{26}$$

respectively so that $\rho \to \alpha\rho$ on D_2^e and $\rho \to \rho/\alpha$ on D_2^o. Hence, the discrete Lelieuvre normal is uniquely determined by prescribing the magnitude of $\mathcal{V}$ or, more precisely, ρ, at one vertex. This degree of freedom does not affect the Gaussian and mean curvatures defined on the quadrilaterals, which is geometrically evident from the definitions (19) and (21) or seen algebraically by inspecting the representations (22) and (24), but it gives rise to a uniform scaling of the vertex Gaussian curvature on the even and odd sublattices. It is observed in passing that the standard definition of the Gaussian curvature of "circular nets", which discretise curvature coordinates on surfaces, likewise depends on the choice of a normal at one vertex [2,3].

3.1. Discrete pseudospherical surfaces

The compatibility of the discrete theory developed in the preceding with the standard discrete models of the classes of (classical) surfaces discussed in Section 2 will now be examined. We begin with the discretisation of pseudospherical surfaces as independently proposed by Sauer [26] and Wunderlich [27] and investigated in detail by Bobenko and Pinkall [28] in the context of integrable systems theory. To this end, it is recalled that a hyperbolic surface Σ is pseudospherical if and only if the asymptotic lines on Σ form a Chebyshev net [6]. Thus, discrete pseudospherical surfaces are discrete asymptotic nets which possess the discrete Chebyshev property. By definition, the quadrilaterals of a discrete Chebyshev net constitute skew parallelograms, that is, opposite edges of any quadrilateral are of the same length so that

$$\Delta_2(\Delta_1\boldsymbol{r})^2 = 0, \quad \Delta_1(\Delta_2\boldsymbol{r})^2 = 0. \tag{27}$$

By symmetry, this implies that the angle between adjacent normals is the same for opposite edges of any quadrilateral so that the orientation of the unit normal $\boldsymbol{N}$ may be chosen in such a manner that

$$\Delta_2(\boldsymbol{N}_1 \cdot \boldsymbol{N}) = 0, \quad \Delta_1(\boldsymbol{N}_2 \cdot \boldsymbol{N}) = 0. \tag{28}$$

On use of the discrete Lelieuvre formulae (16), the conditions (27) therefore simplify to

$$\mathcal{V}_1^2 = \mathcal{V}_2^2, \quad \mathcal{V}_{12}^2 = \mathcal{V}^2. \tag{29}$$

If we now take the "square" of the discrete Moutard equation in the form

$$\mathcal{V}_{12} = -\mathcal{V} + H(\mathcal{V}_1 + \mathcal{V}_2) \tag{30}$$

then relation $(29)_2$ leads to

$$H = 2\frac{\mathcal{V} \cdot (\mathcal{V}_1 + \mathcal{V}_2)}{(\mathcal{V}_1 + \mathcal{V}_2)^2}. \tag{31}$$

Hence, the inner products of (30) with $\mathcal{V}_1$ and $\mathcal{V}_2$ reduce to

$$\Delta_2(\mathcal{V}_1 \cdot \mathcal{V}) = 0, \quad \Delta_1(\mathcal{V}_2 \cdot \mathcal{V}) = 0.$$

The latter combined with (28) result in the two conditions

$$\Delta_2(\rho_1\rho) = 0, \quad \Delta_1(\rho_2\rho) = 0 \tag{32}$$

which are equivalent since $\rho_1^2 = \rho_2^2$ and $\rho_{12}^2 = \rho^2$. As a consequence, the Gaussian curvature (24) of discrete pseudospherical surfaces is constant.

The conditions (29) and (32) show that $\rho_1 = \sigma\rho_2$ and $\rho_{12} = \sigma\rho$, where $[\sigma(n_1, n_2)]^2 = 1$. Since the orientation of the unit normal $\boldsymbol{N}$ is not uniquely determined by the geometric conditions (28), we can adjust the orientation in such a manner that $\sigma = 1$. Hence, we are left with

$$\rho_1 = \rho_2, \quad \rho_{12} = \rho$$

which means that ρ is constant on the even and odd sublattices. Accordingly, the discrete Moutard equation combined with the expression (31) for the coefficient H becomes the integrable discrete nonlinear sigma model

$$\boldsymbol{N}_{12} + \boldsymbol{N} = \frac{\boldsymbol{N}\cdot(\boldsymbol{N}_1 + \boldsymbol{N}_2)}{1 + \boldsymbol{N}_1\cdot\boldsymbol{N}_2}(\boldsymbol{N}_1 + \boldsymbol{N}_2), \quad \boldsymbol{N}^2 = 1$$

which is known to encode the "Gauss map" of discrete pseudospherical surfaces (see, e.g., [29,30] and references therein). Furthermore, the above-mentioned gauge freedom of the discrete Lelieuvre normal may be exploited to scale $\mathcal{V}$ in such a manner that ρ is constant on the entire lattice so that the associated vertex Gaussian curvature $\mathcal{K}^v$ is likewise constant.

3.2. Discrete affine spheres

A canonical discrete model of affine spheres has been proposed in [31] and its integrability has been demonstrated by relating it to the integrable discrete Tzitzéica equation set down in [32]. Subsequently, in [10], it was shown that discrete affine spheres Σ and their polar-reciprocals Σ^* with position vectors $\boldsymbol{r}$ and $\mathcal{V}$ respectively are essentially encoded in the pair of discrete Lelieuvre formulae

$$\begin{aligned} \Delta_1\boldsymbol{r} &= \mathcal{V}_1 \times \mathcal{V}, \qquad \Delta_2\boldsymbol{r} = \mathcal{V} \times \mathcal{V}_2 \\ \Delta_1\mathcal{V} &= \boldsymbol{r} \times \boldsymbol{r}_1 \qquad \Delta_2\mathcal{V} = \boldsymbol{r}_2 \times \boldsymbol{r}. \end{aligned} \tag{33}$$

As in the continuous setting, it is readily verified that the above pair is equivalent to the discrete Lelieuvre formulae $(33)_{1,2}$ subject to the constraint

$$\boldsymbol{r}\cdot\mathcal{V} = 1.$$

It is therefore natural to refer to

$$D = \boldsymbol{r}\cdot\mathcal{V}$$

as the affine distance between the vertex $\boldsymbol{r}$ of a discrete asymptotic net and the origin. Hence, as in the classical case, discrete affine spheres are characterised by the affine distance D being constant. In (33), the position vector $\boldsymbol{r}$ has been scaled so that $D = 1$. As in the continuous case, the associated discrete Tzitzéica system [32] admits the compact "tri-linear" form

$$\begin{vmatrix} \tau & \tau_1 & \tau_{11} \\ \tau_2 & \tau_{12} & \tau_{112} \\ \tau_{22} & \tau_{122} & \tau_{1122} \end{vmatrix} + c\tau_{12}^3 = 0$$

which constitutes a natural discretisation of (9).

It is emphasised that the affine distance is not invariant with respect to the scaling $\mathcal{V} \to \alpha\mathcal{V}$ on the even sublattice and $\mathcal{V} \to \mathcal{V}/\alpha$ on the odd sublattice. However, the natural discrete analogue

$$D^{(4)} = -\frac{dd_1d_2d_{12}}{\mathcal{K}}, \quad d = \boldsymbol{r}\cdot\boldsymbol{N}$$

of the quantity (7) considered by Tzitzéica in the classical setting is independent of the choice of the discrete Lelieuvre normal since

$$D^{(4)} = DD_1D_2D_{12}$$

and $D \to \alpha D$ on the even sublattice and $D \to D/\alpha$ on the odd sublattice. The sense in which D and $D^{(4)}$ may be considered affine quantities may be revealed by setting

$$\mathcal{V} = \frac{\Delta_1\boldsymbol{r} \times \Delta_2\boldsymbol{r}}{f}$$

and evaluating the discrete Lelieuvre formulae (16). We obtain

$$f_1 f = |\Delta_1\boldsymbol{r}_1, \Delta_2\boldsymbol{r}_1, \Delta_2\boldsymbol{r}|, \quad f_2 f = |\Delta_1\boldsymbol{r}_2, \Delta_2\boldsymbol{r}_2, \Delta_1\boldsymbol{r}|,$$

where we have exploited the fact that the stars of discrete asymptotic nets are planar, so that

$$D_1D = \frac{|\Delta_1\boldsymbol{r}_1, \Delta_2\boldsymbol{r}_1, \boldsymbol{r}_1||\Delta_1\boldsymbol{r}, \Delta_2\boldsymbol{r}, \boldsymbol{r}|}{|\Delta_1\boldsymbol{r}_1, \Delta_2\boldsymbol{r}_1, \Delta_2\boldsymbol{r}|} \qquad D_2D = \frac{|\Delta_1\boldsymbol{r}_2, \Delta_2\boldsymbol{r}_2, \boldsymbol{r}_2||\Delta_1\boldsymbol{r}, \Delta_2\boldsymbol{r}, \boldsymbol{r}|}{|\Delta_1\boldsymbol{r}_2, \Delta_2\boldsymbol{r}_2, \Delta_1\boldsymbol{r}|}.$$

Hence, under a centro-affine transformation $\boldsymbol{r} \rightarrow A\boldsymbol{r}$, where A is a non-singular constant matrix, the quantities D_kD transform according to

$$D_kD \rightarrow (\det A)D_kD. \tag{34}$$

Thus, up to a constant multiple, these quantities constitute centro-affine invariants and so does $D^{(4)}$ since

$$D^{(4)} \rightarrow (\det A)^2 D^{(4)}$$

in analogy with Tzitzéica's discovery in the classical case. Moreover, the transformation formulae (34) imply that the affine distance D is a centro-affine invariant in the same sense modulo the non-uniqueness of the discrete Lelieuvre normal $\mathcal{V}$.

3.3. Discrete generalised affine spheres

As in the continuous case, the discrete Lelieuvre normal $\mathcal{V}$ may be scaled in such a manner that the scaled Lelieuvre normal $\mathcal{V}^*$ may be interpreted as the position vector of another discrete asymptotic net Σ^*. Indeed, for instance, taking the cross-product of the Lelieuvre equation $(16)_1$ with $\boldsymbol{r}$ leads to

$$\boldsymbol{r}_1 \times \boldsymbol{r} = (\boldsymbol{r} \cdot \mathcal{V}_1)\mathcal{V} - (\boldsymbol{r} \cdot \mathcal{V})\mathcal{V}_1 = (\boldsymbol{r}_1 \cdot \mathcal{V}_1)\mathcal{V} - (\boldsymbol{r} \cdot \mathcal{V})\mathcal{V}_1$$

by virtue of $\mathcal{V}_1 \cdot \Delta_1\boldsymbol{r} = 0$. Thus, discrete asymptotic nets Σ are encoded in the pairs

$$\begin{aligned} \Delta_1\boldsymbol{r} &= D_1D\mathcal{V}_1^* \times \mathcal{V}^*, \qquad \Delta_2\boldsymbol{r} = D_2D\mathcal{V}^* \times \mathcal{V}_2^* \\ \Delta_1\mathcal{V}^* &= \frac{1}{D_1D}\boldsymbol{r} \times \boldsymbol{r}_1, \qquad \Delta_2\mathcal{V}^* = \frac{1}{D_2D}\boldsymbol{r}_2 \times \boldsymbol{r}, \end{aligned} \tag{35}$$

where

$$\mathcal{V}^* = \frac{\mathcal{V}}{D}.$$

Indeed, if one sets aside the genesis of the above relations then it is readily verified that $\boldsymbol{r} \cdot \mathcal{V}^* = 1$ is implied and, hence, D may be interpreted as the affine distance associated with $\boldsymbol{r}$, where $\mathcal{V} = D\mathcal{V}^*$ constitutes a corresponding discrete Lelieuvre normal. Furthermore,

$$\boldsymbol{r}^* = \frac{\boldsymbol{r}}{D}$$

is a discrete Lelieuvre normal for the discrete asymptotic net Σ^* represented by $\mathcal{V}^*$ and the associated affine distance is given by

$$D^* = \boldsymbol{r}^* \cdot \mathcal{V}^* = \frac{1}{D}.$$

In order to make contact with the discrete generalised affine spheres proposed in [10,23], we now derive the "Gauss–Weingarten equations" for discrete asymptotic nets in the setting of centro-affine geometry. We first observe that the planarity of stars of a discrete net Σ may be expressed as

$$\begin{aligned} \Delta_1\boldsymbol{r}_1 &= a^{11}\Delta_1\boldsymbol{r} + a^{12}\Delta_2\boldsymbol{r}_1 \\ \Delta_2\boldsymbol{r}_2 &= a^{22}\Delta_2\boldsymbol{r} + a^{21}\Delta_1\boldsymbol{r}_2 \end{aligned} \tag{36}$$

for some functions a^{ik}. Moreover, the discrete Lelieuvre formulae $(35)_{3,4}$ for the pair $(\mathcal{V}^*, \boldsymbol{r}^*)$ imply via compatibility that $\boldsymbol{r}^*$ obeys a discrete Moutard equation so that

$$\frac{\boldsymbol{r}_{12}}{D_{12}} + \frac{\boldsymbol{r}}{D} = H\left(\frac{\boldsymbol{r}_1}{D_1} + \frac{\boldsymbol{r}_2}{D_2}\right) \tag{37}$$

for another function H. The compatibility conditions $(\boldsymbol{r}_{11})_2 = (\boldsymbol{r}_{12})_1$ and $(\boldsymbol{r}_{22})_1 = (\boldsymbol{r}_{12})_2$ associated with the discrete Gauss–Weingarten equations (36) and (37) then lead to discrete "Gauss–Mainardi–Codazzi equations" for the functions a^{ik}, H and D. Discrete generalised affine spheres have originally been generated by iterative application of the Tzitzéica transformation to classical affine spheres [31] and may be characterised by a scaled discrete Moutard equation of the form

$$\boldsymbol{r}_{12} + b^{(0)}\boldsymbol{r} = \hat{H}(b^{(2)}\boldsymbol{r}_1 + b^{(1)}\boldsymbol{r}_2),$$

where

$$b^{(1)} = b^{(1)}(n_1), \quad b^{(2)} = b^{(2)}(n_2), \quad b^{(0)} = b^{(1)}b^{(2)}.$$

Comparison with the scaled discrete Moutard equation (37) shows that a surface Σ constitutes a discrete generalised affine sphere if and only if the affine distance D is constrained by

$$\frac{D_2}{D_1} = \frac{b^{(2)}}{b^{(1)}}, \quad \frac{D_{12}}{D} = b^{(0)}.$$

If we parametrise $b^{(1)}$ and $b^{(2)}$ by

$$b^{(1)} = \frac{\beta_1^{(1)}}{\beta^{(1)}}, \quad b^{(2)} = \frac{\beta_2^{(2)}}{\beta^{(2)}} \tag{38}$$

and set

$$\mathcal{D} = \beta^{(1)}(n_1)\beta^{(2)}(n_2) \tag{39}$$

then we obtain

$$\frac{D_2}{\mathcal{D}_2} = \frac{D_1}{\mathcal{D}_1}, \quad \frac{D_{12}}{\mathcal{D}_{12}} = \frac{D}{\mathcal{D}}$$

so that

$$D = \mathcal{D}$$

by virtue of the gauge freedom of the discrete Lelieuvre normal $\mathcal{V}$ and the fact that the functions $\beta^{(1)}$ and $\beta^{(2)}$ in (38) are only defined up to constant multiples. Accordingly, a discrete asymptotic net constitutes a discrete generalised affine sphere if there exists a discrete Lelieuvre normal $\mathcal{V}$ such that the associated affine distance D factorises as in (39), that is,

$$\frac{D_{12}D}{D_1D_2} = 1. \tag{40}$$

The latter constitutes the natural analogue of the condition (11) characterising generalised affine spheres since

$$\frac{\Delta_{12}D}{D} - \frac{(\Delta_1 D)(\Delta_2 D)}{D^2} = 0.$$

It is observed that the discrete "harmonicity condition" (40) is also satisfied for the dual affine distance $D^* = 1/D$ and, hence, as in the classical case, discrete generalised affine spheres come in pairs $(\boldsymbol{r}, \mathcal{V}^*)$. A connection between continuous and discrete (generalised) affine spheres and self-dual Einstein spaces via the underlying integrable systems has been established in [32,33].

3.4. Discrete minimal surfaces

According to the representation (23) of the discrete mean curvature $\mathcal{H}$, discrete asymptotic nets are minimal ($\mathcal{H} = 0$) if and only if the function ρ constitutes a scalar solution of the discrete Moutard equation (17), that is,

$$\rho_{12} + \rho = H(\rho_1 + \rho_2). \tag{41}$$

Hence, as in the continuous case, if we set $\mathsf{V} = (\mathcal{V}, \rho) \in \mathbb{R}^{3,1}$ then it is seen that discrete minimal surfaces are governed by the integrable discrete nonlinear sigma model

$$\mathsf{V}_{12} + \mathsf{V} = H(\mathsf{V}_1 + \mathsf{V}_2), \quad \langle \mathsf{V}, \mathsf{V} \rangle = 0.$$

In view of Bonnet's classical observation discussed in Section 2.4, we now consider pairs of discrete nets $(\Sigma, \bar{\Sigma})$, the position vectors of which are related by the natural analogue of (14), that is,

$$\Delta_1 \bar{\boldsymbol{r}} = \boldsymbol{N}^{(1)} \times \Delta_1 \boldsymbol{r}, \quad \Delta_2 \bar{\boldsymbol{r}} = \boldsymbol{N}^{(2)} \times \Delta_2 \boldsymbol{r}, \tag{42}$$

where the edge normals $\boldsymbol{N}^{(k)}$ are defined by (20). Evaluation of the above relations by virtue of the discrete Lelieuvre formulae (16) yields the discrete analogue

$$\Delta_1 \bar{\boldsymbol{r}} = \rho_1 \rho \Delta_1 \boldsymbol{N}, \quad \Delta_2 \bar{\boldsymbol{r}} = -\rho_2 \rho \Delta_2 \boldsymbol{N} \tag{43}$$

of (15) or, equivalently,

$$\Delta_1 \bar{\boldsymbol{r}} = \rho \mathcal{V}_1 - \rho_1 \mathcal{V}, \quad \Delta_2 \bar{\boldsymbol{r}} = \rho_2 \mathcal{V} - \rho \mathcal{V}_2. \tag{44}$$

The latter pair is essentially of the same algebraic form as the discrete Lelieuvre formulae and, hence, ρ must satisfy the same discrete Moutard equation as $\mathcal{V}$. In fact, by definition (21) of the discrete mean curvature,

$$\Delta_2(\Delta_1 \bar{\boldsymbol{r}}) - \Delta_1(\Delta_2 \bar{\boldsymbol{r}}) = \mathcal{H}[(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)]$$

which indeed shows that the discrete net $\bar{\Sigma}$ exists if and only if the discrete asymptotic net Σ is minimal.

In order to examine the geometric properties of the discrete net $\bar{\Sigma}$, we first observe that the discrete net $\Sigma_\circ$ on the unit sphere defined by $\boldsymbol{N}$ is composed of planar quadrilaterals since combination of the discrete Moutard equations (17) and (41) produces the discrete Darboux equation

$$\rho_{12}(\boldsymbol{N}_{12} - \boldsymbol{N}) = H[\rho_1(\boldsymbol{N}_1 - \boldsymbol{N}) + \rho_2(\boldsymbol{N}_2 - \boldsymbol{N})].$$

Thus, by definition, $\Sigma_\circ$ constitutes a discrete conjugate net [1]. In fact, it is easy to see that a discrete asymptotic net Σ is minimal if and only if the Gauss map $\Sigma_\circ$ forms a discrete conjugate net. Furthermore, the relations (43) show that corresponding edges of the discrete nets $\bar\Sigma$ and $\Sigma_\circ$ are parallel. Accordingly, $\bar\Sigma$ constitutes a discrete Combescure transform [34] of the discrete conjugate net $\Sigma_\circ$. Since the discrete Combescure transformation evidently preserves angles between edges, the quadrilaterals of $\bar\Sigma$ are likewise inscribed in circles so that, by definition, $\bar\Sigma$ turns out to be a discrete curvature net (circular net) [1] and $\bar{\boldsymbol N} = \boldsymbol N$ is an associated vertex normal. Finally, the mapping between $\boldsymbol r$ and $\bar{\boldsymbol r}$ given by (42) may be inverted by taking the cross-product with the edge normals $\boldsymbol N^{(k)}$ to obtain

$$\Delta_1 \boldsymbol r = \Delta_1 \bar{\boldsymbol r} \times \frac{\bar{\boldsymbol N}_1 + \bar{\boldsymbol N}}{2}, \quad \Delta_2 \boldsymbol r = \Delta_2 \bar{\boldsymbol r} \times \frac{\bar{\boldsymbol N}_2 + \bar{\boldsymbol N}}{2}. \tag{45}$$

The associated compatibility condition reads

$$0 = \Delta_1(\Delta_2 \boldsymbol r) - \Delta_2(\Delta_1 \boldsymbol r) = \frac{(\bar{\boldsymbol r}_{12} - \bar{\boldsymbol r}) \times (\bar{\boldsymbol N}_2 - \bar{\boldsymbol N}_1) + (\bar{\boldsymbol N}_{12} - \bar{\boldsymbol N}) \times (\bar{\boldsymbol r}_2 - \bar{\boldsymbol r}_1)}{2}$$

which reveals that the discrete net $\bar\Sigma$ is minimal since the mean curvature of a discrete curvature net is given by [2,3]

$$\bar{\mathcal H} = -\frac{(\bar{\boldsymbol r}_{12} - \bar{\boldsymbol r}) \times (\bar{\boldsymbol N}_2 - \bar{\boldsymbol N}_1) + (\bar{\boldsymbol N}_{12} - \bar{\boldsymbol N}) \times (\bar{\boldsymbol r}_2 - \bar{\boldsymbol r}_1)}{2(\bar{\boldsymbol r}_{12} - \bar{\boldsymbol r}) \times (\bar{\boldsymbol r}_2 - \bar{\boldsymbol r}_1)}.$$

It is recalled that the Gaussian and mean curvatures of a discrete curvature net $\bar\Sigma$ may be obtained by considering Steiner's formula for parallel discrete nets $\bar\Sigma^\parallel$ defined by $\bar{\boldsymbol r}^\parallel = \bar{\boldsymbol r} + \mu \bar{\boldsymbol N}$, namely [3]

$$\frac{(\bar{\boldsymbol r}^\parallel_{12} - \bar{\boldsymbol r}^\parallel) \times (\bar{\boldsymbol r}^\parallel_2 - \bar{\boldsymbol r}^\parallel_1)}{(\bar{\boldsymbol r}_{12} - \bar{\boldsymbol r}) \times (\bar{\boldsymbol r}_2 - \bar{\boldsymbol r}_1)} = 1 - 2\mu \bar{\mathcal H} + \mu^2 \bar{\mathcal K}. \tag{46}$$

Here, $\bar{\boldsymbol N}$ represents a discrete Combescure transform $\bar\Sigma_\circ$ of $\bar\Sigma$ on the unit sphere. Since the above line of arguments may be reversed in that any minimal discrete curvature net gives rise to a minimal discrete asymptotic net via (45), the existence of conjugate pairs of minimal surfaces in the discrete setting has been established.

It turns out that the above pairs of discrete surfaces are also minimal with respect to the definition of mean curvature in the theory of edge-constraint nets [5]. However, in the current setting, one has the additional feature that, as in the continuous case, the Gaussian curvatures of the conjugate discrete minimal surfaces coincide. To make good this assertion, we first recall that the Gaussian curvature of a discrete curvature net $\bar\Sigma$ is given by (cf. (46))

$$\bar{\mathcal K} = \frac{(\bar{\boldsymbol N}_{12} - \bar{\boldsymbol N}) \times (\bar{\boldsymbol N}_2 - \bar{\boldsymbol N}_1)}{(\bar{\boldsymbol r}_{12} - \bar{\boldsymbol r}) \times (\bar{\boldsymbol r}_2 - \bar{\boldsymbol r}_1)}.$$

In the case of the above conjugate discrete minimal surfaces, the diagonals of the quadrilaterals of $\bar\Sigma$ are reciprocal-parallel to the diagonals of the corresponding quadrilaterals of the Gauss map $\Sigma_\circ$ since the relations (44) imply that

$$\bar{\boldsymbol r}_{12} - \bar{\boldsymbol r} = H\rho_1\rho_2(\boldsymbol N_1 - \boldsymbol N_2), \quad \bar{\boldsymbol r}_2 - \bar{\boldsymbol r}_1 = \frac{\rho\rho_{12}}{H}(\boldsymbol N - \boldsymbol N_{12})$$

Hence, we conclude that

$$\bar{\mathcal K} = -\frac{1}{\rho\rho_1\rho_2\rho_{12}} = \mathcal K$$

by virtue of $\bar{\boldsymbol N} = \boldsymbol N$ as claimed.

4. (Discrete) Bianchi surfaces

In 1892, Bianchi [6,35] showed that a conjugate net on a surface $\hat\Sigma$ is isometrically deformable if and only if there exists a reciprocal-parallel asymptotic net (on a surface) Σ, the Gaussian curvature of which is constrained by

$$\left(\sqrt{-\frac{1}{\mathcal K}}\right)_{xy} = 0.$$

Surfaces Σ of the this type are known as Bianchi surfaces and may be characterised by the constraint

$$(\mathcal V^2)_{xy} = 0 \tag{47}$$

by virtue of the representation (4) of the Gaussian curvature. The above constraint may be used to determine the function h in the Moutard equation (3), leading to the generalised nonlinear sigma model

$$\mathcal V_{xy} + \frac{\mathcal V_x \cdot \mathcal V_y}{\mathcal V^2}\mathcal V = \boldsymbol 0 \tag{48}$$

which, in fact, implies the Bianchi constraint (47). The class of Bianchi surfaces is integrable and includes pseudospherical surfaces [8,36,37]. The latter correspond to classical Voss surfaces $\hat{\Sigma}$ [6] which are known to admit isometric deformations which preserve privileged geodesic conjugate nets on $\hat{\Sigma}$.

It may be shown that infinitesimal isometric deformations of conjugate nets of second order are, in fact, finite. This has been exploited in the discrete setting by defining a discrete Bianchi surface Σ as a discrete asymptotic net for which the reciprocal-parallel discrete conjugate nets admit an isometric deformation of second order [24]. It turns out that discrete Bianchi surfaces are likewise integrable and characterised by the condition [29,38]

$$\Delta_{12}[(\mathcal{V}_{12}+\mathcal{V})\cdot(\mathcal{V}_1+\mathcal{V}_2)]=0,\quad \Delta_{12}=\Delta_1\Delta_2 \tag{49}$$

which reduces to (47) in the formal continuum limit.

In order to link the constraint (49) to the vertex Gaussian curvature proposed in the preceding, we now examine the properties of the Lelieuvre normals of a discrete Bianchi surface associated with the odd and even sublattices (26). To this end, we formulate the discrete Moutard equations corresponding to four adjacent quadrilaterals as

$$\begin{aligned}\mathcal{V}_1+\mathcal{V}_2&=\frac{1}{H}(\mathcal{V}+\mathcal{V}_{12}),\qquad \mathcal{V}_{112}+\mathcal{V}_{122}=\frac{1}{H_{12}}(\mathcal{V}_{1122}+\mathcal{V}_{12})\\ \mathcal{V}_{112}+\mathcal{V}_1&=H_1(\mathcal{V}_{11}+\mathcal{V}_{12}),\qquad \mathcal{V}_{122}+\mathcal{V}_2=H_2(\mathcal{V}_{22}+\mathcal{V}_{12}).\end{aligned} \tag{50}$$

Elimination of the left-hand sides leads to a linear equation for the Lelieuvre normals on D_2^e or D_2^o only, namely

$$\Box\mathcal{V}=4\mathsf{H}\mathcal{V}_{12}, \tag{51}$$

where the operator $\Box$ and the function H are given by

$$\Box=\frac{1}{H_{12}}(T_{1122}-T_{12})+\frac{1}{H}(T_0-T_{12})-H_1(T_{11}-T_{12})-H_2(T_{22}-T_{12})$$

and

$$\mathsf{H}=\frac{1}{2}\left(H_1+H_2-\frac{1}{H_{12}}-\frac{1}{H}\right).$$

Here, the T_* denote shift operators with T_0 being the identity operator. Accordingly, the operator $\Box$ may be interpreted as a weighted discretisation of the d'Alembert operator $\partial_u^2-\partial_v^2$ with the conjugate coordinates u and v being defined by

$$x=u+v,\quad y=u-v$$

so that the linear equation (51) constitutes a discretisation of the Moutard equation (3) in the form

$$(\partial_u^2-\partial_v^2)\mathcal{V}=4h\mathcal{V}.$$

It is noted that $H\to 1$ in the formal continuum limit.

Multiplication of $(50)_1$ by $\mathcal{V}-\mathcal{V}_{12}$ and the remaining discrete Moutard equations by the analogous differences leads to

$$\begin{aligned}(\mathcal{V}_1+\mathcal{V}_2)\cdot(\mathcal{V}-\mathcal{V}_{12})&=\frac{1}{H}(|\mathcal{V}|^2-|\mathcal{V}_{12}|^2)\\ (\mathcal{V}_{112}+\mathcal{V}_{122})\cdot(\mathcal{V}_{1122}-\mathcal{V}_{12})&=\frac{1}{H_{12}}(|\mathcal{V}_{1122}|^2-|\mathcal{V}_{12}|^2)\\ (\mathcal{V}_{112}+\mathcal{V}_1)\cdot(\mathcal{V}_{11}-\mathcal{V}_{12})&=H_1(|\mathcal{V}_{11}|^2-|\mathcal{V}_{12}|^2)\\ (\mathcal{V}_{122}+\mathcal{V}_2)\cdot(\mathcal{V}_{22}-\mathcal{V}_{12})&=H_2(|\mathcal{V}_{22}|^2-|\mathcal{V}_{12}|^2).\end{aligned}$$

Comparison with the condition (49) for discrete Bianchi surfaces shows that, once again, the left-hand sides may be eliminated to obtain

$$\Box\mathcal{V}^2=0 \tag{52}$$

which constitutes a discrete version of the Bianchi constraint (47) in the form

$$(\partial_u^2-\partial_v^2)\mathcal{V}^2=0.$$

In fact, in terms of the vertex Gaussian curvature (25), this may be expressed as

$$\Box\left(\sqrt{-\frac{1}{\mathcal{K}^v}}\right)=0$$

which completes the analogy with the classical case.

It is observed that, given vectors $\mathcal{V}$ on D_2^e (or D_2^o) which satisfy the discrete Moutard equation (51) for some function H, corresponding vectors $\mathcal{V}$ on D_2^o (or D_2^e) may be constructed by means of the compatible equations (50) so that $\mathcal{V}$ on the complete lattice constitutes a discrete Lelieuvre normal. In particular, the discrete Moutard equation (51) on one of

the sublattices subject to the constraint (52) encapsulates discrete Bianchi surfaces on the entire lattice. In fact, one may even interpret the pair (51), (52) as encoding a discretisation of Bianchi surfaces parametrised in terms of the conjugate coordinates u and v. Thus, given a solution of the discrete Moutard equation (51) on D_2^e, say, a lattice $\boldsymbol{r} : D_2^o \to \mathbb{R}^3$ may be defined up to a global translation by the compatible equations (cf. (16))

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = \frac{1}{H}\mathcal{V} \times \mathcal{V}_{12}, \qquad \boldsymbol{r}_{112} - \boldsymbol{r}_{122} = \frac{1}{H_{12}}\mathcal{V}_{1122} \times \mathcal{V}_{12}$$
$$\boldsymbol{r}_{112} - \boldsymbol{r}_1 = H_1\mathcal{V}_{11} \times \mathcal{V}_{12}, \qquad \boldsymbol{r}_2 - \boldsymbol{r}_{122} = H_2\mathcal{V}_{22} \times \mathcal{V}_{12}.$$

Indeed, elimination of the left-hand sides leads to the consistency condition

$$\Box\mathcal{V} \times \mathcal{V}_{12} = \boldsymbol{0}.$$

Moreover, the quadrilaterals $[\boldsymbol{r}_1, \boldsymbol{r}_{112}, \boldsymbol{r}_{122}, \boldsymbol{r}_2]$ are planar with $\mathcal{V}_{12}$ being a corresponding normal so that $\boldsymbol{r}$ represents a discrete conjugate net.

It is also worth pointing out that the discrete Moutard equation (51) formulated in terms of the "face" normal (18) normalised to

$$\mathcal{N} = \sqrt{H}(\mathcal{V}_1 + \mathcal{V}_2) = \frac{1}{\sqrt{H}}(\mathcal{V}_{12} + \mathcal{V})$$

adopts the compact form

$$\frac{1}{\sqrt{H_{12}}}\mathcal{N}_{12} + \frac{1}{\sqrt{H}}\mathcal{N} = \sqrt{H_1}\,\mathcal{N}_1 + \sqrt{H_2}\,\mathcal{N}_2 \tag{53}$$

and the discrete Bianchi constraint (49) becomes

$$\Delta_{12}\mathcal{N}^2 = 0. \tag{54}$$

Whether the quantity $\mathcal{N}^2$ gives rise to another measure of discrete Gaussian curvature is currently being investigated. It should be noted that the alternative integrable weighted discretisation (53) of the Moutard equation together with the constraint (54) have essentially been proposed in [38] in the context of discretisations of the Bianchi system and the Ernst equation of general relativity. The latter connection is revealed by considering Bianchi surfaces in a three-dimensional Minkowski space so that $\mathcal{V} \in \mathbb{R}^{2,1}$. If one then introduces the parametrisation

$$\mathcal{V} = \frac{\sqrt{\varrho}}{\mathcal{E} + \bar{\mathcal{E}}}\begin{pmatrix} i(\mathcal{E} - \bar{\mathcal{E}}) \\ \mathcal{E}\bar{\mathcal{E}} - 1 \\ \mathcal{E}\bar{\mathcal{E}} + 1 \end{pmatrix},$$

the vectorial Bianchi equation (48) reduces to the Ernst equation

$$\mathcal{E}_{xy} + \frac{1}{2}\frac{\varrho_y}{\varrho}\mathcal{E}_x + \frac{1}{2}\frac{\varrho_x}{\varrho}\mathcal{E}_y = \frac{\mathcal{E}_x\mathcal{E}_y}{\Re(\mathcal{E})}, \quad \varrho_{xy} = 0 \tag{55}$$

for colliding gravitational waves in general relativity [39]. The above discretisation procedure is, of course, not sensitive to the signature of the ambient space and therefore also applies to the hyperbolic Ernst equation (55).

5. (Discrete) torsions

There exists an intimate connection between the Gaussian curvature of a hyperbolic surface and the torsion of its asymptotic lines. The torsion τ of a curve $x \mapsto \boldsymbol{r}(x) \in \mathbb{R}^3$ measures its degree of "non-planarity" and may be expressed in terms of the change of the binormal along the curve. The binormal $\boldsymbol{b}$ is a unit vector which is orthogonal to the (osculating) plane spanned by the unit tangent $\boldsymbol{t} \sim \boldsymbol{r}_x$ and the principal normal $\boldsymbol{n} \sim \boldsymbol{t}_x$ [6]. This implies that $\boldsymbol{b}_x \times \boldsymbol{b}$ is tangent to the curve and, accordingly, the torsion of the curve may be defined by

$$\boldsymbol{b}_x \times \boldsymbol{b} = -\tau\boldsymbol{r}_x.$$

The Lelieuvre formulae (2) reveal that the binormals to the asymptotic lines on a hyperbolic surface coincide with the unit normal $\boldsymbol{N}$ and the associated torsions are

$$\tau^{(x)} = -\frac{1}{\rho^2}, \quad \tau^{(y)} = \frac{1}{\rho^2}. \tag{56}$$

Hence, we retrieve the classical relations [6]

$$\mathcal{K} = -[\tau^{(x)}]^2 = -[\tau^{(y)}]^2 = \tau^{(x)}\tau^{(y)}. \tag{57}$$

The binormal $\boldsymbol{b}$ to a discrete curve (polygon) $n_1 \mapsto \boldsymbol{r}(n_1) \in \mathbb{R}^3$ is naturally taken to be a unit normal which is orthogonal to adjacent edges of the discrete curve, that is,

$$\boldsymbol{b} = \frac{\Delta_1 \boldsymbol{r} \times \Delta_1 \boldsymbol{r}_{\bar{1}}}{|\Delta_1 \boldsymbol{r} \times \Delta_1 \boldsymbol{r}_{\bar{1}}|}.$$

Hence, as in the continuous case, the discrete torsion τ may be defined via the parallelity property

$$\boldsymbol{b}_1 \times \boldsymbol{b} = -\tau \Delta_1 \boldsymbol{r}.$$

Thus, the sine of the angle between two consecutive "osculating" planes relative to the length of the corresponding edge is a measure of the discrete torsion. This coincides with the definition of discrete torsion proposed in [27]. Comparison with the discrete Lelieuvre formulae (16) shows that the binormals to the coordinate polygons of a discrete asymptotic net coincide with the unit normal $\boldsymbol{N}$ and the associated discrete torsions are given by

$$\tau^{(1)} = -\frac{1}{\rho_1 \rho}, \quad \tau^{(2)} = \frac{1}{\rho_2 \rho}$$

in analogy with the continuous expressions (56). Accordingly, we conclude that

$$\mathcal{K} = -\tau_2^{(1)} \tau^{(1)} = -\tau_1^{(2)} \tau^{(2)}$$

which constitutes the natural symmetric discretisation of the classical relations $(57)_{1,2}$.

In conclusion, we point out that one may define a "twist" of a quadrilateral as the torsion of a zigzag curve composed of a diagonal and two adjacent edges. There exist four twists of a quadrilateral which are encoded in the relations

$$\boldsymbol{r}_{12} - \boldsymbol{r} = -\frac{\boldsymbol{N}_2 \times \boldsymbol{N}_1}{\tau^{\mathsf{Z}}} = -\frac{\boldsymbol{N}_1 \times \boldsymbol{N}_2}{\tau^{\mathsf{H}}}$$
$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = -\frac{\boldsymbol{N} \times \boldsymbol{N}_{12}}{\tau^{\mathsf{N}}} = -\frac{\boldsymbol{N}_{12} \times \boldsymbol{N}}{\tau^{\mathsf{S}}}.$$

The discrete Lelieuvre formulae and the discrete Moutard equation then imply that

$$\tau^{\mathsf{N}} = -\tau^{\mathsf{S}} = -\frac{H}{\rho \rho_{12}}, \quad \tau^{\mathsf{Z}} = -\tau^{\mathsf{H}} = \frac{1}{H \rho_1 \rho_2}$$

so that, for instance,

$$\mathcal{K} = \tau^{\mathsf{N}} \tau^{\mathsf{Z}}$$

which is reminiscent of the classical relation $(57)_3$.

Acknowledgements

This work was supported by the Australian Research Council (Grant No. DP110102186). The author is indebted to Tim Hoffmann and Emanuel Huhnen-Venedey of the DFG Collaborative Research Centre SFB/TRR 109 *Discretization in Geometry and Dynamics* for numerous enlightening discussions.

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