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Minimal representations of simple real Lie groups of non Hermitian type

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ABSTRACT

In the recent paper (Achab and Faraut, 2012), we introduced an analysis of the Brylinski–Kostant model for spherical minimal representations for simple real Lie groups of non Hermitian type. We generalize here that analysis and give a unified geometric realization to a family of unitary irreducible representations of such groups.

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0. Introduction

In [1] the following principles were established:

- Given a semisimple complex Jordan algebra V and a homogeneous polynomial Q on V, there exists a finite covering K of the conformal group $\mathrm{Conf}(V, Q)$ and a cocycle $\mu : K \times V \to \mathbb{C}$ such that the corresponding cocycle representation of K on the polynomial functions on V leaves the space $\mathcal{W}$, spanned by Q and its translates $z \mapsto Q(z - a)$ with $a \in V$, invariant, producing an irreducible representation κ of K on $\mathcal{W}$ (see Proposition 1.1 and Corollary 1.2 in [1]).

- Assuming the polynomial Q of degree 4, $\mathrm{Lie}(K) \oplus \mathcal{W}$ carries the structure of a complex simple Lie algebra $\mathfrak{g}$ (see theorem 3.1 and Theorem 3.3 in [1]). One constructs a non compact real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$, of non Hermitian type, starting with a Euclidean real form $V_{\mathbb{R}}$ of V. The construction also yields a compact real form $K_{\mathbb{R}}$ of the group K. It turns out that we obtain (see Table 1 in [1]) all the simple real Lie algebras of non Hermitian type which possess a strongly minimal real nilpotent orbit, i.e. such that $\mathcal{O}_{min} \cap \mathfrak{g}_{\mathbb{R}} \neq \emptyset$, where $\mathcal{O}_{min}$ is the minimal nilpotent adjoint orbit of $\mathfrak{g}$ (see tables 4.7 and 4.8 in [2]). Recall that condition $\mathcal{O}_{min} \cap \mathfrak{g}_{\mathbb{R}} \neq \emptyset$ is natural, since it is necessary for a simple real Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ to admit an irreducible unitary representation with associated complex nilpotent orbit $\mathcal{O}_{min}$ (see Theorem 8.4 in [3]).

In [4], we established the following:

- The K-orbit $\Xi = K.Q \subset \mathcal{W}$ carries the structure of a complex line bundle over the conformal compactification $\bar{V}$ of V. Its restriction Ξ_0 to V is trivial and the K-action on Ξ yields a left regular representation π of K on the space $\mathcal{O}(\Xi)$ of holomorphic functions on Ξ.

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- The subspaces $\mathcal{O}_m(\Xi)$ of fixed degree of homogeneity m in the fiber direction are irreducible K-modules whose elements restrict to polynomials on Ξ_0. Homogeneity in the fiber direction allows to identify $\mathcal{O}_m(\Xi)$ with a space $\tilde{\mathcal{O}}_m(V)$ of polynomials on V.

- There exists a $K_{\mathbb{R}}$-invariant inner product on $\tilde{\mathcal{O}}_m(V)$ which has a reproducing kernel H^m, where H is obtained from Q by polarization. Adding these kernels with arbitrarily chosen non-negative weights yields a multiplicity free unitary $K_{\mathbb{R}}$-representation on a Hilbert subspace of $\mathcal{O}(\Xi)$.

- The polynomials $p \in \mathcal{W}$ act on $\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m\in\mathbb{N}} \mathcal{O}_m(\Xi)$ by multiplication and differentiation such that the corresponding maps $\mathcal{M}, \mathcal{D} : \mathcal{W} \to \text{End}(\mathcal{O}(\Xi)_{\text{fin}})$ are K-equivariant.

- The elements $E = Q$ and $F = 1$ in $\mathcal{W}$ can be complemented by $H = [E, F] \in \mathfrak{g}$ to an sl_2-triple which allows to determine a specific condition (T) depending on the pair (V, Q) under which, there exists a unique sequence $\delta = (\delta_m)_{m\in\mathbb{N}}$ acting diagonally on $\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m\in\mathbb{N}} \mathcal{O}_m(\Xi)$ such that the map $\rho : \mathcal{W} \to \text{End}(\mathcal{O}(\Xi)_{\text{fin}})$, $p \mapsto \mathcal{M}(p) - \delta \circ \mathcal{D}(p)$ complements $d\pi$ to a representation $d\pi + \rho$ of $\mathfrak{g} = \text{Lie}(K) + \mathcal{W}$ on $\mathcal{O}(\Xi)_{\text{fin}}$ (*).

Since $\mathcal{O}_0(\Xi)$ is reduced to the constants and then consists of K-fixed vectors, this representation is spherical. A list of the cases satisfying condition (T) is given and that determine the non Hermitian simple real Lie groups $\mathfrak{g}_{\mathbb{R}}$ which admit such spherical minimal representations.

- When the condition (T) is satisfied, given the sequence δ, it is possible to determine a set of weights such that the restriction of $d\pi + \rho$ to $\mathfrak{g}_{\mathbb{R}}$ is infinitesimally unitary. Moreover, Nelson's criterion can be used to show that this representation integrates to the simply connected Lie group $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{g}_{\mathbb{R}}$.

As the geometric setting in [4] allowed to give realizations of only spherical minimal representations, we consider in this paper a more general geometric setting which allows to realize a large class of minimal representations which covers those obtained in [2,5,6]. We resolve this problem by applying the preceding principles separately for the simple summands V_i of the Jordan algebra V which means that Q gets factored as a product of powers of Jordan determinants. Then, (neglecting coverings) K is a product of the analogous groups K_i for the simple summands V_i and on the level of K representations, one has a tensor product situation. The variety Ξ is a quotient of $\widetilde{\Xi} := \prod_{i=1}^s \Xi_i$, where s is the number of simple summands, $\Xi_i = K_i \cdot \Delta_i$ is the K_i-orbit of the Jordan determinant Δ_i. This results in an $\mathbb{N}^s$-grading rather than an $\mathbb{N}$-grading. The technical heart of this paper then is to make principle (*) work. To this end, one has to consider holomorphic functions on $\widetilde{\Xi}$ instead of Ξ, and replace $\mathcal{O}(\Xi)_{\text{fin}}$ by spaces $\mathcal{O}(\widetilde{\Xi})_{q,\text{fin}} = \sum_{m\in\mathbb{N}} \otimes_{i=1}^s \mathcal{O}_{k_i m+q_i}(\Xi_i)$ where $(k_1, \ldots, k_s)$ are the multiplicities given by $Q(z) = \prod_{i=1}^s \Delta_i^{k_i}(z_i)$ and $q = (q_1, \ldots, q_s) \in \mathbb{N}^s$. Then the strategy of proof given in [4] can be used to prove principle (*) also in the product setup. The extra freedom of the parameter q turns out to be enough to produce all the representations constructed by Brylinski–Kostant. The representations so obtained are realized on a Hilbert space $\mathcal{F}_q(\widetilde{\Xi})$ of holomorphic functions on $\widetilde{\Xi}$. There is an explicit formula for the reproducing kernel of $\mathcal{F}_q(\widetilde{\Xi})$ involving a hypergeometric function ${}_1F_2$. The space $\mathcal{F}_q(\widetilde{\Xi})$ is a weighted Bergman space. The norm is given by an integral on $\prod_{i=1}^s (\mathbb{C} \times V_i)$, with a weight taking in general both positive and negative values, involving a Meijer G-function $G_{1,3}^{3,0}$. This result permits to get an analogy with the classical Fock space.

A similar theory can be developed on Fock-type models for minimal representations of simple real Lie groups of Hermitian type (See [A12]).

In [7], the authors obtained Fock-type and Schrödinger-type models for minimal representations of real Lie groups of Hermitian type, which arise as conformal groups of simple real Jordan algebras. Also, in a series of papers, A. Dvorsky and S. Sahi obtained explicit Hilbert spaces for unipotent representations of such groups (see [8–10]).

1. Construction process of complex simple Lie algebras and of simple real Lie algebras of non Hermitian type

Let V be a semi-simple Jordan algebra and Q a homogeneous polynomial on V. Define

$$L = \text{Str}(V, Q) = \{g \in \text{GL}(V) \mid \exists \gamma = \gamma(g), Q(g \cdot x) = \gamma(g) Q(x)\}.$$

The conformal group $\text{Conf}(V, Q)$ is the group of rational transformations g of V generated by: the translations $z \mapsto z + a$ $(a \in V)$, the dilations $z \mapsto \ell \cdot z$ $(\ell \in L)$, and the inversion $j : z \mapsto -z^{-1}$ (see [11]). A transformation $g \in \text{Conf}(V, Q)$ is conformal in the sense that the differential $Dg(z)$ belongs to $\text{Str}(V, Q)$ at any point z where g is defined.

Let $\mathcal{W}$ be the space of polynomials on V generated by the translated $Q(z - a)$ of Q, with $a \in V$. Let κ be the cocycle representation of $\text{Conf}(V, Q)$ or of a covering of order two of it on $\mathcal{W}$, defined in [1] and [4] as follows:

Case 1. In case there exists a character χ of $\text{Str}(V, Q)$ such that $\chi^2 = \gamma$, then let $K = \text{Conf}(V, Q)$. Define the cocycle

$$\mu(g, z) = \chi(Dg(z)^{-1}) \quad (g \in K,\ z \in V),$$

and the representation κ of K on $\mathcal{W}$,

$$(\kappa(g)p)(z) = \mu(g^{-1}, z) p(g^{-1} \cdot z).$$

The function $\kappa(g)p$ belongs actually to $\mathcal{W}$ (see [12], Proposition 6.2). The cocycle $\mu(g, z)$ is a polynomial in z of degree $\leq \deg Q$ and

$$(\kappa(\tau_a)p)(z) = p(z - a) \quad (a \in V),$$
$$(\kappa(\ell)p)(z) = \chi(\ell) p(\ell^{-1} \cdot z) \quad (\ell \in L),$$
$$(\kappa(j)p)(z) = Q(z) p(-z^{-1}).$$

Case 2. Otherwise the group K is defined as the set of pairs (g, μ) with $g \in \mathrm{Conf}(V, Q)$, and μ is a rational function on V such that

$$\mu(z)^2 = \gamma(Dg(z))^{-1}.$$

We consider on K the product $(g_1, \mu_1)(g_2, \mu_2) = (g_1 g_2, \mu_3)$ with $\mu_3(z) = \mu_1(g_2 \cdot z)\mu_2(z)$. For $\tilde{g} = (g, \mu) \in K$, define $\mu(\tilde{g}, z) := \mu(z)$. Then $\mu(\tilde{g}, z)$ is a cocycle: $\mu(\tilde{g}_1\tilde{g}_2, z) = \mu(\tilde{g}_1, \tilde{g}_2 \cdot z)\mu(\tilde{g}_2, z)$, where $\tilde{g} \cdot z = g \cdot z$ by definition.

Then

$$\big(\kappa(\tilde{g})p\big)(z) = \mu(\tilde{g}^{-1}, z)p(\tilde{g}^{-1} \cdot z).$$

Recall from [1] that the representation κ of K on $\mathcal{W}$ is irreducible, The inverse in K of $\sigma = (j, Q(z))$ is $\sigma^{-1} = (j, Q(-z))$ and then equals σ if the degree of Q is even.

From now on, assume that the polynomial Q has degree 4. Then there exists $H \in \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{l} = \mathrm{Lie}(L)$ which defines gradings of $\mathfrak{k} = \mathrm{Lie}(K)$ and of $\mathfrak{p} = \mathcal{W}$:

$$\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1, \quad \mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2,$$

with

$$\mathfrak{k}_j = \{X \in \mathfrak{k} \mid \mathrm{ad}(H)X = jX\}, \quad (j = -1, 0, 1),$$
$$\mathfrak{k}_{-1} \simeq V, \quad \mathfrak{k}_0 = Lie(L), \quad \mathrm{Ad}(\sigma) : \mathfrak{k}_j \to \mathfrak{k}_{-j},$$

and where $\mathfrak{p}_j = \{p \in \mathfrak{p} \mid \mathrm{d}\kappa(H)p = jp\}$ is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j + 2$. Furthermore $\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}$, and

$$\mathfrak{p}_{-2} = \mathbb{C}, \quad \mathfrak{p}_2 = \mathbb{C}\,Q, \quad \mathfrak{p}_{-1} \simeq V, \quad \mathfrak{p}_1 \simeq V.$$

Then for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $E = Q$, $F = 1$, we established (see Theorem 3.1 in [1]) the existence on $\mathfrak{g}$ of a unique simple Lie algebra structure such that:

(i) $[X, X'] = [X, X']_{\mathfrak{k}}$ $(X, X' \in \mathfrak{k})$,
(ii) $[X, p] = \mathrm{d}\kappa(X)p$ $(X \in \mathfrak{k}, p \in \mathfrak{p})$,
(iii) $[E, F] = H$.

Recall now the real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ which will be considered in the sequel. It has been introduced in [1] and [4] . We fix a Euclidean real form $V_{\mathbb{R}}$ of the complex Jordan algebra V, denote by $z \mapsto \bar{z}$ the conjugation of V with respect to $V_{\mathbb{R}}$, and then consider the involution $g \mapsto \bar{g}$ of $\mathrm{Conf}(V, Q)$ given by: $\bar{g} \cdot z = \overline{g \cdot \bar{z}}$. For $(g, \mu) \in K$ define

$$\overline{(g, \mu)} = (\bar{g}, \bar{\mu}), \text{ where } \bar{\mu}(z) = \overline{\mu(\bar{z})}.$$

The involution α defined by $\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1}$ is a Cartan involution of K (see Proposition 1.1. in [P02]), and $K_{\mathbb{R}} = \{g \in K \mid \alpha(g) = g\}$ is a compact real form of K.

Let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{u} = \mathfrak{k}_{\mathbb{R}}$, the Lie algebra of $K_{\mathbb{R}}$, denote by $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p} \cap (i\mathfrak{u})$. Then, the real Lie algebra defined by

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}} \qquad (*)$$

is a real form of $\mathfrak{g}$ and the decomposition $(*)$ is its Cartan decomposition. Looking at the subalgebra $\mathfrak{g}^0$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ generated by the triple (E, F, H), one sees that $H \in i\mathfrak{k}_{\mathbb{R}}$, $E + F \in \mathfrak{p}_{\mathbb{R}}$ and $i(E - F) \in \mathfrak{p}_{\mathbb{R}}$. It follows that $\mathfrak{p}_{\mathbb{R}} = \mathcal{U}(\mathfrak{k}_{\mathbb{R}})(E + F) + \mathcal{U}(\mathfrak{k}_{\mathbb{R}})(i(E - F))$. Since the complexification of the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and since $\mathfrak{p}$ is a simple $\mathfrak{k}$-module, it follows that the simple real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is of non Hermitian type. In [4], we have established that $\mathfrak{p}_{\mathbb{R}} = \{p \in \mathfrak{p} \mid \beta(p) = p\}$, where we defined for a polynomial $p \in \mathfrak{p}$, $\bar{p} = \overline{p(\bar{z})}$, and considered the antilinear involution β of $\mathfrak{p}$ given by $\beta(p) = \kappa(\sigma)\bar{p}$.

Since $E + F$ and $i(E - F)$ belong to $\mathfrak{p}_{\mathbb{R}}$ and iH belongs to $\mathfrak{k}_{\mathbb{R}}$ and since

$$[E + F, i(E - F)] = -2i[E, F] = -2iH,$$
$$[iH, E + F] = 2i(E - F), \quad [iH, i(E - F)] = -2(E + F),$$

then the real Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$ generated by iH, $E + F$, $i(E - F)$ is isomorphic to $\mathfrak{su}(1, 1)$.

Another real form of the Lie algebra $\mathfrak{g}$ can be obtained naturally by considering the real analogous of $\mathfrak{g}$, given by $\tilde{\mathfrak{g}}_{\mathbb{R}} = \tilde{\mathfrak{k}}_{\mathbb{R}} + \tilde{\mathfrak{p}}_{\mathbb{R}}$, where $\tilde{\mathfrak{k}}_{\mathbb{R}} = \mathrm{Lie}(\mathrm{Str}(V_{\mathbb{R}}))$, $\tilde{\mathfrak{p}}_{\mathbb{R}} = \tilde{\mathcal{W}}_{\mathbb{R}}$ is the real subspace of $\mathfrak{p}$ generated by the polynomials $Q(x - a)$ on $V_{\mathbb{R}}$ with $a \in V_{\mathbb{R}}$ in such a way that for

$$\tilde{\mathfrak{p}}_j = \{p \in \tilde{\mathfrak{p}}_{\mathbb{R}} \mid \mathrm{d}\kappa(H)p = jp\},$$
$$\tilde{\mathfrak{p}}_{\mathbb{R}} = \tilde{\mathfrak{p}}_{-2} \oplus \tilde{\mathfrak{p}}_{-1} \oplus \tilde{\mathfrak{p}}_0 \oplus \tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2$$

is the eigen-space decomposition of $\tilde{\mathfrak{p}}_{\mathbb{R}}$ under $\mathrm{ad}(\tilde{H}) = d\kappa(\tilde{H})$, where we notice that the element H of $\mathfrak{z}(\mathfrak{k})$ belongs to $\mathfrak{z}(\tilde{\mathfrak{k}}_{\mathbb{R}})$ too.

Since E and F belong to $\tilde{\mathfrak{p}}_{\mathbb{R}}$ and H belongs to $\tilde{\mathfrak{k}}_{\mathbb{R}}$ and since

$$[E, F] = H, [H, E] = 2E, [H, F] = -2F,$$

then the real Lie subalgebra of $\tilde{\mathfrak{g}}_{\mathbb{R}}$ generated by H, E, F is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

We give in Table 1 the classification of the simple real Lie algebras $\mathfrak{g}_{\mathbb{R}}$ obtained in this way. It turns out that they are exactly the simple real Lie algebras of non Hermitian type which satisfy the condition $\mathcal{O}_{min} \cap \mathfrak{g} \neq \emptyset$ (see Tables 4.7 and 4.8 in [2]). It also gives the real Lie algebras $\tilde{\mathfrak{g}}_{\mathbb{R}}$.

We have used the notation:

$$\varphi_p(z) = z_1^2 + \cdots + z_p^2, \quad (z \in \mathbb{C}^p).$$

In case of an exceptional Lie algebra $\mathfrak{g}$, the real form $\mathfrak{g}_{\mathbb{R}}$ has been identified by computing the Cartan signature (see Table 1).

Table 1

V	Q	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$	$\tilde{\mathfrak{g}}_{\mathbb{R}}$
$\mathbb{C}$	z^4	$\mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{R})$
$\mathbb{C}^p$	$\varphi_p(z)^2$	$\mathfrak{so}(p+2, \mathbb{C})$	$\mathfrak{sl}(p+2, \mathbb{C})$	$\mathfrak{sl}(p+2, \mathbb{R})$	$\mathfrak{sl}(p+2, \mathbb{R})$
$\mathbb{C} \oplus \mathbb{C}$	$z_1^2 z_2^2$	$\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$	$\mathfrak{so}(6, \mathbb{C})$	$\mathfrak{so}(3, 3)$	$\mathfrak{so}(6)$
$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$z_1^2 z_2 z_3$	$\mathfrak{sl}(2, \mathbb{C})^{\oplus 3}$	$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(7)$
$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$z_1 z_2 z_3 z_4$	$\mathfrak{sl}(2, \mathbb{C})^{\oplus 4}$	$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(4, 4)$	$\mathfrak{so}(8)$
$\mathbb{C}^p \oplus \mathbb{C}$	$\varphi_p(z) z'^2$	$\mathfrak{so}(p+2, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$	$\mathfrak{so}(p+5, \mathbb{C})$	$\mathfrak{so}(p+2, 3)$	$\mathfrak{so}(p+5)$
$\mathbb{C}^p \oplus \mathbb{C} \oplus \mathbb{C}$	$\varphi_p(z) z' z''$	$\mathfrak{so}(p+2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}^{\oplus 2})$	$\mathfrak{so}(p+6, \mathbb{C})$	$\mathfrak{so}(p+2, 4)$	$\mathfrak{so}(p+6)$
$\mathbb{C}^{p_1} \oplus \mathbb{C}^{p_2}$	$\varphi_{p_1}(z) \varphi_{p_2}(z')$	$\oplus_{i=1}^2 \mathfrak{so}(p_i+2, \mathbb{C})$	$\mathfrak{so}(p_1+p_2+4, \mathbb{C})$	$\mathfrak{so}(p_1+2, p_2+2)$	$\mathfrak{so}(p_1+p_2+4)$
Sym$(4, \mathbb{C})$	$\det z$	$\mathfrak{sp}(8, \mathbb{C})$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(6)}$
M$(4, \mathbb{C})$	$\det z$	$\mathfrak{sl}(8, \mathbb{C})$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(7)}$
Skew$(8, \mathbb{C})$	Pfaff(z)	$\mathfrak{so}(16, \mathbb{C})$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{8(8)}$
Sym$(3, \mathbb{C}) \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{sp}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{f}_4$	$\mathfrak{f}_{4(4)}$	$\mathfrak{f}_{4(4)}$
$M(3, \mathbb{C}) \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{6(6)}$
Skew$(6, \mathbb{C}) \oplus \mathbb{C}$	Pfaff$(z) \cdot z'$	$\mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{7(7)}$
Herm$(3, \mathbb{O})_{\mathbb{C}} \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{e}_7 \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{8(8)}$
$\mathbb{C} \oplus \mathbb{C}$	$z^3 \cdot z'$	$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{g}_2$	$\mathfrak{g}_{2(2)}$	$\mathfrak{g}_{2(2)}$

The Jordan algebra V is a direct sum of simple ideals:

$$V = \bigoplus_{i=1}^{s} V_i,$$

and

$$Q(z) = \prod_{i=1}^{s} \Delta_i(z_i)^{k_i} \quad (z = (z_1, \ldots, z_s)),$$

where Δ_i is the determinant polynomial of the simple Jordan algebra V_i and the k_i are positive integers. The degree of Q is equal to $\sum_{i=1}^{s} k_i r_i$, where r_i is the rank of V_i. Define

$$L_i = \mathrm{Str}(V_i, \Delta_i) = \{g \in \mathrm{GL}(V_i) \mid \exists \gamma_i(g) \in \mathbb{C}^*, \Delta_i(g \cdot x) = \gamma_i(g) \Delta_i(x)\}.$$

In a similar manner, we consider the conformal groups $\mathrm{Conf}(V_i, \Delta_i)$, the corresponding groups K_i analogous of K, the spaces $\mathfrak{p}_i$ of polynomials on V_i generated by the polynomials $\Delta_i(z_i - a_i)$ with $a_i \in V_i$ and the cocycle representations κ_i of K_i, analogous of κ, on the space $\mathfrak{p}_i$ given by $\big(\kappa_i(g_i)p\big)(z_i) = \mu_i(g_i^{-1}, z_i) p(g_i^{-1} \cdot z_i)$. More precisely, $K_i = \mathrm{Conf}(V_i, \Delta_i)$ when there exists a character χ_i of L_i satisfying $\chi_i^2 = \gamma_i$ and $K_i = \{\tilde{g}_i = (g_i, \mu_i) \mid g_i \in \mathrm{Conf}(V_i, \Delta_i), \mu_i(z_i)^2 = \gamma_i(Dg_i(z_i))^{-1}\}$ if the character χ_i does not exist.

Since the differential of the inversion $j : V_i \to V_i, z_i \mapsto -z_i^{-1}$ is given by $Dj(z_i) = P_i(z_i)^{-1}$ where P_i is the quadratic representation of V_i and since $P_i(z_i)$ belongs to L_i and satisfies $\Delta_i(P_i(z_i)x_i) = \Delta_i^2(z_i)\Delta_i(x_i)$ then $\gamma_i(P_i(z_i)) = \Delta_i^2(z_i)$ and the element $\sigma_i = (j, \Delta_i(z_i))$ belongs to K_i. The inverse in K_i of $\sigma_i = (j, \Delta_i(z_i))$ is $\sigma_i^{-1} = (j, \Delta_i(-z_i)) = (j, (-1)^{r_i}\Delta_i(z_i))$.

Furthermore, when χ_i exists, the representation κ_i of K_i is given by $(\kappa_i(g_i)p)(z_i) = \mu_i(g_i^{-1}, z_i) p(g_i^{-1} \cdot z_i)$, for $\mu_i(g_i, z_i) = \chi_i(Dg_i(z_i)^{-1})$. Then

$$(\kappa_i(\tau_{a_i})p)(z_i) = p(z_i - a_i) \quad (a_i \in V_i),$$
$$(\kappa_i(\ell_i)p)(z_i) = \chi_i(\ell_i) p(\ell_i^{-1} \cdot z_i) \quad (\ell_i \in L_i),$$
$$(\kappa_i(j)p)(z_i) = \Delta_i(z_i) p(-z_i^{-1}).$$

And, when χ_i does not exist, it is $\big(\kappa_i(\tilde{g}_i)p\big)(z_i) = \mu_i(\tilde{g}_i^{-1}, z_i) p(\tilde{g}_i^{-1} \cdot z_i)$, where $\tilde{g}_i = (g_i, \mu_i(z_i))$ and $\mu_i(\tilde{g}_i, z_i) = \mu_i(z_i)$. In particular $(\kappa_i(\sigma_i)p)(z_i) = \Delta_i(z_i) p(-z_i^{-1})$ and $(\kappa_i(\sigma_i^{-1})p)(z_i) = (-1)^{r_i}\Delta_i(z_i) p(-z_i^{-1})$.

Let $\mathcal{W}_i^{(k_i)}$ be the vector space generated by the polynomials $\Delta_i^{k_i}(z_i - a_i)$ for $a_i \in V_i$. Then, the group K_i acts on $\mathcal{W}_i^{(k_i)}$ by the representation $\kappa_i^{(k_i)}$ given by $(\kappa_i^{(k_i)}(g_i)p)(z_i) = \mu_i(g_i^{-1}, z_i)^{k_i} p(g_i^{-1} \cdot z_i)$.

Observe that $L := \prod_{i=1}^s L_i$ acting on V by $g \cdot z = (g_i \cdot z_i)$, for $g = (g_i)$, $z = (z_i)$, is a subgroup of L, with same Lie algebra $\mathfrak{l} = \sum_{i=1}^s \mathfrak{l}_i$ with $\mathfrak{l}_i = \mathrm{Lie}(L_i)$, and that $\gamma(g) = \prod_{i=1}^s \gamma_i^{k_i}(g_i)$.

Also, the groups $\widetilde{K} := \prod_{i=1}^s K_i$ and K have same Lie algebra $\mathfrak{k} = \sum_{i=1}^s \mathfrak{k}_i$ with $\mathfrak{k}_i = \mathrm{Lie}(K_i) \simeq V_i \oplus \mathfrak{l}_i \oplus V_i$.

Proposition 1.1. *The space $\mathfrak{p}$ is the tensor product of the $\mathcal{W}_i^{(k_i)}$.*

Proof. In fact, let ψ be the multilinear map $\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_s \to \mathfrak{p}$, $(p_1, \ldots, p_s) \mapsto p$, with $p(z) = p_1(z_1) \ldots p_s(z_s)$ for $z = \sum_{i=1}^s z_i$. Since $\kappa(\tau_a)Q(z) = \prod_{i=1}^s (\kappa_i^{(k_i)}(\tau_{a_i})\Delta_i^{k_i})(z_i)$, then the complex vector space $\mathfrak{p}$ is generated by $\psi(\mathcal{W}_1^{(k_1)} \times \cdots \times \mathcal{W}_s^{(k_s)})$. It remains to prove that for a multilinear map $f : \mathcal{W}_1^{(k_1)} \times \cdots \times \mathcal{W}_s^{(k_s)} \to U$, there is a linear map $\tilde{f} : \mathfrak{p} \to U$ such that $\tilde{f} \circ \psi = f$. Recall that $\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2$, $\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$ with $\mathfrak{k}_{-1} \simeq V \simeq \mathfrak{k}_1 = \kappa(\sigma)(\mathfrak{k}_{-1})$, $\mathfrak{p}_{-2} = \mathbb{C} \cdot 1$, $\mathfrak{p}_2 = \mathbb{C} \cdot Q$ and that for every $p \in \mathfrak{p}_1$, $q \in \mathfrak{p}_{-1}$, there is a unique $X \in \mathfrak{k}_{-1}$ and a unique $Y \in \mathfrak{k}_{-1}$ such that $p = d\kappa(X)Q$ and $q = \kappa(\sigma)d\kappa(Y)Q$ (see [1], Lemma 1.1).

Let us define the linear map $\tilde{f}$ on $\mathfrak{p}$ as follows:

- On $\mathfrak{p}_2$ and $\mathfrak{p}_{-2}$: $\tilde{f}(Q) = f(\Delta_1^{k_1}, \ldots, \Delta_s^{k_s})$; $\tilde{f}(1) = f(1, \ldots, 1)$.
- On $\mathfrak{p}_1$: for $p \in \mathfrak{p}_1$, let $X \in \mathfrak{k}_{-1}$ be such that $p = [X, Q]$, then put

$$\tilde{f}(p) = \tilde{f}(d\kappa(X)Q) = f(d\kappa_1^{(k_1)}(X_1)\Delta_1^{k_1}, \ldots, d\kappa_s^{(k_s)}(X_s)\Delta_s^{k_s})$$

where $\exp(X) = \tau_a$, $a = \sum_{i=1}^s a_i$ and $X_i \in \mathfrak{k}_i$ such that $\exp(X_i) = \tau_{a_i}$.

- On $\mathfrak{p}_{-1}$: for $q \in \mathfrak{p}_{-1}$, let $Y \in \mathfrak{k}_{-1}$ such that $q = \kappa(\sigma)[Y, Q]$, then put

$$\tilde{f}(q) = f(\kappa_1^{(k_1)}(\sigma_1)\mathrm{d}\kappa_1^{(k_1)}(Y_1)\Delta_1^{k_1}, \ldots, \kappa_s^{(k_s)}(\sigma_s)\mathrm{d}\kappa_s^{(k_s)}(Y_s)\Delta_s^{k_s}).$$

where $\exp(Y) = \tau_b$, $b = \sum_{i=1}^s b_i$ and $Y_i \in \mathfrak{k}_i$ such that $\exp(Y_i) = \tau_{b_i}$.

- On $\mathfrak{p}_0$: for $\tilde{p} \in \mathfrak{p}_0$, there is $(X, p) \in \mathfrak{k}_{-1} \times \mathfrak{p}_1$ such that $\tilde{p} = [X, p]$, and, since there is $Y \in \mathfrak{k}_{-1}$ such that $p = [Y, Q]$ then $\tilde{p} = [X, [Y, Q]]$. Let $\{p_1, \ldots, p_{n_0}\}$ be a basis of the space $\mathfrak{p}_0$ where $p_j = [X^{(j)}, [Y^{(j)}, Q]]$ with $X^{(j)}, Y^{(j)} \in \mathfrak{k}_{-1}$. Denote by $a^{(j)} = \sum_{i=1}^s a_i^{(j)}$, $b^{(j)} = \sum_{i=1}^s b_i^{(j)}$ the elements such that $\exp(X^{(j)}) = \tau_{a^{(j)}}$ and $\exp(Y^{(j)}) = \tau_{b^{(j)}}$ and let $X_i^{(j)}$, $Y_i^{(j)} \in \mathfrak{k}_i$ be such that $\exp(X_i^{(j)}) = \tau_{a_i^{(j)}}$ and $\exp(Y_i^{(j)}) = \tau_{b_i^{(j)}}$.

We define $\tilde{f}$ on this basis by

$$\tilde{f}(p_j) = f\left([X_1^{(j)}, [Y_1^{(j)}, \Delta_1^{k_1}]] \prod_{i=2}^s \Delta_i^{k_i}, \ldots, [X_s^{(j)}, [Y_s^{(j)}, \Delta_s^{k_s}]] \prod_{i=1}^{s-1} \Delta_i^{k_i}\right).$$

Since

$$(\mathrm{d}\kappa(X)Q)(z) = (\mathrm{d}\kappa_1^{(k_1)}(X_1)\Delta_1^{k_1})(z_1) \ldots (\mathrm{d}\kappa_s^{(k_s)}(X_s)\Delta_s^{k_s})(z_s),$$
$$(\kappa(\sigma)\mathrm{d}\kappa(Y)Q)(z) = (\kappa_1^{(k_1)}(\sigma_1)\mathrm{d}\kappa_1^{(k_1)}(Y_1)\Delta_1^{k_1})(z_1) \ldots (\kappa_s^{(k_s)}(\sigma_s)\mathrm{d}\kappa_s^{(k_s)}(Y_s)\Delta_s^{k_s})(z_s),$$

and

$$([X^{(j)}, [Y^{(j)}, Q]])(z) = \sum_{l=1}^s ([X_l^{(j)}, [Y_l^{(j)}, \Delta_l^{k_l}]])(z_l) \prod_{i \neq l}^s \Delta_i^{k_i}(z_i),$$

then

$$\tilde{f} \circ \psi = f. \qquad \square$$

Let $\kappa^{(k_1,\ldots,k_s)} = \kappa_1^{(k_1)} \otimes \cdots \otimes \kappa_s^{(k_s)}$ be the tensor product of the representations $\kappa_i^{(k_i)}$. The infinitesimal representations $d\kappa^{(k_1,\ldots,k_s)}$ and $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$ are equal and the structure of simple Lie algebra described above on $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ can be obtained by considering the action $\kappa^{(k_1,\ldots,k_s)}$ of the product group $\widetilde{K} := \prod_{i=1}^s K_i$ (instead of κ) on $\mathfrak{p} = \otimes_{i=1}^s \mathcal{W}_i^{(k_i)}$.

The orbit Ξ of Q under the action of the group $\widetilde{K} := \prod_{i=1}^s K_i$, has dimension $\dim(V) + 1$ and is a conical variety : for $\xi \in \Xi$, $\lambda \in \mathbb{C}^*$, $\lambda\xi$ belongs to Ξ.

Let Ξ_i be the K_i-orbit of Δ_i in $\mathcal{W}_i$ under κ_i:

$$\Xi_i = \{\kappa_i(g_i)\Delta_i \mid g_i \in K_i\}.$$

Then Ξ_i is conical. It can be seen as a line bundle over the conformal compactification of V_i. A polynomial $\xi_i \in \mathcal{W}_i$ can be written

$$\xi_i(v) = w_i\Delta_i(v) + \text{ terms of degree } < r_i \quad (w_i \in \mathbb{C}),$$

and $w_i = w_i(v_i)$ is a linear form on $\mathcal{W}_i$ which is semi-invariant under the preimage in K_i of the maximal parabolic subgroup $P_{\max}^{(i)} = L_i \ltimes N_i$, where N_i is the group of translations $z_i \in V_i \mapsto z_i + a_i$, for $a_i \in V_i$. The set $\Xi_{i,0} = \{\xi_i \in \Xi_i \mid w_i(\xi_i) \neq 0\}$ is open

and dense in Ξ_i. A polynomial $\xi_i \in \Xi_{i,0}$ can be written $\xi_i(v_i) = w_i\Delta_i(v_i - z_i) \quad (w_i \in \mathbb{C}^*, z_i \in V_i)$. Hence we get a coordinate system $(w_i, z_i) \in \mathbb{C}^* \times V_i$ for $\Xi_{i,0}$.

In this coordinate system, the cocycle action of K_i is given by

$$\kappa_i(g_i) : (w_i, z_i) \mapsto \big(\mu_i(g_i, z_i)w_i, g_i \cdot z_i\big).$$

Denote by $\Xi_i^{(k_i)}$ the K_i-orbit of $\Delta_i^{k_i}$ in $\mathcal{W}_i^{(k_i)}$ under $\kappa_i^{(k_i)}$:

$$\Xi_i^{(k_i)} = \{\kappa_i^{(k_i)}(g_i)\Delta_i^{k_i} \mid g_i \in K_i\}.$$

Since $\kappa_i^{(k_i)}(g_i)\Delta_i^{k_i} = (\kappa_i(g_i)\Delta_i)^{k_i}$, then $\Xi_i^{(k_i)} = \{\xi_i^{k_i} \mid \xi_i \in \Xi_i\}$. Furthermore, a polynomial $\xi_i^{(k_i)} \in \mathcal{W}_i^{(k_i)}$ can be written

$$\xi_i^{(k_i)}(v) = \tilde{w}_i\Delta_i^{k_i}(v) + \text{ terms of degree } < k_i r_i \quad (\tilde{w}_i \in \mathbb{C}),$$

the set $\Xi_{i,0}^{(k_i)} = \{\xi_i^{(k_i)} \in \Xi_i^{(k_i)} \mid \tilde{w}_i(\xi_i^{(k_i)}) \neq 0\}$ is open and dense in $\Xi_i^{(k_i)}$ and a polynomial $\xi_i^{(k_i)} \in \Xi_{i,0}^{(k_i)}$ can be written $\xi_i^{(k_i)}(v_i) = \tilde{w}_i\Delta_i^{k_i}(v_i - z_i) \quad (\tilde{w}_i \in \mathbb{C}^*, z_i \in V_i)$. Hence we get a coordinate system $(\tilde{w}_i, z_i) \in \mathbb{C}^* \times V_i$ for $\Xi_{i,0}^{(k_i)}$.

In this coordinate system, the cocycle action of K_i is given by

$$\kappa_i^{(k_i)}(g_i) : (\tilde{w}_i, z_i) \mapsto \big(\mu_i^{k_i}(g_i, z_i)\tilde{w}_i, g_i \cdot z_i\big).$$

Define on $\widetilde{\Xi} = \prod_{i=1}^s \Xi_i$ the equivalence relation

$$(\xi_1, \ldots, \xi_s) \sim (\lambda_1\xi_1, \ldots, \lambda_s\xi_s), \text{ for } (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s, \prod_{i=1}^s \lambda_i^{k_i} = 1.$$

Proposition 1.2. *The map $\widetilde{\Xi}/\sim \;\to \Xi, (\xi_1, \ldots, \xi_s) \mapsto \xi_1^{k_1} \ldots \xi_s^{k_s}$ is a diffeomorphism.*
The group K_i acts on the space $\mathcal{O}(\Xi_i)$ of holomorphic functions on Ξ_i by:

$$\big(\pi_i(g_i)f_i\big)(\xi_i) = f_i\big(\kappa_i(g_i^{-1})\xi_i\big).$$

If $\xi_i(v_i) = w_i\Delta_i(v_i - z_i)$, and $f_i \in \mathcal{O}(\Xi_i)$, we denote by $f_i(\xi_i) = \phi_i(w_i, z_i)$ the restriction of f_i to $\Xi_{i,0}$. Then $\phi_i \in \mathcal{O}(\mathbb{C}^* \times V_i)$. In the coordinates (w_i, z_i), the representation π_i is given by

$$(\pi_i(g_i)\phi_i)(w_i, z_i) = \phi_i(\mu_i(g_i^{-1}, z_i)w_i, g_i^{-1} \cdot z_i).$$

The group K_i acts on the space $\mathcal{O}(\Xi_i^{(k_i)})$ by:

$$\big(\pi_i^{(k_i)}(g_i)f_i^{(k_i)}\big)(\xi_i^{(k_i)}) = f_i^{(k_i)}\big(\kappa_i^{(k_i)}(g_i^{-1})\xi_i^{(k_i)}\big).$$

If $\xi_i^{(k_i)}(v_i) = \tilde{w}_i\Delta_i^{k_i}(v_i - z_i)$, and $f_i \in \mathcal{O}(\Xi_i^{(k_i)})$, *we denote by $f_i^{(k_i)}(\xi_i^{(k_i)}) = \phi_i^{(k_i)}(\tilde{w}_i, z_i)$ the restriction of $f_i^{(k_i)}$ to $\Xi_{i,0}^{(k_i)}$. Then $\phi_i^{(k_i)} \in \mathcal{O}(\mathbb{C}^* \times V_i)$. In the coordinates $(\tilde{w}_i, z_i)$, the representation $\pi_i^{(k_i)}$ is given by*

$$(\pi_i^{(k_i)}(g_i)\phi_i^{(k_i)})(\tilde{w}_i, z_i) = \phi_i^{(k_i)}(\mu_i^{k_i}(g_i^{-1}, z_i)\tilde{w}_i, g_i^{-1} \cdot z_i).$$

Observe that for every $f_i^{(k_i)} \in \mathcal{O}(\Xi_i^{(k_i)})$, one can associate a unique $f_i \in \mathcal{O}(\Xi_i)$ such that $f_i(\xi_i) = f_i^{(k_i)}(\xi_i^{k_i})$ in such a way that

$$\big(\pi_i^{(k_i)}(g_i)f_i^{(k_i)}\big)(\xi_i^{k_i}) = \big(\pi_i(g_i)f_i\big)(\xi_i).$$

The group $\widetilde{K} = \prod_{i=1}^s K_i$ acts on the tensor product space $\otimes_{i=1}^s \mathcal{O}(\Xi_i)$ by the tensor product representation $\pi = \otimes_{i=1}^s \pi_i$ given by:

$$\begin{aligned}\big(\pi(g)(f_1 \otimes \cdots \otimes f_s)\big)(\xi_1, \ldots, \xi_s) &= \big(\pi_1(g_1)f_1\big) \otimes \cdots \otimes \big(\pi_s(g_s)f_s\big)(\xi_1, \ldots, \xi_s)\\ &= f_1(\kappa_1(g_1^{-1})\xi_1) \ldots f_s(\kappa_s(g_s^{-1})\xi_s) \quad (g = (g_i)).\end{aligned}$$

In the coordinates (w_i, z_i), the representation π is given by

$$(\pi(g)(\phi_1 \otimes \cdots \otimes \phi_s))((w_1, z_1), \ldots, (w_s, z_s)) = \prod_{i=1}^s \phi_i(\mu_i(g_i^{-1}, z_i)w_i, g_i^{-1} \cdot z_i).$$

2. $\widetilde{K}$-invariant Hilbert subspaces of $\mathcal{O}(\widetilde{\Xi})$

For every $m_i, p_i \in \mathbb{Z}$, we denote by $\mathcal{O}_{p_i}(\Xi_i^{(k_i)})$ and $\mathcal{O}_{m_i}(\Xi_i)$ the spaces of holomorphic functions $f_i^{(k_i)}$ on $\Xi_i^{(k_i)}$ and f_i on Ξ_i such that for every $\lambda \in \mathbb{C}^*$,

$$f_i^{(k_i)}(\lambda\xi_i^{(k_i)}) = \lambda^{p_i} f_i(\xi_i^{(k_i)}) \text{ and } f_i(\lambda\xi_i) = \lambda^{m_i} f_i(\xi_i).$$

The space $\mathcal{O}_{m_i}(\Xi_i)$ (resp. $\mathcal{O}_{p_i}(\Xi_i^{(k_i)})$) is invariant under the representation π_i (resp. $\pi_i^{(k_i)}$).

The embedding $f_i^{(k_i)} \in \mathcal{O}_{p_i}(\Xi_i^{(k_i)}) \mapsto f_i \in \mathcal{O}_{k_i p_i}(\Xi_i)$, where $f_i(\xi_i) = f_i^{(k_i)}(\xi_i^{k_i})$ intertwines the restrictions $\pi_{i,p_i}^{(k_i)}$ and $\pi_{i,k_i p_i}$ of the representations $\pi_i^{(k_i)}$ and π_i to $\mathcal{O}_{p_i}(\Xi_i^{(k_i)})$ and $\mathcal{O}_{k_i p_i}(\Xi_i)$.

From now on, we essentially work with the spaces $\mathcal{O}_{m_i}(\Xi_i)$. The representations $(\pi_{i,p_i}^{(k_i)}, \mathcal{O}_{p_i}(\Xi_i^{(k_i)}))$ and $(\pi_{i,k_i p_i}, \mathcal{O}_{k_i p_i}(\Xi_i))$ of K_i are irreducible and equivalent.

If $f_i \in \mathcal{O}_{m_i}(\Xi_i)$, then its restriction ϕ_i to $\Xi_{i,0}$ can be written $\phi_i(w_i, z_i) = w_i^{m_i}\psi_i(z_i)$ where ψ_i is a holomorphic function on V_i. We will write $\widetilde{\mathcal{O}}_{m_i}(V_i)$ for the space of the functions ψ_i corresponding to the functions $f_i \in \mathcal{O}_{m_i}(\Xi_i)$, and denote by $\tilde{\pi}_{i,m_i}$ the representation of K_i on $\widetilde{\mathcal{O}}_{m_i}(V_i)$ corresponding to the restriction π_{i,m_i} of π to $\mathcal{O}_{m_i}(\Xi_i)$. It is given by

$$(\tilde{\pi}_{i,m_i}(g_i)\psi_i)(z_i) = \mu_i(g_i^{-1}, z_i)^{m_i}\psi_i(g_i^{-1}\cdot z_i).$$

In particular,

$$(\tilde{\pi}_{i,m_i}(\sigma_i)\psi_i)(z_i) = \Delta_i^{m_i}(z_i)\psi_i(-z_i^{-1}).$$

Theorem 2.1. (i) $\mathcal{O}_{m_i}(\Xi_i) = \{0\}$ *if* $m_i < 0$.

(ii) *The space* $\mathcal{O}_{m_i}(\Xi_i)$ *is finite dimensional, and the representation* π_{i,m_i} *is irreducible.*

(iii) $\widetilde{\mathcal{O}}_{m_i}(V_i)$ *is a space of holomorphic polynomials on* V_i *with degree* $\leq m_i r_i$.

The proof of (i), (ii), is similar to that of Theorem 2.2 in [4]. It remains to prove (iii). Let $\psi_i \in \widetilde{\mathcal{O}}_{m_i}(V_i)$. Since $\pi_{i,m_i}(\sigma_i)\psi_i \in \mathcal{O}(V_i)$, then ψ_i is a polynomial and since $(\pi_{i,m_i}(\sigma_i)\psi_i)(z_i) = \Delta_i^{m_i}(z_i)\psi_i(-z_i^{-1})$, then the degree of ψ_i is $\leq$ to the degree of $\Delta_i^{m_i}$ which is $m_i r_i$. □

For $(m_1, \ldots, m_s) \in \mathbb{N}^s$, we denote by $\mathcal{O}_{(m_1,\ldots,m_s)}(\widetilde{\Xi}) = \otimes_{i=1}^s \mathcal{O}_{m_i}(\Xi_i)$. It is invariant under the tensor product representation $\pi = \otimes_{i=1}^s \pi_i$.

For $f = f_1 \otimes \cdots \otimes f_s \in \mathcal{O}_{(m_1,\ldots,m_s)}(\widetilde{\Xi})$, and if $\phi_i(w_i, z_i)$ is the restriction of f_i to $\Xi_{i,0}$, then $\phi_i(w_i, z_i) = w_i^{m_i}\psi_i(z_i)$ and

$$(\phi_1 \otimes \cdots \otimes \phi_s)((w_1, z_1), \ldots, (w_s, z_s)) = w_1^{m_1} \ldots w_s^{m_s}\psi_1(z_1)\ldots\psi_s(z_s).$$

We write $\widetilde{\mathcal{O}}_{(m_1,\ldots,m_s)}(V) = \otimes_{i=1}^s \widetilde{\mathcal{O}}_{m_i}(V_i)$ and $\tilde{\pi}_{(m_1,\ldots,m_s)} = \otimes_{i=1}^s \tilde{\pi}_{m_i}$. Then

$$\big(\tilde{\pi}_{(m_1,\ldots,m_s)}(g)(\psi_1 \otimes \cdots \otimes \psi_s)\big)(z_1, \ldots, z_s) = \prod_{i=1}^s \mu_i(g_i^{-1}, z_i)^{m_i}\psi_i(g_i^{-1}\cdot z_i).$$

For $\sigma = (\sigma_i)$, $(\tilde{\pi}_{(m_1,\ldots,m_s)}(\sigma)1)(z_1, \ldots, z_s) = \Delta_1^{m_1}(z_1)\ldots.\Delta_s^{m_s}(z_s)$.

We now consider the compact real forms $(K_i)_{\mathbb{R}}$ analogous of $K_{\mathbb{R}}$ for the groups K_i and define on the space $\mathcal{O}_{(m_1,\ldots,m_s)}(\widetilde{\Xi})$ a $\prod_{i=1}^s (K_i)_{\mathbb{R}}$-invariant inner product. Define the subgroup $K_{i,0}$ of K_i as $K_{i,0} = L_i$ in Case 1, and the preimage of L_i in Case 2, relatively to the covering map $K_i \to \mathrm{Conf}(V_i, \Delta_i)$, and also $(K_{i,0})_{\mathbb{R}} = K_{i,0} \cap (K_i)_{\mathbb{R}}$. The coset space $M_i = (K_i)_{\mathbb{R}}/(Ki, 0)_{\mathbb{R}}$, is a compact Hermitian space and is the conformal compactification of V_i. There is on M_i a $(K_i)_{\mathbb{R}}$-invariant probability measure, for which $M_i \setminus V_i$ has measure 0. Its restriction $m_{i,0}$ to V_i is a probability measure with a density with respect to the Lebesgue measure $m(dz_i)$ on V_i.

Let $H_i(z_i, z_i')$ be the polynomial on $V_i \times V_i$, holomorphic in z_i, anti-holomorphic in z_i' such that $H_i(x_i, x_i) = \Delta_i(e_i + x_i^2)$ for $x_i \in (V_i)_{\mathbb{R}}$. Put $H_i(z_i) = H_i(z_i, z_i)$. If z_i is invertible, then $H_i(z_i) = \Delta_i(\bar{z}_i)\Delta_i(\bar{z}_i^{-1} + z_i)$.

The measure $m_{i,0}$ is given by

$$m_{i,0}(dz_i) = \frac{1}{C_{i,0}} H_{i,0}(z_i)m(dz_i),$$

with

$$H_{i,0}(z_i) = H_i(z_i)^{-2\frac{n_i}{r_i}}, \quad C_{i,0} = \int_{V_i} H_{i,0}(z_i)m(dz_i)$$

where the Lebesgue measure $m(dz_i)$ will be chosen such that $C_{i,0} = 1$.

Proposition 2.2. *For* $g_i \in (K_i)_{\mathbb{R}}$,

$$H_i(g_i\cdot z_i, g_i\cdot z_i)'\mu_i(g_i, z_i)\overline{\mu_i(g_i, z_i')} = H_i(z_i, z_i'),$$
$$H_i(g_i\cdot z_i)|\mu_i(g_i, z_i)|^2 = H_i(z_i).$$

We define the norms of $\psi_i \in \widetilde{\mathcal{O}}_{m_i}(V_i)$ and of $\psi \in \widetilde{\mathcal{O}}_{(m_1,\ldots,m_s)}(V)$ by

$$\|\psi_i\|_{i,m_i}^2 = \frac{1}{a_{i,m_i}} \int_{V_i} |\psi_i(z_i)|^2 H_i(z_i)^{-m_i} m_{i,0}(dz_i),$$

$$\|\psi\|_{(m_1,\ldots,m_s)}^2 = \frac{1}{a_{(m_1,\ldots,m_s)}} \int_{\prod_{i=1}^s V_i} |\psi(z_1, \ldots, z_s)|^2 \prod_{i=1}^s H_i(z_i)^{-m_i} m_{i,0}(dz_i),$$

$$a_{i,m_i} = \int_{V_i} H_i(z_i)^{-m_i} m_{i,0}(dz_i), \quad a_{(m_1,\dots,m_s)} = \prod_{i=1}^{s} a_{i,m_i}.$$

Proposition 2.3. (i) *The norm $\|\cdot\|_{i,m_i}$ is $(K_i)_{\mathbb{R}}$-invariant. Hence, $\widetilde{\mathcal{O}}_{m_i}(V_i)$ is a Hilbert subspace of $\mathcal{O}(V_i)$. The reproducing kernel of $\widetilde{\mathcal{O}}_{m_i}(V_i)$ is given by $\tilde{\mathcal{K}}_{i,m_i}(z_i, z_i') = H_i(z_i, z_i')^{m_i}$.*
(ii) *The norm $\|\cdot\|_{(m_1,\dots,m_s)}$ is $\prod_{i=1}^{s}(K_i)_{\mathbb{R}}$-invariant and $\widetilde{\mathcal{O}}_{(m_1,\dots,m_s)}(V)$ is a Hilbert subspace of $\mathcal{O}(\prod_{i=1}^{s} V_i)$ with reproducing kernel*

$$\tilde{\mathcal{K}}_{(m_1,\dots,m_s)}((z_1,\dots,z_s),(z_1',\dots,z_s')) = \prod_{i=1}^{s} H_i(z_i, z_i')^{m_i}.$$

Since $\mathcal{O}_{m_i}(\Xi_i)$ is isomorphic to $\widetilde{\mathcal{O}}_{m_i}(V_i)$, then $\mathcal{O}_{m_i}(\Xi_i)$ becomes an invariant Hilbert subspace of $\mathcal{O}(\Xi_i)$, with reproducing kernel given by $\mathcal{K}_{i,m_i}(\xi_i, \xi_i') = \Phi_i(\xi_i, \xi_i')^{m_i}$, where $\Phi_i(\xi_i, \xi_i') = H_i(z_i, z_i')w_i\overline{w_i'}$. Hence $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ is an invariant Hilbert subspace of $\mathcal{O}(\prod_{i=1}^{s}\Xi_i)$ with reproducing kernel

$$\mathcal{K}_{(m_1,\dots,m_s)}((\xi_1,\dots,\xi_s),(\xi_1',\dots,\xi_s')) = \prod_{i=1}^{s} \Phi_i(\xi_i, \xi_i')^{m_i}.$$

Theorem 2.4. (i) *The group $(K_i)_{\mathbb{R}}$ acts multiplicity free on the space $\mathcal{O}(\Xi_i)$. The irreducible $(K_i)_{\mathbb{R}}$-invariant subspaces of $\mathcal{O}(\Xi_i)$ are the spaces $\mathcal{O}_{m_i}(\Xi_i)$ with $m_i \in \mathbb{N}$. If $\mathcal{H}_i \subset \mathcal{O}(\Xi_i)$ is a $(K_i)_{\mathbb{R}}$-invariant Hilbert subspace, its reproducing kernel can be written $\mathcal{K}_i(\xi_i, \xi_i') = \sum_{m_i\in\mathbb{N}} c_{i,m_i}\Phi_i(\xi_i, \xi_i')^{m_i}$, where (c_{i,m_i}) is a sequence of positive numbers such that the series $\sum_{m_i\in\mathbb{N}} c_{i,m_i}\Phi_i(\xi_i, \xi_i')^{m_i}$ converges uniformly on compact subsets in Ξ_i.*
(ii) *The group $\prod_{i=1}^{s}(K_i)_{\mathbb{R}}$ acts multiplicity free on the space $\otimes_{i=1}^{s}\mathcal{O}(\Xi_i)$. The irreducible $\prod_{i=1}^{s}(K_i)_{\mathbb{R}}$-invariant subspaces of $\otimes_{i=1}^{s}\mathcal{O}(\Xi_i)$ are the spaces $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ with $(m_1,\dots,m_s) \in \mathbb{N}^s$. The tensor product space $\mathcal{H} = \otimes_{i=1}^{s}\mathcal{H}_i \subset \otimes_{i=1}^{s}\mathcal{O}(\Xi_i)$ of $(K_i)_{\mathbb{R}}$-invariant Hilbert spaces is a $\prod_{i=1}^{s}(K_i)_{\mathbb{R}}$-invariant Hilbert subspace and the reproducing kernel of $\mathcal{H}$ is*

$$\mathcal{K}((\xi_1,\dots,\xi_s),(\xi_1',\dots,\xi_s')) = \prod_{i=1}^{s} \mathcal{K}_i(\xi_i, \xi_i').$$

The proof of (i) is similar to that of Theorem 2.5 in [4]. The (ii) is then deduced from (i) for the tensor product situation.

In case of a weighted Bergman space there is an integral formula for the numbers c_{i,p_i}. For a positive function $\nu_i(\xi_i)$ on Ξ_i, consider the subspace $\mathcal{H}_i \subset \mathcal{O}(\Xi_i)$ of functions ϕ_i such that

$$\|\phi_i\|_i^2 = \int_{\mathbb{C}\times V_i} |\phi_i(w_i, z_i)|^2 \nu_i(w_i, z_i) m(dw_i) m_{i,0}(dz_i) \ < \infty,$$

and then consider the subspace $\mathcal{H} = \otimes\mathcal{H}_i \subset \otimes_{i=1}^{s}\mathcal{O}(\Xi_i)$ of ϕ such that

$$\|\phi\|^2 = \int_{\prod_{i=1}^{s}\mathbb{C}\times V_i} |\phi((w_1, z_1),\dots,(w_s, z_s))|^2 \prod_{i=1}^{s} \nu_i(w_i, z_i) m(dw_i) m_{i,0}(dz_i) \ < \infty.$$

Theorem 2.5. *Let F_i be a positive function on $[0, \infty[$. For*

$$\nu_i(w_i, z_i) = F_i(H_i(z_i)|w_i|^2)H_i(z_i).$$

The space $\mathcal{H}_i$ is $(K_i)_{\mathbb{R}}$-invariant. If $\phi_i(w_i, z_i) = \sum_{m_i\in\mathbb{N}} w_i^{m_i}\psi_{i,m_i}(z_i)$, then

$$\|\phi_i\|_i^2 = \sum_{m_i\in\mathbb{N}} \frac{1}{c_{i,m_i}} \|\psi_{i,m_i}\|_{i,m_i}^2$$

with

$$\frac{1}{c_{i,m_i}} = \pi a_{i,m_i} \int_{[0,+\infty[} F_i(u_i) u_i^{m_i} du_i.$$

(iii) *The reproducing kernel of $\mathcal{H}_i$ is given by*

$$\mathcal{K}_i(\xi_i, \xi_i') = \sum_{m_i\in\mathbb{N}} c_{i,m_i}\Phi_i(\xi_i, \xi_i')^{m_i}.$$

(iv) *The space $\mathcal{H}$ is $\prod_{i=1}^{s}(K_i)_{\mathbb{R}}$-invariant. If $\phi = \phi_1 \otimes \cdots \otimes \phi_s$, then*

$$\|\phi\|^2 = \sum_{(m_1,\dots,m_s)\in\mathbb{N}^s} \frac{1}{c_{(m_1,\dots,m_s)}} \prod_{i=1}^{s} \|\psi_{i,m_i}\|_{i,m_i}^2,$$

with $c_{(m_1,\dots,m_s)} = \prod_{i=1}^{s} c_{i,m_i}$. *The reproducing kernel is given by*

$$\mathcal{K}((\xi_1,\dots,\xi_s),(\xi_1',\dots,\xi_s')) = \prod_{i=1}^{s} \mathcal{K}_i(\xi_i,\xi_i').$$

The proof is similar to that of Theorem 2.6. in [4] and (iv) follows by applying (i), (ii) and (iii) to the tensor product situation.

Proposition 2.6. (i) *The polynomial* $\Delta_i^{k_i}$ *satisfies the following Bernstein identity*

$$\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\Delta_i^{k_i\alpha} = B_i(\alpha)\Delta_i^{k_i\alpha-k_i} \quad (z_i \in V_i),$$

with

$$B_i(\alpha) = b_i(k_i\alpha)b_i(k_i\alpha-1)\dots b_i(k_i\alpha-k_i+1),$$

where b_i *is the Bernstein polynomial relative to the determinant* Δ_i.

(ii) *Furthermore*

$$\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)H_i(z)^{k_i\alpha} = B_i(\alpha)\overline{\Delta_i^{k_i}(z_i)}H_i(z_i)^{k_i(\alpha-1)}.$$

Proposition 2.7.

$$a_{i,m_i} = \frac{\Gamma_{\Omega_i}(2\frac{n_i}{r_i})}{\Gamma_{\Omega_i}(\frac{n_i}{r_i})}\prod_{i=1}^{s}\frac{\Gamma_{\Omega_i}(m_i+\frac{n_i}{r_i})}{\Gamma_{\Omega_i}(m_i+2\frac{n_i}{r_i})},$$

where Γ_{Ω_i} *is the Gindikin gamma function of the symmetric cone* Ω_i *in the Euclidean Jordan algebra* $(V_i)_{\mathbb{R}}$.

The proof is similar to that of Proposition 3.2 in [4], in the situation of Δ_i instead of Q.

3. Irreducible representations of the Lie algebra

In the sequel, we construct irreducible representations ρ_q of the Lie algebra $\mathfrak{g}$, which will be the infinitesimal versions of minimal representations. Recall that the group $\prod_{i=1}^{s} K_i$ acts on $\otimes_{i=1}^{s}\mathcal{O}(\Xi_i)$ by $\pi = \otimes_{i=1}^{s}\pi_i$,

$$\big(\pi(g)f\big)(\xi_1,\dots,\xi_s) = \prod_{i=1}^{s} f_i\big(\kappa_i(g_i)^{-1}\xi_i\big), \quad (g=(g_i), f = f_1\otimes\cdots\otimes f_s)$$

which is given in the coordinates $(w,z) = ((w_1,z_1),\dots,(w_s,z_s))$ by

$$\pi(g)\phi((w_1,z_1),\dots,(w_s,z_s)) = \prod_{i=1}^{s}\phi_i(\mu_i(g_i^{-1},z_i)w_i, g_i^{-1}\cdot z_i).$$

This leads to a representation $\mathrm{d}\pi$ of the Lie algebra $\mathfrak{k}$ in $\otimes_{i=1}^{s}\mathcal{O}(\Xi_i)$.

Following the method of R. Brylinski and B. Kostant, we will construct a representation ρ of $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ on some subspace of the space of finite sums

$$\mathcal{O}(\widetilde{\Xi})_{\mathrm{fin}} = \sum_{(m_1,\dots,m_s)\in\mathbb{N}^s}\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi}),$$

such that,

$$\forall X\in\mathfrak{k},\ \rho(X) = \mathrm{d}\pi(X).$$

We define first a representation ρ of the subalgebra generated by E, F, H, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In particular

$$\rho(H) = \mathrm{d}\pi(H) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi(\exp tH).$$

Hence, for $\phi\in\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$,

$$\rho(H)\phi = \Big(\mathcal{E} - \frac{(m_1r_1+\cdots+m_sr_s)}{2}\Big)\phi$$

where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi((w_1,z_1),\dots,(w_s,z_s)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi((w_1,e^tz_1),\dots,(w_s,e^tz_s)).$$

The space $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ can be identified to the space of functions $\phi_{(m_1,\dots,m_s)}((w_1,z_1),\dots,(w_s,z_s)) = \prod_{i=1}^{s} w_i^{m_i}\psi_i(z_i)$ with $\psi_i \in \widetilde{\mathcal{O}}_{m_i}(V_i)$. Then, each $\phi \in \mathcal{O}_{\text{fin}}(\widetilde{\Xi})$ can be written

$$\phi((w_1,z_1),\dots,(w_s,z_s)) = \sum_{(m_1,\dots,m_s)} \phi_{(m_1,\dots,m_s)}((w_1,z_1),\dots,(w_s,z_s)).$$

Let $\mathbb{D}(V)^L$ denote the algebra of L-invariant holomorphic differential operators on the open set of invertible elements of V. One can show that this algebra is isomorphic to the algebra $\mathbb{D}(\Omega)^G$ of G-invariant differential operators on Ω, the symmetric cone associated to the Euclidean real form $V_{\mathbb{R}}$ of V, where G is the connected component of the identity of the group $G(\Omega) = \{g \in GL(V) \mid g\Omega = \Omega\}$. In [4] we established the following:

(1) If V is simple with rank r, degree d and dimension n, and $Q = \Delta$, the determinant polynomial, then $\mathbb{D}(V)^L$ is isomorphic to the algebra $\mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r}$ of symmetric polynomials in r variables. The map

$$D \mapsto \tilde{\gamma}(D), \quad \mathbb{D}(V)^L \to \mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r},$$

is the Harish-Chandra isomorphism, with $\tilde{\gamma}(D)$ determined by

$$\tilde{\gamma}(D_\alpha)(\lambda) = \prod_{j=1}^{r}(\lambda_j - \alpha + \frac{d}{4}(r-1))$$

where $D_\alpha = \Delta(z)^{1+\alpha}\Delta(\frac{\partial}{\partial z})\Delta(z)^{-\alpha}$. (Recall that $\frac{n}{r} = 1 + (r-1)\frac{d}{2}$).

(2) If $V = \sum_{i=1}^{s} V_i$ is semisimple and $Q = \prod_{i=1}^{s}\Delta_i^{k_i}$, $\mathbb{D}(V)^L$ is isomorphic to the algebra $\prod_{i=1}^{s}\mathcal{P}(\mathbb{C}^{r_i})^{\mathfrak{S}_{r_i}}$. The isomorphism is given by $D \mapsto \tilde{\gamma}(D) = \big(\tilde{\gamma}_1(D),\dots,\tilde{\gamma}_s(D)\big)$, where $\tilde{\gamma}_i$ is the Harish-Chandra isomorphism relative to the algebra $\mathbb{D}(V_i)^{L_i}$.

And, for $D \in \mathbb{D}(V)^L$, we defined the adjoint D^* by $D^* = J \circ D \circ J$, where $Jf(z) = f \circ j(z) = f(-z^{-1})$ and established that $\tilde{\gamma}(D^*)(\lambda) = \tilde{\gamma}(D)(-\lambda)$.

In the present setting we define the Maass differential operator $\mathbf{D}_\alpha^{(i)}$ as

$$\begin{aligned}\mathbf{D}_\alpha^{(i)} &= \Delta_i(z_i)^{k_i+\alpha}\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\Delta_i(z_i)^{-\alpha}\\ &= \prod_{j=1}^{k_i}\Delta_i(z_i)^{\alpha+k_i-j+1}\Delta_i\Big(\frac{\partial}{\partial z_i}\Big)\Delta_i(z_i)^{-(\alpha+k_i-j)}.\end{aligned}$$

It is a L_i-invariant holomorphic differential operator on V_i. We write

$$\gamma_\alpha^{(i)}(\lambda^{(i)}) = \tilde{\gamma}_i(\mathbf{D}_\alpha^{(i)})(\lambda^{(i)}).$$

Then

$$\gamma_\alpha^{(i)}(\lambda^{(i)}) = \prod_{j=1}^{r}\Big[\lambda_j^{(i)} - \alpha + \frac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big)\Big]_{k_i}$$

where n_i is the dimension of V_i and where $[a]_k = a(a-1)\dots(a-k+1)$.

Observe that the spaces $\widetilde{\mathcal{O}}_{m_i}(V_i)$, are stable under the operator $\mathbf{D}_\alpha^{(i)}$.

Lemma 3.1. *For $\psi_i \in \widetilde{\mathcal{O}}_{m_i}(V_i)$, the polynomial $\Delta_i\Big(\frac{\partial}{\partial z_i}\Big)\psi_i$ belongs to $\widetilde{\mathcal{O}}_{m_i-1}(V_i)$.*

Proof. Let ψ_i be an element of $\widetilde{\mathcal{O}}_{m_i}(V_i)$. We have to show that the function $(\pi_{i,m_i-1}(\sigma_i)(\Delta_i\Big(\frac{\partial}{\partial z_i}\Big)\psi_i))$ is holomorphic on V_i. In fact, it is given by

$$\begin{aligned}(\pi_{i,m_i-1}(\sigma_i)(\Delta_i\Big(\frac{\partial}{\partial z_i}\Big)\psi_i))(z_i) &= \Delta_i^{(m_i-1)}(z_i)(\Delta_i\Big(\frac{\partial}{\partial z_i}\Big)\psi_i)(-z_i^{-1})\\ &= \Delta_i^{m_i}(z_i)(\Delta_i\Delta_i\Big(\frac{\partial}{\partial z_i}\Big)\psi_i)(-z_i^{-1}) = \Delta_i^{m_i}(z_i)(\mathbf{D}_0^{(i)}\psi_i)(-z_i^{-1})\\ &= \pi_{i,m_i}(\sigma_i)(\mathbf{D}_0^{(i)}\psi_i)(z_i)\end{aligned}$$

and since the polynomial function $\mathbf{D}_0^{(i)}\psi_i$ belongs to $\widetilde{\mathcal{O}}_{m_i}(V_i)$, it follows that $\pi_{i,m_i}(\sigma_i)(\mathbf{D}_0^{(i)}\psi_i)(z_i)$ is holomorphic on V_i. □

We introduce a multiplication operator $\mathcal{M}$ and a differential operator $\mathcal{D}$ on the space $\mathcal{O}(\widetilde{\Xi})_{\text{fin}}$ defined as follows : for $\phi_{(m_1,\dots,m_s)}$ in $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$

$$(\mathcal{M}\phi_{(m_1,\dots,m_s)})((w_1,z_1),\dots,(w_s,z_s)) = \prod_{i=1}^{s} w_i^{m_i+k_i}\psi_i(z_i),$$

and

$$(\mathcal{D}\phi_{(m_1,\dots,m_s)})((w_1,z_1),\dots,(w_s,z_s)) = \prod_{i=1}^{s} w_i^{m_i-k_i}\left(\Delta_i^{k_i}\left(\frac{\partial}{\partial z_i}\right)\psi_i\right)(z_i)$$

where

$$\phi_{(m_1,\dots,m_s)}((w_1,z_1),\dots,(w_s,z_s)) = \prod_{i=1}^{s} w_i^{m_i}\psi_i(z_i).$$

Using the item (v) of Theorem 2.1, and Lemma 3.1, one can see that, the operator $\mathcal{M}$ maps the space $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ into $\mathcal{O}_{(m_1+k_1,\dots,m_s+k_s)}(\widetilde{\Xi})$ and that the operator $\mathcal{D}$ maps $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ into $\mathcal{O}_{(m_1-k_1,\dots,m_s-k_s)}(\widetilde{\Xi})$ if every m_i is $\geq k_i$ and into $\{0\}$ if there is m_i such that $m_i < k_i$.

We denote by $\mathcal{M}^\sigma$ and $\mathcal{D}^\sigma$ the conjugate operators:

$$\mathcal{M}^\sigma = \pi(\sigma)\mathcal{M}\pi(\sigma)^{-1}, \quad \mathcal{D}^\sigma = \pi(\sigma)\mathcal{D}\pi(\sigma)^{-1}.$$

Then

$$\begin{aligned}(\mathcal{M}^\sigma\phi_{(m_1,\dots,m_s)})(w,z) &= (\pi(\sigma)\mathcal{M}\pi(\sigma)\phi_{(m_1,\dots,m_s)})(w,z)\\ &= (\mathcal{M}\pi(\sigma)\phi_{(m_1,\dots,m_s)})((\Delta_1(z_1)w_1,-z_1^{-1}),\dots,(\Delta_s(z_s)w_s,-z_s^{-1})),\end{aligned}$$

and since

$$\begin{aligned}(\pi(\sigma)\phi_{(m_1,\dots,m_s)})(w,z) &= \phi_{(m_1,\dots,m_s)}((\Delta_1(z_1)w_1,-z_1^{-1}),\dots,(\Delta_s(z_s)w_s,-z_s^{-1}))\\ &= \prod_{i=1}^{s} w_i^{m_i}\Delta_i^{m_i}(z_i)\psi_i(-z_i^{-1}) = \prod_{i=1}^{s} w_i^{m_i}\Delta_i^{m_i}(z_i)(\psi_i\circ j_i)(z_i)\end{aligned}$$

it follows that

$$(\mathcal{M}\pi(\sigma)\phi_{(m_1,\dots,m_s)})(w,z) = \prod_{i=1}^{s} w_i^{m_i+k_i}\Delta_i^{m_i}(z_i)(\psi_i\circ j_i)(z_i)$$

and

$$\begin{aligned}&(\mathcal{M}\pi(\sigma)\phi_{(m_1,\dots,m_s)})((\Delta_1(z_1)w_1,-z_1^{-1}),\dots,(\Delta_s(z_s)w_s,-z_s^{-1}))\\ &\quad = \Delta_i^{(m_i+k_i)}(z_i)\Delta_i^{m_i}(-z_i^{-1})(\psi_i\circ j_i)(-z_i^{-1}).\end{aligned}$$

It follows that

$$(\mathcal{M}^\sigma\phi_{(m_1,\dots,m_s)})((w_1,z_1),\dots,(w_s,z_s)) = \prod_{i=1}^{s} w_i^{m_i+k_i}\Delta_i^{k_i}(z_i)\psi_i(z_i).$$

Furthermore, if there is m_i such that $p_i < k_i$, then $(\mathcal{D}^\sigma\phi_{(m_1,\dots,m_s)}) = 0$, and if every m_i is $\geq k_i$, then

$$\begin{aligned}(\mathcal{D}^\sigma\phi_{(m_1,\dots,m_s)})(w,z) &= (\pi(\sigma)\mathcal{D}\pi(\sigma)\phi_{(m_1,\dots,m_s)})(w,z)\\ &= (\mathcal{D}\pi(\sigma)\phi_{(m_1,\dots,m_s)})((\Delta_1(z_1)w_1,-z_1^{-1}),\dots,(\Delta_s(z_s)w_s,-z_s^{-1})),\end{aligned}$$

$$\begin{aligned}(\pi(\sigma)\phi_{(m_1,\dots,m_s)})(w,z) &= \phi_{(m_1,\dots,m_s)}((\Delta_1(z_1)w_1,-z_1^{-1}),\dots,(\Delta_s(z_s)w_s,-z_s^{-1}))\\ &= \prod_{i=1}^{s} w_i^{m_i}\Delta_i^{m_i}(z_i)\psi_i(-z_i^{-1}) = \prod_{i=1}^{s} w_i^{m_i}\Delta_i^{m_i}(z_i)(\psi_i\circ j_i)(z_i),\end{aligned}$$

and

$$(\mathcal{D}\pi(\sigma)\phi_{(m_1,\dots,m_s)})(w,z) = \prod_{i=1}^{s} w_i^{m_i-k_i}\Delta_i^{k_i}\left(\frac{\partial}{\partial z_i}\right)(\Delta_i^{m_i}(\psi_i\circ j_i))(z_i).$$

Then

$$\begin{aligned}(\mathcal{D}^\sigma\phi_{(m_1,\dots,m_s)})(w,z) &= \prod_{i=1}^{s} w_i^{m_i-k_i}(\Delta_i^{k_i-m_i}\Delta_i^{k_i}\left(\frac{\partial}{\partial z_i}\right)\Delta_i^{m_i}(\psi_i\circ j_i))\circ j_i(z_i)\\ &= \prod_{i=1}^{s} w_i^{m_i-k_i}((\mathbf{D}_{-m_i}^{(i)})^*\psi_i)(z_i).\end{aligned}$$

Given a sequence $(\delta_{(m_1,\dots,m_s)})_{(m_1,\dots,m_s)\in\mathbb{N}^s}$, one defines the diagonal operator δ on $\mathcal{O}(\widetilde{\Xi})_{\text{fin}}$ by

$$\delta\Big(\sum_{(m_1,\dots,m_s)\in\mathbb{N}^s}\phi_{(m_1,\dots,m_s)}\Big)=\sum_{(m_1,\dots,m_s)\in\mathbb{N}^s}\delta_{(m_1,\dots,m_s)}\phi_{(m_1,\dots,m_s)},$$

$$\rho(F)=\mathcal{M}-\delta\circ\mathcal{D},\quad \rho(E)=\pi(\sigma)\rho(F)\pi(\sigma)^{-1}=\mathcal{M}^{\sigma}-\delta\circ\mathcal{D}^{\sigma}.$$

Lemma 3.2. $[\rho(H),\rho(E)]=2\rho(E),\quad [\rho(H),\rho(F)]=-2\rho(F).$

Proof. Since $\rho(H)\mathcal{M}$ and $\mathcal{M}\rho(H)$ map $\prod_{i=1}^s w_i^{m_i}\psi_i(z_i)$ to respectively
$(\mathcal{E}-\sum_{i=1}^s\frac{(m_i+k_i)r_i}{2})\prod_{i=1}^s w_i^{m_i+k_i}\psi_i(z_i)$, and $(\mathcal{E}-\sum_{i=1}^s\frac{m_ir_i}{2})\prod_{i=1}^s w_i^{m_i+k_i}\psi_i(z_i)$,
one obtains $[\rho(H),\mathcal{M}]=-\frac{\sum_{i=1}^s k_ir_i}{2}\mathcal{M}=-2\mathcal{M}$.
- If $m_i\geq k_i$, then $\rho(H)\delta\circ\mathcal{D}$ and $\delta\circ\mathcal{D}\rho(H)$ map $\prod_{i=1}^s w_i^{m_i}\psi_i(z_i)$ to

$$\delta_{(m_1-k_1,\dots,m_s-k_s)}\Big(\mathcal{E}-\sum_{i=1}^s\frac{(m_i-k_i)r_i}{2}\Big)\prod_{i=1}^s\Big(\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\psi_i(z_i)w_i^{m_i-k_i}\Big),$$

and

$$\delta_{(m_1-k_1,\dots,m_s-k_s)}\Big(\prod_{i=1}^s\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\Big)\Big(\mathcal{E}-\sum_{i=1}^s\frac{m_ir_i}{2}\Big)\Big(\prod_{i=1}^s\psi_i(z_i)w_i^{m_i-k_i}\Big).$$

Then, by using the identity

$$\Big[\prod_{i=1}^s\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big),\mathcal{E}\Big]=4\prod_{i=1}^s\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big),$$

one obtains

$$[\rho(H),\delta\circ\mathcal{D}]\Big(\prod_{i=1}^s w_i^{m_i}\psi_i(z_i)\Big)=-2\delta_{(m_1-k_1,\dots,m_s-k_s)}\prod_{i=1}^s\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\psi_i(z_i)w_i^{m_i-k_i}$$

i.e.

$$[\rho(H),\rho(F)]\phi_{(m_1,\dots,m_s)}=-2\rho(F)\phi_{(m_1,\dots,m_s)}.$$

- If there is i such that $m_i<k_i$, then

$$\rho(H)\delta\circ\mathcal{D}\phi_{(m_1,\dots,m_s)}=0,\quad \delta\circ\mathcal{D}\rho(H)\phi_{(m_1,\dots,m_s)}=0,$$
$$[\rho(H),\rho(F)]\phi_{(m_1,\dots,m_s)}=[\rho(H),\mathcal{M}]\phi_{(m_1,\dots,m_s)}=-2\rho(F)\phi_{(m_1,\dots,m_s)}.$$

Finally, one gets

$$[\rho(H),\rho(F)]=-2\rho(F).$$

Furthermore, since the δ commutes with $\pi(\sigma)$, and $\pi(\sigma)\rho(H)\pi(\sigma)^{-1}=-\rho(H)$, then $[\rho(H),\rho(E)]=2\rho(E)$. □

Proposition 3.3. *The subspaces $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ are invariant under $[\rho(E),\rho(F)]$ and the restriction of $[\rho(E),\rho(F)]$ to $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ commutes with the $(\prod_{i=1}^s L_i)$-action:*

$$[\rho(E),\rho(F)]:\psi(z)\prod_{i=1}^s w_i^{m_i}\mapsto(P_{(m_1,\dots,m_s)}\psi)(z)\prod_{i=1}^s w_i^{m_i},$$

where $\psi(z)=\prod_{i=1}^s\psi_i(z_i)$ and $P_{(m_1,\dots,m_s)}$ is an $\prod_{i=1}^s L_i$-invariant holomorphic differential operator on V given as follows:
- If every m_i is $\geq k_i$, then

$$P_{(m_1,\dots,m_s)}=\delta_{(m_1,\dots,m_s)}\Big(\prod_{i=1}^s\mathbf{D}_{-1}^{(i)}-\prod_{i=1}^s(\mathbf{D}_{-m_i-k_i}^{(i)})^*\Big)$$
$$+\delta_{(m_1-k_1,\dots,m_s-k_s)}\Big(\prod_{i=1}^s(\mathbf{D}_{-m_i}^{(i)})^*-\prod_{i=1}^s\mathbf{D}_0^{(i)}\Big).$$

- If there is i such that $m_i<k_i$, then

$$P_{(m_1,\dots,m_s)}=\delta_{(m_1,\dots,m_s)}\Big(\Big(\prod_{i=1}^s\mathbf{D}_{-1}^{(i)}-\prod_{i=1}^s(\mathbf{D}_{-m_i-k_i}^{(i)})^*\Big)\Big).$$

Proof. - If every m_i is $\geq k_i$, then the restrictions to $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ are given by
$\mathcal{M}^\sigma\mathcal{D} = \prod_{i=1}^s \mathbf{D}_0^{(i)}$, $\mathcal{D}\mathcal{M}^\sigma = \prod_{i=1}^s \mathbf{D}_{-1}^{(i)}$,
$\mathcal{M}\mathcal{D}^\sigma = \prod_{i=1}^s (\mathbf{D}_{-m_i}^{(i)})^*$, $\mathcal{D}^\sigma\mathcal{M} = \prod_{i=1}^s (\mathbf{D}_{-m_i-k_i}^{(i)})^*$.
Then the restriction of $[\rho(E), \rho(F)]$ to $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ is given by

$$\begin{aligned}
[\rho(E), \rho(F)] &= [\mathcal{M}^\sigma - \delta\circ\mathcal{D}^\sigma, \mathcal{M} - \delta\circ\mathcal{D}] \\
&= [\mathcal{M}, \delta\circ\mathcal{D}^\sigma] + [\delta\circ\mathcal{D}, \mathcal{M}^\sigma] \\
&= \mathcal{M}\delta\mathcal{D}^\sigma - \delta\mathcal{D}^\sigma\mathcal{M} + \delta\mathcal{D}\mathcal{M}^\sigma - \mathcal{M}^\sigma\delta\circ\mathcal{D} \\
&= \delta_{(m_1,\dots,m_s)}(\mathcal{D}\mathcal{M}^\sigma - \mathcal{D}^\sigma\mathcal{M}) + \delta_{(m_1-k_1,\dots,m_s-k_s)}(\mathcal{M}\mathcal{D}^\sigma - \mathcal{M}^\sigma\mathcal{D}) \\
&= \delta_{(m_1,\dots,m_s)}\Big(\prod_{i=1}^s \mathbf{D}_{-1}^{(i)} - \prod_{i=1}^s (\mathbf{D}_{-m_i-k_i}^{(i)})^*\Big) \\
&\quad + \delta_{(m_1-k_1,\dots,m_s-k_s)}\Big(\prod_{i=1}^s (\mathbf{D}_{-m_i}^{(i)})^* - \prod_{i=1}^s \mathbf{D}_0^{(i)}\Big).
\end{aligned}$$

Then, by the Harish-Chandra isomorphism the operator $P_{(m_1,\dots,m_s)}$ corresponds to the polynomial $p_{(m_1,\dots,m_s)} = \tilde{\gamma}(P_{(m_1,\dots,m_s)})$,

$$\begin{aligned}
p_{(m_1,\dots,m_s)}(\lambda) = \delta_{(m_1,\dots,m_s)}&\Big(\prod_{i=1}^s \gamma_{-1}^{(i)}(\lambda^{(i)}) - \prod_{i=1}^s \gamma_{-m_i-k_i}^{(i)}(-\lambda^{(i)})\Big) \\
&+ \delta_{(m_1-k_1,\dots,m_s-k_s)}\Big(\prod_{i=1}^s \gamma_{-m_i}^{(i)}(-\lambda^{(i)}) - \prod_{i=1}^s \gamma_0^{(i)}(\lambda^{(i)})\Big).
\end{aligned}$$

- If there is i such that $m_i < k_i$, then the restrictions to $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ are:

$$\mathcal{M}^\sigma\mathcal{D} = 0,\ \mathcal{D}\mathcal{M}^\sigma = \prod_{i=1}^s \mathbf{D}_{-1}^{(i)},\ \mathcal{M}\mathcal{D}^\sigma = 0,\ \mathcal{D}^\sigma\mathcal{M} = \prod_{i=1}^s (\mathbf{D}_{-m_i-k_i}^{(i)})^*.$$

It follows that the restriction of $[\rho(E), \rho(F)]$ to $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$ is given by

$$\begin{aligned}
[\rho(E), \rho(F)] &= \delta\circ\mathcal{D}\mathcal{M}^\sigma - \delta\circ\mathcal{D}^\sigma\mathcal{M} \\
&= \delta_{(m_1,\dots,m_s)}\Big[\prod_{i=1}^s \mathbf{D}_{-1}^{(i)} - \prod_{i=1}^s (\mathbf{D}_{-m_i-k_i}^{(i)})^*\Big].
\end{aligned}$$

Then, by the Harish-Chandra isomorphism the operator $P_{(m_1,\dots,m_s)}$ corresponds to the polynomial $p_{(m_1,\dots,m_s)} = \tilde{\gamma}(P_{(m_1,\dots,m_s)})$,

$$p_{(m_1,\dots,m_s)}(\lambda) = \delta_{(m_1,\dots,m_s)}\Big(\prod_{i=1}^s \gamma_{-1}^{(i)}(\lambda^{(i)}) - \prod_{i=1}^s \gamma_{-m_i-k_i}^{(i)}(-\lambda^{(i)})\Big)$$

where

$(\lambda = (\lambda^{(1)}, \dots, \lambda^{(s)})$ with $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_{r_i}^{(i)}) \in \mathbb{C}^{r_i})$.

The question is now whether it is possible to choose the sequence $(\delta_{(p_1,\dots,p_s)})$ in such a way that $[\rho(E), \rho(F)] = \rho(H)$.
Recall that, restricted to $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi})$,

$$\rho(H) = \mathcal{E} - \frac{(m_1r_1 + \cdots + m_sr_s)}{2},$$

where $\mathcal{E}$ is the Euler operator

$$(\mathcal{E}\phi_{(m_1,\dots,m_s)})((w_1,z_1),\dots,(w_s,z_s)) = \Big(\prod_{i=1}^s w_i^{m_i}\Big)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(\prod_{i=1}^s \psi_i(e^t z_i)\Big).$$

Then it amounts to checking that,

$$p_{(m_1,\dots,m_s)}(\lambda) = \tilde{\gamma}(\mathcal{E})(\lambda) - \frac{(m_1r_1 + \cdots + m_sr_s)}{2} \qquad (S).$$

From now on, we introduce the subspaces $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ of $\mathcal{O}(\widetilde{\Xi})_{\mathrm{fin}}$, for $q = (q_1, \dots, q_s) \in \mathbb{N}^s$ defined by

$$\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}} = \sum_{m\in\mathbb{N}} \mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi}) \qquad (m_i = k_im + q_i).$$

The subspaces $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ which are obviously stable under $\rho(H)$, are stable under the operators $\rho(E)$ and $\rho(F)$ iff $\exists i$, $q_i < k_i$.

In fact, for every q, the space $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ is stable under the operators $\mathcal{M}$ and $\pi(\sigma)$. But, stability under the operator $\mathcal{D}$ (and $\mathcal{D}^\sigma$) requires the condition $\mathcal{D}(\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}) = \{0\}$ for $m = 0$, which requires the existence of i such that $\Delta^{k_i}(\frac{\partial}{\partial z_i})(\tilde{\mathcal{O}}_{q_i}(V_i)) = 0$. Using (iii) of Theorem 2.1, one gets $q_i < k_i$ for some i.

From now on, assume that there is q_i such that $q_i < k_i$ and denote by $\rho_q(H)$, $\rho_q(E)$ and $\rho_q(F)$ the restrictions of $\rho(H)$, $\rho(E)$ and $\rho(F)$ to $\mathcal{O}_{q,\mathrm{fin}}(\widetilde{\Xi})$.

Theorem 3.4. (i) *For every $m \in \mathbb{N}^*$, one can choose $(\delta_{(k_1m+q_1,\dots,k_sm+q_s)})$ such that* $[\rho_q(H), \rho_q(E)] = 2\rho_q(E)$, $[\rho_q(H), \rho_q(F)] = -2\rho_q(F)$, $[\rho_q(E), \rho_q(F)] = \rho_q(H)$ *iff*

$$\exists \eta_q \text{ such that for all } i, \frac{q_i}{k_i} + \frac{n_i}{k_i r_i} = \eta_q \quad (T)$$

and then, for $A = \prod_{i=1}^s k_i^{k_i r_i}$,

$$\delta_{(k_1m+q_1,\dots,k_sm+q_s)} = \frac{1}{A(m+\eta_q)(m+\eta_q+1)}.$$

(ii) *For $m = 0$, one can choose $(\delta_{(q_1,\dots,q_s)})$ such that* $[\rho_q(H), \rho_q(E)] = 2\rho_q(E)$, $[\rho_q(H), \rho_q(F)] = -2\rho_q(F)$, $[\rho_q(E), \rho_q(F)] = \rho_q(H)$ *iff q is as in Table 2 and δ_q as obtained below.*

Proof. (i) Assume $m \neq 0$. First, recall (from Theorem 2.1) that for $m_i = k_im + q_i$, the space $\mathcal{O}_{(m_1,\dots,m_s)}(\widetilde{\Xi}) = \mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$, consists in polynomials $\prod_{i=1}^s \psi_{m,i}(z_i) w_i^{k_im+q_i}$, the $\psi_{m,i}$ have degree $\leq k_ir_im + q_ir_i$. Moreover, one has

$$\begin{aligned}\prod_{i=1}^s \gamma_\alpha^{(i)}(\lambda^{(i)}) &= \prod_{i=1}^s \prod_{j=1}^{r_i} \Big[\lambda_j^{(i)} - \alpha + \frac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big)\Big]_{k_i} \\ &= \prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} \Big(\lambda_j^{(i)} - \alpha + \frac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big) - (k-1)\Big) \\ &= A \prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} \Big(\frac{\lambda_j^{(i)}}{k_i} - \frac{\alpha}{k_i} + \frac{1}{2k_i}\Big(\frac{n_i}{r_i} - 1\Big) - \frac{k-1}{k_i}\Big).\end{aligned}$$

Denote by

$$X_{jk}^{(i)} = \frac{\lambda_j^{(i)}}{k_i} + \frac{1}{2k_i}\Big(\frac{n_i}{r_i} - 1\Big) - \frac{k-1}{k_i},$$

and

$$b_{k_im+q_i}^{(i)} = m + \frac{q_i}{k_i} + \frac{n_i}{k_ir_i} - 1.$$

Then

$$\begin{aligned}p_{(k_1m+q_1,\dots,k_sm+q_s)}(\lambda) &= A\delta_{(k_1m+q_1,\dots,k_sm+q_s)} \\ &\quad \times \Big(\prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)} + 1) - \prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)} - b_{k_im+q_i}^{(i)} - 1)\Big) \\ &\quad + A\delta_{(k_1(m-1)+q_1,\dots,k_s(m-1)+q_s)} \\ &\quad \times \Big(\prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)} - b_{k_im+q_i}^{(i)}) - \prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)})\Big),\end{aligned}$$

$$\tilde{\gamma}(\mathcal{E})(\lambda) = \sum_{i=1}^s \sum_{j=1}^{r_i} \sum_{k=1}^{k_i} X_{jk}^{(i)} - \frac{1}{2} \sum_{i=1}^s \sum_{j=1}^{r_i} \sum_{k=1}^{k_i} b_{k_im+q_i}^{(i)}.$$

The restriction of $\rho(H)$ to $\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$, is given by

$$\rho_q(H) = \mathcal{E} - \sum_{i=1}^s \frac{k_ir_im + q_ir_i}{2} = \mathcal{E} - 2m - \sum_{i=1}^s \frac{q_ir_i}{2}.$$

The identity (S) is then proved by using the following lemma:

Lemma 3.5. *To a partition $p = (p_1, \dots, p_\ell)$ of 4 and length ℓ: $p_1 + \cdots + p_\ell = 4$, and numbers γ_{ij} $(1 \le i \le \ell, 1 \le j \le p_i - 1)$, one associates the polynomial F in the ℓ variables $T_1, \dots, T_\ell$: $F(T_1, \dots, T_\ell) = \prod_{i=1}^{\ell} T_i \prod_{j=1}^{p_i-1}(T_i + \gamma_{ij})$. Given $\alpha, \beta, c \in \mathbb{R}$, and $b_1, \dots, b_\ell \in \mathbb{R}$, then*

$$\alpha\big(F(T_1+1,\dots,T_\ell+1) - F(T_1-b_1-1,\dots,T_\ell-b_\ell-1)\big) + \beta\big(F(T_1-b_1,\dots,T_\ell-b_\ell) - F(T_1,\dots,T_\ell)\big) = \sum_{i=1}^{\ell} T_i + c$$

is an identity in the variables $T_1, \dots, T_\ell$ iff there exists $b \neq 0$ such that $b_1 = \cdots = b_\ell = b$, $\alpha = \frac{1}{(b+1)(b+2)}$, $\beta = \frac{1}{b(b+1)}$, $c = \sum_{i=1}^{\ell}\sum_{j=1}^{k_i-1}\gamma_{ij} - 2b$.

By this lemma, we see that for every m, the $b^{(i)}_{k_i m+q_i}$ must be equal for every i, i.e. there must exist η_q such that for every i, $\eta_q = \frac{q_i}{k_i} + \frac{n_i}{k_i r_i}$.

(ii) Now, consider the situation of $m = 0$ and determine the cases where the constants $\delta_{(q_1,\dots,q_s)}$ exist. The $\psi_{0,i}$ have degree $\le q_i r_i$. Let us consider the restriction of the operators $\mathcal{D}\mathcal{M}^\sigma$, $\mathcal{D}^\sigma\mathcal{M}$, $\mathcal{M}\mathcal{D}^\sigma$, $\mathcal{M}^\sigma\mathcal{D}$ and $\rho(H)$ to the space $\mathcal{O}_{(q_1,\dots,q_s)}(\widetilde{\Xi})$. The restriction of $\rho(H)$ to $\mathcal{O}_{(q_1,\dots,q_s)}(\widetilde{\Xi})$, is given by

$$\rho_q(H) = \mathcal{E} - \sum_{i=1}^{s} \frac{q_i r_i}{2}.$$

By Proposition 3.3, the restriction of $[\rho(E), \rho(F)]$ to $\mathcal{O}_{(q_1,\dots,q_s)}(\widetilde{\Xi})$, is

$$\delta_{(q_1,\dots,q_s)}\Big[\Big(\prod_{i=1}^{s}\mathbf{D}^{(i)}_{-1} - \prod_{i=1}^{s}(\mathbf{D}^{(i)}_{-p_i-k_i})^*\Big)\Big]$$

then

$$p_{(q_1,\dots,q_s)}(\lambda) = A\delta_{(q_1,\dots,q_s)}\Big(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X^{(i)}_{jk}+1) - \prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X^{(i)}_{jk} - b^{(i)}_{q_i} - 1)\Big),$$

$$\tilde{\gamma}(\mathcal{E})(\lambda) = \sum_{i=1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{k_i} X^{(i)}_{jk} - \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{k_i} b^{(i)}_{q_i}.$$

Let us check the identity (S) case by case.

(1) $\widetilde{\mathrm{SL}}(3,\mathbb{R}) : V = \mathbb{C}, Q(z) = z^4, r = 1, k = 4, q < 4, \eta_q = \frac{q}{4} + \frac{1}{4}$. The functions ϕ_q are given by $\phi_q(w,z) = z^\alpha w^q$ with $\alpha \le q$. The identity (S) becomes

$$\delta_q\Big(\prod_{j=1}^{4}(\alpha+j) - \prod_{j=1}^{4}(q-\alpha+j)\Big) = \alpha - \frac{q}{2}.$$

One has to determine δ_q such that this identity is available for every $\alpha \le q$ such that $\alpha \ge q - \alpha$, i.e. for $\frac{q}{2} \le \alpha \le q$. It follows that for $q = 0$, the identity is trivial and δ_0 is arbitrarily, for $q = 1, \alpha = 1$ and $\delta_1 = \frac{1}{2(120-24)}$, for $q = 2, \alpha = 2$ and δ_2 is arbitrarily and for $q = 3, \alpha = 2$ or $\alpha = 3$ then (S) is impossible.

(2) $\widetilde{\mathrm{SL}}(p+2,\mathbb{R}) : V = \mathbb{C}^p, Q(z) = (z_1^2 + \cdots + z_p^2)^2, r = 2, k = 2, q \in \{0,1\}, \eta_q = \frac{q}{2} + \frac{p}{4}$. The functions ϕ_q are given by $\phi_q(w,z) = \Delta(z)^\alpha w^q$ with $\alpha \le q$. The identity (S) becomes

$$\delta_q\Big((\alpha+2)(\alpha+1) - (q-\alpha+2)(q-\alpha+1)\Big) = 2\alpha - q.$$

One has to determine δ_q such that (S) occurs for every α such that $\frac{q}{2} \le \alpha \le q$. If $q = 0$, the identity is trivial and δ_0 is arbitrarily, and if $q = 1$, it becomes $\delta_1(8\alpha - 4) = 2\alpha - 1$, then $\delta_1 = \frac{1}{4}$.

(3) $\widetilde{\mathrm{SO}}(3,3) : V = \mathbb{C}\oplus\mathbb{C}, Q(z) = z_1^2 z_2^2, r_1 = r_2 = 1, k_1 = k_2 = 2, q_1 = q_2 =: \tilde{q} \in \{0,1\}, \eta_q = \frac{\tilde{q}}{2} + \frac{1}{2}$. The functions ϕ_q are given by $\phi_{(q_1,q_2)}((w_1,z_1),(w_2,z_2)) = z_1^{\alpha_1} z_2^{\alpha_2} w_1^{q_1} w_2^{q_2}$ with $\alpha_1 \le 1$ and $\alpha_2 \le 1$. The identity (S) becomes

$$\delta_{(q_1,q_2)}(\alpha_1+2)(\alpha_1+1)(\alpha_2+2)(\alpha_2+1) - \delta_{(q_1,q_2)}(q_1-\alpha_1+2)(q_1-\alpha_1+1)(q_2-\alpha_2+2)(q_2-\alpha_2+1) = \alpha_1+\alpha_2-\tilde{q}.$$

One has to determine $\delta_q = \delta_{(q_1,q_2)}$ such that this identity is available for every α_i such that $\frac{q_i}{2} \leq \alpha_i \leq q_i$. It follows that, if $\tilde{q} = 0$, then $\alpha_i = 0$ and $\delta_{(0,0)}$ is arbitrarily, and if $\tilde{q} = 1$, then $\alpha_1 = \alpha_2 = 1$ and $\delta_{(1,1)} = \frac{1}{36-4}$.

(4) $\widetilde{\mathrm{SO}}(4,4) : V = \oplus^4\mathbb{C}$, $Q(z) = z_1z_2z_3z_4$, $r_i = 1 = k_i$, $q_i = \tilde{q} = 0$, $\eta_q = \tilde{q} + 1 = 1$. Then $\phi_q = \phi_0$ are constants, (S) is trivial.

(5) $\widetilde{\mathrm{SO}}(p+2,3) : V = \mathbb{C}^p \oplus \mathbb{C}$, $Q(z) = (z_1^2 + \cdots + z_p^2)z'^2$, p odd, $r_1 = 2$, $r_2 = 1$, $k_1 = 1$, $k_2 = 2$, $q_1 = 0$, $q_2 = \frac{p-1}{2}$, $\eta_q = \frac{p}{2}$. The functions ϕ_q are given by $\phi_{(q_1,q_2)}((w,z),(w',z')) = (z')^{\alpha_2}(w')^{q_2}$ with $\alpha_2 \leq q_2$. The identity (S) becomes

$$\delta_{(0,q_2)}[(\alpha_2+2)(\alpha_2+1) - (q_2-\alpha_2+2)(q_2-\alpha_2+1)] = \alpha_2 - \frac{p-1}{4}$$

i.e., $\delta_{(0,q_2)}(p+5)(\alpha_2 - \frac{p-1}{4}) = \alpha_2 - \frac{p-1}{4}$.

One has to determine $\delta_{(0,q_2)}$ such that this identity is available for every α_2 such that $\frac{q_2}{2} \leq \alpha_2 \leq q_2$. It follows that $\delta_{(0,q_2)} = \frac{1}{p+5}$.

(6) $\widetilde{\mathrm{SO}}(p+2,4) : V = \mathbb{C}^p \oplus \mathbb{C} \oplus \mathbb{C}$, $Q(z) = (z_1^2 + \cdots + z_p^2)z'z''$, p even, $r_1 = 2, r_2 = r_3 = 1, k_i = 1$, $(q_1,q_2,q_3) = (0, \frac{p}{2}-1, \frac{p}{2}-1)$, $\eta_q = \frac{p}{2}$. The ϕ_q are given by $\phi_{(q_1,q_2,q_3)}((w,z),(w',z'),(w'',z'')) = (z')^{\alpha_2}(z'')^{\alpha_3}(w')^{q_2}(w'')^{q_3}$, with $\alpha_2 \leq q_2$, $\alpha_3 \leq q_3$. The identity (S) becomes

$$\delta_{(0,q_2,q_3)}[(\alpha_2+1)(\alpha_3+1) - (q_2-\alpha_2+1)(q_3-\alpha_3+1)] = \alpha_2 + \alpha_3 - (\frac{p}{2}-1),$$

i.e. $\delta_{(0,q_2,q_3)}(\frac{p}{2}+1)[\alpha_2+\alpha_3-(\frac{p}{2}-1)] = \alpha_2+\alpha_3-(\frac{p}{2}-1)$. One has to determine $\delta_{(0,q_2,q_3)}$ such that this identity is available for every α_2 and α_3 such that $\frac{q_i}{2} \leq \alpha_i \leq q_i$. It follows that $\delta_{(0,q_2,q_3)} = \frac{1}{\frac{p}{2}+1}$.

(7) $\widetilde{\mathrm{SO}}(p_1+2,p_2+2) : V = \mathbb{C}^{p_1} \oplus \mathbb{C}^{p_2}$, $Q = (z_1^2 + \cdots + z_{p_1}^2)({z'_1}^2 + \cdots + {z'_{p_2}}^2)$,

$r_1 = r_2 = 2, k_1 = k_2 = 1$, $p_1 - p_2 \geq 0$ even, $q_1 = 0$, $q_2 = \frac{p_1-p_2}{2}$, $\eta_q = \frac{p_1}{2} = \frac{p_1}{2}$. The ϕ_q are given by $\phi_{(q_1,q_2)}((w_1,z),(w_2,z')) = \Delta(z')^{\alpha_2}w_1^{q_1}w_2^{q_2}$, with $\alpha_2 \leq q_2$. The identity (S) becomes

$$\delta_{(0,q_2)}[(\alpha_2+1) - (q_2-\alpha_2+1)] = 2\alpha_2 - \frac{p_1-p_2}{2}$$

i.e., $\delta_{(0,q_2)}(2\alpha_2 - \frac{p_1-p_2}{2}) = 2\alpha_2 - \frac{p_1-p_2}{2}$. One has to determine $\delta_{(0,q_2)}$ such that this identity is available for every α_2 such that $\frac{q_2}{2} \leq \alpha_2 \leq q_2$. It follows that if $p_1 = p_2$, then $q = 0$ and δ_0 is arbitrarily, and if $p_1 \neq p_2$, then $\delta_{(0,\frac{p_1-p_2}{2})} = 1$.

(8) $E_{6(6)}, E_{7(7)}, E_{8(8)} : (V, Q(z))$ is respectively the pair.

$$(\mathrm{Sym}(4,\mathbb{C}), \det z);\ (M(4,\mathbb{C}), \det z);\ (\mathrm{Skew}(8,\mathbb{C}), \mathrm{Pfaff}(z)),$$

$r = 4, k = 1, q = 0, \eta_q = q + \frac{n}{4} = 1 + \frac{3d}{2}$, with $d = 1, 2$ or 4 respectively.

The functions ϕ_q are given by $\phi_q(w,z) = c$, the identity (S) is trivial.

(9) $F_{4(4)}, E_{6(2)}, E_{7(-5)}, E_{8(-24)} : (V, Q(z))$ is respectively the pair

$$(\mathrm{Sym}(3,\mathbb{C}) \oplus \mathbb{C}, \det(z_1)\cdot z_2), \quad (M(3,\mathbb{C}) \oplus \mathbb{C}, \det(z_1)\cdot z_2)$$
$$(\mathrm{Skew}(6,\mathbb{C}) \oplus \mathbb{C}, \mathrm{Pfaff}(z_1)\cdot z_2), \quad (\mathrm{Herm}(3,\mathbb{C}) \oplus \mathbb{C}, \det(z_1)\cdot z_2),$$

$r_1 = 3, r_2 = 1, k_1 = k_2 = 1, q_1 = 0, q_2 = \frac{n_1}{3} - 1, \eta_q = q_1 + \frac{n_1}{3} = 1 + d_1$.

The functions ϕ_q are given by $\phi_{(q_1,q_2)}((w_1,z_1),(w_2,z_2)) = z_2^{\alpha_2}w_2^{q_2}$ with $\alpha_2 \leq q_2$. The identity (S) becomes

$$\delta_{(0,q_2)}[(\alpha_2+1) - (q_2-\alpha_2+1)] = \alpha_2 - \frac{q_2}{2}$$

i.e. $\delta_{(0,q_2)}(2\alpha_2 - q_2) = \alpha_2 - \frac{q_2}{2}$. One has to determine $\delta_{(0,q_2)}$ such that this identity is available for every α_2 such that $\frac{q_2}{2} \leq \alpha_2 \leq q_2$. Then $\delta_{(0,q_2)} = \frac{1}{2}$.

(10) $G_{2(2)} : V = \mathbb{C} \oplus \mathbb{C}$, $Q = z_1^3z_2$, $r_1 = r_2 = 1$, $k_1 = 3, k_2 = 1$, $q_1 = 2, q_2 = 0$, $\eta_q = \frac{q_1}{3} + \frac{1}{3} = 1$. The functions ϕ_q are given by $\phi_{(2,0)}((w_1,z_1),(w_2,z_2)) = z_1^{\alpha_1}w_1^{q_1}$ with $\alpha_1 \leq 2$. The identity (S) becomes

$$\delta_{(2,0)}[(\alpha_1+1) - (q_1-\alpha_1+1)] = \alpha_1 - \frac{q_1}{2}$$

i.e. $\delta_{(2,0)}(2\alpha_1 - 2) = \alpha_1 - 1$. One has to determine $\delta_{(2,0)}$ such that this identity is available for every α_1 such that $\frac{q_1}{2} = 1 \leq \alpha_1 \leq q_1 = 2$. Then $\delta_{(2,0)} = \frac{1}{2}$.

Now, we extend ρ to a linear map from $\mathfrak{p}$ to the space $\mathrm{End}(\mathcal{O}(\widetilde{\Xi})_{\mathrm{fin}})$.

Proposition 3.6. (i) *There is a unique linear map* $\mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\widetilde{\Xi})_{\mathrm{fin}}\big)$, $p \mapsto \mathcal{M}(p)$, *such that* $\mathcal{M}(1) = \mathcal{M}$, $\mathcal{M}(Q) = \mathcal{M}^\sigma$ *and, for* $X \in \mathfrak{k}$, $\mathcal{M}\big([X,p]\big) = [\mathrm{d}\pi(X), \mathcal{M}(p)]$.

(ii) *There is a unique linear map* $\mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\widetilde{\Xi})_{\mathrm{fin}}\big)$, $p \mapsto \mathcal{D}(p)$, *such that* $\mathcal{D}(1) = \mathcal{D}$, $\mathcal{D}(Q) = \mathcal{D}^\sigma$ *and, for* $X \in \mathfrak{k}$, $\mathcal{D}\big([X,p]\big) = [\mathrm{d}\pi(X), \mathcal{D}(p)]$.

(iii) $\mathcal{M}\big(\tilde{\kappa}(g)p\big) = \pi(g)\mathcal{M}(p)\pi(g^{-1})$ *and* $\mathcal{D}\big(\tilde{\kappa}(g)p\big) = \pi(g)\mathcal{D}(p)\pi(g^{-1})$.

Hence we get a map $\rho : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\widetilde{\Xi})_{\mathrm{fin}}\big)$, where

$$\rho(X) := \mathrm{d}\pi(X) \quad (X \in \mathfrak{k}), \qquad \rho(p) := \mathcal{M}(p) - \delta \circ \mathcal{D}(p) \quad (p \in \mathfrak{p}).$$

Since the subspaces $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ are stable under ρ, we denote by ρ_q the restriction of ρ to $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$.

Theorem 3.7. *Assume that the constant η_q of Theorem 3.4 exists. Fix $(\delta_{(k_1m+q_1,\dots,k_sm+q_s)})$ as in Theorem 3.4.*
(i) *ρ_q is a representation of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$.*
(ii) *The representation ρ_q is irreducible.*

Proof. (i) Since π is a representation of $\widetilde{K} := \prod_{i=1}^{s} K_i$, for $X, X' \in \mathfrak{k}$, $\rho_q([X, X']) = [\rho_q(X), \rho_q(X')]$. It follows from Proposition 3.3 that for $X \in \mathfrak{k}, p \in \mathfrak{p}$, $\rho_q([X, p]) = [\rho_q(X), \rho_q(p)]$. It remains to show that for $p, p' \in \mathfrak{p}$, $\rho_q([p, p']) = [\rho_q(p), \rho_q(p')]$. From Theorem 3.4, $[\rho_q(E), \rho_q(F)] = \rho_q([E, F])$. Then this follows from [BK95 , Lemma 3.6]. Consider the map $\tau_q : \bigwedge^2 \mathfrak{p} \to \mathrm{End}(\mathcal{O}_{q,\mathrm{fin}})$, defined by $\tau_q(p \wedge p') = [\rho_q(p), \rho_q(p')] - \rho_q([p, p'])$. We know that $\tau_q(E \wedge F) = 0$. We have to show that $\tau_q \equiv 0$ on the space $\bigwedge^2 \mathfrak{p}$. Since the group K acts on $\mathfrak{p}$ by the representation $\tilde{\kappa} = \otimes \kappa_i$ and that $d\tilde{\kappa} = d\kappa$. It follows that K acts on the space $\bigwedge^2 \mathfrak{p}$ by the representation (still denoted $\tilde{\kappa}$) given by $\tilde{\kappa}(g)(p \wedge p') = (\tilde{\kappa}(g)p) \wedge (\tilde{\kappa}(g)p')$ and by differentiation, the Lie algebra $\mathfrak{k}$ acts on the space $\bigwedge^2 \mathfrak{p}$ by the representation (still denoted $d\kappa$) given by $d\kappa(X)(p \wedge p') = (\mathrm{d}\kappa(X)p) \wedge p' + p \wedge (d\kappa(X)p')$. Since the representation $\mathrm{d}\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible and since E and F are highest weight and lowest weight vectors with respect to $\mathfrak{k}_0 + \mathfrak{k}_1$ and $\mathfrak{k}_0 + \mathfrak{k}_{-1}$ respectively and since $\mathfrak{p}_1 = \mathrm{d}\kappa(\mathfrak{k}_{-1})E, \mathfrak{p}_{-1} = \mathrm{d}\kappa(\mathfrak{k}_1)F$ and $\mathfrak{p}_0 = \mathrm{d}\kappa(\mathfrak{k}_{-1})(d\kappa(\mathfrak{k}_{-1}))E = d\kappa(\mathfrak{k}_1)(d\kappa(\mathfrak{k}_1))F$, it follows that $E \wedge F$ is cyclic for the action of $\mathfrak{k}$ on $\bigwedge^2 \mathfrak{p}$, i.e. $\bigwedge^2 \mathfrak{p} = \mathcal{U}(\mathfrak{k})(E \wedge F)$.

Furthermore, using Proposition 3.3, one can write for every $X \in \mathfrak{k}, p \in \mathfrak{p}$, $\rho_q(d\kappa(X)p) = [\mathrm{d}\pi(X), \rho_q(p)]$. Then, for every $X \in \mathfrak{k}, p, p' \in \mathfrak{p}$,

$$\begin{aligned}
\tau_q(\mathrm{d}\kappa(X)(p \wedge p')) &= \tau_q(\mathrm{d}\kappa(X)p \wedge p') + \tau_q(p \wedge \mathrm{d}\kappa(X)p') \\
&= [\rho_q(\mathrm{d}\kappa(X)p), \rho_q(p')] - \rho_q([\mathrm{d}\kappa(X)p, p']) \\
&\quad + [\rho_q(p), \rho_q(\mathrm{d}\kappa(X)p')] - \rho_q([p, \mathrm{d}\kappa(X)p']) \\
&= [[\mathrm{d}\pi(X), \rho_q(p)], \rho_q(p')] - \rho_q([[X, p], p']) \\
&\quad + [\rho_q(p), [\mathrm{d}\pi(X), \rho_q(p')]] - \rho_q([p, [X, p']]) \\
&= [\mathrm{d}\pi(X), [\rho_q(p), \rho_q(p')]] - \rho_q([X, [p, p']]) \\
&= [\mathrm{d}\pi(X), [\rho_q(p), \rho_q(p')]] - [\mathrm{d}\pi(X), \rho_q([p, p'])] \\
&= [\mathrm{d}\pi(X), \tau_q(p \wedge p')].
\end{aligned}$$

We deduce that for every $X \in \mathfrak{k}$, $\tau_q(\mathrm{d}\kappa(X)(E \wedge F)) = 0$ and, since $E \wedge F$ is cyclic, it follows that $\tau_q \equiv 0$.

(ii) Let $\mathcal{V} \neq \{0\}$ be a $\rho_q(\mathfrak{g})$-invariant subspace of $\mathcal{O}_{q,\mathrm{fin}}(\widetilde{\Xi})$. Then $\mathcal{V}$ is $\rho(\mathfrak{k})$-invariant. As $\mathcal{O}_{q,\mathrm{fin}}(\widetilde{\Xi}) = \sum_{m\in\mathbb{N}} \mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$ and as the subspaces $\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$ are $\rho_q(\mathfrak{k})$-irreducible, then there exists $\mathcal{J} \subset \mathbb{N}$ $(\mathcal{J} \neq \emptyset)$ such that $\mathcal{V} = \sum_{m\in\mathcal{J}} \mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$.

Observe now that if $\mathcal{V}$ contains $\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$ then it contains $\mathcal{O}_{(k_1(m+1)+q_1,\dots,k_s(m+1)+q_s)}(\widetilde{\Xi})$. In fact, denote by $\phi_{m,q} = \otimes_{i=1}^{s} \phi_{k_im+q_i}$ with $\phi_{k_im+q_i} \in \mathcal{O}_{k_im+q_i}(\Xi_i)$ given by $\phi_{k_im+q_i}(w_i, z_i) = w_i^{k_im+q_i}$. As $\mathcal{D}\phi_{m,q} = 0$, it follows that $\rho_q(F)\phi_{m,q} = \mathcal{M}\phi_{m,q} = \phi_{m+1,q}$ belongs to $\mathcal{O}_{(k_1(m+1)+q_1,\dots,k_s(m+1)+q_s)}(\widetilde{\Xi})$. Then $\mathcal{O}_{(k_1(m+1)+q_1,\dots,k_s(m+1)+q_s)}(\widetilde{\Xi}) \subset \mathcal{V}$. Let m_0 be the minimum of the m such that $\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi}) \subset \mathcal{V}$, then $\mathcal{V} = \sum_{m=m_0}^{\infty} \mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$. The function $\phi = \otimes_{i=1}^{s} \phi_i$ with $\phi_i(w_i, z_i) = \Delta_i^{k_im+q_i} w_i^{k_im+q_i}$ belongs to $\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$ and for $(w, z) = ((w_1, z_1), \dots, (w_s, z_s))$,

$$\begin{aligned}
\rho_q(F)\phi(w, z) &= \prod_{i=1}^{s} \Delta_i^{(k_im+q_i)}(z_i) w_i^{k_im+q_i+p_i} \\
&\quad - \delta_{(k_1(m-1)+q_1,\dots,k_s(m-1)+q_s)} \prod_{i=1}^{s} \Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big) \Delta_i^{k_im+q_i}(z_i) w_i^{k_im+q_i-p_i}.
\end{aligned}$$

By the Bernstein identity (Proposition 2.6)

$$\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big) \Delta_i^{k_i(m+\frac{q_i}{k_i})} = B_i\Big(m + \frac{q_i}{k_i}\Big) \Delta_i^{k_i(m+\frac{q_i}{k_i})-k_i}$$

with $B_i(m + \frac{q_i}{k_i}) = b_i(k_im + q_i)b_i(k_im + q_i - 1) \dots b_i(k_im + q_i - k_i + 1)$, where b_i is the Bernstein polynomial relative to the determinant Δ_i.

Since $B_i(m + \frac{q_i}{k_i}) > 0$ for $m > 0$, then, if $\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi}) \subset \mathcal{V}$, then $\mathcal{O}_{(k_1(m-1)+q_1,\dots,k_s(m-1)+q_s)}(\widetilde{\Xi}) \subset \mathcal{V}$. Therefore $m_0 = 0$ and $\mathcal{V} = \mathcal{O}_{q,\mathrm{fin}}(\widetilde{\Xi})$. □

We give in the following table, all the cases where ρ_q is a representation of the Lie algebra $\mathfrak{g}$.

Table 2

(1) $\widetilde{\mathrm{SL}}(3,\mathbb{R}) : V=\mathbb{C}, Q(z)=z^4, q\in\{0,1,2\}, \eta_q=\frac{q}{4}+\frac{1}{4}$.

(2) $\widetilde{\mathrm{SL}}(p+2,\mathbb{R}) : V=\mathbb{C}^p, Q(z)=(z_1^2+\cdots+z_p^2)^2, q\in\{0,1\}, \eta_q=\frac{q}{2}+\frac{p}{4}$.

(3) $\widetilde{\mathrm{SO}}(3,3) : V=\mathbb{C}\oplus\mathbb{C}, Q(z)=z_1^2z_2^2, q_1=q_2=:\tilde{q}\in\{0,1\}, \eta_q=\frac{\tilde{q}}{2}+\frac{1}{2}$.

(4) $\widetilde{\mathrm{SO}}(4,4) : V=\oplus^4\mathbb{C}, Q(z)=z_1z_2z_3z_4, q_i=\tilde{q}=0, \eta_q=1$.

(5) $\widetilde{\mathrm{SO}}(p+2,3) : V=\mathbb{C}^p\oplus\mathbb{C}, Q(z)=(z_1^2+\cdots+z_p^2)z'^2, p$ odd
$q_1=0, q_2=\frac{p-1}{2}, \eta_q=\frac{p}{2}$.

(6) $\widetilde{\mathrm{SO}}(p+2,4) : V=\mathbb{C}^p\oplus\mathbb{C}\oplus\mathbb{C}, Q(z)=(z_1^2+\cdots+z_p^2)z'z'',$
p even, $q_1=0, q_2=q_3=\frac{p}{2}-1, \eta_q=\frac{p}{2}$.

(7) $\widetilde{\mathrm{SO}}(p_1+2,p_2+2) : (V,Q(z))$ is the pair
$\mathbb{C}^{p_1}\oplus\mathbb{C}^{p_2}, (z_1^2+\cdots+z_{p_1}^2)({z'_1}^2+\cdots+{z'_{p_2}}^2)$,
$p_1-p_2\geq 0$ even, $q_1=0, q_2=\frac{p_1-p_2}{2}, \eta_q=\frac{p_1}{2}$.

(8) $E_{6(6)}, E_{7(7)}, E_{8(8)} : (V,Q(z))$ is respectively the pair
$(\mathrm{Sym}(4,\mathbb{C}),\det z)\,;(M(4,\mathbb{C}),\det z)\,;(\mathrm{Skew}(8,\mathbb{C}),\mathrm{Pfaff}(z))$,
$q=0, \eta_q=1+\frac{3d}{2}$, with $d=1,2$ or 4 respectively.

(9) $F_{4(4)}, E_{6(2)}, E_{7(-5)}, E_{8(-24)} : (V,Q(z))$ is respectively the pair
$(\mathrm{Sym}(3,\mathbb{C})\oplus\mathbb{C},\det(z_1)\cdot z_2)$
$(M(3,\mathbb{C})\oplus\mathbb{C},\det(z_1)\cdot z_2)$
$(\mathrm{Skew}(6,\mathbb{C})\oplus\mathbb{C},\mathrm{Pfaff}(z_1)\cdot z_2)$
$(\mathrm{Herm}(3,\mathbb{C})\oplus\mathbb{C},\det(z_1)\cdot z_2)$,
$q_1=0, q_2=\frac{n_1}{3}-1, \eta_0=1+d_1$, with $d_1=1,2,4,8$.

(10) $G_{2(2)} : V=\mathbb{C}\oplus\mathbb{C}, Q=z_1^3z_2, q_1=2, q_2=\frac{q_1-2}{3}=0, \eta_q=\frac{q_1}{3}+\frac{1}{3}=1.$
$(p,p_1,p_2\geq 2)$

4. Irreducible unitary representations of the corresponding real Lie group

We consider, for a sequence $(c_{(k_1m+q_1,\ldots,k_sm+q_s)})$ of positive numbers, an inner product on $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ such that

$$\|\phi\|^2=\sum_{m\in\mathbb{N}}\frac{1}{c_{(k_1m+q_1,\ldots,k_sm+q_s)}}\|\psi_{(k_1m+q_1,\ldots,k_sm+q_s)}\|^2_{(k_1m+q_1,\ldots,k_sm+q_s)},$$

for

$$\phi(w,z)=\sum_{m\in\mathbb{N}}\psi_{(k_1m+q_1,\ldots,k_sm+q_s)}(z_1,\ldots,z_s)w_1^{k_1m+q_1}\ldots w_s^{k_sm+q_s}.$$

This inner product is invariant under $\widetilde{K}_{\mathbb{R}}=\prod_{i=1}^s(K_i)_{\mathbb{R}}$. We assume that the constant η_0 of Theorem 3.4. and the constants $\delta_{(k_1m+q_1,\ldots,k_sm+q_s)}$ exist, and we will determine the sequence $(c_{(k_1m+q_1,\ldots,k_sm+q_s)})$ such that this inner product is invariant under the representation ρ_q restricted to $\mathfrak{g}_{\mathbb{R}}$. We denote by $\mathcal{F}_q(\widetilde{\Xi})$ the Hilbert space completion of $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ with respect to this inner product. We will assume $c_{(q_1,\ldots,q_s)}=1$.

Recall from Proposition 2.6, the Bernstein identity

$$\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\Delta_i(z_i)^{k_im+q_i}=B_i\Big(m+\frac{q_i}{k_i}\Big)\Delta_i(z_i)^{k_im+q_i-k_i}$$

where

$$B_i(\alpha)=b_i(k_i\alpha)b_i(k_i\alpha-1)\ldots b_i(k_i\alpha-k_i+1),$$

and b_i is the Bernstein polynomial relative to the determinant polynomial Δ_i, given by $b_i(\alpha)=\alpha(\alpha+\frac{d_i}{2})\ldots(\alpha+(r_i-1)\frac{d_i}{2})$ with d_i is the degree of the simple summand V_i, $(\frac{n_i}{r_i}=1+(r_i-1)\frac{d_i}{2})$. The Bernstein polynomial $B(\alpha)=\prod_{i=1}^s B_i(\alpha)$ is of degree 4, and vanishes at 0. The roots of B are: $\frac{x}{k_i}-y\frac{d_i}{2k_i}$ with $1\leq i\leq s, x,y\in\mathbb{N},\quad 0\leq x\leq k_i-1,\quad 0\leq y\leq r_i-1$.

We have two cases:

First case: $B(1-\eta_q)=0$. This is the case if $\exists i, q_i=0$. Let us denote by a_0 and b_0 the two other roots of $B(\alpha)$, such that $B(\alpha)=A\alpha(\alpha+\eta_q-1)(\alpha-a_0)(\alpha-b_0)$.

Second case: $B(1-\eta_q)\neq 0$. This is the case if $\forall i, q_i\neq 0$. Let us denote by a'_0, b'_0, c'_0 the three other zeros of B, in such a way that $B(\alpha)=A\alpha(\alpha-a'_0)(\alpha-b'_0)(\alpha-c'_0)$.

The remaining roots of the Bernstein polynomial are given in the following tables, Table 3 for the case 1, and Table 4 for the case 2.

Table 3

	q	η_q	$1-\eta_q$	a_0	b_0
(1)	$q=0$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$
(2)	$q=0$	$\frac{p}{4}$	$1-\frac{p}{4}$	$\frac{1}{2}$	$\frac{1}{2}-\frac{p}{4}$
(3)	$q_1=q_2=0$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
(4)	$q_1=q_2=q_3=q_4=0$	1	0	0	0
(5)	$q_1=0, q_2=p-1$	$\frac{p}{2}$	$1-\frac{p}{2}$	$\frac{1}{2}$	0
(6)	$q_1=0, p$ even, $q_2=q_3=\frac{p}{2}-1$	$\frac{p}{2}$	$1-\frac{p}{2}$	0	0
(7)	$q_1=0, p_1+p_2$ even, $q_2=\frac{p_1-p_2}{2}$	$\frac{p_1}{2}$	$1-\frac{p_1}{2}$	0	$1-\frac{p_2}{2}$
(8)	$1+\frac{3d}{2}$	$-\frac{3d}{2}$	$-2\frac{d}{2}$	$-\frac{d}{2}$	
(9)	$q_1=0, q_2=d_1$	$1+d_1$	$-d_1$	$-\frac{d_1}{2}$	0
(10)	$q_1=2, q_2=0$	1	$-q_2=0$	$\frac{1}{3}$	$\frac{2}{3}$

Table 4

	q	η_q	c_0'	a_0'	b_0'
(1)	$q\in\{1,2\}$	$\frac{q}{4}+\frac{1}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$
(2)	$q=1, p\neq 2$	$\frac{1}{2}+\frac{p}{4}$	$1-\frac{p}{4}$	$\frac{1}{2}$	$\frac{1}{2}-\frac{p}{4}$
(3)	$\tilde{q}=1, q_1=q_2=1$	$\frac{\tilde{q}}{2}+\frac{1}{2}=1$	$\frac{1}{2}$	0	$\frac{1}{2}$

Theorem 4.1. *Case* 1 : *Assume* $B(1-\eta_q)=0$. *Then*
(i) *The inner product of* $\mathcal{F}_q(\widetilde{\Xi})$ *is* $\mathfrak{g}_{\mathbb{R}}$*-invariant iff*

$$c_{(k_1m+q_1,\dots,k_sm+q_s)}=\frac{(\eta_q+1)_m}{(\eta_q+a_0)_m(\eta_q+b_0)_m}\frac{1}{m!}.$$

(ii) *The reproducing kernel of* $\mathcal{F}_q(\widetilde{\Xi})$ *is given by*

$$\mathcal{K}(\xi,\xi')={}_1F_2\Big(\eta_q+1;\eta_q+a_0,\eta_q+b_0;\prod_{i=1}^{s}H_i(z_i,z_i')^{k_i}(w_i\overline{w_i'})^{k_i}\Big)\mathcal{K}_q(\xi,\xi').$$

Case 2 : *Assume* $B(1-\eta_q)\neq 0$. *Then*
(i) *The inner product of* $\mathcal{F}_q(\widetilde{\Xi})$ *is* $\mathfrak{g}_{\mathbb{R}}$*-invariant iff*

$$c_{(k_1m+q_1,\dots,k_sm+q_s)}=\frac{(\eta_q+1)_m(1)_m}{(\eta_q+a_0')_m(\eta_q+b_0')_m(\eta_q+c_0')_m}\frac{1}{m!}.$$

(ii) *The reproducing kernel of* $\mathcal{F}_q(\widetilde{\Xi})$ *is given by*

$$\mathcal{K}(\xi,\xi')={}_2F_3\Big(\eta_q+1,1;\eta_q+a_0',\eta_q+b_0',\eta_q+c_0';\prod_{i=1}^{s}H_i(z_i,z_i')^{k_i}(w_i\overline{w_i'})^{k_i}\Big)\mathcal{K}_q(\xi,\xi')$$

where $\xi=((w_1,z_1),\dots,(w_s,z_s))$, $\xi'=((w_1',z_1'),\dots,(w_s',z_s'))$, *and*

$$\mathcal{K}_q(\xi,\xi')=\prod_{i=1}^{s}H_i(z_i,z_i')^{q_i}(w_i\overline{w_i'})^{q_i}$$

is the reproducing kernel of the space $\mathcal{O}_{(q_1,\dots,q_s)}(\widetilde{\Xi})$.

(Considering the values of η_q+1, η_q+a_0', η_q+b_0', η_q+c_0' *occurring in Table 4, the function* ${}_2F_3$ *becomes* ${}_1F_2$*).*

Proof. (i) Recall that $\mathfrak{p}_{\mathbb{R}}=\{p\in\mathfrak{p}\mid\beta(p)=p\}$, where β is the conjugation of $\mathfrak{p}$, we introduced at the end of Section 1. Recall also that $\beta(\kappa(g)p)=\kappa\big(\alpha(g)\big)\beta(p)$.

The inner product of $\mathcal{F}_q(\widetilde{\Xi})$ is $\mathfrak{g}_{\mathbb{R}}$-invariant if and only if, for every $p\in\mathfrak{p}$,

$$\rho_q(p)^*=-\rho_q\big(\beta(p)\big).$$

But this is equivalent to the single condition $\rho_q(E)^*=-\rho_q(F)$ i.e, for $\phi\in\mathcal{O}_{(k_1(m+1)+q_1,\dots,k_s(m+1)+q_s)}(\widetilde{\Xi})$, $\phi'\in\mathcal{O}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$,

$$\frac{1}{c_{(k_1(m+1)+q_1,\dots,k_s(m+1)+q_s)}}(\phi\mid\mathcal{M}^{\sigma}\phi')=\frac{\delta_{(k_1m+q_1,\dots,k_sm+q_s)}}{c_{(k_1m+q_1,\dots,k_sm+q_s)}}(\mathcal{D}\phi\mid\phi').$$

Recall that the norm of $\psi = \otimes_{i=1}^{s}\psi_i \in \tilde{\mathcal{O}}_{(k_1m+q_1,\dots,k_sm+q_s)}(\widetilde{\Xi})$ can be written

$$\|\psi\|^2_{(k_1m+q_1,\dots,k_sm+q_s)} = \frac{1}{a_{(k_1m+q_1,\dots,k_sm+q_s)}}\prod_{i=1}^{s}\int_{V_i}|\psi_i(z_i)|^2 H_i(z_i)^{-k_im-q_i-2\frac{n_i}{r_i}}\,m(dz_i).$$

Then, the required condition of invariance becomes

$$\begin{aligned}&\frac{1}{c_{(m+1)}a_{(m+1)}}\int_{\prod_{i=1}^{s}V_i}\prod_{i=1}^{s}\psi_i(z_i)\Delta_i(z_i)^{k_i}\overline{\psi_i'(z)}H_i(z_i)^{-k_i(m+1)-q_i-2\frac{n_i}{r_i}}\,m(dz_i)\\ &\quad=\frac{\delta_{(m)}}{c_{(m+1)}a_{(m+1)}}\int_{\prod_{i=1}^{s}V_i}\prod_{i=1}^{s}\Big(\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\psi_i\Big)(z_i)\overline{\psi_i'(z_i)}H_i(z_i)^{-k_im-q_i-2\frac{n_i}{r_i}}\,m(dz_i)\end{aligned}$$

where we denote

$$x_{(m)} = x_{(k_1m+q_1,\dots,k_sm+q_s)}.$$

By integrating by parts:

$$\begin{aligned}&\int_{\prod_{i=1}^{s}V_i}\prod_{i=1}^{s}\Big(\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\psi_i\Big)(z)\overline{\psi_i'(z)}H_i(z_i)^{-k_im-q_i-2\frac{n_i}{r_i}}\,m(dz_i)\\ &\quad=\int_{\prod_{i=1}^{s}V_i}\prod_{i=1}^{s}\psi_i(z_i)\overline{\psi_i'(z_i)}\Big(\Big(\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)\Big)H_i(z_i)^{-k_im-q_i-2\frac{n_i}{r_i}}\Big)m(dz_i),\end{aligned}$$

and, by the relation

$$\Delta_i^{k_i}\Big(\frac{\partial}{\partial z_i}\Big)H_i(z)^{-k_im-q_i-2\frac{n_i}{r_i}} = B_i\Big(-m-\frac{q_i}{k_i}-2\frac{n_i}{k_ir_i}\Big)\overline{\Delta_i^{k_i}(z_i)}H_i(z_i)^{-k_i(m+1)-q_i-2\frac{n_i}{r_i}},$$

the invariance condition can be written

$$\frac{c_{(m)}}{c_{(m+1)}} = \frac{a_{(m+1)}}{a_{(m)}}\delta_{(m)}\prod_{i=1}^{s}B_i\Big(-m-\frac{q_i}{k_i}-2\frac{n_i}{k_ir_i}\Big).$$

From Proposition 2.7 it follows that

$$\frac{a_{(m+1)}}{a_{(m)}} = \frac{\prod_{i=1}^{s}B_i(-m-\frac{q_i}{k_i}-\frac{n_i}{k_ir_i})}{\prod_{i=1}^{s}B_i(-m-\frac{q_i}{k_i}-2\frac{n_i}{k_ir_i})}.$$

Then we obtain

$$\frac{c_{(m)}}{c_{(m+1)}} = \delta_{(m)}\prod_{i=1}^{s}B_i\Big(-m-\frac{q_i}{k_i}-\frac{n_i}{k_ir_i}\Big),$$

$$\begin{aligned}\frac{c_{(m+1)}}{c_{(m)}} &= \frac{(m+\eta_q)(m+\eta_q+1)}{B(-m-\eta_0)}\\ &= \frac{(m+\eta_q+1)}{(m+\eta_q+a_0')(m+\eta_q+b_0')(m+\eta_q+c_0')}.\end{aligned}$$

Since $c_{(q_1,\dots,q_s)} = 1$, we get
in case 1, i.e. if $c_0' = 1-\eta_q$, $a_0' = a_0$, $b_0' = b_0$, then

$$c_{(m)} = \frac{(\eta_q+1)_m}{(\eta_q+a_0)_m(\eta_q+b_0)_m}\frac{1}{m!}$$

and in case 2, we obtain :

$$c_{(m)} = \frac{(\eta_q+1)_m(1)_m}{(\eta_q+a_0')_m(\eta_q+b_0')_m(\eta_q+c_0')_m}\frac{1}{m!}.$$

(ii) By Theorem 2.5 the reproducing kernel of $\mathcal{F}_q(\widetilde{\Xi})$ is in case 1 given by

$$\begin{aligned}\mathcal{K}(\xi,\xi') &= \sum_{m\in\mathbb{N}} c_{(k_1m+q_1,\dots,k_sm+q_s)} \prod_{i=1}^{s} H_i(z_i,z_i')^{k_im+q_i}(w_i\overline{w_i'})^{k_im+q_i}\\ &= {}_1F_2\Big(\eta_q+1;\eta_q+a_0,\eta_q+b_0;\prod_{i=1}^{s} H_i(z_i,z_i')^{k_i}(w_i\overline{w_i'})^{k_i}\Big)\mathcal{K}_q(\xi,\xi'),\end{aligned}$$

and in case 2 it is given by

$$\begin{aligned}\mathcal{K}(\xi,\xi') &= \sum_{m\in\mathbb{N}} c_{(k_1m+q_1,\dots,k_sm+q_s)} \prod_{i=1}^{s} H_i(z_i,z_i')^{k_im+q_i}(w_i\overline{w_i'})^{k_im+q_i}\\ &= {}_2F_3\Big(1,\eta_q+1;\eta_q+a_0',\eta_q+b_0',\eta_q+c_0';\prod_{i=1}^{s} H_i(z_i,z_i')^{k_i}(w_i\overline{w_i'})^{k_i}\Big)\mathcal{K}_q(\xi,\xi'),\\ &\quad(\xi=((w_1,z_1),\dots,(w_s,z_s)),\xi'=((w_1',z_1'),\dots,(w_s',z_s'))).\qquad\square\end{aligned}$$

(See [13] for the notations used here concerning the hypergeometric functions ${}_1F_2$, and ${}_2F_3$.)

The formulas obtained in Theorem 4.1, for the reproducing kernel of the Hilbert space $\mathcal{F}_q(\widetilde{\Xi})$, by considering the cases in Tables 3 and 4, agree exactly with the formulas given by Theorem 8.1 and Table 6.9 in [2] for the minimal representations (one has to replace r_0 by η_q, a by η_q+a_0, b by η_q+b_0) and in [6] for the $SL(3,\mathbb{R})$-case.

In the special case of $\widetilde{\mathrm{SL}}(p+2,\mathbb{R})$ (with $p\geq 3$), the reproducing kernels of the two representations corresponding to $q=0$ and 1 are respectively

$${}_1F_2\Big(\frac{5}{4};\frac{3}{4},\frac{1}{2};(\phi_p(1+z\overline{z'}))^2(w\overline{w'})^2\Big),$$
$${}_1F_2\Big(\frac{p}{4}+\frac{3}{2};\frac{3}{2},\frac{p}{4};(\phi_p(1+z\overline{z'}))^2(w\overline{w'})^2\Big)$$

where $\xi=(w,z)\in\mathbb{C}\times\mathbb{C}^p$ and $\mathcal{K}_q(\xi,\xi')=(\phi_p(1+z\overline{z'}))^q(w\overline{w'})^q$.

In the special case of $\widetilde{\mathrm{SO}}(3,3)\simeq\widetilde{\mathrm{SL}}(4,\mathbb{R})$, the reproducing kernels of the two representations corresponding to $q_1=q_2=0$ and $q_1=q_2=1$ are respectively given by

$${}_1F_2\Big(\frac{3}{2};\frac{1}{2},1;(1+z_1\overline{z_1'})(1+z_2\overline{z_2'})\Big(w_1\overline{w_1'}(w_2\overline{w_2'})\Big)\Big),$$
$${}_1F_2\Big(2;\frac{3}{2},\frac{3}{2};(1+z_1\overline{z_1'})(1+z_2\overline{z_2'})\Big(w_1\overline{w_1'}(w_2\overline{w_2'})\Big)\Big)$$

where $\xi=(w,z)=((w_1,z_1),(w_2,z_2)),\xi'=(w',z')=((w_1',z_1'),(w_2',z_2'))\in(\mathbb{C}\times\mathbb{C})^2$ and $\mathcal{K}_{(q_1,q_2)}(\xi,\xi')=[(1+z_1\overline{z_1'})(w_1\overline{w_1'})]^{q_1}[(1+z_2\overline{z_2'})(w_2\overline{w_2'})]^{q_2}$.

One can observe that the case $((1),q=0)$ agrees with the case $((2),q=0,p=1)$, the case $((1),q=2)$ agrees with the case $((2),q=1,p=1)$, the case $((3),q_1=q_2=0)$ agrees with the case $((2),q=0,p=2)$ and the case $((3),q_1=q_2=1)$ agrees with the case $((2),q=1,p=2)$.

In the following, we will see that the Hilbert space $\mathcal{F}_q(\widetilde{\Xi})$ is a pseudo-weighted Bergman space. It means that the norm of $\phi\in\mathcal{F}_q(\widetilde{\Xi})$ is given by an integral of $|\phi|^2$ with respect to a weight taking both positive and negative values. The weight involves a Meijer G-function:

$$G(u)=G^{4,0}_{2,4}\Big(u\Big|^{\eta_q}_{\beta_1}\ {}^{\eta_q-1}_{\beta_2}\ {}_{\beta_3}\ {}_{\beta_4}\Big).$$

Theorem 4.2. *For $\phi\in\mathcal{F}_q(\widetilde{\Xi})$,*

$$\|\phi\|^2=\int_{\prod_{i=1}^{s}(\mathbb{C}\times V_i)}|\phi(w,z)|^2p(w,z)\prod_{i=1}^{s}m(dw_i)m_{i,0}(dz_i),$$

with $p(w,z)=CG\big(\prod_{i=1}^{s}|w_i|^2H_i(z_i)\big)\prod_{i=1}^{s}H_i(z_i)$.

In fact, from the proof of Theorem 4.1 it follows that

$$\frac{c_{(m)}a_{(m)}}{c_{(m+1)}a_{(+1)}} = \delta_{(m)}\prod_{i=1}^{s} B_i\Big(-m-\frac{q_i}{k_i}-2\frac{n_i}{k_i r_i}\Big) = \frac{\prod_{i=1}^{s} B_i(-m-\eta_q-\frac{n_i}{k_i r_i})}{(m+\eta_q)(m+\eta_q+1)}.$$

Then

$$\begin{aligned}\frac{1}{c_{(m+1)}a_{(m+1)}} &= \frac{\prod_{i=1}^{s} B_i(-m-\eta_q-\frac{n_i}{k_i r_i})}{(m+\eta_q)(m+\eta_q+1)}\frac{1}{(c\cdot a)_{(k_1 m+q_1,\dots,k_s m+q_s)}}\\ &= \frac{(m+\eta_q+\alpha_1')(m+\eta_q+\alpha_2')(m+\eta_q+\alpha_3')(m+\eta_q+\alpha_4')}{(m+\eta_q)(m+\eta_q+1)}\frac{1}{c_{(m)}a_{(m)}}\end{aligned}$$

with $\tilde{B}(\alpha)=\prod_{i=1}^{s} B_i(\alpha-\frac{n_i}{k_i r_i}) = A(\alpha-\alpha_1')(\alpha-\alpha_2')(\alpha-\alpha_3')(\alpha-\alpha_4')$. Then

$$\begin{aligned}\frac{1}{c_{(m)}a_{(m)}} &= \frac{(\eta_q+\alpha_1')_m(\eta_q+\alpha_2')_m(\eta_q+\alpha_3')_m(\eta_q+\alpha_4')_m}{(\eta_q)_m(\eta_q+1)_m}\\ &= \frac{\Gamma(\eta_q+\alpha_1'+m)\Gamma(\eta_q+\alpha_2'+m)\Gamma(\eta_q+\alpha_3'+m)\Gamma(\eta_q+\alpha_4'+m)}{\Gamma(\eta_q+m)\Gamma(\eta_q+1+m)}\\ &= \frac{\Gamma(\eta_q+\alpha_1'+m)\Gamma(\eta_q+\alpha_2'+m)\Gamma(\eta_q+\alpha_3'+m)\Gamma(\eta_q+\alpha_4'+m)}{\Gamma(\eta_q+m)\Gamma(\eta_q+1+m)}\\ &= C\int_0^{\infty} G(u)u^m du\end{aligned}$$

where

$$G(u) = G_{2,4}^{4,0}\Big(u\Big|_{\beta_1\ \ \beta_2\ \ \beta_3\ \ \beta_4}^{\eta_q\ \ \eta_q-1}\Big).$$

$$\beta_1=\eta_q+\alpha_1'-1,\ \beta_2=\eta_q+\alpha_2'-1,\ \beta_3=\eta_q+\alpha_3'-1,\ \beta_4=\eta_q+\alpha_4'-1.$$

Observe that in the case $q=0$, $\tilde{B}(\alpha)=B(\alpha-\eta_q)$ and, since the Bernstein polynomial B vanishes at 0 and $1-\eta_q$, then

$$\alpha_1'=\eta_q,\quad \alpha_2'=1,\quad \alpha_3'=\eta_q+a_0,\quad \alpha_4'=\eta_q+b_0$$

then $\beta_1=2\eta_q-1$, $\beta_2=\eta_q$, $\beta_3=2\eta_q+a_0-1$ and $\beta_4=2\eta_q+b_0-1$
and the G-function is given by

$$G(u) = G_{1,3}^{3,0}\Big(u\Big|_{2\eta_q-1\ \ 2\eta_q+a_0-1\ \ 2\eta_q+b_0-1}^{\eta_q-1}\Big),$$

and these formulas agree with those obtained in [4].

(See Tables 5 and 6 for these constants and [13] for the notation $G_{2,4}^{4,0}$ and $G_{1,3}^{3,0}$ of G-functions.)

Let $G_{\mathbb{R}}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$.

Theorem 4.3. (i) *There is a unique unitary irreducible representation π_q of $G_{\mathbb{R}}$ on $\mathcal{F}_q(\widetilde{\Xi})$ such that $d\pi_q=\rho_q$.*

(ii) *The representation π_q is spherical if and only if $q=0$. In that case, the space $\mathcal{O}_{(0,\dots,0)}(\widetilde{\Xi})$ reduces to the constant functions, the K-fixed vectors.*

The proof of (i) is the same than the proof of Theorem 6.3 in [4].

Table 5

	α_1'	α_2'	α_3'	α_4'
(1)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
(2)	$\frac{p}{4}$	$\frac{1}{2}$	$\frac{p}{4}+\frac{1}{2}$	1
(3)	$\frac{1}{2}$	1	$\frac{1}{2}$	1
(4)	1	1	1	1
(5)	$\frac{p}{2}$	1	$\frac{1}{2}$	1
(6)	$\frac{p}{2}$	1	1	1
(7)	$\frac{p_1}{2}$	1	$\frac{p_2}{2}$	1
(8)	$1+\frac{3d}{2}$	$1+d$	$1+\frac{d}{2}$	1
(9)	$1+d_1$	$1+\frac{d_1}{2}$	1	1
(10)	$\frac{1}{3}$	$\frac{2}{3}$	1	1

Table 6

	$\eta_q - 1$	η_q	β_1	β_2	β_3	β_4
(1)	$\frac{q}{4} - \frac{3}{4}, q \neq 3$	$\frac{q}{4} + \frac{1}{4}$	$\frac{q}{4} - \frac{1}{2}$	$\frac{q}{4} - \frac{1}{4}$	$\frac{q}{4}$	$\frac{q}{4} + \frac{1}{4}$
(2)	$\frac{q}{2} + \frac{p}{4} - 1, q \in \{0, 1\}$	$\frac{q}{2} + \frac{p}{4}$	$\frac{q}{2} + \frac{p}{2} - 1$	$\frac{q}{2} + \frac{p}{4} - \frac{1}{2}$	$\frac{q}{2} + \frac{p}{2} - \frac{1}{2}$	$\frac{q}{2} + \frac{p}{4}$
(3)	$\frac{\tilde{q}}{2} - \frac{1}{2}, \tilde{q} \in \{0, 1\}$	$\frac{\tilde{q}}{2} + \frac{1}{2}$	$\frac{\tilde{q}}{2}$	$\frac{\tilde{q}}{2}$	$\frac{\tilde{q}}{2} + \frac{1}{2}$	$\frac{\tilde{q}}{2} + \frac{1}{2}$
(4)	0	1	1	1	1	1
(5)	$\frac{p}{2} - 1$	$\frac{p}{2}$	$p - 1$	$\frac{p}{2}$	$\frac{p}{2} - \frac{1}{2}$	$\frac{p}{2}$
(6)	$\frac{p}{2} - 1$	$\frac{p}{2}$	$p - 1$	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{p}{2}$
(7)	$\frac{p_1}{2} - 1$	$\frac{p_1}{2}$	$p_1 - 1$	$\frac{p_1}{2}$	$\frac{p_1+p_2}{2} - 1$	$\frac{p_1}{2}$
(8)	$\frac{3d}{2}$	$\frac{3d}{2} + 1$	$3d + 1$	$\frac{5d}{2} + 1$	$2d + 1$	$\frac{3d}{2} + 1$
(9)	d_1	$d_1 + 1$	$2d_1 + 1$	$\frac{3d_1}{2} + 1$	$d_1 + 1$	$d_1 + 1$
(10)	0	1	$\frac{1}{3}$	$\frac{2}{3}$	1	1

Observe that one of the α_i' always equals 1, and then one of the β_i always equals η_q, therefore the involved G-function is always $G^{3,0}_{1,3}$.

5. The $\mathfrak{sl}(3, \mathbb{R})$-case

In the special case $s = 1, r = 1, k = 4, V = \mathbb{C}$, the polynomials Δ and Q are given by $\Delta(z) = z$ and $Q(z) = z^4$. We denote by $E = Q \in \mathcal{V} := \mathfrak{p}_2$ and $F = 1 \in \mathcal{V}^\sigma := \mathfrak{p}_{-2}$. For $H \in \mathfrak{k}_0$ such that $\exp(tH)$ is the dilation $z \in \mathbb{C} \mapsto e^{-t}z \in \mathbb{C}$, we have $[H, E] = 2E$ and $[H, F] = -2F$, and we consider the Lie algebra structure on $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$ such that $[E, F] = H$. In this case $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbb{C})$ and the real form $\mathfrak{g}_\mathbb{R}$ is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. Then $\widetilde{L} = \mathrm{Str}(V, \Delta) = \mathbb{C}^*$ acting by dilations l_λ, and, since for $\lambda \in \mathbb{C}^*$, $\Delta(\lambda \cdot z) = \lambda\Delta(z)$, the character χ does not exist, then $\widetilde{K} = \{(g, \mu) \mid g \in \mathrm{Conf}(V, \Delta), \mu(z)^2 = z\}$, with $\mathrm{Conf}(V, \Delta) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$. The orbit $\Xi = \Xi^{(4)}$ of Q under $\widetilde{K}$ acting by $\kappa^{(4)}$ and the orbit $\widetilde{\Xi}$ of Δ under $\widetilde{K}$ acting by κ are given by $\Xi = \{w^4 \cdot (v - z)^4 \mid (w, z) \in \mathbb{C} \times \mathbb{C}\}$ and $\widetilde{\Xi} = \{w \cdot (v - z) \mid (w, z) \in \mathbb{C} \times \mathbb{C}\}$ in such a way that the map $\widetilde{\Xi} \to \Xi, \xi \mapsto \xi^4$ is a covering and Ξ is diffeomorphic to $\widetilde{\Xi}/Z_4$, where Z_4 is the group of the 4-roots of unity.

One has $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}} = \sum_{m\in\mathbb{N}} \mathcal{O}_{4m+q}(\widetilde{\Xi})$, with

$$\mathcal{O}_{4m+q}(\widetilde{\Xi}) \simeq \{\phi \in \mathcal{O}(\mathbb{C}^2) \mid \phi(w, z) = w^{4m+q}\psi(z), \psi \in \widetilde{\mathcal{O}}_{4m+q}(V)\}.$$

The norm on the space $\mathcal{O}_{4m+q}(\widetilde{\Xi})$ is given for $\phi(w, z) = w^{4m+q}\psi(z)$ by

$$\|\phi\|^2_{4m+q} = \frac{\pi}{4m + q + 1} \int_\mathbb{C} |\psi(z)|(1 + |z|^2)^{-(4m+q+2)} dz.$$

In particular, for $\Phi_{4m+q}(w, z) := w^{4m+q}$, one has $\|\Phi_{4m+q}\|_{4m+q} = 1$. The reproducing kernel of the space $(\widetilde{\mathcal{O}}_{4m+q}(\mathbb{C}), \|\cdot\|_{4m+q})$ is given by

$$\tilde{\mathcal{K}}(z, z') = (1 + z\bar{z}')^{4m+q}.$$

The representation π of K on $\mathcal{O}(\widetilde{\Xi}) = \mathcal{O}(\mathbb{C}^2)$ is given by

$$(\pi(g)\phi)(w, z) = \phi(\mu(g^{-1}, z)w, g^{-1} \cdot z).$$

It follows that for $\lambda = e^{-t}$, $(\pi(l_\lambda)\phi)(w, z) = \phi(e^{-2t}w, e^t \cdot z)$,

$$(d\pi(H)\phi)(w, z) = \frac{d}{dt}_{|t=0} \phi(e^{-2t}w, e^t \cdot z) = -2\phi(w, z) + (\mathcal{E}\phi)(w, z).$$

The operators $\mathcal{M}, \mathcal{D}, \mathcal{M}^\sigma, \mathcal{D}^\sigma$ are given by

$$(\mathcal{M}\phi)(w, z) = w^4\phi(w, z), \quad (\mathcal{D}\phi)(w, z) = w^{-4}\frac{\partial^4}{\partial z^4}\phi(w, z).$$
$$\mathcal{M}^\sigma(w^{4m+q}\psi(z)) = w^{4(m+1)+q}z^4\phi(w, z),$$
$$(\mathcal{D}^\sigma\phi)(w, z) = w^{4(m-1)+q}(\mathbf{D}_{-4m-q})^*\psi(z)$$

($(\mathbf{D}_{-4m-q})^*$ is the adjoint of the operator $\mathbf{D}_{-4m-q} = z^{4-4m-q}\frac{\partial^4}{\partial z^4}z^{4m+q}$).

The representation ρ_q is given by

$$(\rho_q(E))(w^{4m+q}\psi(z)) = w^{4(m+1)+q}\psi(z) - \delta_{4(m-1)+q}w^{4(m-1)+q}\frac{\partial^4}{\partial z^4}\psi(z),$$
$$\rho_q(F)(w^{4m+q}\psi(z)) = w^{4(m+1)+q}z^4\psi(z) - \delta_{4(m-1)+q}w^{4(m-1)+q}(\mathbf{D}_{-4m-q})^*\psi(z),$$
$$\rho_q(H)(w^{4m+q}\psi(z)) = d\pi(H)(w^{4m+q}\psi(z)) = w^{4m+q}(-2\psi + \tilde{\mathcal{E}}\psi)(z)$$

with

$$(\tilde{\mathcal{E}}\psi)(z) = \frac{d}{dt}_{|t=0} \psi(e^t z)$$

and

$$\delta_{4m+q} = \frac{1}{4^4(m+\frac{q}{4}+\frac{1}{4})(m+\frac{q}{4}+\frac{1}{4}+1)}.$$

The invariance condition $\rho_q(F)^* = -\rho_q(E)$ is equivalent to:

$$\frac{1}{a_{4(m+1)+q}c_{4(m+1)+q}} \int_{\mathbb{C}} z^4 \psi(z)\overline{\psi'(z)}(1+|z|^2)^{-(4(m+1)+q+2)}dz$$

$$= -\frac{\delta_{4m+q}}{a_{4m+q}c_{4m+q}} \int_{\mathbb{C}} \psi(z)\frac{\partial^4}{\partial z^4}\overline{\psi'(z)}(1+|z|^2)^{-(4m+q+2)}dz$$

which is equivalent to

$$\frac{1}{a_{4(m+1)+q}c_{4(m+1)+q}} \int_{\mathbb{C}} z^4 \psi(z)\overline{\psi'(z)}(1+|z|^2)^{-(4m+q+6)}dz$$

$$= -\frac{\delta_{4m+q}}{a_{4m+q}c_{4m+q}} \int_{\mathbb{C}} \psi(z)\overline{\psi'(z)}\frac{\partial^4}{\partial z^4}(1+|z|^2)^{-(4m+q+2)}dz.$$

Since

$$\frac{\partial^4}{\partial \bar{z}^4}(1+|z|^2)^{-(4m+q+2)} = (4m+q+2)(4m+q+3)(4m+q+4)(4m+q+5)z^4(1+|z|^2)^{-(4(m+1)+q+2)},$$

$$\delta_{4m+q} = \frac{1}{4^4(m+\frac{q}{4}+\frac{1}{4})(m+\frac{q}{4}+\frac{1}{4}+1)},$$

and $a_m = \frac{\pi}{4m+q+1}$, it follows, after an integration by parts, that

$$\frac{1}{a_{4(m+1)+q}c_{4(m+1)+q}} = \frac{(4m+q+2)(4m+q+3)(4m+q+4)(4m+q+5)}{4^4(m+\frac{q}{4}+\frac{1}{4})(m+\frac{q}{4}+\frac{1}{4}+1)}\frac{1}{a_{4m+q}c_{4m+q}}$$

i.e.

$$(4m+q+5)\frac{1}{c_{4(m+1)+q}} = \frac{(4m+q+2)(4m+q+3)(4m+q+4)(4m+q+5)(4m+q+1)}{4^4(m+\frac{q}{4}+\frac{1}{4})(m+\frac{q}{4}+\frac{1}{4}+1)}\frac{1}{c_{4m+q}}$$

i.e.

$$\frac{1}{c_{4(m+1)+q}} = \frac{(m+\frac{q}{4}+\frac{2}{4})(m+\frac{q}{4}+\frac{3}{4})(m+\frac{q}{4}+1)}{m+1+\frac{q}{4}+\frac{1}{4}}\frac{1}{c_{4m+q}}$$

and for $c_q = 1$, one obtains

$$c_{4m} = \frac{(\frac{5}{4})_m}{(\frac{2}{4})_m(\frac{3}{4})_m}\frac{1}{m!} \quad (q=0)$$

$$c_{4m+1} = \frac{(\frac{6}{4})_m}{(\frac{3}{4})_m(\frac{5}{4})_m}\frac{1}{m!} \quad (q=1)$$

$$c_{4m+2} = \frac{(\frac{7}{4})_m}{(\frac{5}{4})_m(\frac{6}{4})_m}\frac{1}{m!} \quad (q=2).$$

The representation $d\pi + \rho_q$ integrates to a unitary representation of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ in a Hilbert space $\mathcal{F}_q(\widetilde{\Xi})$, the completion of $\mathcal{O}(\widetilde{\Xi})_{q,\mathrm{fin}}$ with respect to the norm for $\phi = \sum_{m\in\mathbb{N}}\phi_{4m+q}$, by $\|\phi\|^2 = \sum_{m\in\mathbb{N}}\frac{1}{c_{4m+q}}\|\phi_{4m+q}\|^2_{4m+q}$, and the reproducing kernels

of $\mathcal{F}_q(\widetilde{\Xi})$ for $q = 0, 1, 2$ are respectively:

$$
\begin{aligned}
&{}_1F_2\Big(\frac{5}{4}; \frac{3}{4}, \frac{1}{2}; (1+z\overline{z'})^4(w_i\overline{w_i'})^4\Big)\mathcal{K}_0(\xi,\xi') \quad (q=0)\\
&{}_1F_2\Big(\frac{3}{2}; \frac{5}{4}, \frac{3}{4}; (1+z\overline{z'})^4(w_i\overline{w_i'})^4\Big)\mathcal{K}_1(\xi,\xi') \quad (q=1)\\
&{}_1F_2\Big(\frac{7}{4}; \frac{6}{4}, \frac{5}{4}; (1+z\overline{z'})^4(w\overline{w'})^4\Big)\mathcal{K}_2(\xi,\xi') \quad (q=2),
\end{aligned}
$$

where $\xi = (w, z), \xi' = (w', z') \in \mathbb{C}^2$ and $\mathcal{K}_q(\xi,\xi') = (1+z\overline{z'})^q(w\overline{w'})^q$.

Furthermore the Hilbert space $\mathcal{F}_q(\widetilde{\Xi})$ is a weighted Bergman space and its norm is given by

$$\|\phi\|^2 = \int_{\mathbb{C}^2} |\phi(w,z)|^2 p(w,z) m(dw)(1+|z|^2)^{-2} dz,$$

with $p(w,z) = G\big(|w|^2(1+|z|^2)\big)(1+|z|^2)$ and the G-function is given by

$$
\begin{aligned}
G(u) &= G^{3,0}_{1,3}\Big(u\Big|^{-\frac{3}{4}}_{-\frac{2}{4}} \quad 0 \quad -\frac{1}{4}\Big) \quad (q=0),\\
G(u) &= G^{3,0}_{1,3}\Big(u\Big|^{-\frac{1}{2}}_{-\frac{1}{4}} \quad 0 \quad \frac{1}{4}\Big) \quad (q=1),\\
G(u) &= G^{3,0}_{1,3}\Big(u\Big|^{-\frac{1}{4}}_{0} \quad \frac{1}{4} \quad -\frac{1}{2}\Big) \quad (q=2).
\end{aligned}
$$

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