



Mass functions of a compact manifold

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ABSTRACT

Let M be a compact manifold of dimension n . In this paper, we introduce the *Mass Function* $a \geq 0 \mapsto X_+^M(a)$ (resp. $a \geq 0 \mapsto X_-^M(a)$) which is defined as the supremum (resp. infimum) of the masses of all metrics on M whose Yamabe constant is larger than a and which are flat on a ball of radius 1 and centered at a point $p \in M$. Here, the mass of a metric flat around p is the constant term in the expansion of the Green function of the conformal Laplacian at p . We show that these functions are well defined and have many properties which allow to obtain applications to the Yamabe invariant (i.e. the supremum of Yamabe constants over the set of all metrics on M).

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1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and denote by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)}s_g : C^\infty(M) \rightarrow C^\infty(M)$$

the conformal Laplace operator of g , where s_g is the scalar curvature of g and Δ_g is the Laplace–Beltrami operator with non-negative spectrum. Assume that the metric g is flat on an open neighborhood of a point $p \in M$ and that all eigenvalues of L_g are strictly positive. Then it is well-known that there exists a unique Green function G_g of L_g at p , i.e. in the sense of distributions we have $L_g G_g = \delta_p$, the function G_g is smooth and strictly positive on $M \setminus \{p\}$ and as $x \rightarrow p$ we have

$$G_g(x) = \frac{1}{(n-2)\omega_{n-1}r(x)^{n-2}} + m(g, p) + o(1)$$

where ω_{n-1} is the volume of the standard sphere of dimension $n-1$, the function r denotes the Riemannian distance from p and $m(g, p) \in \mathbb{R}$ is a number called the mass of g at p . This quantity is related to the so-called ADM mass of an asymptotically flat Riemannian manifold. The study of the mass has led to many interesting results in geometric analysis and General Relativity. An example is an application to the so-called conformal Yamabe constant of (M, g) defined by

$$Y(M, g) := \inf_M \int_M s_g dv^g,$$

where the inf is taken over the set of all Riemannian metrics on M which have unit volume and are conformal to g . Namely, in a famous article [8], Richard Schoen used the positivity of the mass $m(g, p)$ to prove that $Y(M, g) < Y(S^n, g_{\text{can}})$ if (M, g) satisfies the assumptions above and is not conformally diffeomorphic to the standard sphere (S^n, g_{can}) .

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In this article we consider the dependence of the mass on the Yamabe constant $Y(M, g)$. We define two functions $a \mapsto X_+^M(a)$ and $a \mapsto X_-^M(a)$ whose values are a sup and an inf of masses $m(g, p)$ respectively taken over the set of all Riemannian metrics g with $Y(M, g) > a$ which are flat on a ball of radius 1 centered at $p \in M$ (see Definition 3.1). We prove that for small values of a the values $X_+^M(a)$ and $X_-^M(a)$ decrease and increase respectively under surgery of codimension at least 3 (see Theorem 5.1). Finally, we give an application to the smooth Yamabe invariant of M defined by

$$\sigma(M) := \sup Y(M, g)$$

where the sup is taken over the set of all Riemannian metrics on M . The question of whether for a given smooth manifold M one has $\sigma(M) < \sigma(S^n)$ is open in general. We prove that if $X_+^M(\sigma(M)) > 0$ then we have $\sigma(M) < \sigma(S^n)$. The precise statement is given in Theorem 6.1.

In the proofs of these theorems we use a surgery result obtained by the second author together with Ammann and Dahl [1] and a variational characterization of the mass $m(g, p)$ obtained by the two authors of the present article [6].

2. Notation

Let M be a compact manifold of dimension $n \geq 3$. The set of Riemannian metrics on M will be denoted by $\mathcal{M}_M$. For $g \in \mathcal{M}_M$, we denote by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g : C^\infty(M) \rightarrow C^\infty(M)$$

the conformal Laplace operator of g , where s_g is the scalar curvature of g and Δ_g is the Laplace–Beltrami operator with non-negative spectrum. Moreover, we write $N := \frac{2n}{n-2}$ and we denote by

$$Y(g) := Y(M, g) = \inf \left\{ \frac{\int_M u L_g u \, dv^g}{\|u\|_{L^N}^2} \mid u \in C^\infty(M) \setminus \{0\} \right\}$$

the (conformal) Yamabe constant of g and by $\sigma(M) := \sup_{g \in \mathcal{M}_M} Y(g)$ the (smooth) Yamabe invariant of M . We will write $Y(g)$ instead of $Y(M, g)$ since M will always be clear from the context. We have $Y(g) > 0$ if and only if all eigenvalues of L_g are strictly positive. In the following, we will always assume that $\sigma(M) > 0$ which is equivalent to the fact that M admits a metric of positive scalar curvature. We define, for any $a \in [0, \sigma(M)]$,

$$Z_M(a) := \{g \in \mathcal{M}_M \mid Y(g) > a\}$$

$$\Omega_M^a := \{(g, p) \in Z_M(a) \times M \mid B_p^g(1) \text{ is isometric to } \mathbb{B}\}.$$

where $B_p^g(1)$ stands for the ball with center p and radius 1 with respect to the metric g and where $\mathbb{B}$ is the standard Euclidean unit ball of dimension n . Note that these sets are not empty as soon as $\sigma(M) > 0$ (see the first item of Proposition 4.1).

Let η be a smooth function on M such that $\eta \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B_p^g(\frac{1}{2})$ and $\text{supp}(\eta) \subset B_p^g(1)$, where ω_{n-1} denotes the volume of S^{n-1} with the standard metric. If $(g, p) \in \Omega_M^a$ for some $a \geq 0$ then there exists a unique Green function G_g of L_g at p and we have for all $x \in M \setminus \{p\}$:

$$G_g(x) = \eta(x)r(x)^{2-n} + m(g, p) + \alpha(x), \quad (1)$$

where $r(x)$ denotes the Riemannian distance of x and p with respect to g , α is a smooth function defined on all of M which is harmonic on $B_p^g(\frac{1}{2})$ and satisfies $\alpha(p) = 0$ and $m(g, p) \in \mathbb{R}$ is a number called the mass of g at p .

We recall that $m(g, p)$ has a variational characterization established in [6] Namely, the function $F_\eta : M \rightarrow \mathbb{R}$ defined by

$$F_\eta(x) := \begin{cases} \Delta_g(\eta r^{2-n})(x), & x \neq p \\ 0, & x = p \end{cases}$$

has support contained in $\text{supp}(\nabla \eta)$ and is smooth on M . For every $u \in C^\infty(M)$ we define

$$J_p^g(u) := \int_{M \setminus \{p\}} \eta r^{2-n} F_\eta \, dv^g + 2 \int_M u F_\eta \, dv^g + \int_M u L_g u \, dv^g.$$

Then, it was proven in [6] that

$$-m(g, p) = \inf \{J_p^g(u) \mid u \in C^\infty(M)\} \quad (2)$$

and that the infimum is attained for the smooth function θ defined by

$$\theta(x) := m(g, p) + \alpha(x) \quad (3)$$

with the function α as in (1). We say that a compact manifold satisfies PMT (for Positive Mass Theorem) if for every metric g on M and for all points $p \in M$ such that g is flat on an open neighborhood of p and $Y(g) > 0$ we have $m(g, p) \geq 0$. It is conjectured that every compact manifold satisfies PMT. This conjecture has been proved in some special cases (see e.g. [10, 11, 14]). A complete proof of the conjecture has been announced by Lohkamp [7] and Schoen–Yau [9].

3. Upper and lower mass functions of M

Definition 3.1. The upper (resp. lower) mass function

$$X_+^M : [0, \sigma(M)] \rightarrow \mathbb{R} \cup \{\pm\infty\} \quad (\text{resp. } X_-^M : [0, \sigma(M)] \rightarrow \mathbb{R} \cup \{\pm\infty\})$$

is defined as follows:

$$X_+^M(a) := \begin{cases} \sup\{m(g, p) \mid (g, p) \in \Omega_M^0\}, & a = 0 \\ \limsup_{\varepsilon \rightarrow 0} \sup\{m(g, p) \mid (g, p) \in \Omega_M^{a-\varepsilon}\}, & a \in (0, \sigma(M)] \end{cases}$$

$$X_-^M(a) := \begin{cases} \inf\{m(g, p) \mid (g, p) \in \Omega_M^0\}, & a = 0 \\ \liminf_{\varepsilon \rightarrow 0} \inf\{m(g, p) \mid (g, p) \in \Omega_M^{a-\varepsilon}\}, & a \in (0, \sigma(M)] \end{cases}$$

The goal of this paper is to establish several properties of $X_{\pm}^M(a)$.

4. Basic properties of $X_{\pm}^M$

Proposition 4.1. Let M be a compact manifold of dimension $n \geq 3$ such that $\sigma(M) > 0$. Then the following hold:

- (1) X_+^M and X_-^M are well defined for all $a \in [0, \sigma(M)]$ as soon as $\sigma(M) > 0$;
- (2) For all $a \in [0, \sigma(M)]$, $X_+^M(a) \geq X_-^M(a)$;
- (3) X_+^M is a decreasing function of a and X_-^M is an increasing function of a and they are both left continuous;
- (4) For all $a \in [0, \sigma(M)]$, $0 \geq X_-^M(a) > -\infty$;
- (5) For all $a > 0$, $X_+^M(a) < +\infty$;
- (6) The following statements are equivalent:
 - (a) $X_-^M(0) = 0$;
 - (b) $X_-^M(a) = 0$ for all $a \in [0, \sigma(M)]$;
 - (c) M possesses the property PMT.
- (7) Let M, N be compact manifolds of dimension $n \geq 3$ with positive Yamabe invariant. Then, for all $a \in (0, \min(\sigma(M), \sigma(N)))$ we have

$$X_+^{M \sqcup N}(a) = \max(X_+^M(a), X_+^N(a)) \quad \text{and} \quad X_-^{M \sqcup N}(a) = \min(X_-^M(a), X_-^N(a)).$$

Proof. (1) Let $0 < a < \sigma(M)$. It suffices to show that Ω_M^a is not empty. By definition of $\sigma(M)$ we have a metric g on M with $Y(g) > a$. Let $\xi = \sum_i dx_i^2$, where $(x_1, \dots, x_n)$ is a system of normal coordinates at some $p \in M$, be a flat metric around p and let $g_\varepsilon := (1 - \eta_\varepsilon)g + \eta_\varepsilon\xi$, where $\eta_\varepsilon : M \rightarrow [0, 1]$ is a cut-off function equal to 1 on $B_p^g(\varepsilon)$, equal to 0 outside $B_p^g(2\varepsilon)$ and such that $|\eta_\varepsilon| \leq \frac{2}{\varepsilon}$ and $|\nabla^2 \eta_\varepsilon| \leq \frac{2}{\varepsilon^2}$. By Lemma A.1 we have $\lim_{\varepsilon \rightarrow 0} Y(g_\varepsilon) = Y(g)$. Now, the metric $h_\varepsilon = \frac{1}{\varepsilon^2} g_\varepsilon$ is flat on $B_p^{h_\varepsilon}(1)$. If ε is small enough then $Y(h_\varepsilon) = Y(g_\varepsilon) > a$ which implies that $h_\varepsilon \in \Omega_M^a$.

(2) and (3) are clear from the definitions.

(4) Let u be any nonzero smooth function compactly supported in the Euclidean unit ball $\mathbb{B}$. Let $(g, p) \in \Omega_M^a$ for some a . From the definition of Ω_M^a , we can identify $(B_p(1), g)$ with $\mathbb{B}$ so that u can be considered as a test function in the variational characterization (2) which provides

$$-m(g, p) \leq J_g^p(u).$$

The inequality $X_-^M(a) > -\infty$ follows by noticing that $J_g^p(u)$ does not depend on the choice of $a \geq 0$ nor on the choice of $(g, p) \in \Omega_M^a$.

Let us prove now that $X_-^M(a) \leq 0$. It comes from the facts that if $(g, p) \in \Omega_M^a$ then $(bg, p) \in \Omega_M^a$ for any $b \geq 1$, and also that for any such b

$$m(bg, p) = b^{1-\frac{n}{2}} m(g, p).$$

(5) Let $a > 0$ and let $(g, p) \in \Omega_M^a$. We have to show that $m(g, p)$ is bounded by a constant which depends only on a but not on (g, p) . Let $u \in C^\infty(M)$. In what follows, $C > 0$ denotes a positive constant which might depend on a but not on (g, p) . By the variational characterization (2), choose u so that

$$-m(g, p) + 1 \geq J_g^p(u)$$

From the definition of J_g^p , one has

$$J_g^p(u) \geq -C + 2 \int_M u F_\eta dv^g + \int_M u L_g u dv^g.$$

Using that fact that $Y(g) \geq a$ and using Hölder inequality, one gets

$$J_g^p(u) \geq -C - 2\|F_\eta\|_{L^\infty} \left(\int_{B_p^g(1)} |u|^N dv^g \right)^{\frac{1}{N}} \text{vol}(B_p^g(1))^{\frac{N-1}{N}} + a \left(\int_M |u|^N, dv^g \right)^{\frac{2}{N}}.$$

Set now

$$X_g = \left(\int_M |u|^N dv^g \right)^{\frac{1}{N}},$$

we obtain that there exists some $C', C'' > 0$ independent of (g, p) such that

$$J_g^p(u) \geq C - C'X_g + C''X_g^2. \quad (4)$$

This quantity is bounded from below independently of (g, p) . This shows that $m(g, p)$ is bounded from above by a constant independent of $(g, p) \in \Omega_M^a$. This implies that for all $a > 0$, $X_+^M(a) < +\infty$.

(6) Clearly the property PMT for a manifold is equivalent to $X_-^M(0) \geq 0$. Since $X_-^M(0) \leq 0$ by assertion (4), the result follows.

(7) Let $a \in (0, \sigma(M \sqcup N)] = (0, \min(\sigma(M), \sigma(N))]$ and $\varepsilon > 0$. On the one hand, let $(g, p) \in \Omega_{M \sqcup N}^{a-\varepsilon}$ where $p \in M$. Then g decomposes as $g = g_M \sqcup g_N$ where $g_M \in \mathcal{M}_M$ and $g_N \in \mathcal{M}_N$. We have $a - \varepsilon < Y(g) = \min(Y(g_M), Y(g_N))$. Since $p \in M$, this implies that $(g_M, p) \in \Omega_M^{a-\varepsilon}$. Let $\rho > 0$. If ε is small enough, it follows from the definition of $X_+^M(a)$ that $m(g, p) = m(g_M, p) \leq X_+^M(a) + \rho \leq \max(X_+^M(a), X_+^N(a)) + \rho$. In the same way, if $p \in N$, $m(g, p) \leq \max(X_+^M(a), X_+^N(a)) + \rho$. From these inequalities, we obtain

$$X_+^{M \sqcup N}(a) \leq \max(X_+^M(a), X_+^N(a)) + \rho$$

and since ρ is arbitrary

$$X_+^{M \sqcup N}(a) \leq \max(X_+^M(a), X_+^N(a)).$$

On the other hand, let $(g_M, p) \in \Omega_M^{a-\varepsilon}$ and let g_N any metric on N with $Y(g_N) \geq a - \varepsilon$. If ε is small enough, then $m(g_M, p) = m(g_M \sqcup g_N, p) \leq X_+^{M \sqcup N}(a) + \rho$. Hence $X_+^M(a) \leq X_+^{M \sqcup N}(a) + \rho$. The same holds for N and the result follows.

The proof for $X_-^M(a)$ is similar. $\square$

5. $X_\pm^M(a)$ and surgery

In this section, we first establish the following theorem, whose proof is a consequence of the results in [6] and [1].

Theorem 5.1. *Let M be a compact manifold of dimension $n \geq 3$ and $M^\sharp$ be obtained from M by a surgery of dimension $k \leq n - 3$. Then, there exist constants $\Lambda_{n,k} > 0$ depending only on n and k but not on M such that for all $a \in [0, \sigma(M)]$, one has*

$$X_+^M(a) \leq X_+^{M^\sharp}(\min(a, \Lambda_{n,k})) \text{ and } X_-^M(a) \geq X_-^{M^\sharp}(\min(a, \Lambda_{n,k})).$$

In the case $k = 0$ one may take $\Lambda_{n,0} = \sigma(S^n)$ for all n .

We remark that some known and conjectured lower estimates for the constants $\Lambda_{n,k}$, $1 \leq k \leq n - 3$, can be found in [1]. In this section, we also prove the following consequence of Theorem 5.1.

Corollary 5.2. *Let M_0 be any compact non spin (resp. spin) simply connected manifold of dimension $n \geq 5$ such that $\sigma(M_0) > 0$ and let $a > 0$. Then, for all compact (resp. compact spin) manifolds M of the same dimension as M_0 and with $\sigma(M) > 0$ one has*

$$0 \geq X_-^M(\min(a, \Lambda_n)) \geq X_-^{M_0}(\min(a, \Lambda_n)) \text{ and } X_+^M(\min(a, \Lambda_n)) \leq X_+^{M_0}(\min(a, \Lambda_n))$$

where $\Lambda_n := \min_{1 \leq k \leq n-3} \Lambda_{n,k} > 0$.

This has the following obvious consequence:

Corollary 5.3. *Let M_0, M_1 be two compact non spin (resp. spin) simply connected manifolds of dimension $n \geq 5$ such that $\sigma(M_0), \sigma(M_1) > 0$ and let $a \in (0, \Lambda_n)$. Then we have*

$$X_\pm^{M_0}(a) = X_\pm^{M_1}(a).$$

Remark 5.4. By Corollary C in [5], if M is a compact simply connected non-spin manifold of dimension at least 5 then $\sigma(M) > 0$. By [1], when M is simply connected and $\sigma(M) > 0$, it holds that

$$\sigma(M) \geq \min\{\Lambda_n, \sigma(W_1), \dots, \sigma(W_k)\}$$

where $W_1, \dots, W_k$ are generators of the oriented cobordism group in dimension n . For every n this group is finitely generated and generators are well known [13]. When M is spin and $\sigma(M) > 0$, then it is in the Kernel of the index map $\alpha : \Omega_n^{\text{spin}} \rightarrow KO_*$ which is all also finitely generated by some spin manifolds $V_1, \dots, V_l$ (see [12]) and again, we get

$$\sigma(M) \geq \min\{\Lambda_n, \sigma(V_1), \dots, \sigma(V_l)\} > 0.$$

Remark 5.5. This corollary allows to recover a result in [6]: if M_0 not spin, simply connected of dimension $n \geq 5$ satisfies PMT, then all the manifolds of the same dimension satisfy PMT. Indeed, assume that M_0 satisfies PMT then $X_-^{M_0}(0) = 0$ (see Proposition 4.1) and hence $X_-^M(0) = 0$ which implies PMT. Note that Lohkamp [7] and Schoen and Yau [9] recently announced a complete proof of the Positive Mass Theorem (i.e. all manifolds satisfy PMT).

Another consequence is the following:

Corollary 5.6. Assume that M is compact and simply connected, that $\sigma(M) > 0$ and that $a < \sigma(M)$ if M satisfies PMT or $a < \min(\Lambda_n, \sigma(M))$ otherwise. Then we have $X_+^M(a) > 0$.

Proof. Assume first that M satisfies PMT. Then, let g a metric such that $Y(g) \in (a, \sigma(M))$. Let ξ be a flat metric around some point $p \in M$ and let $0 \leq \chi_\varepsilon \leq 1$ be a cut-off decreasing function satisfying $\chi_\varepsilon(x) = 1$ if $x \in B_p^\varepsilon(\varepsilon)$, $\chi_\varepsilon(x) = 0$ if $x \notin B_p^\varepsilon(2\varepsilon)$ and $|\chi_\varepsilon'(x)| \leq \frac{2}{\varepsilon}$. As in Lemma A.1 one obtains metrics g_ε on M which are flat around p and such that $Y(g_\varepsilon)$ tends to $Y(g)$ as ε tends to 0. Hence, replacing g by g_ε , we can assume that g is flat around some point p . Also, multiplying g by a large constant, we can assume that it is flat on $B_p^\varepsilon(1)$. By the Positive Mass Theorem, its mass at p is positive and hence $X_+^M(a) > 0$. If M does not satisfy PMT, it is not spin. It then follows from Corollary 5.2 that $X_+^M(a) \geq X_+^N(a)$ for any compact manifold N of the same dimension such that $\sigma(N) > 0$. In particular, if $N = S^n$, we have $X_+^N(a) > 0$ and consequently, $X_+^M(a) > 0$. $\square$

Remark 5.7. Again, if the proof of the Positive Mass Theorem by Lohkamp in [7] or by Schoen and Yau announced in [9] is confirmed then, for all M and all $a < \sigma(M)$ we deduce from Corollary 5.6 that $X_+^M(a) > 0$.

5.1. Proof of Theorem 5.1

Let $g \in \Omega_M^a$. In [6], we constructed a sequence of metrics g_j on $M^\sharp$ such that $\lim_j m(g_j, p) = m(g, p)$. In the construction, the metric g_j can be made isometric to g in $B_p^\varepsilon(1)$ (as soon as $B_p^\varepsilon(1)$ is topologically trivial). Moreover, we used exactly the same metrics as in the main result of [1] where it was proved that

$$\lim_j Y(g_j) \geq \min(\Lambda_{n,k}, Y(g))$$

where $\Lambda_{n,0} = \sigma(S^n)$ and $\Lambda_{n,k} > 0$ depends only on n and k . This proves that for all $\varepsilon > 0$ we have $g_j \in \Omega_{M^\sharp}^{\min(a, \Lambda_{n,k}) - \varepsilon}$ as soon as j is large enough. Theorem 5.1 easily follows.

5.2. Proof of Corollary 5.2

(1) Let M_0 be a compact non-spin (resp. spin) simply connected manifold with $\sigma(M_0) > 0$ and M any compact (resp. compact spin) manifold of the same dimension. By Proposition 4.1 we have for all a

$$X_+^{M \sqcup (-M)}(a) = X_+^M(a),$$

where $(-M)$ is M equipped with the opposite orientation. Theorem 5.1 then shows that for all a

$$X_+^M(\min(a, \Lambda_n)) = X_+^{M \sqcup (-M)}(\min(a, \Lambda_n)) \leq X_+^{M \sharp (-M)}(\min(a, \Lambda_n)) \quad (5)$$

where $\sharp$ denotes the connected sum. Here, we used that the connected sum is a surgery of dimension 0.

(2) The manifolds $M \sharp (-M)$ and $M_0 \sharp (-M_0)$ are oriented (resp. spin) cobordant since they are both oriented (resp. spin) cobordant to S^n . Since $M_0 \sharp (-M_0)$ is simply connected and not spin (resp. spin), it is obtained from $M \sharp (-M)$ by a finite sequence of surgeries of dimension $k \leq n - 3$ (see the proofs of Theorem B and Theorem C in the article [5] by Gromov–Lawson). Theorem 5.1 then implies that

$$X_+^{M \sharp (-M)}(\min(a, \Lambda_{n,k})) \leq X_+^{M_0 \sharp (-M_0)}(\min(a, \Lambda_{n,k})). \quad (6)$$

Inequality (5) remains true when M is replaced by M_0 . As a consequence, we get from Proposition 4.1 that

$$\begin{aligned} X_+^{M_0 \# (-M_0)}(\min(a, \Lambda_n)) &= \max(X_+^{M_0 \# (-M_0)}(\min(a, \Lambda_n)), X_+^{M_0}(\min(a, \Lambda_n))) \\ &= X_+^{M_0 \# (-M_0) \natural M_0}(\min(a, \Lambda_n)). \end{aligned}$$

Using Theorem 5.1, we obtain

$$X_+^{M_0 \# (-M_0)}(\min(a, \Lambda_{n,k})) \leq X_+^{M_0 \# (-M_0) \natural M_0}(\min(a, \Lambda_{n,k}))$$

Now, $M_0 \# (-M_0) \natural M_0$ is oriented (resp. spin) cobordant to M_0 and M_0 is simply connected and not spin (resp. spin): by the same argument as above, M_0 is obtained from $M_0 \# (-M_0) \natural M_0$ by a finite sequence of surgeries of dimension $k \leq n - 3$. This proves that

$$X_+^{M_0 \# (-M_0)}(\min(a, \Lambda_n)) \leq X_+^{M_0 \# (-M_0) \natural M_0}(\min(a, \Lambda_n)) \leq X_+^{M_0}(\min(a, \Lambda_n)).$$

Together with Inequalities (5) and (6), we obtain the desired inequality

$$X_+^M(\min(a, \Lambda_n)) \leq X_+^{M_0}(\min(a, \Lambda_n)).$$

(3) The argument for $X_-^M(a)$ is similar.

6. Application to the Yamabe invariant

Theorem 6.1.

1. For any compact manifold M with $\sigma(M) > 0$, one has $X_+^M(0) = +\infty$.
2. For every $n \geq 3$ there exists a constant $d_n > 0$ such that for all compact manifolds M of dimension n with $\sigma(M) > 0$ and for all $a \in (0, \sigma(M))$ we have

$$X_+^M(a) \leq d_n \frac{(\sigma(S^n) - a)^{1/n}}{a}. \quad (7)$$

Corollary 6.2. Let d_n be the constant in part 1 of Theorem 6.1 and suppose that M is a compact manifold of dimension n with $\sigma(M) > 0$ such that

$$\frac{d_n}{\sigma(S^n)} < \limsup_{\varepsilon \rightarrow 0} \frac{X_+^M(\sigma(M) - \varepsilon)}{\varepsilon^{1/n}}.$$

Then we have $\sigma(M) < \sigma(S^n)$.

Note that the hypothesis of Corollary 6.2 is satisfied if $X_+^M(\sigma(M)) > 0$ since the function $a \mapsto X_+^M(a)$ is decreasing and thus the right hand side in the assumption of Corollary 6.2 is ∞ . This fact leads to a natural question: is this possible that $X_+^M(\sigma(M)) > 0$? The answer is given by

Example 6.3. It holds that

$$X_+^{\mathbb{R}P^3}(\sigma(\mathbb{R}P^3)) > 0 \text{ and } X_+^{S^n}(\sigma(S^n)) = 0.$$

Namely, the fact that $X_+^{S^n}(\sigma(S^n)) = 0$ is an immediate consequence of Corollary 6.2. Besides, it was proven by Bray and Neves [4] that $\sigma(\mathbb{R}P^3) < \sigma(S^3)$ is attained by the standard metric. Since the standard metric of $\mathbb{R}P^3$ is locally conformally flat, one can choose a metric g in its conformal class such that $B_p^g(1)$ is flat where $p \in \mathbb{R}P^3$ is fixed. Then, since $\mathbb{R}P^3$ satisfies PMT, one has

$$X_+^{\mathbb{R}P^3}(\sigma(\mathbb{R}P^3)) \geq m(g, p) > 0.$$

Proof of Corollary 6.2. Assume that $\sigma(M) = \sigma(S^n)$. Using Theorem 6.1 we get

$$\frac{d_n}{\sigma(S^n)} < \limsup_{\varepsilon \rightarrow 0} \frac{d_n \varepsilon^{1/n}}{(\sigma(S^n) - \varepsilon) \varepsilon^{1/n}} = \limsup_{\varepsilon \rightarrow 0} \frac{d_n}{\sigma(S^n) - \varepsilon}$$

which is a contradiction. $\square$

Proof of Theorem 6.1. For the first statement, let $p \in M$ and $g_m \in \Omega_M^0$ be a sequence of metrics converging in C^2 to some metric g_∞ such that $Y(g_\infty) = 0$. To see the existence of such a sequence, it suffices to construct g_∞ . For this, just consider any metric $(g, p) \in \Omega_M^a$ for some $a > 0$. It is standard (see [2], p.388) that one can modify g locally outside $B_p^g(1)$ to get a metric h such that $Y(h) < 0$. Then, set $g_t = tg + (1 - t)h$ for $t \in [0, 1]$. Let

$$t_\infty := \max\{t \in [0, 1] \mid Y(g_t) \leq 0\}.$$

We can then set $g_\infty := g_{t_\infty}$ and $g_m := g_{t_\infty + \frac{1}{m}}$. It was then proven in [6] and in [3] that $\lim_m m(g_m, p) = +\infty$ which proves that $X_+^M(0) = +\infty$.

The second statement is much harder to prove. Let $a \in (0, \sigma(M))$ and let $(g_\varepsilon)_{\varepsilon \in (0, a)}$ be a sequence of Riemannian metrics on M such that for all ε we have $g_\varepsilon \in \Omega_M^{a-\varepsilon}$ and

$$A_\varepsilon := m(g_\varepsilon, p) \geq X_+^M(a) - \varepsilon.$$

For every ε let θ_ε be the smooth function on M such that $-A_\varepsilon = J_p^g(\theta_\varepsilon)$. We put

$$X_\varepsilon := \left(\int_M |\theta_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{1}{N}}$$

where $N = \frac{2n}{n-2}$. By Hölder's inequality and by the definition of $Y(g_\varepsilon)$ we have

$$\begin{aligned} -A_\varepsilon &= J_p^{g_\varepsilon}(\theta_\varepsilon) = \int_{M \setminus \{p\}} \eta r^{2-n} F_\eta dv^{g_\varepsilon} + 2 \int_M \theta_\varepsilon F_\eta dv^{g_\varepsilon} + \int_M \theta_\varepsilon L_g \theta_\varepsilon dv^{g_\varepsilon} \\ &\geq \int_{M \setminus \{p\}} \eta r^{2-n} F_\eta dv^{g_\varepsilon} - 2 \left(\int_M |F_\eta|^{\frac{N}{N-1}} dv^{g_\varepsilon} \right)^{\frac{N-1}{N}} X_\varepsilon + (a - \varepsilon) X_\varepsilon^2. \end{aligned}$$

We put

$$C_n := \int_{M \setminus \{p\}} \eta r^{2-n} F_\eta dv^g, \quad D_n := 2 \left(\int_M |F_\eta|^{\frac{N}{N-1}} dv^g \right)^{\frac{N-1}{N}}.$$

The numbers C_n, D_n are independent of (M, g) since g is flat on the supports of η and F_η . We get

$$-A_\varepsilon \geq C_n - D_n X_\varepsilon + (a - \varepsilon) X_\varepsilon^2. \quad (8)$$

We have $C_n \geq 0$ since for $M = S^n$ with the standard metric g_{can} we have

$$0 = -m(g_{\text{can}}, p) \leq J_p^{g_{\text{can}}}(0) = C_n.$$

We also have $C_n + A_\varepsilon \geq 0$ for all ε since

$$-A_\varepsilon = m(g_\varepsilon, p) \leq J_p^{g_\varepsilon}(0) = C_n.$$

Due to (8) the quadratic function $f(x) := A_\varepsilon + C_n - D_n x + (a - \varepsilon)x^2$ satisfies $f(X_\varepsilon) \leq 0$ and attains its minimum value at $x_0 = \frac{D_n}{2(a-\varepsilon)}$. We get

$$0 \geq f(x_0) = A_\varepsilon + C_n - \frac{D_n^2}{4(a-\varepsilon)}$$

and thus

$$X_+^M(a)\varepsilon \leq A_\varepsilon \leq \frac{D_n^2}{4(a-\varepsilon)} - C_n \leq \frac{D_n^2}{4(a-\varepsilon)}.$$

As $\varepsilon \rightarrow 0$ we obtain

$$X_+^M(a) \leq \frac{D_n^2}{4a} \quad (9)$$

Moreover, since $f(X_\varepsilon) \leq 0$, the number X_ε is less than or equal to the largest root of the equation $f(x) = 0$. Using that $A_\varepsilon + C_n \geq 0$ we get

$$X_\varepsilon \leq \frac{D_n + \sqrt{D_n^2 - 4(A_\varepsilon + C_n)(a-\varepsilon)}}{2(a-\varepsilon)} \leq \frac{D_n}{a-\varepsilon}.$$

By definition of θ_ε and by Eqs. (3), (1), the function θ_ε is harmonic on $B_p^{g_\varepsilon}(\frac{1}{2})$. Thus for all $x \in B_p^{g_\varepsilon}(\frac{1}{3})$ we have

$$\theta_\varepsilon(x) = \frac{6^n}{\text{vol}(\mathbb{B})} \int_{B_x^{g_\varepsilon}(\frac{1}{6})} \theta_\varepsilon(y) dv^{g_\varepsilon},$$

where $\mathbb{B}$ denotes the Euclidean unit ball, since $B_x^{g_\varepsilon}(\frac{1}{6}) \subset B_p^{g_\varepsilon}(\frac{1}{2})$. Using Hölder's inequality we get for all $x \in B_p^{g_\varepsilon}(\frac{1}{3})$:

$$\begin{aligned} |\theta_\varepsilon(x)| &\leq \frac{6^n}{\text{vol}(\mathbb{B})} \text{vol}\left(B_x^{g_\varepsilon}\left(\frac{1}{6}\right)\right)^{\frac{N-1}{N}} \left(\int_{B_x^{g_\varepsilon}(\frac{1}{6})} |\theta_\varepsilon|^N \right)^{\frac{1}{N}} \\ &\leq 6^{\frac{n}{N}} \text{vol}(\mathbb{B})^{-\frac{1}{N}} X_\varepsilon \end{aligned}$$

$$\begin{aligned} &\leq 6^{\frac{n}{N}} \text{vol}(\mathbb{B})^{-\frac{1}{N}} \frac{D_n}{a - \varepsilon} \\ &=: \frac{B_n}{a - \varepsilon}. \end{aligned} \quad (10)$$

In the following we always assume $\varepsilon \in (0, \frac{a}{2})$. For every n and a we choose $\delta_{n,a} > 0$ small enough such that for all $\varepsilon \in (0, \frac{a}{2})$ the number ρ_ε defined by

$$\rho_\varepsilon^{n-2} := \delta_{n,a} \frac{|A_\varepsilon|}{(B_n + |A_\varepsilon|)^2} \quad (11)$$

satisfies $\rho_\varepsilon < \frac{1}{6}$. This is possible since the function $x \mapsto \frac{|x|}{(B_n + |x|)^2}$ is bounded on $\mathbb{R}$. Then for every $\varepsilon \in (0, \frac{a}{2})$ we choose $h_\varepsilon \in C^\infty(M)$ such that $0 \leq h_\varepsilon \leq 1$, $h_\varepsilon \equiv 1$ on $B_p^{g_\varepsilon}(\rho_\varepsilon)$, $h_\varepsilon \equiv 0$ on $M \setminus B_p^{g_\varepsilon}(2\rho_\varepsilon)$ and $|dh_\varepsilon| \leq \frac{2}{\rho_\varepsilon}$. Moreover, for every ε , we have $A_\varepsilon = \theta_\varepsilon(p)$ and we set $\alpha_\varepsilon := \theta_\varepsilon - A_\varepsilon$ so that the Green function G_ε of L_{g_ε} at p satisfies

$$G_\varepsilon(x) = \eta(x)r(x)^{2-n} + A_\varepsilon + \alpha_\varepsilon(x)$$

(cf. Eqs. (1), (3)). Note that $\alpha_\varepsilon \in C^\infty(M)$ is harmonic on $B_p^{g_\varepsilon}(\frac{1}{2})$.

Step 1. For ε close enough to 0 (not necessarily uniformly in a), we have

$$\int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv^{g_\varepsilon} \leq \frac{|A_\varepsilon|}{4}.$$

Since α_ε is harmonic on the support of dh_ε , we can use Identity (3) in [6] and Hölder inequality to write

$$\int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv^{g_\varepsilon} = \int_{C_\varepsilon} |dh_\varepsilon|^2 \alpha_\varepsilon^2 dv^{g_\varepsilon} \leq \left(\int_{C_\varepsilon} |dh_\varepsilon|^n dv_{g_\varepsilon} \right)^{\frac{2}{n}} \left(\int_{C_\varepsilon} |\alpha_\varepsilon|^N dv_{g_\varepsilon} \right)^{\frac{2}{N}}$$

where $C_\varepsilon := B_p^{g_\varepsilon}(2\rho_\varepsilon) \setminus B_p^{g_\varepsilon}(\rho_\varepsilon)$ is the support of dh_ε . Observe that the definition of h_ε and the fact that the volume of the support of dh_ε is bounded by $C\rho_\varepsilon^n$ with C independent of ε imply that there exists $\alpha_0 > 0$ which is independent of ε such that

$$\left(\int_{C_\varepsilon} |dh_\varepsilon|^n dv_{g_\varepsilon} \right)^{\frac{2}{n}} \leq \alpha_0.$$

Hence, for all ε small enough,

$$\int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv^{g_\varepsilon} \leq \alpha_0 \left(\int_{C_\varepsilon} |\alpha_\varepsilon|^N dv_{g_\varepsilon} \right)^{\frac{2}{N}}. \quad (12)$$

Now, since $\alpha_\varepsilon = \theta_\varepsilon - A_\varepsilon$ and using Eqs. (10) and (11) we get for ε close to 0:

$$\begin{aligned} \left(\int_{C_\varepsilon} |\alpha_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{1}{N}} &\leq \left(\int_{C_\varepsilon} |\theta_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{1}{N}} + |A_\varepsilon| \text{vol}(C_\varepsilon)^{\frac{1}{N}} \\ &\leq \left(\frac{B_n}{a - \varepsilon} + |A_\varepsilon| \right) \text{vol}(C_\varepsilon)^{\frac{1}{N}} \\ &= \left(\frac{B_n}{a - \varepsilon} + |A_\varepsilon| \right) (2^n - 1)^{\frac{1}{N}} \rho_\varepsilon^{\frac{n-2}{2}} \text{vol}(\mathbb{B})^{\frac{1}{N}} \\ &\leq \left(\frac{B_n}{a - \varepsilon} + |A_\varepsilon| \right) (2^n - 1)^{\frac{1}{N}} \delta_{n,a}^{\frac{1}{2}} \frac{\sqrt{|A_\varepsilon|}}{B_n + |A_\varepsilon|} \text{vol}(\mathbb{B})^{\frac{1}{N}} \\ &\leq E_{n,a} \sqrt{|A_\varepsilon|} \end{aligned}$$

for some $E_{n,a} > 0$ which is independent of ε for ε close to 0. By choosing $\delta_{n,a}$ in Eq. (11) smaller we may assume that $E_{n,a}^2 \alpha_0 \leq \frac{1}{4}$. Therefore the assertion of Step 1 follows from Eq. (12).

For every $\varepsilon \in (0, \frac{a}{2})$, we define

$$\beta_\varepsilon := \zeta_n |A_\varepsilon|^{\frac{1}{2}} \rho_\varepsilon^{\frac{n}{2}} \quad (13)$$

where $\zeta_n > 0$ will be fixed later. We have for all ε :

$$\beta_\varepsilon = \zeta_n |A_\varepsilon|^{\frac{1}{2}} \rho_\varepsilon^{\frac{n-2}{2}} \rho_\varepsilon = \zeta_n \delta_{n,a}^{\frac{1}{2}} \frac{|A_\varepsilon|}{B_n + |A_\varepsilon|} \rho_\varepsilon \leq \zeta_n \delta_{n,a}^{\frac{1}{2}} \rho_\varepsilon. \quad (14)$$

We define $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ by

$$u_\varepsilon(x) := \left(\frac{\beta_\varepsilon}{\beta_\varepsilon^2 + |x|^2} \right)^{\frac{n-2}{2}}$$

and define $\psi_\varepsilon \in C^0(M)$ by

$$\psi_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } r(x) \leq \rho_\varepsilon \\ \ell_\varepsilon(G_\varepsilon - h_\varepsilon \alpha_\varepsilon)(x) & \text{if } \rho_\varepsilon \leq r(x) \leq 2\rho_\varepsilon \\ \ell_\varepsilon G_\varepsilon(x) & \text{if } r(x) \geq 2\rho_\varepsilon \end{cases}$$

where $r(x)$ denotes the Riemannian distance of x to p and where

$$\ell_\varepsilon := u_\varepsilon(\rho_\varepsilon) \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right)^{-1}$$

is a real number depending on ε chosen so that ψ_ε is continuous. Furthermore, ψ_ε is smooth everywhere except possibly at $r = \rho_\varepsilon$. Also note that in the definition of ψ_ε we have identified $B_p^{g_\varepsilon}(\rho_\varepsilon)$ with the Euclidean ball of radius ρ_ε centered at 0.

Step 2. Conclusion.

For all $\varepsilon \in (0, \frac{a}{2})$ we define

$$E_\varepsilon := \int_M \left(|d\psi_\varepsilon|^2 + \frac{n-2}{4(n-1)} s_{g_\varepsilon} |\psi_\varepsilon|^2 \right) dv^{g_\varepsilon}$$

and

$$D_\varepsilon := \left(\int_M |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}}.$$

Let $Q_\varepsilon := Q_{g_\varepsilon}$ be the Yamabe functional of g_ε defined in (21). Then we have

$$Q_\varepsilon(\psi_\varepsilon) = \frac{E_\varepsilon}{D_\varepsilon}. \quad (15)$$

Let $B_\varepsilon := B_p^{g_\varepsilon}(\rho_\varepsilon)$ which is isometric to the Euclidean ball of radius ρ_ε . We write

$$E_\varepsilon = E_1 + E_2$$

where, taking into account that the scalar curvature of g_ε vanishes on B_ε ,

$$E_1 := \int_{B_\varepsilon} \left(|d\psi_\varepsilon|^2 + \frac{n-2}{4(n-1)} s_{g_\varepsilon} |\psi_\varepsilon|^2 \right) dv^{g_\varepsilon} = \int_{B_\varepsilon} |du_\varepsilon|^2 dv^{g_\varepsilon}$$

and

$$E_2 := \int_{M \setminus B_\varepsilon} \left(|d\psi_\varepsilon|^2 + \frac{n-2}{4(n-1)} s_{g_\varepsilon} |\psi_\varepsilon|^2 \right) dv^{g_\varepsilon}.$$

On B_ε we have

$$\Delta_{g_\varepsilon} u_\varepsilon = n(n-2)u_\varepsilon^{N-1}$$

and we get from multiplying this equation by u_ε and integrating by parts that

$$\int_{B_\varepsilon} |du_\varepsilon|^2 dv^{g_\varepsilon} - \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} d\alpha^{g_\varepsilon} = n(n-2) \int_{B_\varepsilon} |u_\varepsilon|^N dv^{g_\varepsilon}. \quad (16)$$

One also has

$$\sigma(S^n) = \frac{\int_{\mathbb{R}^n} |du_\varepsilon|^2 dx}{\left(\int_{\mathbb{R}^n} |u_\varepsilon|^N dx \right)^{\frac{2}{N}}} = n(n-2) \left(\int_{\mathbb{R}^n} |u_\varepsilon|^N dx \right)^{\frac{2}{n}}$$

which leads to

$$n(n-2) \left(\int_{B_\varepsilon} |u_\varepsilon|^N dx \right)^{\frac{2}{n}} \leq \sigma(S^n).$$

Plugging this estimate into Eq. (16), we obtain

$$E_1 \leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} + \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} d\alpha^{g_\varepsilon}. \quad (17)$$

Now, we evaluate E_2 . For this, we integrate by parts:

$$E_2 = \int_{M \setminus B_\varepsilon} \psi_\varepsilon L_{g_\varepsilon} \psi_\varepsilon dv_{g_\varepsilon} + E_3$$

where E_3 is a boundary term which will be computed below. Note that since ψ_ε is not necessarily differentiable at $r = \rho_\varepsilon$, the term E_3 and the boundary term in (17) do not cancel in general. Since $L_{g_\varepsilon} G_\varepsilon = 0$ on $M \setminus B_\varepsilon$, we obtain

$$E_2 = \ell_\varepsilon^2 \int_{B'_\varepsilon \setminus B_\varepsilon} L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon)(G_\varepsilon - h_\varepsilon \alpha_\varepsilon) dv_{g_\varepsilon} + E_3$$

where $B'_\varepsilon := B_p^{g_\varepsilon}(2\rho_\varepsilon)$. Note that since α_ε is harmonic and h_ε is constant on B_ε , one has on B_ε ,

$$L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) = \Delta_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) = 0.$$

Hence, by definition of the Green function G_ε , one has:

$$\int_{B'_\varepsilon \setminus B_\varepsilon} L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) G_\varepsilon dv_{g_\varepsilon} = \int_M L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) G_\varepsilon dv_{g_\varepsilon} = -\alpha_\varepsilon(p) = 0.$$

We also have

$$\int_{B'_\varepsilon \setminus B_\varepsilon} L_{g_\varepsilon}(h_\varepsilon \alpha_\varepsilon)(h_\varepsilon \alpha_\varepsilon) dv_{g_\varepsilon} = \int_M L_{g_\varepsilon}(h_\varepsilon \alpha_\varepsilon)(h_\varepsilon \alpha_\varepsilon) dv_{g_\varepsilon} = \int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv_{g_\varepsilon}.$$

Since, by Step 1 we have

$$\int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv_{g_\varepsilon} \leq \frac{|A_\varepsilon|}{4} =: \gamma_\varepsilon \quad (18)$$

we obtain:

$$E_2 \leq \ell_\varepsilon^2 \gamma_\varepsilon + E_3.$$

So let us evaluate E_3 :

$$\begin{aligned} E_3 &= -\ell_\varepsilon^2 \int_{\partial B_\varepsilon} (G_\varepsilon - h_\varepsilon \alpha_\varepsilon) \frac{\partial(G_\varepsilon - h_\varepsilon \alpha_\varepsilon)}{\partial r} d\mathbf{a}^{g_\varepsilon} \\ &= -\ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) \left(-\frac{1}{\omega_{n-1}} \rho_\varepsilon^{1-n} \right) \text{vol}(\partial B_\varepsilon) \\ &= \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right). \end{aligned}$$

Combining this with (17)

$$\begin{aligned} E_1 + E_2 &\leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv_{g_\varepsilon} \right)^{\frac{2}{N}} \\ &\quad + \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} d\mathbf{a}^{g_\varepsilon} + \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) + \ell_\varepsilon^2 \gamma_\varepsilon. \end{aligned} \quad (19)$$

It remains to compute

$$E_4 := \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} d\mathbf{a}^{g_\varepsilon}.$$

Using the fact that on ∂B_ε and the definition of ℓ_ε , we have

$$u_\varepsilon = \ell_\varepsilon \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right)$$

and

$$\frac{\partial u_\varepsilon}{\partial r} = -(n-2) \frac{\rho_\varepsilon}{\beta_\varepsilon^2 + \rho_\varepsilon^2} u_\varepsilon$$

we obtain

$$E_4 = -(n-2)\omega_{n-1} \frac{\rho_\varepsilon^n}{\beta_\varepsilon^2 + \rho_\varepsilon^2} \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right)^2$$

which together with the definition (13) of β_ε leads to

$$\begin{aligned} &\int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} d\mathbf{a}^{g_\varepsilon} + \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) \\ &= \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) \left(1 - \frac{\rho_\varepsilon^n}{\beta_\varepsilon^2 + \rho_\varepsilon^2} (\rho_\varepsilon^{2-n} + (n-2)\omega_{n-1} A_\varepsilon) \right) \end{aligned}$$

$$\begin{aligned}
&= \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) \frac{\beta_\varepsilon^2 - \rho_\varepsilon^n (n-2)\omega_{n-1} A_\varepsilon}{\beta_\varepsilon^2 + \rho_\varepsilon^2} \\
&= \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) \frac{(\zeta_n^2 |A_\varepsilon| - (n-2)\omega_{n-1} A_\varepsilon) \rho_\varepsilon^n}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^n + \rho_\varepsilon^2} \\
&= \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} + A_\varepsilon \rho_\varepsilon^{n-2} \right) \frac{\zeta_n^2 |A_\varepsilon| - (n-2)\omega_{n-1} A_\varepsilon}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^{n-2} + 1}
\end{aligned}$$

By assumption we have

$$A_\varepsilon \geq X_+^M(a) - \varepsilon.$$

If $X_+^M(a) \leq 0$, then the assertion (7) of Theorem 6.1 holds trivially. Thus we may assume that $A_\varepsilon > 0$ for all $\varepsilon > 0$ sufficiently close to 0. We put

$$\zeta_n := \frac{\sqrt{(n-2)\omega_{n-1}}}{2} \quad (20)$$

and we get

$$\frac{\zeta_n^2 |A_\varepsilon| - (n-2)\omega_{n-1} A_\varepsilon}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^{n-2} + 1} = -3A_\varepsilon \left(A_\varepsilon \rho_\varepsilon^{n-2} + \frac{4}{(n-2)\omega_{n-1}} \right)^{-1}.$$

If ρ_ε is small enough, we obtain

$$\int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} da^{g_\varepsilon} + \ell_\varepsilon^2 \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right) \leq -\ell_\varepsilon^2 \frac{A_\varepsilon}{2}.$$

Inserting this into Eq. (19) and using the definition (18) of γ_ε we get

$$E_1 + E_2 \leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - \frac{\ell_\varepsilon^2 A_\varepsilon}{4}.$$

Moreover, if ρ_ε is small enough, we obtain

$$\begin{aligned}
\ell_\varepsilon^2 &= \left(\frac{\beta_\varepsilon}{\beta_\varepsilon^2 + \rho_\varepsilon^2} \right)^{n-2} \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right)^{-2} \\
&= \left(\frac{\zeta_n |A_\varepsilon|^{1/2} \rho_\varepsilon^{n/2}}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^n + \rho_\varepsilon^2} \right)^{n-2} \left(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \right)^{-2} \\
&= \left(\frac{\zeta_n |A_\varepsilon|^{1/2} \rho_\varepsilon^{n/2}}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^{n-2} + 1} \right)^{n-2} \rho_\varepsilon^{4-2n} \left(\frac{1}{(n-2)\omega_{n-1}} + A_\varepsilon \rho_\varepsilon^{n-2} \right)^{-2} \rho_\varepsilon^{2n-4} \\
&\geq \left(\frac{\zeta_n |A_\varepsilon|^{1/2}}{2} \right)^{n-2} \rho_\varepsilon^{\frac{n(n-2)}{2}} \frac{(n-2)^2 \omega_{n-1}^2}{4}.
\end{aligned}$$

From this and the definitions (20) and (11) of ζ_n and ρ_ε it follows that

$$\begin{aligned}
E_1 + E_2 &\leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - 4^{-n} ((n-2)\omega_{n-1})^{\frac{n+2}{2}} |A_\varepsilon|^{\frac{n}{2}} \rho_\varepsilon^{\frac{n(n-2)}{2}} \\
&= \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - 4^{-n} ((n-2)\omega_{n-1})^{\frac{n+2}{2}} \delta_{n,a}^{\frac{n}{2}} \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n} \\
&=: \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - E'_n \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n}.
\end{aligned}$$

We have

$$D_\varepsilon \geq \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}}$$

and therefore

$$Q_\varepsilon(\psi_\varepsilon) = \frac{E_1 + E_2}{D_\varepsilon} \leq \sigma(S^n) - E'_n \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n} \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{-\frac{2}{N}}.$$

Moreover, using the substitution $r = \beta_\varepsilon s$ we get

$$\begin{aligned} \int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} &= \omega_{n-1} \int_0^{\rho_\varepsilon} \left(\frac{\beta_\varepsilon}{\beta_\varepsilon^2 + r^2} \right)^n r^{n-1} dr \\ &= \omega_{n-1} \int_0^{\rho_\varepsilon/\beta_\varepsilon} \left(\frac{1}{1+s^2} \right)^n s^{n-1} ds \\ &\leq \omega_{n-1} \int_0^\infty \left(\frac{1}{1+s^2} \right)^n s^{n-1} ds \\ &=: F_n. \end{aligned}$$

It follows that

$$Q_\varepsilon(\psi_\varepsilon) \leq \sigma(S^n) - \frac{E'_n}{F_n^{2/N}} \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n}$$

and therefore with $G_n := (E'_n F_n^{-2/N})^{1/n}$ we get

$$a - \varepsilon < Y(g_\varepsilon) \leq \sigma(S^n) - (G_n)^n \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n}.$$

We take the limit $\varepsilon \rightarrow 0$ and we get

$$a \leq \sigma(S^n) - (G_n)^n \frac{X_+^M(a)^n}{(B_n + X_+^M(a))^n} = \sigma(S^n) - (G_n)^n \frac{1}{\left(\frac{B_n}{X_+^M(a)} + 1\right)^n}$$

It follows that

$$\frac{B_n}{X_+^M(a)} + 1 \geq \frac{G_n}{(\sigma(S^n) - a)^{1/n}}$$

We put $\alpha_n := \sigma(S^n) - \frac{(G_n)^n}{2^n}$ and we distinguish two cases.

(a) If $a \geq \alpha_n$ we get

$$\frac{B_n}{X_+^M(a)} \geq \frac{1}{2} \frac{G_n}{(\sigma(S^n) - a)^{1/n}}$$

and since $a < \sigma(S^n)$ we get

$$X_+^M(a) \leq \frac{2B_n}{G_n} (\sigma(S^n) - a)^{1/n} \leq \frac{2B_n}{G_n} \frac{\sigma(S^n)}{a} (\sigma(S^n) - a)^{1/n}$$

(b) If $a \leq \alpha_n$ we have

$$\sigma(S^n) - a \geq \sigma(S^n) - \alpha_n$$

and therefore using Eq. (9)

$$X_+^M(a) \leq \frac{D_n^2}{4a} \leq \frac{D_n^2 (\sigma(S^n) - a)^{1/n}}{4(\sigma(S^n) - \alpha_n)^{1/n} a} = \frac{D_n^2 (\sigma(S^n) - a)^{1/n}}{2G_n a}.$$

This shows the second statement. $\square$

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Appendix. A lemma on the Yamabe constant

Let M be a compact manifold of dimension $n \geq 3$. For any Riemannian metric g on M we denote by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} S_g$$

the Yamabe operator of g and by

$$Q_g : C^\infty(M) \rightarrow C^\infty(M), \quad Q_g(u) := \frac{\int_M u L_g u dv^g}{\left(\int_M |u|^N dv^g\right)^{2/N}} \quad (21)$$

the Yamabe functional of g where $N := \frac{2n}{n-2}$. We denote by

$$Y(g) := \inf\{Q_g(u) \mid u \in C^\infty(M), u \neq 0\}$$

the Yamabe constant of (M, g) . In this Appendix, we prove the following Lemma which is used several times in the paper.

Lemma A.1. *Let (M, g) be a compact Riemannian manifold with $Y(g) > 0$. Let $p \in M$, let $\varepsilon > 0$ and let h be a Riemannian metric defined on $B_p^g(2\varepsilon)$. We assume that g and h coincide at p . Let $\chi_\varepsilon \in C^\infty(M)$, $0 \leq \chi_\varepsilon \leq 1$ be a cut-off function equal to 1 on $B_p^g(\varepsilon)$, equal to 0 outside $B_p^g(2\varepsilon)$ and which satisfies $|d\chi_\varepsilon|_g \leq \frac{C}{\varepsilon}$ and $|\nabla^2 \chi_\varepsilon| \leq \frac{C}{\varepsilon^2}$. Define*

$$g_\varepsilon := \chi_\varepsilon h + (1 - \chi_\varepsilon)g.$$

Then we have $\lim_{\varepsilon \rightarrow 0} Y(g_\varepsilon) = Y(g)$.

Proof. Note that by Taylor expansion we have on $B_p^g(2\varepsilon)$

$$|g - h|_g \leq C\varepsilon \quad \text{and thus} \quad |g - g_\varepsilon|_g \leq C\varepsilon$$

for some C independent of ε . Thus for all $u \in C^\infty(M)$ we have

$$\begin{aligned} \left| \int_M u dv^{g_\varepsilon} - \int_M u dv^g \right| &\leq C\varepsilon \int_{B_p^g(2\varepsilon)} |u| dv^g, \\ |du|_{g_\varepsilon}^2 - |du|_g^2 &= |(g_\varepsilon^{ij} - g^{ij}) \partial_i u \partial_j u| \leq C\varepsilon |du|_g^2 \end{aligned}$$

and using that $|\nabla^2 \chi_\varepsilon| |g - h|_g \leq \frac{C}{\varepsilon}$ we get

$$|s_{g_\varepsilon} - s_g| \leq \frac{C}{\varepsilon} \quad \text{and} \quad |s_{g_\varepsilon}| \leq \frac{C}{\varepsilon}.$$

Let $(u_\varepsilon)_{\varepsilon > 0}$ be a sequence of smooth functions on M such that $\int_M |u_\varepsilon|^N dv^g$ and $Q_g(u_\varepsilon)$ are bounded independently of ε . We prove that

$$\lim_{\varepsilon \rightarrow 0} (Q_{g_\varepsilon}(u_\varepsilon) - Q_g(u_\varepsilon)) = 0. \quad (22)$$

By the triangle inequality we have

$$\begin{aligned} &\left| \int_M s_{g_\varepsilon} u_\varepsilon^2 dv^{g_\varepsilon} - \int_M s_g u_\varepsilon^2 dv^g \right| \\ &\leq \left| \int_M s_{g_\varepsilon} u_\varepsilon^2 dv^{g_\varepsilon} - \int_M s_{g_\varepsilon} u_\varepsilon^2 dv^g \right| + \int_M |s_{g_\varepsilon} - s_g| |u_\varepsilon|^2 dv^g \end{aligned}$$

By the Hölder inequality we get

$$\int_M |s_{g_\varepsilon} - s_g| |u_\varepsilon|^2 dv^g \leq \frac{C}{\varepsilon} \text{vol}(B_p^g(2\varepsilon))^{\frac{2}{N}} \left(\int_M |u_\varepsilon|^N dv^g \right)^{\frac{2}{N}} \leq C\varepsilon \left(\int_M |u_\varepsilon|^N dv^g \right)^{\frac{2}{N}},$$

and

$$\begin{aligned} &\left| \int_M s_{g_\varepsilon} u_\varepsilon^2 dv^{g_\varepsilon} - \int_M s_{g_\varepsilon} u_\varepsilon^2 dv^g \right| \leq C\varepsilon \int_{B_p^g(2\varepsilon)} |s_{g_\varepsilon}| |u_\varepsilon|^2 dv^g \\ &\leq C \text{vol}(B_p^g(2\varepsilon))^{\frac{2}{N}} \left(\int_M |u_\varepsilon|^N dv^g \right)^{\frac{2}{N}} \leq C\varepsilon^2 \left(\int_M |u_\varepsilon|^N dv^g \right)^{\frac{2}{N}}. \end{aligned}$$

Moreover, we get

$$\begin{aligned} &\left| \int_M |du_\varepsilon|_{g_\varepsilon}^2 dv^{g_\varepsilon} - \int_M |du_\varepsilon|_g^2 dv^g \right| \\ &\leq \left| \int_M |du_\varepsilon|_{g_\varepsilon}^2 dv^{g_\varepsilon} - \int_M |du_\varepsilon|_{g_\varepsilon}^2 dv^g \right| + \int_M ||du_\varepsilon|_{g_\varepsilon}^2 - |du_\varepsilon|_g^2| dv^g \\ &\leq C\varepsilon \int_M |du_\varepsilon|_g^2 dv^g \\ &= C\varepsilon \left(\int_M u_\varepsilon L_g u_\varepsilon dv^g - \int_M s_g |u_\varepsilon|^2 dv^g \right) \\ &\leq C\varepsilon \left(\int_M u_\varepsilon L_g u_\varepsilon dv^g + C \left(\int_M |u_\varepsilon|^N dv^g \right)^{\frac{2}{N}} \right). \end{aligned}$$

Eq. (22) follows easily from these estimates. Now let $(u_\varepsilon)_\varepsilon$ be a sequence in $C^\infty(M)$ such that $\int_M |u_\varepsilon|^N dv^g = 1$ and

$$Q_{g_\varepsilon}(u_\varepsilon) = Y(g_\varepsilon) + \varepsilon$$

for all ε . Using (22) we get

$$Y(g) \leq \liminf_{\varepsilon \rightarrow 0} Q_g(u_\varepsilon) = \liminf_{\varepsilon \rightarrow 0} Q_{g_\varepsilon}(u_\varepsilon) = \liminf_{\varepsilon \rightarrow 0} Y(g_\varepsilon).$$

On the other hand let $u \in C^\infty(M)$ such that $\int_M |u|^N dv^g = 1$ and

$$Q_g(u) = Y(g) + \delta$$

where $\delta > 0$. Using (22) we get

$$\limsup_{\varepsilon \rightarrow 0} Y(g_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} Q_{g_\varepsilon}(u) = \limsup_{\varepsilon \rightarrow 0} Q_g(u) = Y(g) + \delta.$$

As $\delta \rightarrow 0$ we get $\limsup_{\varepsilon \rightarrow 0} Y(g_\varepsilon) \leq Y(g)$. $\square$

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