



Minimal representations of filiform Lie algebras and their application for construction of Leibniz algebras

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ABSTRACT

In this paper we find minimal faithful representations of several classes of filiform Lie algebras by means of strictly upper-triangular matrices. We investigate Leibniz algebras whose corresponding Lie algebras L/I are filiform Lie algebras such that the action $I \times L/I \rightarrow I$ gives rise to a minimal faithful representation of a filiform Lie algebra, where I is the ideal generated by the squares of the elements of the algebra L . The classification up to isomorphism of such Leibniz algebras is given for low-dimensional cases.

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1. Introduction

According to Ado's Theorem, given any finite-dimensional complex Lie algebra \mathfrak{g} , there exists a matrix algebra isomorphic to \mathfrak{g} . In this way, every finite-dimensional complex Lie algebra can be represented as a Lie subalgebra of the complex general linear algebra $\mathfrak{gl}(n, \mathbb{C})$, formed by all the complex $n \times n$ matrices, for some $n \in \mathbb{N}$. We consider the following integer valued invariant of \mathfrak{g} :

$$\mu(\mathfrak{g}) = \min\{\dim(M) \mid M \text{ is a faithful } \mathfrak{g}\text{-module}\}$$

It follows from the proof of Ado's Theorem that $\mu(\mathfrak{g})$ can be bounded by a function depending on only n . This value is also equal to the minimal value n such that $\mathfrak{gl}(n, \mathbb{C})$ contains a subalgebra isomorphic to \mathfrak{g} .

Given a Lie algebra \mathfrak{g} , a representation of \mathfrak{g} in \mathbb{C}^n is a homomorphism of Lie algebras $f: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(n, \mathbb{C})$. The natural integer n is called dimension (or degree) of this representation. We consider faithful representations because such representations allow us to identify a given Lie algebra with its image under the representation, which is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. Representations can be also defined by using arbitrary n -dimensional vector spaces V (see [10]). In such a case, a representation would be a homomorphism of Lie algebras from \mathfrak{g} to the Lie algebra of endomorphisms of the vector space V , $\mathfrak{gl}(V)$, which is called a \mathfrak{g} -module. However, it is sufficient to consider representations on \mathbb{C}^n because there always exists a unique $n \in \mathbb{N}$ such that V is isomorphic to \mathbb{C}^n .

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Many works are devoted to finding the value $\mu(\mathfrak{g})$ of several finite-dimensional Lie algebras. In [6], the value of $\mu(\mathfrak{g})$ for abelian Lie algebras and Heisenberg algebras is found and, additionally, the estimated value of $\mu(\mathfrak{g})$ for filiform Lie algebras is given. In the works [4,8,11] the authors find a matrix representation of some low-dimensional Lie algebras.

In the paper [8] a minimal faithful representation of the filiform Lie algebra \mathcal{L}_n is shown. Let \mathfrak{g}_n be the nilpotent matrix algebra of $n \times n$ strictly upper-triangular matrices. In [8] the authors introduce

$$\overline{\mu}(\mathfrak{g}) = \min\{n \in \mathbb{N} \mid \text{there exists a subalgebra of } \mathfrak{g}_n \text{ isomorphic to } \mathfrak{g}\},$$

and prove the next proposition.

Proposition 1 ([8]). *Let \mathfrak{g} be an n -dimensional filiform Lie algebra. Then $\overline{\mu}(\mathfrak{g}) \geq n$.*

Leibniz algebras are a non-antisymmetric generalization of Lie algebras, and they were introduced in 1965 by Bloh in [5], who called them D -algebras. In 1993 Loday [13] rediscovered these algebras called them Leibniz algebras.

An algebra $(L, [-, -])$ over a field \mathbb{F} is called a *Leibniz algebra* if for any $x, y, z \in L$, the so-called Leibniz identity

$$[x, y], z = [x, z], y + [x, [y, z]]$$

holds.

One of the methods of classification of Leibniz algebras is the study of those with given corresponding Lie algebras. In the papers [2,7,15,16], Leibniz algebras whose corresponding Lie algebras are naturally graded filiform Lie algebras \mathcal{L}_n , Heisenberg algebras, simple Lie \mathfrak{sl}_2 and Diamond Lie algebras are studied. Let L be a Leibniz algebra. The ideal I generated by the squares of elements of the algebra L , that is $\langle [x, x] \mid x \in L \rangle$, plays an important role in the theory since it determines the (possible) non-Lie character of L . Observe that we can write $L = (L/I) \oplus I$ as a direct sum of vector spaces. From the Leibniz identity, it is easy to see that this ideal belongs to the right annihilator of L , that is

$$[L, I] = 0.$$

Clearly, the quotient algebra L/I is a Lie algebra, called the *corresponding Lie algebra* of L .

The map $I \times (L/I) \rightarrow I$ defined as $(v, \bar{x}) \mapsto [v, x]$, $v \in I$, $x \in L$, endows I with a structure of right (L/I) -module. If we consider the direct sum of vector spaces $(L/I) \oplus I$, then the structure of right (L/I) -module defines a Leibniz algebra structure on $(L/I) \oplus I$, known as the *hemisemidirect product* of L/I with I (see [1] or [12]), with the following multiplication

$$\begin{aligned} \bar{x} + v, \bar{y} + w &:= [\bar{x}, \bar{y}] + [v, y], \quad \text{that is,} \\ [\bar{x}, \bar{y}] &= [\bar{x}, \bar{y}], \quad [v, \bar{x}] = [v, x], \quad [\bar{x}, v] = 0, \quad [v, w] = 0, \quad x, y \in L, \quad v, w \in I. \end{aligned}$$

Therefore, given a Lie algebra \mathfrak{g} and a \mathfrak{g} -module V , we can construct a Leibniz algebra (\mathfrak{g}, V) by the above construction, and so, the principal question appearing in this relation is the description of the Leibniz algebras L , such that the associated Leibniz algebra $(L/I) \oplus I$ is isomorphic to an earlier given Leibniz algebra (\mathfrak{g}, V) .

The theory of representations of Leibniz algebras was introduced in [14]. Let L be a Leibniz algebra and V a vector space over the field \mathbb{F} . A *representation* of the Leibniz algebra L on the vector space V is a pair (λ, ρ) of linear maps $\lambda, \rho: L \rightarrow \mathfrak{gl}(V)$ satisfying the following properties:

$$\begin{aligned} \rho_{[x,y]} &= \rho_y \rho_x - \rho_x \rho_y, \\ \lambda_{[x,y]} &= \rho_y \lambda_x - \lambda_x \rho_y, \\ \lambda_{[x,y]} &= \rho_y \lambda_x + \lambda_x \rho_y, \quad \text{for all } x, y \in L, \text{ where } \lambda_x = \lambda(x) \text{ and } \rho_x = \rho(x). \end{aligned}$$

If \mathfrak{g} is a Lie algebra, then its Lie representation $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ becomes a Leibniz representation with $\lambda = \varphi$ and $\rho = -\varphi$, or also by taking $\lambda = 0$ and $\rho = -\varphi$.

Notice that the concepts of representations of Lie algebras and Leibniz algebras are different. Ado's theorem on the existence of faithful representations is a relevant theorem in the theory of Lie algebras. The analogous in the case of Leibniz algebras was proved in [3] in an easier way and gives a stronger result. That is because the kernel of the Leibniz algebra representation is the intersection of the kernels of λ and ρ , $\ker(\lambda, \rho) := \ker \lambda \cap \ker \rho = \{x \in L \mid \lambda_x = 0 = \rho_x\}$, which are in general different, in contrast to the representations of Lie algebras, where these kernels are the same. Therefore, a faithful representation of Leibniz algebras can be obtained more easily than a faithful representation in the case of Lie algebras. The representation theories of Leibniz algebras and their corresponding Lie algebras are very related. In fact, in [9] the authors use representations of the Lie algebra L/I to construct representations of the Leibniz algebra L .

The paper is devoted to finding minimal representations of some classes of filiform Lie algebras of dimension n . Specifically, in Section 2 we find minimal faithful representations of the filiform Lie algebras \mathcal{Q}_{2n} , \mathcal{R}_n and \mathcal{W}_n . Finally, in Section 3 we construct Leibniz algebras with corresponding Lie algebras \mathcal{Q}_6 , \mathcal{R}_7 and \mathcal{W}_5 , by using obtained representations of these algebras.

2. Minimal representation of filiform Lie algebras

Now we recall the definitions of nilpotent and filiform Lie algebras.

For a Lie algebra L consider the following lower central series:

$$L^1 = L, \quad L^{k+1} = [L^1, L^k], \quad k \geq 1.$$

Definition 2. A Lie algebra L is called nilpotent if there exists $s \in \mathbb{N}$ such that $L^s = 0$.

The smallest number s is called the *index of nilpotency* or *nilindex* of L . The index of nilpotency of an n -dimensional Lie algebra is at most n . Among the nilpotent Leibniz algebras, we distinguish the filiform algebras (nilindex n).

Definition 3. An n -dimensional Lie algebra L is said to be filiform if $\dim L^i = n - i$, $2 \leq i \leq n$.

We list some classes of n -dimensional filiform Lie algebras with basis $\{e_1, \dots, e_n\}$.

1. Let \mathcal{L}_n be the Lie algebra whose brackets in the basis are defined by

$$[e_1, e_i] = -[e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

2. Let \mathcal{Q}_{2s} ($n = 2s$) be the Lie algebra defined by

$$\begin{aligned} [e_1, e_i] &= -[e_i, e_1] = e_{i+1}, & 2 \leq i \leq 2s-2, \\ [e_{2s+1-i}, e_i] &= -[e_i, e_{2s+1-i}] = (-1)^i e_{2s}, & 2 \leq i \leq s. \end{aligned}$$

3. Let \mathcal{R}_n be the Lie algebra defined by

$$\begin{aligned} [e_1, e_i] &= -[e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_2, e_i] &= -[e_i, e_2] = e_{i+2}, & 3 \leq i \leq n-2. \end{aligned}$$

4. Let \mathcal{W}_n be the Lie algebra defined by

$$[e_i, e_j] = -[e_j, e_i] = (j-i)e_{i+j}, \quad i+j \leq n.$$

Notice that the algebras \mathcal{L}_n and \mathcal{Q}_{2n} are naturally graded model filiform Lie algebras. The algebra \mathcal{W}_n is the finite-dimensional Witt algebra.

Proposition 4. A minimal faithful representation of the algebra \mathcal{Q}_{2n} is

$$a_1 e_1 + a_2 e_2 + \dots + a_{2n} e_{2n} \mapsto \begin{pmatrix} 0 & a_2 & -a_3 & \dots & a_{2n-2} & -a_{2n-1} & -2a_{2n} \\ 0 & 0 & a_1 & \dots & 0 & 0 & a_{2n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_1 & a_3 \\ 0 & 0 & 0 & \dots & 0 & 0 & a_2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Consider the linear map $\varphi: \mathcal{Q}_{2n} \rightarrow \mathfrak{gl}(2n, \mathbb{C})$ given by

$$\varphi(e_1) = \sum_{k=2}^{2n-2} E_{k,k+1}, \quad \varphi(e_i) = (-1)^i E_{1,i} + E_{2n-i+1,2n} \quad 2 \leq i \leq 2n-1, \quad \varphi(e_{2n}) = -2E_{1,2n},$$

where $E_{i,j}$ is the matrix with (i,j) th entry equal to 1 and all others zero.

By checking that $[\varphi(e_i), \varphi(e_j)] = \varphi(e_i)\varphi(e_j) - \varphi(e_j)\varphi(e_i)$ for all $1 \leq i, j \leq 2n$, we verify that φ is an injective homomorphism of Lie algebras. Then by Proposition 1 we obtain that it is minimal. \square

Let us denote by $V = \mathbb{C}^{2n}$ the natural $\varphi(\mathcal{Q}_{2n})$ -module and endow it with a \mathcal{Q}_{2n} -module structure $V \times \mathcal{Q}_{2n} \rightarrow V$, given by

$$(x, e) := x\varphi(e),$$

where $x \in V$ and $e \in \mathcal{Q}_{2n}$.

Then we obtain

$$\left\{ \begin{aligned} (x_i, e_1) &= x_{i+1}, & 2 \leq i \leq 2n-2, \\ (x_1, e_i) &= (-1)^i x_i, & 2 \leq i \leq 2n-1, \\ (x_{2n+1-i}, e_i) &= x_{2n}, & 2 \leq i \leq 2n-1, \\ (x_1, e_{2n}) &= -2x_{2n}, \end{aligned} \right. \quad (1)$$

and the remaining products are zero.

Proposition 5. A minimal faithful representation of the algebra \mathcal{R}_n is

$$a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mapsto \begin{pmatrix} 0 & a_1 & a_2 & 0 & \cdots & 0 & 0 & 0 & a_n \\ 0 & 0 & a_1 & a_2 & \cdots & 0 & 0 & 0 & a_{n-1} \\ 0 & 0 & 0 & a_1 & \cdots & 0 & 0 & 0 & a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_1 & a_2 & a_4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. We take the linear map $\psi: \mathcal{R}_n \rightarrow \mathfrak{gl}(n, \mathbb{C})$ given by

$$\psi(e_1) = \sum_{i=1}^{n-2} E_{i,i+1}, \quad \psi(e_2) = \sum_{i=1}^{n-3} E_{i,i+2} + E_{n-1,n}, \quad \psi(e_i) = E_{n+1-i,n}, \quad 3 \leq i \leq n.$$

By checking that $[\psi(e_i), \psi(e_j)] = \psi(e_i)\psi(e_j) - \psi(e_j)\psi(e_i)$ for all $1 \leq i, j \leq n$, we verify that ψ is an injective homomorphism of Lie algebras. Then by Proposition 1 we obtain that it is minimal. \square

Now, we construct a \mathcal{R}_n -module structure $V = \mathbb{C}^n$, $V \times \mathcal{R}_n \rightarrow V$, given by

$$(x, e) := x\psi(e),$$

where $x \in V$ and $e \in \mathcal{R}_n$.

Then we obtain

$$\begin{cases} (x_i, e_1) = x_{i+1}, & 1 \leq i \leq n-2, \\ (x_i, e_2) = x_{i+2}, & 1 \leq i \leq n-3, \\ (x_{n+1-j}, e_j) = x_n, & 2 \leq j \leq n, \end{cases}$$

and the remaining products in the action being zero.

Denote by $C_m^n = \binom{m}{n}$ the binomial coefficient.

Proposition 6. A minimal faithful representation of the algebra \mathcal{W}_n is given by $\varphi: \mathcal{W}_n \rightarrow \mathfrak{gl}(n, \mathbb{C})$, where

$$\begin{aligned} \varphi(e_1) &= \sum_{k=1}^{n-2} E_{k,k+1}, & \varphi(e_2) &= \sum_{k=1}^{n-3} \frac{1}{n-k} E_{k,k+2} + E_{n-1,n}, \\ \varphi(e_i) &= \frac{1}{(i-2)!} \left(\sum_{k=1}^{n-i-1} \left(\sum_{s=0}^{i-2} \frac{(-1)^{i+s} C_{i-2}^s}{n-k-s} \right) E_{k,k+i} + E_{n+1-i,n} \right), & 3 \leq i \leq n. \end{aligned}$$

Proof. We take the injective homomorphism $\varphi: \mathcal{W}_n \rightarrow \mathfrak{gl}(n, \mathbb{C})$ such that

$$\varphi(e_1) = \sum_{k=1}^{n-2} E_{k,k+1}, \quad \varphi(e_2) = \sum_{s=1}^{n-3} \alpha_s E_{s,s+2} + E_{n-1,n}.$$

Now we consider

$$\begin{aligned} \varphi(e_3) &= [\varphi(e_1), \varphi(e_2)] = \varphi(e_1)\varphi(e_2) - \varphi(e_2)\varphi(e_1) \\ &= \left(\sum_{k=1}^{n-2} E_{k,k+1} \right) \left(\sum_{s=1}^{n-3} \alpha_s E_{s,s+2} + E_{n-1,n} \right) - \left(\sum_{s=1}^{n-3} \alpha_s E_{s,s+2} + E_{n-1,n} \right) \left(\sum_{k=1}^{n-2} E_{k,k+1} \right) \\ &= \sum_{k=1}^{n-4} (\alpha_{k+1} - \alpha_k) E_{k,k+3} + E_{n-2,n}, \end{aligned}$$

Let us prove the following equality:

$$\varphi(e_i) = \frac{1}{(i-2)!} \left(\sum_{p=1}^{n-i-1} \left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+s} \right) E_{p,p+i} + E_{n+1-i,n} \right), \quad 3 \leq i \leq n.$$

We suppose that the above equality is true for k and let us prove it for $k + 1$.

$$\begin{aligned}
 \varphi(e_{k+1}) &= \frac{1}{k-1} [\varphi(e_1), \varphi(e_k)] = \frac{1}{k-1} (\varphi(e_1)\varphi(e_k) - \varphi(e_k)\varphi(e_1)) \\
 &= \frac{1}{k-1} \left(\frac{1}{(k-2)!} \left(\sum_{t=1}^{n-2} E_{t,t+1} \right) \left(\sum_{p=1}^{n-k-1} \left(\sum_{s=0}^{k-2} (-1)^{k+s} C_{k-2}^s \alpha_{p+s} \right) E_{p,p+k} + E_{n+1-k,n} \right) \right. \\
 &\quad \left. - \frac{1}{(k-2)!} \left(\sum_{p=1}^{n-k-1} \left(\sum_{s=0}^{k-2} (-1)^{k+s} C_{k-2}^s \alpha_{p+s} \right) E_{p,p+k} + E_{n+1-k,n} \right) \left(\sum_{t=1}^{n-2} E_{t,t+1} \right) \right) \\
 &= \frac{1}{(k-1)!} \left(\sum_{p=1}^{n-k-2} \left(\sum_{s=0}^{k-2} (-1)^{k+s} C_{k-2}^s (\alpha_{p+s+1} - \alpha_{p+s}) \right) E_{p,p+k+1} + E_{n-k,n} \right) \\
 &= \frac{1}{(k-1)!} \left(\sum_{p=1}^{n-k-2} \left(\sum_{s=0}^{k-1} (-1)^{k+s+1} C_{k-1}^s \alpha_{p+s} \right) E_{p,p+k+1} + E_{n-k,n} \right).
 \end{aligned}$$

By using the multiplications, where $i + j \leq n$, we get

$$\begin{aligned}
 [\varphi(e_i), \varphi(e_j)] &= \varphi(e_i)\varphi(e_j) - \varphi(e_j)\varphi(e_i) \\
 &= \frac{1}{(i-2)!(j-2)!} \left(\left(\sum_{p=1}^{n-i-1} \left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+s} \right) E_{p,p+i} + E_{n+1-i,n} \right) \right. \\
 &\quad \left(\sum_{q=1}^{n-j-1} \left(\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{q+r} \right) E_{q,q+j} + E_{n+1-j,n} \right) \\
 &\quad - \left(\sum_{q=1}^{n-j-1} \left(\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{q+r} \right) E_{q,q+j} + E_{n+1-j,n} \right) \\
 &\quad \left. \left(\sum_{p=1}^{n-i-1} \left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+s} \right) E_{p,p+i} + E_{n+1-i,n} \right) \right) \\
 &= \frac{1}{(i-2)!(j-2)!} \left(\sum_{p=1}^{n-i-j-1} \left(\left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+s} \right) \left(\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{p+i+r} \right) \right. \right. \\
 &\quad \left. - \left(\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{p+r} \right) \left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+j+s} \right) \right) E_{p,p+i+j} \\
 &\quad \left. + \left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{n+s+1-i-j} - \sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{n+r+1-i-j} \right) E_{n+1-i-j,n} \right).
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 [\varphi(e_i), \varphi(e_j)] &= (j-i)\varphi(e_{i+j}) \\
 &= \frac{(j-i)}{(i+j-2)!} \left(\sum_{p=1}^{n-i-j-1} \left(\sum_{s=0}^{i+j-2} (-1)^{i+j+s} C_{i+j-2}^s \alpha_{p+s} \right) E_{p,p+i+j} + E_{n+1-i-j,n} \right).
 \end{aligned}$$

Next, we have the following system of equations

$$\begin{aligned}
 &\left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+s} \right) \left(\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{p+i+r} \right) - \left(\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{p+r} \right) \\
 &\left(\sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{p+j+s} \right) + \frac{(i-j)(i-2)!(j-2)!}{(i+j-2)!} \sum_{s=0}^{i+j-2} (-1)^{i+j+s} C_{i+j-2}^s \alpha_{p+s} = 0,
 \end{aligned} \tag{2}$$

where $1 \leq p \leq n - i - j - 1$, $i + j \leq n - 2$.

$$\sum_{r=0}^{j-2} (-1)^{j+r} C_{j-2}^r \alpha_{n+r+1-i-j} - \sum_{s=0}^{i-2} (-1)^{i+s} C_{i-2}^s \alpha_{n+s+1-i-j} = \frac{(i-j)(i-2)!(j-2)!}{(i+j-2)!}, \quad i + j \leq n. \tag{3}$$

One of the solutions of the system of Eqs. (2) and (3) is

$$\alpha_i = \frac{1}{n-i}, \quad 1 \leq i \leq n-3.$$

Now we check it by using the next property of binomial coefficients

$$\sum_{k=0}^m \frac{(-1)^k C_m^k}{x+k} = \frac{m!}{x(x+1)\cdots(x+m)}, \quad x \notin \{0, -1, \dots, -m\}. \quad (4)$$

By putting all values of α_i in the system (2)–(3), and by using the property (4), we get

$$\begin{aligned} & \frac{(i-2)!}{(n-i-p+2)(n-i-p+3)\cdots(n-p)} \cdot \frac{(j-2)!}{(n-i-j-p+2)(n-i-j-p+3)\cdots(n-i-p)} \\ & - \frac{(j-2)!}{(n-j-p+2)(n-j-p+3)\cdots(n-p)} \cdot \frac{(i-2)!}{(n-i-j-p+2)(n-i-j-p+3)\cdots(n-j-p)} \\ & + \frac{(i-j)(i-2)!(j-2)!}{(n-i-j-p+2)(n-i-j-p+3)\cdots(n-p)} = 0, \quad 1 \leq p \leq n-i-j-1, \quad i+j \leq n-2, \end{aligned}$$

and

$$\frac{(j-2)!}{(i+1)(i+2)\cdots(i+j-1)} - \frac{(i-2)!}{(j+1)(j+2)\cdots(i+j-1)} = \frac{(i-j)(i-2)!(j-2)!}{(i+j-2)!}, \quad i+j \leq n.$$

So, the values of α_i satisfy the system of equations (2)–(3).

By Proposition 1 we get that this representation is minimal. \square

Now, we construct a \mathcal{W}_n -module structure $V = \mathbb{C}^n$, $V \times \mathcal{W}_n \rightarrow V$, given by

$$(x, e) := x\varphi(e),$$

where $x \in V$ and $e \in \mathcal{W}_n$.

Then we obtain

$$\begin{cases} (x_i, e_1) = x_{i+1}, & 1 \leq i \leq n-2, \\ (x_i, e_2) = \frac{1}{n-i} x_{i+2}, & 1 \leq i \leq n-3, \\ (x_i, e_j) = \frac{1}{(j-2)!} \sum_{s=0}^{j-2} \frac{(-1)^{j+s} C_{j-2}^s}{n-i-s} x_{i+j}, & 3 \leq j \leq n-2, \quad 1 \leq i \leq n-j-1, \\ (x_{n+1-j}, e_j) = \frac{1}{(j-2)!} x_n, & 2 \leq j \leq n, \end{cases}$$

and the remaining products in the action are zero.

3. Leibniz algebras constructed by minimal faithful representations of Lie algebra

Now we investigate Leibniz algebras L such that $L/I \cong \mathcal{Q}_{2n}$ and $I = V$ as a \mathcal{Q}_{2n} -module.

Further we define the multiplications $[e_i, e_j]$ for $1 \leq i, j \leq 2n$. We put

$$[e_i, e_j] = \begin{cases} e_{i+1} + \sum_{k=1}^{2n} \alpha_{i,1}^k x_k, & i = 1, \quad 2 \leq j \leq 2n-2, \\ -e_{j+1} + \sum_{k=1}^{2n} \alpha_{1,j}^k x_k, & j = 1, \quad 2 \leq i \leq 2n-2, \\ (-1)^j e_{2n} + \sum_{k=1}^{2n} \alpha_{i,j}^k x_k, & i = 2n-j+1, \quad 2 \leq j \leq n, \\ (-1)^{j+1} e_{2n} + \sum_{k=1}^{2n} \alpha_{i,j}^k x_k, & j = 2n-i+1, \quad 2 \leq i \leq n, \\ \sum_{k=1}^{2n} \alpha_{i,j}^k x_k, & \text{otherwise.} \end{cases} \quad (5)$$

In the multiplication table (5), by taking the basis transformation

$$e'_1 = e_1 - \sum_{k=2}^{2n-2} \alpha_{1,1}^{k+1} x_k - (\alpha_{1,2}^{2n} + \alpha_{2,1}^{2n}) x_{2n-1}, \quad e'_2 = e_2 - \sum_{k=2}^{2n-2} (\alpha_{1,2}^{k+1} + \alpha_{2,1}^{k+1}) x_k,$$

$$e'_i = [e'_1, e'_{i-1}], \quad 3 \leq i \leq 2n-1, \quad e'_{2n} = [e'_{2n-1}, e'_2],$$

we obtain

$$\begin{aligned} [e_1, e_1] &= \alpha_{1,1}^1 x_1 + \alpha_{1,1}^2 x_2 + \alpha_{1,1}^{2n} x_{2n}, & [e_2, e_1] &= -e_3 + \alpha_{2,1}^1 x_1 + \alpha_{2,1}^2 x_2, \\ [e_1, e_i] &= e_{i+1}, \quad 2 \leq i \leq 2n-2, & [e_{2n-1}, e_2] &= e_{2n}. \end{aligned} \quad (6)$$

There are difficulties to classify the general case, therefore we classify low-dimensional Leibniz algebras of such type. It is well known that \mathcal{Q}_2 is abelian and $\mathcal{L}_4 \cong \mathcal{Q}_4$, therefore we start by classifying Leibniz algebras such that $L/I \cong \mathcal{Q}_6$.

By using the multiplications (1)–(6), and by checking Leibniz identity, we get the following family of algebras denoted by $\lambda(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9)$:

$$\left\{ \begin{array}{lll} [e_1, e_1] = \alpha_1 x_6, & [e_1, e_3] = e_4, & [x_1, e_6] = -2x_6, \\ [e_3, e_1] = -e_4, & [e_5, e_3] = \frac{1}{4} \alpha_3 x_6, & [e_2, e_1] = -e_3 + \alpha_2 x_1 + \alpha_3 x_2, \\ [e_4, e_1] = -e_5, & [x_1, e_3] = -x_3, & [e_2, e_2] = \alpha_5 x_3 + \alpha_7 x_4 + \alpha_8 x_5, \\ [e_5, e_1] = -\alpha_4 x_6, & [x_4, e_3] = x_6, & [e_3, e_2] = 4\alpha_2 x_2 - \alpha_6 x_3 - 2\alpha_7 x_5 - \alpha_9 x_6, \\ [e_6, e_1] = -\frac{1}{4} \alpha_3 x_6, & [e_1, e_4] = e_5, & [e_4, e_2] = -2\alpha_2 x_3 + \frac{1}{2} \alpha_6 x_4, \\ [x_2, e_1] = x_3, & [e_3, e_4] = e_6, & [e_2, e_3] = -3\alpha_2 x_2 + \alpha_6 x_3 - \alpha_5 x_4 + \alpha_7 x_5 + \alpha_9 x_6, \\ [x_3, e_1] = x_4, & [x_1, e_4] = x_4, & [e_3, e_3] = -2\alpha_2 x_3 + \frac{1}{2} \alpha_6 x_4, \\ [x_4, e_1] = x_5, & [x_3, e_4] = x_6, & [e_4, e_3] = -e_6 + 2\alpha_2 x_4 - \frac{1}{2} \alpha_6 x_5, \\ [e_1, e_2] = e_3, & [e_1, e_5] = \alpha_4 x_6, & [e_2, e_4] = 4\alpha_2 x_3 - \frac{3}{2} \alpha_6 x_4 + \alpha_5 x_5, \\ [e_5, e_2] = e_6, & [x_1, e_5] = -x_5, & [e_4, e_4] = -2\alpha_2 x_5 - \frac{1}{2} \alpha_3 x_6, \\ [e_6, e_2] = -\alpha_6 x_6, & [x_2, e_5] = x_6, & [e_2, e_5] = -e_6 - 3\alpha_2 x_4 + \frac{3}{2} \alpha_6 x_5, \\ [x_1, e_2] = x_2, & [e_2, e_6] = \frac{5}{2} \alpha_6 x_6, & [e_3, e_5] = 2\alpha_2 x_5 + \frac{3}{4} \alpha_3 x_6, \\ [x_5, e_2] = x_6, & [e_3, e_6] = -2\alpha_2 x_6, & [e_1, e_6] = -2\alpha_2 x_5 - \frac{3}{4} \alpha_3 x_6. \end{array} \right.$$

Theorem 7. Let L be a 12-dimensional Leibniz algebra such that $L/I \cong \mathcal{Q}_6$ and I a natural L/I -module with a minimal faithful representation. Then L is isomorphic to one of the pairwise non-isomorphic algebras given in Appendix A.

Proof. Let $L(\alpha) := L$ be the 12-dimensional Leibniz algebra given by $\lambda(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9)$. Let $\varphi: L(\alpha) \rightarrow L(\alpha')$ be the isomorphism of Leibniz algebras:

$$\varphi(e_1) = \sum_{k=1}^6 A_k e_k + \sum_{k=1}^6 B_k x_k, \quad \varphi(e_2) = \sum_{k=1}^6 P_k e_k + \sum_{k=1}^6 Q_k x_k, \quad \varphi(x_1) = \sum_{k=1}^6 M_k e_k + \sum_{k=1}^6 R_k x_k,$$

and the other elements of the new basis are obtained as products of the above elements.

Then, we obtain the following restrictions:

$$\begin{aligned} A_1 P_2 R_4 &\neq 0, & A_2 &= B_1 = P_1 = M_i = 0, & 1 &\leq i \leq 6, \\ A_6 &= \frac{-A_4^2 + 2A_3 A_5}{2A_1}, & P_4 &= \frac{P_3^2}{2P_2}, & R_2 &= \frac{A_3 R_1}{A_1}, & R_3 &= -\frac{A_4 R_1}{A_1}, & R_4 &= \frac{A_5 R_1}{A_1}, \\ B_2 &= \frac{2A_2^2 \alpha_2}{A_1}, & B_3 &= \frac{-4\alpha_2 A_3 A_4 - \alpha_6 A_3^2}{2A_1}, & B_4 &= \frac{2\alpha_2 A_4^2 + \alpha_6 A_3 A_4}{2A_1}, \\ B_5 &= \frac{4\alpha_2 P_2^2 (A_1^2 R_6 - A_4 A_5 R_1 - A_1 A_3 R_5) - \alpha_6 A_4^2 P_2^2 R_1 - 4A_3^2 \alpha_7 P_2^2 R_1}{4A_1 P_2^2 R_1} \\ &\quad + \frac{\alpha_3 (4A_1 A_5 P_2 P_3 R_1 - 2A_1 A_4 P_3^2 R_1 + 4A_1 A_3 P_2 P_5 R_1 - 4A_1^2 P_2 P_6 R_1 + 4A_1^2 P_2^2 R_5)}{4A_1 P_2^2 R_1}, \end{aligned}$$

$$\begin{aligned}
Q_1 &= -\alpha_2 P_3, & Q_2 &= \frac{3\alpha_2(A_3 P_3 - A_4 P_2) - \alpha_3 A_1 P_3}{A_1}, \\
Q_3 &= \frac{2\alpha_2(4A_5 P_2^2 - A_4 P_2 P_3 - A_3 P_3^2) + \alpha_3 A_1 P_3^2 + 2\alpha_5 A_3 P_2^2 + 2\alpha_6(A_4 P_2^2 - 2A_3 P_2 P_3)}{2A_1 P_2}, \\
Q_4 &= \frac{\alpha_2(2A_4 P_3^2 R_1 + A_1 P_2^2 R_5 - 3A_5 P_2 P_3 R_1 + 2A_1 P_2 P_6 R_1 - 2A_3 P_2 P_5 R_1)}{A_1 P_2 R_1} \\
&\quad + \frac{-4\alpha_3 A_1 P_2 P_5 - 4\alpha_5 A_4 P_2^2 + \alpha_6(A_3 P_3^2 - 6A_5 P_2^2 + 2A_4 P_2 P_3) + 4\alpha_7 A_3 P_2^2}{4A_1 P_2}, \\
Q_5 &= \frac{\alpha_2(A_5 P_2^2 P_3^2 R_1 - A_4 P_2 P_3^3 R_1 + 2A_3 P_2^2 P_3 P_5 R_1 - 2A_1 P_2^2 P_3 P_6 R_1 - A_1 P_2^3 P_3 R_5)}{A_1 P_2^3 R_1} \\
&\quad + \frac{8\alpha_5 A_5 P_2^4 R_1 + \alpha_6(12A_3 P_2^3 P_5 R_1 - 8A_4 P_2^2 P_3^2 R_1 - 12A_1 P_2^3 P_6 R_1 + 12A_5 P_2^3 P_3 R_1)}{8A_1 P_2^3 R_1} \\
&\quad + \frac{\alpha_3 A_1(P_2^2 P_3 P_5 - P_3^4) + \alpha_7(A_4 P_2^4 - 2A_3 P_2^3 P_3) + \alpha_8 A_3 P_2^4}{A_1 P_2^3},
\end{aligned}$$

and parameters

$$\begin{aligned}
\alpha'_1 &= \frac{\alpha_1}{A_1 P_2^2 R_1}, & \alpha'_2 &= \frac{\alpha_2 A_1 P_2}{R_1}, & \alpha'_3 &= \frac{\alpha_3 A_1}{R_1}, & \alpha'_4 &= \frac{\alpha_4 A_1}{P_2 R_1}, & \alpha'_5 &= \frac{\alpha_5 P_2}{A_1 R_1}, & \alpha'_6 &= \frac{\alpha_6 P_2}{R_1}, \\
\alpha'_7 &= \frac{2\alpha_7 P_2^3 + \alpha_2(P_3^3 - 6P_2^2 P_5)}{2A_1^2 P_2^2 R_1}, & \alpha'_8 &= \frac{4\alpha_8 P_2^3 - \alpha_6(P_3^3 - 6P_2^2 P_5)}{4A_1^3 P_2^2 R_1}, \\
\alpha'_9 &= \frac{\alpha_9}{A_1^2 R_1} + \frac{2\alpha_2(2A_5 P_2^2 P_3 R_1 - A_4 P_2 P_3^2 R_1 - 2A_1 P_2^2 P_6 R_1 + 2A_1 P_2^3 R_5 + 2A_3 P_2^2 P_5 R_1)}{A_1^3 P_2^3 R_1^2} \\
&\quad + \frac{\alpha_3(P_3^3 - 6P_2^2 P_5)}{8A_1^2 P_2^3 R_1}
\end{aligned}$$

Considering all the possible cases, we obtain the families of the algebras listed in the theorem. \square

Now, we will give the classification of the Leibniz algebras L of lowest dimension in the other two types, $L/I \cong \mathcal{W}_n$ and $L/I \cong \mathcal{R}_n$. Taking into account that $\mathcal{W}_1 \cong \mathcal{R}_1 \cong \mathcal{L}_1$, $\mathcal{W}_2 \cong \mathcal{R}_2 \cong \mathcal{L}_2$ are abelian, and $\mathcal{W}_3 \cong \mathcal{R}_3 \cong \mathcal{L}_3$, $\mathcal{W}_4 \cong \mathcal{R}_4 \cong \mathcal{L}_4$, $\mathcal{W}_5 \cong \mathcal{R}_5$, $\mathcal{W}_6 \cong \mathcal{R}_6$, we will study the cases $L/I \cong \mathcal{W}_5$ and $L/I \cong \mathcal{R}_7$. We denote the next families of algebras by $\mu(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$ and $\eta(\beta_1, \beta_2, \beta_3, \beta_4)$:

$$\left\{ \begin{array}{lll}
[e_1, e_1] = \gamma_1 x_5, & [e_2, e_1] = -e_3, & [e_3, e_1] = -2e_4, \\
[e_4, e_1] = -3e_5, & [e_5, e_1] = -\gamma_2 x_5, & [e_2, e_4] = \frac{1}{2}(\gamma_3 x_4 + \gamma_2 x_5), \\
[x_2, e_1] = x_3, & [x_3, e_1] = x_4, & [e_1, e_2] = e_3, \\
[x_2, e_2] = \frac{1}{3}x_4, & [x_3, e_3] = x_5, & [e_4, e_2] = -\frac{1}{2}\gamma_2 x_5, \\
[e_5, e_2] = -\gamma_5 x_5, & [x_1, e_2] = \frac{1}{4}x_3, & [e_2, e_2] = \gamma_3 x_2 + \gamma_4 x_3 + \gamma_6 x_4, \\
[x_4, e_2] = x_5, & [e_1, e_3] = 2e_4, & [e_2, e_3] = e_5 - \gamma_3 x_3 + \gamma_4 x_4 + \gamma_7 x_5, \\
[e_4, e_3] = 3\gamma_5 x_5, & [x_1, e_3] = \frac{1}{12}x_4, & [e_3, e_2] = -e_5 - 2\gamma_4 x_4 - \gamma_7 x_5, \\
[e_1, e_4] = 3e_5, & [x_1, e_1] = x_2, & [e_3, e_4] = -3\gamma_5 x_5, \\
[x_2, e_4] = \frac{1}{2}x_5, & [e_1, e_5] = \gamma_2 x_5, & [e_2, e_5] = \gamma_5 x_5, \\
[x_1, e_5] = \frac{1}{6}x_5.
\end{array} \right.$$

and

$$\left\{ \begin{array}{llll} [e_1, e_1] = \beta_1 x_7, & [e_1, e_2] = e_3, & [e_1, e_3] = e_4, & [e_1, e_4] = e_5, \\ [e_1, e_5] = e_6, & [e_1, e_6] = e_7, & [e_1, e_7] = \beta_2 x_7, & [e_2, e_1] = -e_3, \\ [e_2, e_2] = \beta_3 x_4 + \beta_4 x_6, & [e_2, e_3] = e_5 - \beta_3 x_5, & [e_2, e_4] = e_6 + \beta_3 x_6, & [e_2, e_5] = e_7, \\ [e_2, e_6] = \beta_2 x_7, & [e_3, e_1] = -e_4, & [e_3, e_2] = -e_5, & [e_4, e_1] = -e_5, \\ [e_4, e_2] = -e_6, & [e_5, e_1] = -e_6, & [e_5, e_2] = -e_7, & [e_6, e_1] = -e_7, \\ [e_6, e_2] = -\beta_2 x_7, & [e_7, e_1] = -\beta_2 x_7, & [x_1, e_1] = x_2, & [x_1, e_2] = x_3, \\ [x_1, e_7] = x_7, & [x_2, e_1] = x_3, & [x_2, e_2] = x_4, & [x_2, e_6] = x_7, \\ [x_3, e_1] = x_4, & [x_3, e_2] = x_5, & [x_3, e_5] = x_7, & [x_4, e_1] = x_5, \\ [x_4, e_2] = x_6, & [x_4, e_4] = x_7, & [x_5, e_1] = x_6, & [x_5, e_3] = x_7, \\ [x_6, e_2] = x_7. \end{array} \right.$$

Theorem 8. Let L be a 10-dimensional Leibniz algebra such that $L/I \cong \mathcal{W}_5$ and I a natural L/I -module with a minimal faithful representation. Then L is isomorphic to one of the pairwise non-isomorphic algebras given in [Appendix B](#).

Theorem 9. Let L be a 14-dimensional Leibniz algebra such that $L/I \cong \mathcal{R}_7$ and I a natural L/I -module with a minimal faithful representation. Then L is isomorphic to one of the following pairwise non-isomorphic algebras:

$$\begin{array}{cccccc} \eta(0, 0, 0, 0, 0, 0), & \eta(0, 0, 0, 0, 1), & \eta(0, 0, 1, 0, 0), & \eta(0, 1, 0, 1), & \eta(0, 1, \beta_3, 0)_{\beta_3 \neq 0}, \\ \eta(1, 0, 0, 0, 0, 0), & \eta(1, 0, 0, 1), & \eta(1, 0, 1, 0), & \eta(1, 1, 0, \beta_4), & \eta(1, 1, \beta_3, 0)_{\beta_3 \neq 0}, \end{array}$$

with $\beta_3, \beta_4 \in \mathbb{C}$.

The proofs of [Theorems 8](#) and [9](#) are carried out by applying similar arguments to those used in [Theorem 7](#).

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Appendix A

$\lambda(0,0,0,0,0,0,0,0,0)$	$\lambda(0,0,0,0,0,0,0,0,1)$	$\lambda(0,0,0,0,0,0,1,0)$	$\lambda(0,0,0,1,1,1,\alpha_7,0,\alpha_9)$
$\lambda(0,0,0,0,0,0,1,0,0)$	$\lambda(0,0,0,0,0,0,1,0,1)$	$\lambda(0,0,0,0,0,0,1,1,0)$	$\lambda(1,0,0,0,1,0,1,\alpha_8,\alpha_9)$
$\lambda(0,0,0,0,0,1,0,0,0)$	$\lambda(0,0,0,0,0,1,0,0,1)$	$\lambda(0,0,0,0,0,1,1,0,0)$	$\lambda(1,0,1,0,1,0,\alpha_7,\alpha_8,0)$
$\lambda(0,0,0,0,1,0,0,0,0)$	$\lambda(0,0,0,0,1,0,0,0,1)$	$\lambda(0,0,0,0,1,0,0,1,0)$	$\lambda(0,0,1,1,0,0,1,\alpha_8,0)$
$\lambda(0,0,0,0,0,0,0,1,1)$	$\lambda(0,0,0,1,0,0,0,1,1)$	$\lambda(0,0,0,0,1,1,\alpha_7,0,0)$	$\lambda(0,0,0,0,1,1,\alpha_7,0,1)$
$\lambda(0,0,0,1,0,0,0,0,0)$	$\lambda(0,0,0,1,0,0,0,0,1)$	$\lambda(0,0,0,1,0,0,0,1,0)$	$\lambda(0,0,0,0,1,0,1,\alpha_8,1)$
$\lambda(0,0,0,1,0,0,1,0,0)$	$\lambda(0,0,0,1,0,0,1,0,1)$	$\lambda(0,0,0,1,0,0,1,1,\alpha_9)$	$\lambda(1,0,0,1,1,0,\alpha_7,\alpha_8,\alpha_9)$
$\lambda(0,0,0,1,0,1,0,0,1)$	$\lambda(0,0,0,1,0,1,0,0,0)$	$\lambda(0,0,0,1,1,0,0,0,0)$	$\lambda(0,0,0,1,1,0,1,\alpha_8,\alpha_9)$
$\lambda(0,0,0,0,0,0,1,1,1)$	$\lambda(0,0,0,1,1,0,0,0,1)$	$\lambda(0,0,0,0,1,0,1,\alpha_8,0)$	$\lambda(0,0,1,0,0,1,1,\alpha_8,0)$
$\lambda(0,0,1,0,0,0,0,1,0)$	$\lambda(0,0,1,0,0,0,1,0,0)$	$\lambda(0,0,1,0,0,0,1,1,0)$	$\lambda(0,0,1,0,1,1,\alpha_7,\alpha_8,0)$
$\lambda(0,0,1,0,0,1,0,1,0)$	$\lambda(0,0,1,0,0,0,0,0,0)$	$\lambda(0,0,1,0,1,0,0,0,0)$	$\lambda(0,0,1,1,0,1,\alpha_7,0,0)$
$\lambda(0,0,0,0,0,1,1,0,1)$	$\lambda(0,0,1,0,0,1,0,0,0)$	$\lambda(0,0,1,1,0,0,0,0,0)$	$\lambda(1,0,1,1,\alpha_5,0,\alpha_7,\alpha_8,0)$
$\lambda(0,0,0,0,1,0,0,1,1)$	$\lambda(0,0,1,0,1,0,0,1,0)$	$\lambda(1,0,0,0,0,0,1,1,\alpha_9)$	$\lambda(0,0,1,1,1,0,\alpha_7,\alpha_8,0)$
$\lambda(0,1,0,0,0,0,0,0,0)$	$\lambda(0,1,0,0,0,0,0,1,0)$	$\lambda(0,1,0,0,0,1,0,\alpha_8,0)$	$\lambda(0,1,0,0,1,\alpha_6,0,\alpha_8,0)$
$\lambda(0,1,0,1,0,0,0,0,0)$	$\lambda(0,1,0,1,0,0,0,1,0)$	$\lambda(0,1,0,1,0,1,0,\alpha_8,0)$	$\lambda(0,1,0,1,1,\alpha_6,0,\alpha_8,0)$
$\lambda(0,1,1,0,0,0,0,0,0)$	$\lambda(0,1,1,0,0,0,0,1,0)$	$\lambda(0,1,1,0,0,1,0,\alpha_8,0)$	$\lambda(0,1,1,0,1,\alpha_6,0,\alpha_8,0)$
$\lambda(0,0,0,1,0,0,0,1,1)$	$\lambda(1,0,0,0,0,0,0,0,0)$	$\lambda(1,0,0,0,1,0,0,1,\alpha_9)$	$\lambda(1,0,1,1,\alpha_5,\alpha_6,\alpha_7,0,0)$
$\lambda(1,0,0,0,0,0,0,1,1)$	$\lambda(1,0,0,0,0,0,1,0,0)$	$\lambda(1,0,1,0,0,1,\alpha_7,0,0)$	$\lambda(0,0,1,1,1,\alpha_6,\alpha_7,0,0)_{\alpha_6 \neq 0}$
$\lambda(1,0,0,0,0,1,0,0,0)$	$\lambda(1,0,0,0,0,1,0,0,1)$	$\lambda(1,0,0,0,0,1,1,0,\alpha_9)$	$\lambda(1,1,0,1,\alpha_5,\alpha_6,0,\alpha_8,0)$
$\lambda(1,0,0,0,1,0,0,0,1)$	$\lambda(0,0,1,1,0,0,0,1,0)$	$\lambda(0,0,0,1,1,0,0,1,\alpha_9)$	$\lambda(1,0,0,0,1,1,\alpha_7,0,\alpha_9)$
$\lambda(1,0,0,1,0,0,0,0,0)$	$\lambda(1,0,0,1,0,0,0,0,1)$	$\lambda(1,0,0,1,0,0,0,1,\alpha_9)$	$\lambda(1,0,0,1,0,0,1,\alpha_8,\alpha_9)$
$\lambda(1,0,1,0,0,0,0,0,0)$	$\lambda(1,0,1,0,0,0,0,1,0)$	$\lambda(0,0,0,1,0,1,1,0,\alpha_9)$	$\lambda(1,0,0,1,1,\alpha_6,\alpha_7,0,\alpha_9)_{\alpha_6 \neq 0}$
$\lambda(1,0,0,0,1,0,0,0,0)$	$\lambda(1,0,0,0,0,0,1,0,1)$	$\lambda(0,0,1,0,1,0,1,\alpha_8,0)$	$\lambda(1,0,1,0,1,\alpha_6,\alpha_7,0,0)_{\alpha_6 \neq 0}$
$\lambda(1,0,0,0,0,0,0,1,0)$	$\lambda(1,0,0,0,0,0,0,0,1)$	$\lambda(1,1,0,0,0,1,0,\alpha_8,0)$	$\lambda(1,1,0,0,1,\alpha_6,0,\alpha_8,0)$
$\lambda(1,1,0,0,0,0,0,0,0)$	$\lambda(1,1,0,0,0,0,0,1,0)$	$\lambda(1,0,1,0,0,0,1,\alpha_8,0)$	$\lambda(1,1,1,\alpha_4,\alpha_5,\alpha_6,0,\alpha_8,0)$

with $\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9 \in \mathbb{C}$.

Appendix B

$\mu(0,0,0,0,0,0)$	$\mu(0,0,1,0,1,0,0)$	$\mu(0,1,0,0,0,\gamma_6,0)$	$\mu(0,0,1,1,\gamma_5,0,0)_{\gamma_5 \neq 0}$
$\mu(0,0,0,0,0,0,1)$	$\mu(1,0,0,0,0,0,0)$	$\mu(1,0,0,0,1,\gamma_6,0)$	$\mu(0,1,0,1,\gamma_5,\gamma_6,0)_{\gamma_5 \neq 0}$
$\mu(0,0,0,0,0,1,0)$	$\mu(1,0,0,0,0,0,1)$	$\mu(0,1,0,1,0,\gamma_6,\gamma_7)$	$\mu(0,1,1,\gamma_4,\gamma_5,0,0)_{\gamma_5 \neq 0}$
$\mu(0,0,0,0,0,1,1)$	$\mu(0,0,0,1,0,1,\gamma_7)$	$\mu(1,0,0,1,0,\gamma_6,\gamma_7)$	$\mu(1,0,0,1,\gamma_5,\gamma_6,0)_{\gamma_5 \neq 0}$
$\mu(0,0,0,0,1,0,0)$	$\mu(0,0,0,1,\gamma_5,1,0)$	$\mu(1,0,1,\gamma_4,0,0,\gamma_7)$	$\mu(1,0,1,\gamma_4,\gamma_5,0,0)_{\gamma_5 \neq 0}$
$\mu(0,0,0,0,1,1,0)$	$\mu(0,0,1,1,0,0,\gamma_7)$	$\mu(0,1,1,\gamma_4,0,0,\gamma_7)$	$\mu(1,1,0,\gamma_4,\gamma_5,\gamma_6,0)_{\gamma_5 \neq 0}$
$\mu(0,0,0,1,0,0,1)$	$\mu(0,1,0,0,0,\gamma_6,1)$	$\mu(1,1,0,\gamma_4,0,\gamma_6,\gamma_7)$	$\mu(1,1,\gamma_3,\gamma_4,0,0,\gamma_7)_{\gamma_3 \neq 0}$
$\mu(0,0,1,0,0,0,0)$	$\mu(1,0,0,0,0,1,\gamma_7)$	$\mu(0,0,0,1,\gamma_5,0,0)_{\gamma_5 \neq 0}$	$\mu(1,1,\gamma_3,\gamma_4,\gamma_5,0,0)_{\gamma_3 \neq 0, \gamma_5 \neq 0}$
$\mu(0,0,1,0,0,0,1)$	$\mu(0,1,0,0,1,\gamma_6,0)$		

with $\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7 \in \mathbb{C}$.

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