Lie groupoids in classical field theory II: Gauge theories, minimal coupling and Utiyama's theorem

Bruno T. Costa [b],*, Michael Forger [a], Luiz Henrique P. Pêgas [c]

[a] *Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, BR–05315-970 São Paulo, SP, Brazil*
[b] *Departamento de Matemática, Universidade Federal de Santa Catarina, Rua João Pessoa, 2750, BR–89036-256 Blumenau, SC, Brazil*
[c] *Departamento de Matemática, Universidade Federal do Paraná, Caixa Postal 19081, BR–81531-980 Curitiba, PR, Brazil*

ABSTRACT

In the two papers of this series, we initiate the development of a new approach to implementing the concept of symmetry in classical field theory, based on replacing Lie groups/algebras by Lie groupoids/algebroids, which are the appropriate mathematical tools to describe local symmetries when gauge transformations are combined with space-time transformations.

In this second part, we shall adapt the formalism developed in the first paper to the context of gauge theories and deal with minimal coupling and Utiyama's theorem.

© 2021 Elsevier B.V. All rights reserved.

1. Introduction

In the first paper of this series [2], we have initiated an investigation of how to handle symmetries – or more precisely, local symmetries – in classical field theories using the language of Lie groupoids and their actions. However, the formalism developed there is perhaps a bit too general because it allows us to leave the nature of the underlying Lie groupoids and their actions completely unspecified, whereas there can be no doubt that the motivation for the entire program comes predominantly from one single (class of) example(s), namely, gauge theories. Spelling out the details for this case is the main goal of the present paper and is necessary not only because it provides us with a class of examples whose importance can hardly be overestimated but also because it leads to a substantial clarification of the general structure of the theory. Moreover, the results will generalize those of earlier work [4] by extending them from internal symmetries to space-time symmetries.

Let us begin with a few comments on the already traditional geometric formulation of gauge theories (as classical field theories) over a general space-time manifold M; more details can be found in textbooks such as [1,3,5]. The basic input data one has to fix right at the start are an internal symmetry group, which is a Lie group G_0 with Lie algebra $\mathfrak{g}_0$,[1] together with a principal bundle P over M with structure group G_0 and bundle projection $\rho : P \longrightarrow M$: then gauge fields are described in terms of connections in P, which can be viewed as sections of an affine bundle over M, namely, the connection bundle

---

* Corresponding author.

E-mail address: b.t.costa@ufsc.br (B.T. Costa).

[1] Note that we perform a slight change of notation as compared to Ref. [4], where we have denoted the internal symmetry group by G and its Lie algebra by $\mathfrak{g}$: here, we want to reserve these symbols for the basic Lie groupoid and Lie algebroid of the theory.

$CP = JP/G_0$ of P. Moreover, if the theory is to contain not only gauge fields (as in "pure" Yang-Mills theories) but also matter fields, one also has to fix a vector space V equipped with a representation of G_0 or, more generally, a manifold Q equipped with an action of G_0: then matter fields are described by sections of the associated vector bundle $E = P \times_{G_0} V$ (for scalar matter fields) or of its tensor product with some tensor or spinor bundle over M (for tensor or spinor matter fields) or of the associated fiber bundle $E = P \times_{G_0} Q$ (for nonlinear scalar matter fields such as in the nonlinear sigma models). Finally, there is gravity, described by yet another and very special kind of field, namely, a metric tensor g on M. (Some discussion of what sets the metric tensor apart from all other fields can be found in Ref. [6].)

Symmetries in this approach are traditionally described in terms of automorphisms of the principal bundle P and the induced automorphisms of its connection bundle and its associated bundles. To set the stage, recall that an *automorphism* of P is a diffeomorphism of P as a manifold which is G_0-equivariant, i.e., which commutes with the right action of the structure group G_0 on P: since the orbits of this action are precisely the fibers of P, it then follows that it takes points in the same fiber to points in the same fiber and hence induces a diffeomorphism of the base manifold M. Moreover, the automorphism is said to be *strict* if it preserves the fibers, or equivalently, if the induced diffeomorphism on the base manifold is the identity. Automorphisms of P form a group $\mathrm{Aut}(P)$ and strict automorphisms of P form a normal subgroup $\mathrm{Aut}_s(P)$ which is the kernel of a natural group homomorphism

$$\mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(M)$$

that projects each automorphism of P to the diffeomorphism of M it induces. In physics language, strict automorphisms are also called *gauge transformations* and the group $\mathrm{Aut}_s(P)$ is often called the *gauge group* and denoted by $\mathrm{Gau}(P)$, but we prefer the more precise term *group of gauge transformations* so as to avoid the confusion whether by "gauge group" one means the infinite-dimensional group $\mathrm{Gau}(P)$ or the finite-dimensional structure group G_0. Thus strict automorphisms, or gauge transformations, are *internal symmetries* since they do not move points in space-time, whereas general automorphisms will in what follows be referred to as *space-time symmetries*.[2] At any rate, all such symmetry transformations, being represented by automorphisms of P, can be lifted to automorphisms of its jet bundle JP and hence act naturally on the connection bundle $CP = JP/G_0$ of P as well as on any associated vector bundle or fiber bundle E, its jet bundle JE and any tensor or spinor bundle over M, thus providing the appropriate setting for deciding which of them are symmetries of the field theoretical model under consideration.

The main mathematical difficulty within this approach comes from the fact that one is dealing here with infinite-dimensional groups which are notoriously hard to handle from the point of view of Lie theory. Therefore, it is desirable to recast the property of invariance of a field theory under such local symmetries into a form where one deals exclusively with finite-dimensional objects. This program has been initiated in Ref. [4] and implemented there for strict automorphisms (gauge transformations), where it leads naturally to replacing Lie groups by Lie group bundles (and similarly Lie algebras by Lie algebra bundles), making use of the well-known fact that there is a natural isomorphism between the group of strict automorphisms of P and the *group of sections* of the *gauge group bundle* of P, which is the Lie group bundle $P \times_{G_0} G_0$ associated to P via the action of G_0 on itself by conjugation:

$$\mathrm{Aut}_s(P) \cong \Gamma(P \times_{G_0} G_0).$$

In order to extend the resulting analysis from strict automorphisms to general automorphisms, we have to go one step further and replace Lie groups or Lie group bundles by Lie groupoids (and similarly Lie algebras or Lie algebra bundles by Lie algebroids). In this case, the basic observation is that there is a natural isomorphism between the group of automorphisms of P and the *group of bisections* of the *gauge groupoid* of P, which is the Lie groupoid $(P \times P)/G_0$ obtained as the quotient of the cartesian product of two copies of P by the "diagonal" right action of G_0:

$$\mathrm{Aut}(P) \cong \mathrm{Bis}((P \times P)/G_0).$$

Thus our task in what follows will be to extend the results of Ref. [4] by applying the general formalism of Ref. [2] to this specific situation.

When we replace Lie groups by Lie groupoids, or to put it a bit more precisely, actions of Lie groups on manifolds by actions of Lie groupoids on fiber bundles (over the same base manifold), we have to face one important novel feature, namely, that the construction of induced actions will involve changing the Lie groupoid as well. For example, while an action of a Lie group G_0 on a manifold X induces an action of the same Lie group G_0 on its tangent bundle TX, an action of a Lie groupoid G on a fiber bundle E (both over the same base manifold M) induces an action not of the original Lie

---

[2] There is some abuse of language in this simplified terminology because general automorphisms always represent a mixture of "pure" space-time symmetries with internal symmetries. The problem here is that there is in general no natural notion of a "pure" space-time symmetry, since that would require a *lifting* of the group $\mathrm{Diff}(M)$ (or at least of an appropriate subgroup thereof) to realize it as a subgroup (and not only as a quotient group) of $\mathrm{Aut}(P)$, whose elements would then represent the "pure" space-time symmetries. However, such a lifting may not even exist, and even if it does (which happens, e.g., when the principal bundle P is trivial), it is far from unique, so what one means by a "pure" space-time transformation still depends on which lifting is chosen.

groupoid G but rather of its jet groupoid JG on the jet bundle JE of E. (A similar phenomenon already occurs for Lie group bundles, as observed in Ref. [4]). As it turns out, properly dealing with this feature is the key to make the entire theory work out smoothly.

Let us pass to briefly describe the contents of the paper. In Section 2, we present the minimal coupling prescription and the curvature map that enters the formulation of Utiyama's theorem in a very general context, and we show that these constructions are invariant (or perhaps it might be better to say, equivariant) under any action of any Lie groupoid over space-time on the bundle of field configurations over space-time, provided we employ the correct induced actions of the pertinent Lie groupoids derived from the former on the pertinent bundles derived from the latter. We conclude with a series of comments intended to show why, from the point of view of field theory, this approach is excessively general and needs to be adapted to a setting where all bundles are derived from some principal bundle and all connections are derived from principal connections in that principal bundle – which is the standard setup for gauge theories anyway. In Section 3, we collect the technical tools needed to perform this adjustment and to state the main results. The first step here is to recall the definition of the gauge groupoid G of a principal bundle P and of its natural actions on any bundle E associated to P (including P itself). Next, we introduce the (first order) jet groupoid JG of G and use the results of the previous section and of Ref. [2] to write down natural actions of JG on various derived bundles such as the jet bundle JP and the connection bundle CP of P or the jet bundle JE of any bundle E associated to P. We also show how iterating this procedure provides induced actions of the second order jet groupoid J^2G and, more generally, the semiholonomous second order jet groupoid $\bar{J}^2G$ of G on the semiholonomous second order jet bundle $\bar{J}^2P$ and on the (first order) jet bundle $J(CP)$ of the connection bundle CP of P. In Section 4, we then prove the main theorems concerning the invariance (or perhaps it might be better to say, the equivariance) of the minimal coupling prescription and the curvature map under the actions of the pertinent Lie groupoids introduced in the previous section, thus providing the desired extension of the results of Ref. [4] from the setting of Lie group bundles (internal symmetries) to that of Lie groupoids (space-time symmetries).

2. Minimal coupling, curvature and jets

As stated in the introduction, our main goal in this paper is to extend the results of Ref. [4] about invariance of the minimal coupling prescription and of the curvature map (Utiyama's theorem) from the context of Lie group bundles to that of Lie groupoids. To do so, let us begin by recalling the general definition of these two constructions.

The term "minimal coupling" is widely used in mathematical physics to denote a procedure for converting ordinary derivatives to covariant derivatives. Such derivatives apply to "matter fields" on space-time M which in a general geometric framework are sections of some fiber bundle E over M: then their ordinary derivatives are sections of its (first order) jet bundle JE, as a fiber bundle over M, while their covariant derivatives are sections of its linearized (first order) jet bundle

$$\vec{J}E \;\cong\; L(\pi^*(TM), VE) \;\cong\; \pi^*(T^*M) \otimes VE\,, \tag{1}$$

as a fiber bundle over M, where π is the bundle projection from E to M, $\pi^*(TM)$ resp. $\pi^*(T^*M)$ is the pull-back of the tangent resp. cotangent bundle of M to E, VE is the vertical bundle of E and $L(\pi^*(TM), VE)$ denotes the bundle of fiberwise linear maps from $\pi^*(TM)$ to VE. Within this context, the minimal coupling prescription states that the covariant derivative $D\varphi$ of a section φ of E is obtained from its ordinary derivative $\partial\varphi$ by using a connection in E to decompose the tangent bundle TE of (the total space of) E into the direct sum of the vertical bundle VE and horizontal bundle HE and then projecting onto the vertical part. Now if we think of that connection as being given by its horizontal lifting map, which is a section Γ of JE as an affine bundle over E, so that at each point $e \in E$ with $\pi(e) = x$, $\Gamma(e)$ is a linear map from T_xM to T_eE whose image is the horizontal space H_eE at e of the connection, then that projection onto the vertical part is precisely $1 - \Gamma(e) \circ T_e\pi$. Thus if $\varphi \in \Gamma(M, E)$, so that $\partial\varphi \in \Gamma(M, JE)$ and $D\varphi \in \Gamma(M, \vec{J}E)$, then as fiberwise linear maps from TM to TE, $\partial\varphi$ is just the first order jet (or tangent map) of φ, while $D\varphi$ is the difference

$$D\varphi \;=\; \partial\varphi - \Gamma \circ \varphi\,. \tag{2}$$

This rule can be recast in a purely algebraic form, namely, by viewing it as the result of inserting $\partial\varphi$ and $\Gamma \circ \varphi$ into the *difference map* for (first order) jet bundles, i.e., the bundle map

$$- : JE \times_E JE \;\longrightarrow\; L(\pi^*(TM), VE) \;\cong\; \pi^*(T^*M) \otimes VE \tag{3}$$

over E, explicitly constructed as follows: given any point $e \in E$ with $\pi(e) = x$ and any two jets $u_e^1, u_e^2 \in J_eE \subset L(T_xM, T_eE)$, we have $T_e\pi \circ u_e^i = \mathrm{id}_{T_xM}$, for $i = 1, 2$, and hence the difference $u_e^1 - u_e^2$ (in the vector space $L(T_xM, T_eE)$) takes values in the kernel of $T_e\pi$, that is, the vertical space V_eE of E, so it becomes a linear map from T_xM to V_eE.

The construction of the "curvature map" for connections in a given fiber bundle E over M is similar but somewhat more complicated because it involves its semiholonomous second order jet bundle $\bar{J}^2E$. To see how that goes, we proceed as in Ref. [2] by first constructing the iterated jet bundle $J(JE)$ of E and noting that this allows two projections to JE, namely, the iterated jet target projection $\pi_{J(JE)} : J(JE) \longrightarrow JE$ as well as the jet prolongation $J\pi_{JE} : J(JE) \longrightarrow JE$ of the jet target projection $\pi_{JE} : JE \longrightarrow E$: then by definition, $\bar{J}^2E$ is the subset of $J(JE)$ where these two projections coincide. Concretely, for $e \in E$, $u_e \in J_eE$ and $u'_{u_e} \in J_{u_e}(JE)$,

$$(\pi_{J(JE)})_{u_e}(u'_{u_e}) = u_e \ , \ \ (J\pi_{JE})_{u_e}(u'_{u_e}) = T_{u_e}\pi_{JE} \circ u'_{u_e} \, . \tag{4}$$

As it turns out [9, Theorem 5.3.4, p. 174], $\bar{J}^2 E$ is an affine bundle over JE which decomposes naturally into a symmetric part and an antisymmetric part: the former is precisely the usual second order jet bundle J^2E of E (sometimes also called the holonomous second order jet bundle of E) and is an affine bundle over JE, with difference vector bundle equal to the pull-back to JE of the vector bundle $\pi^*(\bigvee^2 T^*M) \otimes VE$ over E by the jet target projection π_{JE}, whereas the latter is a vector bundle over JE, namely the pull-back to JE of the vector bundle $\pi^*(\bigwedge^2 T^*M) \otimes VE$ over E by the jet target projection π_{JE}:

$$
\begin{aligned}
\bar{J}^2 E &\cong J^2 E \times_{JE} \pi^*_{JE}\Big(\pi^*(\textstyle\bigwedge^2 T^*M) \otimes VE\Big), \\
\vec{J}^2 E &\cong \pi^*_{JE}\Big(\pi^*(\textstyle\bigvee^2 T^*M) \otimes VE\Big).
\end{aligned}
\tag{5}
$$

Now the proofs of these statements given in Ref. [9] and elsewhere in the literature all involve local coordinate representations, so it may be of some interest to provide a more direct, global argument. To this end, consider what we shall call the *difference map* for semiholonomous second order jet bundles, i.e., the bundle map

$$- : \bar{J}^2 E \times_{JE} \bar{J}^2 E \ \longrightarrow \ L^2(\pi^*(TM), VE) \cong \pi^*(\textstyle\bigotimes^2 T^*M) \otimes VE \tag{6}$$

over π_{JE}, where $L^2(\pi^*(TM), VE)$ denotes the bundle of fiberwise bilinear maps from $\pi^*(TM)$ to VE, explicitly constructed as follows: given any point $e \in E$ with $\pi(e) = x$, any jet $u_e \in J_e E$ and any two semiholonomous second order jets $u'^1_{u_e}, u'^2_{u_e} \in \bar{J}^2_{u_e}E \subset J_{u_e}(JE) \subset L(T_x M, T_{u_e}(JE))$, we have $T_{u_e}\pi_{JE} \circ u'^i_{u_e} = u_e$, for $i = 1,2$, and hence the difference $u'^1_{u_e} - u'^2_{u_e}$ takes values in the kernel of $T_{u_e}\pi_{JE}$, that is, the vertical space $V^{\text{jt}}_{u_e}(JE)$ of JE with respect to the jet target projection π_{JE} from JE to E. But with respect to this projection, JE is an affine bundle with difference vector bundle $\vec{J}E$, so this vertical space is canonically isomorphic to the corresponding difference vector space,

$$V^{\text{jt}}_{u_e}(JE) \cong L(T_x M, V_e E) \, ,$$

and thus the difference $u'^1_{u_e} - u'^2_{u_e}$ becomes a linear map from $T_x M$ to this vector space, which can be identified with a bilinear map from $T_x M$ to $V_e E$. Obviously, any such bilinear map can be canonically decomposed into its symmetric and its antisymmetric part, and the restriction of the difference map for semiholonomous second order jet bundles to the symmetric part will provide the *difference map* for second order jet bundles, i.e., the bundle map

$$- : J^2 E \times_{JE} J^2 E \ \longrightarrow \ L^2_s(\pi^*(TM), VE) \cong \pi^*(\textstyle\bigvee^2 T^*M) \otimes VE \tag{7}$$

over π_{JE}, where $L^2_s(\pi^*(TM), VE)$ denotes the bundle of fiberwise symmetric bilinear maps from $\pi^*(TM)$ to VE. Moreover, it will provide an *alternator* or *antisymmetrizer* for semiholonomous second order jets, which is an affine bundle map

$$\text{Alt} : \bar{J}^2 E \ \longrightarrow \ L^2_a(\pi^*(TM), VE) \cong \pi^*(\textstyle\bigwedge^2 T^*M) \otimes VE \tag{8}$$

over π_{JE}, where $L^2_a(\pi^*(TM), VE)$ denotes the bundle of fiberwise antisymmetric bilinear maps from $\pi^*(TM)$ to VE, as follows: given any point $e \in E$ with $\pi(e) = x$, any jet $u_e \in J_e E$ and any semiholonomous second order jet $u'_{u_e} \in \bar{J}^2_{u_e}E$, choose any holonomous second order jet $u'^0_{u_e} \in J^2_{u_e}E$ and define $\text{Alt}(u'_{u_e})$ to be the antisymmetric part of the difference $u'_{u_e} - u'^0_{u_e}$, which obviously does not depend on the choice of $u'^0_{u_e}$. It is this construction that we shall use to define the curvature of a connection in E, given, say, in terms of its horizontal lifting map, which is a section Γ of JE as a bundle over E: observing that its jet prolongation $j\Gamma$ will then be a section not just of $J(JE)$ but actually of $\bar{J}^2 E$, again as a bundle over E, since $T\pi_{JE} \circ j\Gamma = T(\pi_{JE} \circ \Gamma) = T\,\text{id}_E = \text{id}_{TE}$, and noting that it will therefore be a section of $\bar{J}^2 E$ along Γ when $\bar{J}^2 E$ is considered as a bundle over JE instead, we can compose it with the alternator to produce a section of $\pi^*_{JE}(\pi^*(\bigwedge^2 T^*M) \otimes VE)$ along Γ, which is just a section of $\pi^*(\bigwedge^2 T^*M) \otimes VE$ and (possibly up to a sign which is a matter of convention) is the curvature

$$\text{curv}(\Gamma) = \text{Alt} \circ j\Gamma \tag{9}$$

of the given connection.

The main statement we want to prove in this section is that these two constructions are invariant (or perhaps it might be better to say, equivariant) under any action of any Lie groupoid G over M on the bundle E over M, provided we employ the correct induced actions of the pertinent Lie groupoids derived from G on the pertinent bundles derived from E.

Thus assume we are given a Lie groupoid G over M, with source projection $\sigma_G : G \longrightarrow M$ and target projection $\tau_G : G \longrightarrow M$, together with an action

$$
\begin{aligned}
\Phi_E: \quad G \times_M E &\longrightarrow E \\
(g, e) &\longmapsto g \cdot e
\end{aligned}
\tag{10}
$$

of G on E. (Cf. equation (44) of Ref. [2]). Then we obtain an induced action

$$
\begin{aligned}
\Phi_{VE}: \quad G \times_M VE &\longrightarrow VE \\
(g, v_e) &\longmapsto g \cdot v_e
\end{aligned}
\tag{11}
$$

of G on the vertical bundle VE of E, defined by

$$
g \cdot v_e = T_e L_g(v_e),
\tag{12}
$$

where TL_g denotes the tangent map to L_g; in other words, left translation by g in VE is just the derivative of left translation by g in E. (Cf. equations (89) and (90) of Ref. [2]). Combining this with the natural action of the linear frame groupoid $GL(TM)$ of the base manifold M on the cotangent bundle T^*M of M, we obtain an induced action of the Lie groupoid $GL(TM) \times_M G$ on the linearized jet bundle $\vec{J}E$ of E,

$$
\begin{aligned}
\left(GL(TM) \times_M G\right) \times_M \vec{J}E &\longrightarrow \vec{J}E \\
\left((a, g), \vec{u}_e\right) &\longmapsto (a, g) \cdot \vec{u}_e
\end{aligned}
\tag{13}
$$

as suggested by the isomorphism of equation (1), defined by

$$
(a, g) \cdot \vec{u}_e = T_e L_g \circ \vec{u}_e \circ a^{-1}.
\tag{14}
$$

(Cf. equations (96) and (98) of Ref. [2]). On the other hand, applying the jet functor to all structural maps that appear in the original action (10), we obtain an induced action

$$
\begin{aligned}
\Phi_{JE}: \quad JG \times_M JE &\longrightarrow JE \\
(u_g, u_e) &\longmapsto u_g \cdot u_e
\end{aligned}
\tag{15}
$$

of the jet groupoid JG of G on the jet bundle JE of E, defined by

$$
u_g \cdot u_e = T_{(g,e)}\Phi_E \circ (u_g, u_e) \circ \pi_{JG}^{\mathrm{fr}}(u_g)^{-1},
\tag{16}
$$

where $T\Phi_E$ denotes the tangent map to Φ_E and $\pi_{JG}^{\mathrm{fr}}: JG \longrightarrow GL(TM)$ is the natural projection of JG to the linear frame groupoid $GL(TM)$ of the base manifold M defined by

$$
\pi_{JG}^{\mathrm{fr}}(u_g) = T_g \tau_G \circ u_g,
\tag{17}
$$

whereas $\pi_{JG}: JG \longrightarrow G$ is the usual jet target projection. (Cf. equations (51), (93) and (94) of Ref. [2]). This definition can also be phrased in terms of (bi)sections, as follows: given any bisection β of G and any section φ of E, concatenate them into a map (β, φ) from M to $G \times_M E$ and compose that with the action Φ_E of G on E to produce a map from M to E which, when precomposed with the inverse of the diffeomorphism $\tau_G \circ \beta$ of M induced by β, gives a new section $\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}$ of E, and Φ_{JE} is then fully characterized by the property that, upon taking the jet prolongations of all these (bi)sections,

$$
\Phi_{JE} \circ (j\beta, j\varphi) \circ (\tau_G \circ \beta)^{-1} = j\left(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}\right).
\tag{18}
$$

Indeed, for any $y \in M$, putting $x = (\tau_G \circ \beta)^{-1}(y) \in M$, we have $(\beta(x), \varphi(x)) \in G \times_M E$, $(j\beta(x), j\varphi(x)) \in JG \times_M JE$ and

$$
\pi_{JG}^{\mathrm{fr}}(j\beta(x))^{-1} = (T_{\beta(x)}\tau_G \circ T_x\beta)^{-1} = (T_x(\tau_G \circ \beta))^{-1} = T_y\left((\tau_G \circ \beta)^{-1}\right),
$$

so

$$
\begin{aligned}
\left(\Phi_{JE} \circ (j\beta, j\varphi) \circ (\tau_G \circ \beta)^{-1}\right)(y) &= \Phi_{JE}(j\beta(x), j\varphi(x)) \\
&= T_{(\beta(x), \varphi(x))}\Phi_E \circ (T_x\beta, T_x\varphi) \circ T_y\left((\tau_G \circ \beta)^{-1}\right) \\
&= T_y\left(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}\right) = j\left(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}\right)(y).
\end{aligned}
$$

Now we have the following statement about compatibility between these various actions:

Proposition 1. *The difference map of equation (3) is equivariant, i.e., the diagram*

$$
\begin{array}{ccc}
JG \times_M (JE \times_E JE) & \longrightarrow & JE \times_E JE \\
{\scriptstyle (\pi_{JG}^{\mathrm{fr}} \times \pi_{JG}, -)} \downarrow & & \downarrow {\scriptstyle -} \\
\left(GL(TM) \times_M G\right) \times_M \vec{J}E & \longrightarrow & \vec{J}E
\end{array}
\tag{19}
$$

commutes.

Proof. Given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi(e) = x$, $u_g \in J_g G$ and $u_e^1, u_e^2 \in J_e E \subset L(T_x M, T_e E)$, we want to prove that

$$u_g \cdot u_e^2 - u_g \cdot u_e^1 \ = \ (\pi_{JG}^{\mathrm{fr}}(u_g), g) \cdot (u_e^2 - u_e^1) \,.$$

Fixing some tangent vector $v \in T_x M$, choose a vertical curve $e(t)$ in E $(\pi(e(t)) = x)$ such that

$$e(t)\big|_{t=0} \ = \ e \quad , \quad \frac{d}{dt} e(t)\Big|_{t=0} \ = \ (u_e^2 - u_e^1)(v) \,.$$

Then

$$T_{(g,e)} \Phi_E \big(u_g(v), u_e^2(v) \big) \ - \ T_{(g,e)} \Phi_E \big(u_g(v), u_e^1(v) \big) \ = \ T_{(g,e)} \Phi_E \big(0, (u_e^2 - u_e^1)(v) \big)$$

$$= \ \frac{d}{dt} \Phi_E(g, e(t))\Big|_{t=0} \ = \ \frac{d}{dt} L_g(e(t))\Big|_{t=0} \ = \ T_e L_g \big((u_e^2 - u_e^1)(v) \big) \,,$$

or using that v was arbitrary,

$$T_{(g,e)} \Phi_E \circ \big(u_g, u_e^2 \big) \ - \ T_{(g,e)} \Phi_E \circ \big(u_g, u_e^1 \big) \ = \ T_{(g,e)} \Phi_E \circ \big(0, u_e^2 - u_e^1 \big) \ = \ T_e L_g \circ (u_e^2 - u_e^1) \,.$$

Precomposing with $\pi_{JG}^{\mathrm{fr}}(u_g)^{-1}$ proves the claim. $\square$

To deal with the second part, we begin by iterating the procedure of applying the jet functor to obtain an induced action

$$\begin{aligned} \Phi_{J(JE)} : \quad J(JG) \times_M J(JE) &\longrightarrow & J(JE) \\ (u'_{u_g}, u'_{u_e}) &\longmapsto & u'_{u_g} \cdot u'_{u_e} \end{aligned} \tag{20}$$

of the iterated jet groupoid $J(JG)$ of G on the iterated jet bundle $J(JE)$ of E, defined by

$$u'_{u_g} \cdot u'_{u_e} \ = \ T_{(u_g, u_e)} \Phi_{JE} \circ (u'_{u_g}, u'_{u_e}) \circ \pi_{J(JG)}^{\mathrm{fr}}(u'_{u_g})^{-1} \,, \tag{21}$$

with the same notation as before; in particular, the definition can again be phrased in terms of (bi)sections. Namely, given any bisection $\tilde{\beta}$ of JG and any section $\tilde{\varphi}$ of JE which (by composition with π_{JG}) project to a bisection β of G and to a section φ of E, respectively, so that $\tau_{JG} \circ \tilde{\beta} = \tau_G \circ \beta$, we have, just as in equation (18) above,

$$\Phi_{J(JE)} \circ (j\tilde{\beta}, j\tilde{\varphi}) \circ (\tau_G \circ \beta)^{-1} \ = \ j \big(\Phi_{JE} \circ (\tilde{\beta}, \tilde{\varphi}) \circ (\tau_G \circ \beta)^{-1} \big) \,. \tag{22}$$

This iterated action admits restrictions to several subgroupoids and subbundles, among which the following will become important to us at some point or another: the natural induced actions

$$\begin{aligned} \Phi_{\bar{J}^2 E} : \quad \bar{J}^2 G \times_M \bar{J}^2 E &\longrightarrow & \bar{J}^2 E \\ (u'_{u_g}, u'_{u_e}) &\longmapsto & u'_{u_g} \cdot u'_{u_e} \end{aligned} \tag{23}$$

of the semiholonomous second order jet groupoid $\bar{J}^2 G$ of G and

$$\begin{aligned} \Phi_{\bar{J}^2 E} : \quad J^2 G \times_M \bar{J}^2 E &\longrightarrow & \bar{J}^2 E \\ (u'_{u_g}, u'_{u_e}) &\longmapsto & u'_{u_g} \cdot u'_{u_e} \end{aligned} \tag{24}$$

of the second order jet groupoid $J^2 G$ of G on the semiholonomous second order jet bundle $\bar{J}^2 E$ of E, as well as the action

$$\begin{aligned} \Phi_{J^2 E} : \quad J^2 G \times_M J^2 E &\longrightarrow & J^2 E \\ (u'_{u_g}, u'_{u_e}) &\longmapsto & u'_{u_g} \cdot u'_{u_e} \end{aligned} \tag{25}$$

of the second order jet groupoid $J^2 G$ of G on the second order jet bundle $J^2 E$ of E, all defined by the same formula:

$$u'_{u_g} \cdot u'_{u_e} \ = \ T_{(u_g, u_e)} \Phi_{JE} \circ (u'_{u_g}, u'_{u_e}) \circ \pi_{JG}^{\mathrm{fr}}(u_g)^{-1} \,. \tag{26}$$

Here, the simplification in the last term on the rhs of equation (26), as compared to that of equation (21), stems from the fact that when $u'_{u_g} \in \bar{J}_{u_g}^2 G$, i.e., $T_{u_g} \pi_{JG} \circ u'_{u_g} = u_g$, then since $\tau_{JG} = \tau_G \circ \pi_{JG}$, we get

$$\pi_{J(JG)}^{\mathrm{fr}}(u'_{u_g}) \ = \ T_{u_g} \tau_{JG} \circ u'_{u_g} \ = \ T_g \tau_G \circ T_{u_g} \pi_{JG} \circ u'_{u_g} \ = \ T_g \tau_G \circ u_g \ = \ \pi_{JG}^{\mathrm{fr}}(u_g) \,.$$

Moreover, if u'_{u_g} and u'_{u_e} are both semiholonomous, then so is $u'_{u_g} \cdot u'_{u_e}$, i.e., we have

$$u'_{u_g} \in \bar{J}_{u_g}^2 G \,, \ u'_{u_e} \in \bar{J}_{u_e}^2 E \implies u'_{u_g} \cdot u'_{u_e} \in \bar{J}_{u_g \cdot u_e}^2 E \,,$$

since in this case, $T_{u_g}\pi_{JG} \circ u'_{u_g} = u_g$ and $T_{u_e}\pi_{JE} \circ u'_{u_e} = u_e$, and using the equality $\pi_{JE} \circ \Phi_{JE} = \Phi_E \circ (\pi_{JG} \times_M \pi_{JE})$, we get

$$
\begin{aligned}
T_{u_g \cdot u_e}\pi_{JE} \circ (u'_{u_g} \cdot u'_{u_e}) &= T_{u_g \cdot u_e}\pi_{JE} \circ T_{(u_g, u_e)}\Phi_{JE} \circ (u'_{u_g}, u'_{u_e}) \circ \pi^{\mathrm{fr}}_{JG}(u_g)^{-1} \\
&= T_{(g,e)}\Phi_E \circ \big(T_{u_g}\pi_{JG} \circ u'_{u_g}, T_{u_e}\pi_{JE} \circ u'_{u_e}\big) \circ \pi^{\mathrm{fr}}_{JG}(u_g)^{-1} \\
&= T_{(g,e)}\Phi_E \circ (u_g, u_e) \circ \pi^{\mathrm{fr}}_{JG}(u_g)^{-1} \\
&= u_g \cdot u_e .
\end{aligned}
$$

Similarly, it is clear that if u'_{u_g} and u'_{u_e} are both holonomous, then so is $u'_{u_g} \cdot u'_{u_e}$, i.e., we have

$$
u'_{u_g} \in J^2_{u_g}G , \ u'_{u_e} \in J^2_{u_e}E \implies u'_{u_g} \cdot u'_{u_e} \in J^2_{u_g \cdot u_e}E ,
$$

since in this case there will exist a local bisection β of G and a local section φ of E, both defined in some open neighborhood U of x, satisfying $g = \beta(x)$, $e = \varphi(x)$, $u_g = j\beta(x) = T_x\beta$, $u_e = j\varphi(x) = T_x\varphi$, $u'_{u_g} = j(j\beta)(x) = T_x(j\beta)$, $u'_{u_e} = j(j\varphi)(x) = T_x(j\varphi)$ and hence, putting $y = (\tau_G \circ \beta)(x)$ and using equation (26), equation (22) with $\tilde\beta = j\beta$, $\tilde\varphi = j\varphi$ and equation (18),

$$
\begin{aligned}
u'_{u_g} \cdot u'_{u_e} &= \Phi_{J(JE)}(j(j\beta)(x), j(j\varphi)(x)) \\
&= \big(\Phi_{J(JE)} \circ (j(j\beta), j(j\varphi)) \circ (\tau_G \circ \beta)^{-1}\big)(y) \\
&= j\big(\Phi_{JE} \circ (j\beta, j\varphi) \circ (\tau_G \circ \beta)^{-1}\big)(y) \\
&= j\big(j\big(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}\big)\big)(y) .
\end{aligned}
$$

Finally, observe that, just like the (first order) jet groupoid JG of G, its iterated jet groupoid $J(JG)$ and, by restriction, its semiholonomous second order jet groupoid $\bar{J}^2G$ and second order jet groupoid J^2G all admit natural projections both to $GL(TM)$ and to G, which are just given by composition of those for JG with the natural projection $\pi_{J(JG)}: J(JG) \longrightarrow JG$ and its respective restrictions $\pi_{\bar{J}^2G}: \bar{J}^2G \longrightarrow JG$ and $\pi_{J^2G}: J^2G \longrightarrow JG$:

$$
\begin{aligned}
\pi^{\mathrm{fr}}_{J(JG)} &= \pi^{\mathrm{fr}}_{JG} \circ \pi_{J(JG)}: J(JG) \longrightarrow GL(TM) \ , \quad \pi_{J(JG),G} = \pi_{JG} \circ \pi_{J(JG)}: J(JG) \longrightarrow G \\
\pi^{\mathrm{fr}}_{\bar{J}^2G} &= \pi^{\mathrm{fr}}_{JG} \circ \pi_{\bar{J}^2G}: \bar{J}^2G \longrightarrow GL(TM) \ , \quad \pi_{\bar{J}^2G,G} = \pi_{JG} \circ \pi_{\bar{J}^2G}: \bar{J}^2G \longrightarrow G \\
\pi^{\mathrm{fr}}_{J^2G} &= \pi^{\mathrm{fr}}_{JG} \circ \pi_{J^2G}: J^2G \longrightarrow GL(TM) \ , \quad \pi_{J^2G,G} = \pi_{JG} \circ \pi_{J^2G}: J^2G \longrightarrow G
\end{aligned}
$$

With this notation, we can now formulate the following statement about compatibility between these various actions:

Proposition 2. *The difference maps of equations (6) and (7) are equivariant, i.e., the diagrams*

$$
\begin{array}{ccc}
\bar{J}^2G \times_M (\bar{J}^2E \times_{JE} \bar{J}^2E) & \longrightarrow & \bar{J}^2E \times_{JE} \bar{J}^2E \\
{\scriptstyle (\pi^{\mathrm{fr}}_{\bar{J}^2G} \times \pi_{\bar{J}^2G,G}, -)} \downarrow & & \downarrow {\scriptstyle -} \\
(GL(TM) \times_M G) \times_M \big(\pi^*(\bigotimes^2 T^*M) \otimes VE\big) & \longrightarrow & \pi^*(\bigotimes^2 T^*M) \otimes VE
\end{array}
\tag{27}
$$

and

$$
\begin{array}{ccc}
J^2G \times_M (J^2E \times_{JE} J^2E) & \longrightarrow & J^2E \times_{JE} J^2E \\
{\scriptstyle (\pi^{\mathrm{fr}}_{J^2G} \times \pi_{J^2G,G}, -)} \downarrow & & \downarrow {\scriptstyle -} \\
(GL(TM) \times_M G) \times_M \big(\pi^*(\bigvee^2 T^*M) \otimes VE\big) & \longrightarrow & \pi^*(\bigvee^2 T^*M) \otimes VE
\end{array}
\tag{28}
$$

commute. Similarly, the alternator or antisymmetrizer map of equation (8) is also equivariant, i.e., the diagram

$$
\begin{array}{ccc}
J^2G \times_M \bar{J}^2E & \longrightarrow & \bar{J}^2E \\
{\scriptstyle (\pi^{\mathrm{fr}}_{J^2G} \times \pi_{J^2G,G}, \mathrm{Alt})} \downarrow & & \downarrow {\scriptstyle \mathrm{Alt}} \\
(GL(TM) \times_M G) \times_M \big(\pi^*(\bigwedge^2 T^*M) \otimes VE\big) & \longrightarrow & \pi^*(\bigwedge^2 T^*M) \otimes VE
\end{array}
\tag{29}
$$

commutes.

Proof. First of all, the statements about commutativity of the last two diagrams are trivial consequences of that about commutativity of the first, together with the fact that the decomposition of rank 2 tensors into their symmetric and anti-symmetric parts is obviously invariant under the action of $GL(TM) \times_M G$. To deal with the first diagram, we shall find it convenient to keep track of the identifications made in the definition of the difference map in equation (6) by momentarily (i.e., just for the remainder of this proof) denoting that difference map by δ. Thus given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi(e) = x$, $u_g \in J_g G$, $u_e \in J_e E$, $u'_{u_g} \in \bar{J}^2_{u_g} G$ and $u'^1_{u_e}, u'^2_{u_e} \in \bar{J}^2_{u_e} E \subset J_{u_e}(JE) \subset L(T_x M, T_{u_e}(JE))$, we want to show that

$$\delta(u'_{u_g} \cdot u'^2_{u_e}, u'_{u_g} \cdot u'^1_{u_e}) = (\pi^{\text{fr}}_{JG}(u_g), g) \cdot \delta(u'^2_{u_e}, u'^1_{u_e}).$$

Note that $\delta(u'^2_{u_e}, u'^1_{u_e}) \in L^2(T_x M, V_e E)$ can be defined explicitly by stating that, for any tangent vector $v \in T_x M$, the standard difference $u'^2_{u_e} - u'^1_{u_e}$, when evaluated on v, gives a tangent vector in $T_{u_e}(JE)$ which, being vertical with respect to the jet target projection π_{JE}, can be realized as that of a straight line in $J_e E$ through u_e, whose direction is $\delta(u'^2_{u_e}, u'^1_{u_e})(v, .) \in L(T_x M, V_e E)$:

$$(u'^2_{u_e} - u'^1_{u_e})(v) = \frac{d}{dt}(u_e + t\,\delta(u'^2_{u_e}, u'^1_{u_e})(v, .))\big|_{t=0}.$$

Similarly, $\delta(u'_{u_g} \cdot u'^2_{u_e}, u'_{u_g} \cdot u'^1_{u_e}) \in L^2(T_y M, V_{g \cdot e} E)$ can be defined explicitly by stating that, for any tangent vector $w \in T_y M$, the standard difference $u'_{u_g} \cdot u'^2_{u_e} - u'_{u_g} \cdot u'^1_{u_e}$, when evaluated on w, gives a tangent vector in $T_{u_g \cdot u_e}(JE)$ which, being vertical with respect to the jet target projection π_{JE}, can be realized as that of a straight line in $J_{g \cdot e} E$ through $u_g \cdot u_e$, whose direction is $\delta(u'_{u_g} \cdot u'^2_{u_e}, u'_{u_g} \cdot u'^1_{u_e})(w, .) \in L(T_y M, V_{g \cdot e} E)$:

$$(u'_{u_g} \cdot u'^2_{u_e} - u'_{u_g} \cdot u'^1_{u_e})(w) = \frac{d}{dt}(u_g \cdot u_e + t\,\delta(u'_{u_g} \cdot u'^2_{u_e}, u'_{u_g} \cdot u'^1_{u_e})(w, .))\big|_{t=0}.$$

On the other hand, putting $v = \pi^{\text{fr}}_{JG}(u_g)^{-1}(w)$, we have

$$
\begin{aligned}
(u'_{u_g} \cdot u'^2_{u_e} &- u'_{u_g} \cdot u'^1_{u_e})(w) \\
&= T_{(u_g, u_e)} \Phi_{JE}(u'_{u_g}(v), u'^2_{u_e}(v)) - T_{(u_g, u_e)} \Phi_{JE}(u'_{u_g}(v), u'^1_{u_e}(v)) \\
&= T_{(u_g, u_e)} \Phi_{JE}(0, (u'^2_{u_e} - u'^1_{u_e})(v)) \\
&= \frac{d}{dt} \Phi_{JE}(u_g, u_e + t\,\delta(u'^2_{u_e}, u'^1_{u_e})(v, .))\big|_{t=0} \\
&= ((\pi^{\text{fr}}_{JG}(u_g), g) \cdot \delta(u'^2_{u_e}, u'^1_{u_e}))(w, .),
\end{aligned}
$$

where in the last step we have used the fact that, as shown in Ref. [2], the action Φ_{JE} is affine along the fibers of JE over E, together with Proposition 1. $\square$

Returning to the formalization of the minimal coupling prescription and the curvature map, we want to emphasize that the context outlined above is a little bit too broad to fit into the theoretical setting of field theory, since general connections in general fiber bundles are *not* fields! This is so because they are not sections of bundles over space-time but rather sections of bundles over some "extended space-time" which is itself the total space of some fiber bundle over ordinary space-time. As such, when expressed in local coordinates and local trivializations, such sections correspond to multiplets of functions which, apart from being functions on space-time, depend on extra "vertical" variables, namely, the local coordinates along the fibers of this bundle, and in the absence of stringent restrictions on that dependence will produce *infinite* multiplets of fields when expanded in an appropriate basis. This situation is familiar from "Kaluza-Klein" type theories, which have been proposed long ago as models for unifying gravity with the other fundamental interactions and where the extended space-time is assumed to be the total space of some principal bundle over ordinary space-time, so that one can use the representation theory of the underlying structure group to control and restrict the dependence of functions on the extra vertical variables.[3] The main problem with these models is that the aforementioned stringent restrictions, needed to weed out the large number of (often unwanted) extra fields, are usually quite artificial and imposed more or less "ad hoc", without any convincing argument as to how they should arise from the dynamics of a fundamental theory in higher dimensions.

---

[3] The simplest such model and one of the most interesting attempts to unify gravity with electromagnetism uses an extended space-time which is the total space of a principal $U(1)$-bundle over ordinary space-time, so the extra vertical variables reduce to a single phase θ, the representations of the structure group are given by its characters $\theta \longmapsto \exp(ik\theta)$, $k \in \mathbb{Z}$, and the expansion of functions on extended space-time is just a Fourier expansion with coefficients that are functions on ordinary space-time: still an infinite multiplet of fields.

Here, these remarks serve merely as a guide to what should be done and what not: we shall completely avoid all these problems by working not with general connections but only with connections that do have a natural interpretation as fields in physics: these are connections whose behavior along the fibers is fixed by some condition, such as linear connections in vector bundles or affine connections in affine bundles, where the connection coefficients are required to be linear or affine functions along the fibers, respectively, or more generally, principal connections, which are required to be equivariant under the action of the structure group on the fibers of the principal bundle and are therefore completely fixed along the entire fiber once they are known at a single point in that fiber.

Thus from this point onward and throughout the rest of the paper, we shall assume that E is not just a general fiber bundle but rather a fiber bundle with structure group, which is a Lie group G_0, with Lie algebra g_0, say, so there is a principal G_0-bundle P to which E is associated (this, by the way, includes the case where E is P itself), and any connection in E to be considered is associated to a principal connection in P. As a result, we have to adapt our formalism to this situation, and of course the Lie groupoid G that appears above, as well as in Ref. [2], but has so far been left unspecified, will now be the gauge groupoid of P.

3. Gauge groupoids, jet groupoids and induced actions

In order to implement the program outlined in the last paragraph of the previous section, we shall first introduce the gauge groupoid of a principal bundle and some of its actions (more specifically, on the principal bundle itself and on any of its associated bundles, as well as on the respective vertical bundles) and then investigate how some of these lift when taking first and second order jet prolongations.

3.1. The gauge groupoid and its actions

To begin with, let us recall the definition of the gauge groupoid of a principal bundle [8]:

Proposition 3. *Given a principal bundle P over a manifold M with structure group G_0, whose bundle projection will be denoted by $\rho : P \longrightarrow M$, let*

$$G = (P \times P)/G_0$$

*denote the orbit space of the cartesian product of P with itself under the diagonal action of G_0 (we shall write its elements as classes $[p_2, p_1]$ of pairs (p_2, p_1) in $P \times P$, where $[p_2 \cdot g_0, p_1 \cdot g_0] = [p_2, p_1]$). Then G is a Lie groupoid over M, called the **gauge groupoid** of P, with source projection $\sigma_G : G \longrightarrow M$, target projection $\tau_G : G \longrightarrow M$, multiplication map $\mu_G : G \times_M G \longrightarrow G$, unit map $1_G : M \longrightarrow G$ and inversion $\iota_G : G \longrightarrow G$ defined as follows:*

- *for $[p_2, p_1] \in G$,*

$$\sigma_G([p_2, p_1]) = \rho(p_1) \ , \quad \tau_G([p_2, p_1]) = \rho(p_2) \,;$$

- *for $[p_2, p_1], [p_3, p_2] \in G$,*

$$[p_3, p_2][p_2, p_1] \equiv \mu_G([p_3, p_2], [p_2, p_1]) = [p_3, p_1] \,;$$

- *for $x \in M$,*

$$(1_G)_x = [p, p] \,,$$

  where p is any element of $\rho^{-1}(x)$;
- *for $[p_2, p_1] \in G$,*

$$[p_2, p_1]^{-1} \equiv \iota_G([p_2, p_1]) = [p_1, p_2] \,.$$

Observe that the gauge group bundle associated with P employed in Ref. [4], also known as the adjoint bundle $\mathrm{Ad}P = P \times_{G_0} G_0$ (where G_0 acts on itself by conjugation), is (up to a canonical isomorphism) just the isotropy subgroupoid of G, that is,

$$P \times_{G_0} G_0 \cong G_{\mathrm{iso}} \,. \tag{30}$$

This isomorphism can be constructed explicitly by noting that the map

$$\begin{aligned} P \times G_0 &\longrightarrow & P \times P \\ (p, g_0) &\longmapsto & (p, p \cdot g_0) \end{aligned}$$

is equivariant under the right action of G_0 on both sides (since it takes $(p \cdot g_0', (g_0')^{-1} g_0 g_0')$ to $(p \cdot g_0', p \cdot g_0 g_0')$) and hence factors to the respective quotients to yield a map

$$
\begin{aligned}
P \times_{G_0} G_0 &\longrightarrow (P \times P)/G_0 \\
[p, g_0] &\longmapsto [p, p \cdot g_0]
\end{aligned}
$$

which is the desired isomorphism onto its image

$$
G_{\mathrm{iso}} = \{[p_2, p_1] \in G \mid \tau_G([p_2, p_1]) = \sigma_G([p_2, p_1])\} = \{[p_2, p_1] \in G \mid \rho(p_2) = \rho(p_1)\}. \tag{31}
$$

Moreover, it is well known that the group of bisections of the gauge groupoid $G = (P \times P)/G_0$ is isomorphic to the group of automorphisms of P,

$$
\mathrm{Bis}(G) \cong \mathrm{Aut}(P), \tag{32}
$$

while the group of sections of the gauge group bundle $G_{\mathrm{iso}} \cong P \times_{G_0} G_0$ is isomorphic to the group of strict automorphisms of P,

$$
\Gamma(G_{\mathrm{iso}}) \cong \mathrm{Aut}_s(P). \tag{33}
$$

Next, let us specify how the gauge groupoid of a principal bundle acts naturally on the principal bundle itself and on any of its associated bundles. To this end, some authors find it convenient to introduce the "difference map" for P, which is the smooth map

$$
\delta_P : P \times_M P \longrightarrow G_0
$$

defined implicitly by the condition that given any two points p and p' in the same fiber of P, $\delta_P(p, p')$ is the unique element of G_0 that transforms p into p':

$$
p \cdot \delta_P(p, p') = p'.
$$

Note that, obviously, $\delta_P(p, p) = 1$ and

$$
\delta_P(p \cdot g_0, p' \cdot g_0) = g_0^{-1} \delta_P(p, p') g_0.
$$

Here, we use this map to write down a natural action

$$
\begin{aligned}
\Phi_P : \quad G \times_M P &\longrightarrow P \\
([p_2, p_1], p) &\longmapsto [p_2, p_1] \cdot p
\end{aligned}
$$

of the gauge groupoid $G = (P \times P)/G_0$ on the principal bundle P itself, defined as follows: given $[p_2, p_1] \in G$ and $p \in P$ such that $\rho(p_1) = \sigma_G([p_2, p_1]) = \rho(p)$, put

$$
[p_2, p_1] \cdot p = p_2 \cdot \delta_P(p_1, p).
$$

Note, however, that we can always adapt the second component in the pair (p_2, p_1) representing the class $[p_2, p_1]$ to be equal to p, which allows us to rewrite the previous two equations in the simplified form

$$
\begin{aligned}
\Phi_P : \quad G \times_M P &\longrightarrow P \\
([p', p], p) &\longmapsto [p', p] \cdot p
\end{aligned} \tag{34}
$$

where

$$
[p', p] \cdot p = p'. \tag{35}
$$

In the sequel, when defining other actions of the gauge groupoid, we shall already perform this kind of simplification right from the start and without further notice, thus dispensing the need to deal with the difference map δ_P altogether. Of course, as the total space of a principal bundle, P also carries a right action of the structure group G_0, and remarkably, these two actions commute,

$$
[p', p] \cdot (p \cdot g_0) = ([p', p] \cdot p) \cdot g_0, \tag{36}
$$

because both sides are equal to $[p' \cdot g_0, p \cdot g_0] \cdot (p \cdot g_0) = p' \cdot g_0$. Thus using the natural projection of G to the pair groupoid $M \times M$ of the base manifold M, we get a commutative diagram:

$$
\begin{array}{ccc}
G \times_M P & \longrightarrow & P \\
\downarrow & & \downarrow \\
(M \times M) \times_M M & \longrightarrow & M
\end{array}
\tag{37}
$$

This procedure can be generalized as follows. First, given any manifold Q, we can introduce a natural action

$$
\Phi_{P \times Q} : \begin{array}{ccc}
G \times_M (P \times Q) & \longrightarrow & P \times Q \\
([p', p], (p, q)) & \longmapsto & [p', p] \cdot (p, q)
\end{array}
\tag{38}
$$

of the gauge groupoid $G = (P \times P)/G_0$ on the product manifold $P \times Q$ (as a fiber bundle over M), defined by letting G act as above on the first factor and trivially on the second factor,

$$
[p', p] \cdot (p, q) = (p', q).
\tag{39}
$$

Now suppose we are also given a left action

$$
\begin{array}{ccc}
G_0 \times Q & \longrightarrow & Q \\
(g_0, q) & \longmapsto & g_0 \cdot q
\end{array}
\tag{40}
$$

of G_0 on the manifold Q, which according to the standard definition of the total space of an associated bundle is extended to a "diagonal" right action

$$
\begin{array}{ccc}
G_0 \times (P \times Q) & \longrightarrow & P \times Q \\
(g_0, (p, q)) & \longmapsto & (p \cdot g_0, g_0^{-1} \cdot q)
\end{array}
\tag{41}
$$

of G_0 on the product manifold $P \times Q$, and once again, these two actions commute,

$$
[p', p] \cdot ((p, q) \cdot g_0) = ([p', p] \cdot (p, q)) \cdot g_0,
\tag{42}
$$

because both sides are equal to $[p' \cdot g_0, p \cdot g_0] \cdot (p \cdot g_0, g_0^{-1} \cdot q) = (p' \cdot g_0, g_0^{-1} \cdot q)$. This implies that the action $\Phi_{P \times Q}$ of G on $P \times Q$ in equation (38) passes to the quotient $P \times_{G_0} Q$, and so we get a natural induced action

$$
\Phi_{P \times_{G_0} Q} : \begin{array}{ccc}
G \times_M (P \times_{G_0} Q) & \longrightarrow & P \times_{G_0} Q \\
([p', p], [p, q]) & \longmapsto & [p', p] \cdot [p, q]
\end{array}
\tag{43}
$$

of the gauge groupoid $G = (P \times P)/G_0$ on the associated bundle $P \times_{G_0} Q$, defined by

$$
[p', p] \cdot [p, q] = [p', q].
\tag{44}
$$

It will be convenient to visualize this construction in terms of the "magical square" for associated bundles, i.e., the commutative diagram

$$
\begin{array}{ccc}
P \times Q & \xrightarrow{\rho_Q} & P \times_{G_0} Q \\
{\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle \pi} \\
P & \xrightarrow{\rho} & M
\end{array}
\tag{45}
$$

in which the horizontal projections define principal G_0-bundles while the vertical projections provide fiber bundles with typical fiber Q (the first of which is of course just the trivial bundle over P) such that ρ_Q is an isomorphism on each fiber and, by definition, is G-equivariant. And again, using the natural projection of G to the pair groupoid $M \times M$ of the base manifold M, we get a commutative diagram:

$$
\begin{array}{ccc}
G \times_M (P \times_{G_0} Q) & \longrightarrow & P \times_{G_0} Q \\
\downarrow & & \downarrow \\
(M \times M) \times_M M & \longrightarrow & M
\end{array}
\tag{46}
$$

Of course, these actions extend the actions of the gauge group bundle $P \times_{G_0} G_0$ on the principal bundle P itself and on the associated bundle $P \times_{G_0} Q$, respectively, considered in Ref. [4].

As a first example of induced actions, consider those of the gauge groupoid of a principal bundle on the vertical bundle of the principal bundle itself and on the vertical bundle of any of its associated bundles, constructed according to the prescription specified in equations (10)–(12) above. These actions can be simplified by making use of the fact that the

vertical bundle of a principal bundle is trivial and that the vertical bundle of an associated bundle is again an associated bundle, i.e., we have canonical isomorphisms

$$VP \cong P \times \mathfrak{g}_0, \tag{47}$$

and

$$V(P \times_{G_0} Q) \cong P \times_{G_0} TQ, \tag{48}$$

both as fiber bundles over M and as vector bundles over the respective total spaces P and $P \times_{G_0} Q$, where in the second case, the action of G_0 on the tangent bundle TQ of Q is the one induced from that on Q. Similarly, we also have canonical isomorphisms

$$\vec{J}P \cong L(\pi^*(TM), (P \times \mathfrak{g}_0)) \cong \pi^*(T^*M) \otimes (P \times \mathfrak{g}_0), \tag{49}$$

and

$$\vec{J}(P \times_{G_0} Q) \cong L(\pi^*(TM), P \times_{G_0} TQ) \cong \pi^*(T^*M) \otimes (P \times_{G_0} TQ), \tag{50}$$

in the same sense. The statement is then that these bundle isomorphisms are equivariant under the action of the gauge groupoid G, in the first two cases, and of the Lie groupoid $GL(TM) \times_M G$, in the last two cases.

For the proof, we need only consider the statements for the vertical bundles, since the corresponding ones for the linearized jet bundles follow directly from them by combining the corresponding actions of the gauge groupoid with that of the linear frame groupoid $GL(TM)$ of the base manifold M on the cotangent bundle T^*M of M. To this end, consider the fundamental vector fields $(X_0)_P$ on P associated to the generators $X_0 \in \mathfrak{g}_0$ through the right action of G_0 on P, and for later use, also the fundamental vector fields $(X_0)_Q$ on Q associated to the generators $X_0 \in \mathfrak{g}_0$ through the left action of G_0 on Q, defined by

$$(X_0)_P(p) = \frac{d}{dt}\big(p \cdot \exp(tX_0)\big)\Big|_{t=0}, \tag{51}$$

and by

$$(X_0)_Q(q) = \frac{d}{dt}\big(\exp(-tX_0) \cdot q\big)\Big|_{t=0}, \tag{52}$$

respectively.[4] Then the isomorphism in equation (47) is given by the mapping that takes the pair (p, X_0) to the vertical vector $(X_0)_P(p)$, and that this is equivariant follows immediately from the following simple calculation:

$$
\begin{aligned}
[p', p] \cdot (X_0)_P(p) &= T_p L_{[p',p]}\Big(\frac{d}{dt}\big(p \cdot \exp(tX_0)\big)\Big|_{t=0}\Big) = \frac{d}{dt}\big([p', p] \cdot (p \cdot \exp(tX_0))\big)\Big|_{t=0} \\
&= \frac{d}{dt}\big([p' \cdot \exp(tX_0), p \cdot \exp(tX_0)] \cdot (p \cdot \exp(tX_0))\big)\Big|_{t=0} \\
&= \frac{d}{dt}\big(p' \cdot \exp(tX_0)\big)\Big|_{t=0} = (X_0)_P(p').
\end{aligned}
$$

Similarly, the isomorphism in equation (48) is given by the mapping (momentarily denoted by ϕ) that takes $[p, \frac{d}{dt}q(t)|_{t=0}] \in (P \times_{G_0} TQ)_{[p,q]}$ to $\frac{d}{dt}[p, q(t)]|_{t=0} \in V_{[p,q]}(P \times_{G_0} Q)$, and that this is equivariant follows immediately from the following simple calculation:

$$
\begin{aligned}
[p', p] \cdot \phi\big([p, \tfrac{d}{dt}q(t)\big|_{t=0}]\big) &= [p', p] \cdot \Big(\frac{d}{dt}[p, q(t)]\Big|_{t=0}\Big) = T_p L_{[p',p]}\Big(\frac{d}{dt}[p, q(t)]\Big|_{t=0}\Big) \\
&= \frac{d}{dt}\big([p', p] \cdot [p, q(t)]\big)\Big|_{t=0} = \frac{d}{dt}[p', q(t)]\Big|_{t=0} = \phi\big([p', \tfrac{d}{dt}q(t)\big|_{t=0}]\big) \\
&= \phi\big([p', p] \cdot [p, \tfrac{d}{dt}q(t)\big|_{t=0}]\big).
\end{aligned}
$$

Similar simplifications occur for the other induced actions considered in the previous section, and this will be discussed in the next two subsections.

---

[4] We recall that the correspondence in equation (51) establishes a canonical linear isomorphism between the Lie algebra $\mathfrak{g}_0$ and the vertical space V_pP of P at p, whereas the extra minus sign in equation (52) is introduced merely for convenience, so as to guarantee consistency of the formulas when we switch between left and right actions.

3.2. First order jet groupoids and induced actions

To begin with, we apply the general procedure developed in Ref. [2] of "differentiating" actions of Lie groupoids on fiber bundles to the natural actions of the gauge groupoid $G = (P \times P)/G_0$ on the principal bundle P itself and on any associated bundle $P \times_{G_0} Q$ to obtain natural induced actions

$$
\Phi_{JP}: \quad
\begin{aligned}
JG \times_M JP &\longrightarrow & JP \\
(u_{[p',p]}, u_p) &\longmapsto & u_{[p',p]} \cdot u_p
\end{aligned}
\tag{53}
$$

and

$$
\Phi_{J(P\times_{G_0}Q)}: \quad
\begin{aligned}
JG \times_M J(P \times_{G_0} Q) &\longrightarrow & J(P \times_{G_0} Q) \\
(u_{[p',p]}, u_{[p,q]}) &\longmapsto & u_{[p',p]} \cdot u_{[p,q]}
\end{aligned}
\tag{54}
$$

derived from the actions Φ_P in equation (34) and $\Phi_{P \times_{G_0} Q}$ in equation (43) by applying the general formula in equation (16) of the previous section.

A more profound understanding of the situation can be obtained by extending the "magical square" for associated bundles in equation (45) to the corresponding jet bundles, considering the commutative diagram

$$
\begin{array}{ccc}
J(P \times Q) & \xrightarrow{\ J\rho_Q\ } & J(P \times_{G_0} Q) \\
{\scriptstyle \pi_{J(P\times Q)}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{J(P\times_{G_0}Q)}} \\
P \times Q & \xrightarrow{\ \rho_Q\ } & P \times_{G_0} Q \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
P & \xrightarrow{\ \rho\ } & M
\end{array}
\tag{55}
$$

and noting that, just like there is a natural action of G on $P \times Q$ derived from that on P such that ρ_Q is an isomorphism on each fiber and is G-equivariant, as discussed in the previous subsection, there is also a natural action of JG on $J(P \times Q)$ derived from that on JP such that $J\rho_Q$, although no longer an isomorphism on each fiber (it is still onto but has a kernel), is JG-equivariant.[5]

To prove these statements, let us pick points $p \in P$ and $q \in Q$ with $\rho(p) = x$ and take tangent maps to the commutative diagram in equation (45) to obtain the commutative diagram

$$
\begin{array}{ccc}
T_p P \oplus T_q Q & \xrightarrow{\ T_{(p,q)}\rho_Q\ } & T_{[p,q]}(P \times_{G_0} Q) \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle T_{[p,q]}\pi} \\
T_p P & \xrightarrow[\ T_p\rho\ ]{} & T_x M
\end{array}
\tag{56}
$$

Since ρ_Q is a submersion and hence its tangent maps are surjective, this means that the tangent spaces $T_{[p,q]}(P \times_{G_0} Q)$ of the orbit space $P \times_{G_0} Q$ can be realized as quotient spaces, namely, the linear maps

$$
T_{(p,q)}\rho_Q : T_p P \oplus T_q Q \longrightarrow T_{[p,q]}(P \times_{G_0} Q)
\tag{57}
$$

induce isomorphisms

$$
T_{[p,q]}(P \times_{G_0} Q) \cong (T_p P \oplus T_q Q)/\ker T_{(p,q)}\rho_Q \,,
\tag{58}
$$

and noting that

$$
J_{(p,q)}(P \times Q) = J_p P \oplus L(T_x M, T_q Q),
\tag{59}
$$

this leads to an analogous realization of the jet spaces $J_{[p,q]}(P \times_{G_0} Q)$ of the orbit space $P \times_{G_0} Q$ as quotient spaces, namely, the affine maps

$$
J_{(p,q)}\rho_Q : J_p P \oplus L(T_x M, T_q Q) \longrightarrow J_{[p,q]}(P \times_{G_0} Q)
\tag{60}
$$

---

5 Note that here, $J(P \times Q)$ is meant to be the jet bundle of $P \times Q$ as a bundle over M, i.e., with respect to the projection $\rho \circ \mathrm{pr}_1$, whereas the previous statement that $P \times Q$ is a trivial bundle refers to its structure as a bundle over P, i.e., to the projection pr_1.

defined by

$$J_{(p,q)}\rho_Q(u_p, u_q) = T_{(p,q)}\rho_Q \circ (u_p, u_q) \tag{61}$$

induce isomorphisms

$$J_{[p,q]}(P \times_{G_0} Q) \cong (J_p P \oplus L(T_x M, T_q Q)) / L(T_x M, \ker T_{(p,q)}\rho_Q). \tag{62}$$

Now using the G-equivariance of ρ_Q, which means that $\Phi_{P \times_{G_0} Q} \circ (\mathrm{id}_G \times_M \rho_Q) = \rho_Q \circ \Phi_{P \times Q} = \rho_Q \circ (\Phi_P \times \mathrm{id}_Q)$ (where in the last equality we have applied the identity $G \times_M (P \times Q) = (G \times_M P) \times Q$), we can prove the JG-equivariance of $J\rho_Q$. To this end, let us also pick a point $[p', p] \in G$, a jet $u_{[p',p]} \in J_{[p',p]}G$ and another jet $u_p \in J_p P$ together with a linear map $u_q \in L(T_x M, T_q Q)$, and calculate

$$u_{[p',p]} \cdot J_{(p,q)}\rho_Q(u_p, u_q)$$

$$= T_{([p',p],[p,q])}\Phi_{P \times_{G_0} Q} \circ (u_{[p',p]}, T_{(p,q)}\rho_Q \circ (u_p, u_q)) \circ \pi_{JG}^{\mathrm{fr}}(u_{[p',p]})^{-1}$$

$$= T_{([p',p],[p,q])}\Phi_{P \times_{G_0} Q} \circ T_{([p',p],(p,q))}(\mathrm{id}_G \times_M \rho_Q) \circ (u_{[p',p]}, (u_p, u_q)) \circ \pi_{JG}^{\mathrm{fr}}(u_{[p',p]})^{-1}$$

$$= T_{(p',q)}\rho_Q \circ T_{([p',p],(p,q))}\Phi_{P \times Q} \circ (u_{[p',p]}, (u_p, u_q)) \circ \pi_{JG}^{\mathrm{fr}}(u_{[p',p]})^{-1}$$

$$= J_{(p',q)}\rho_Q(u_{[p',p]} \cdot (u_p, u_q)) = J_{(p',q)}\rho_Q(u_{[p',p]} \cdot u_p, u_q).$$

For later use, we also note that

$$\ker T_{(p,q)}\rho_Q = \{((X_0)_P(p), (X_0)_Q(q)) \mid X_0 \in \mathfrak{g}_0\} \cong \mathfrak{g}_0 \cong V_p P, \tag{63}$$

where $(X_0)_P$ and $(X_0)_Q$ denote the fundamental vector fields on P and on Q associated to a generator $X_0 \in \mathfrak{g}_0$ via the pertinent actions of G_0, respectively, as defined in equations (51) and (52) above. Moreover, under the projection $T_{(p,q)}\rho_Q$, the vertical spaces of the principal bundle P and of the associated bundle $P \times_{G_0} Q$ are related by

$$V_{[p,q]}(P \times_{G_0} Q) \cong (V_p P \oplus T_q Q) / \ker T_{(p,q)}\rho_Q, \tag{64}$$

while, with respect to any principal connection in P and its associated connection in $P \times_{G_0} Q$, the corresponding horizontal spaces of the principal bundle P and of the associated bundle $P \times_{G_0} Q$ are related by

$$H_{[p,q]}(P \times_{G_0} Q) \cong (H_p P \oplus \{0\}) / \ker T_{(p,q)}\rho_Q. \tag{65}$$

At the end of this subsection, we shall see how to express the correspondence between principal connections in P and their associated connections in $P \times_{G_0} Q$ in terms of jets.

Another important property of the action of JG on JP in equation (53) is that it commutes with the right action of the structure group G_0 on JP: this is essentially obvious because they are induced from an action of G on P and a right action of G_0 on P which commute. But since this is an important fact, let us give a quick formal proof of the pertinent formula,

$$u_{[p',p]} \cdot (w_p \cdot g_0) = (u_{[p',p]} \cdot w_p) \cdot g_0. \tag{66}$$

Indeed, according to equations (16) and (36) (the second of which can be reformulated as stating that $\Phi_P \circ (\mathrm{id}_G \times R_{g_0}) = R_{g_0} \circ \Phi_P$, where R_{g_0} denotes right translation by g_0 in P),

$$u_{[p',p]} \cdot (w_p \cdot g_0) = T_{([p',p],p\cdot g_0)}\Phi_P \circ (u_{[p',p]}, T_p R_{g_0} \circ w_p) \circ \pi_{JG}^{\mathrm{fr}}(u_{[p',p]})^{-1}$$

$$= T_{([p',p],p\cdot g_0)}\Phi_P \circ T_{([p',p],p)}(\mathrm{id}_G \times_M R_{g_0}) \circ (u_{[p',p]}, w_p) \circ \pi_{JG}^{\mathrm{fr}}(u_{[p',p]})^{-1}$$

$$= T_p R_{g_0} \circ T_{([p',p],p)}\Phi_P \circ (u_{[p',p]}, w_p) \circ \pi_{JG}^{\mathrm{fr}}(u_{[p',p]})^{-1}$$

$$= (u_{[p',p]} \cdot w_p) \cdot g_0.$$

This implies that the action Φ_{JP} of JG on JP in equation (53) passes to the quotient

$$CP = JP/G_0, \tag{67}$$

which is an affine bundle over M called the *connection bundle* of P because its sections correspond precisely to the G_0-equivariant sections of JP (as an affine bundle over P), which are exactly the principal connections on P. Thus we get a natural induced action

$$
\begin{array}{rccc}
\Phi_{CP}: & JG \times_M CP & \longrightarrow & CP \\
& (u_{[p',p]}, [w_p]) & \longmapsto & u_{[p',p]} \cdot [w_p]
\end{array}
\tag{68}
$$

of JG on CP. It will be convenient to visualize this construction in terms of the "magical square" for connection bundles, i.e., the commutative diagram

$$
\begin{array}{ccc}
JP & \xrightarrow{\;\rho_C\;} & CP \\
{\scriptstyle \pi_{JP}} \big\downarrow & & \big\downarrow {\scriptstyle \pi_{CP}} \\
P & \xrightarrow[\;\rho\;]{} & M
\end{array}
\tag{69}
$$

in which the horizontal projections define principal G_0-bundles while the vertical projections provide affine bundles such that ρ_C is an isomorphism on each fiber and, by definition, is JG-equivariant.

Now we can formulate the rule that to each principal connection in P assigns its associated connection in $P \times_{G_0} Q$ in terms of a canonical bundle map over $P \times_{G_0} Q$, namely:

$$
\begin{array}{rccc}
\pi^*(CP) & \longrightarrow & J(P \times_{G_0} Q) \\
([p,q],[w_p]) & \longmapsto & J_{(p,q)}\rho_Q(w_p, 0)
\end{array}
\tag{70}
$$

To see that it is well defined, we have to check that, given any point $x \in M$, the result remains unchanged if we pick any $g_0 \in G_0$ to replace the representative $(p,q) \in (P \times Q)_x$ of $[p,q] \in (P \times_{G_0} Q)_x$ by another representative $(p \cdot g_0, g_0^{-1} \cdot q)$ and the representative $w_p \in J_p P$ of $[w_p] \in C_x P$ by another representative $w_{p \cdot g_0}$: writing $R^P_{g_0}$ for right translation by g_0 in P and $L^Q_{g_0^{-1}}$ for left translation by g_0^{-1} in Q, we have $w_{p \cdot g_0} = T_p R^P_{g_0} \circ w_p$ and get

$$
\begin{aligned}
J_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q(w_{p \cdot g_0}, 0) &= T_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q \circ \big(T_p R^P_{g_0} \circ w_p, 0\big) \\
&= T_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q \circ T_{(p,q)}\big(R^P_{g_0} \times L^Q_{g_0^{-1}}\big) \circ (w_p, 0) \\
&= T_{(p,q)}\rho_Q \circ (w_p, 0) = J_{(p,q)}\rho_Q(w_p, 0).
\end{aligned}
$$

Moreover, this bundle map is also JG-equivariant: this follows trivially from the definition of the action of JG on the spaces involved and the JG-equivariance of $J\rho_Q$ that was proved above. And finally, we observe that this bundle map does capture the essence of passing from a principal connection to its associated connection, since if the former is given by a section $\Gamma^P : M \longrightarrow CP$ and the latter by a section $\Gamma^{P \times_{G_0} Q} : P \times_{G_0} Q \longrightarrow J(P \times_{G_0} Q)$, then $\Gamma^{P \times_{G_0} Q}$ is simply the push-forward of the section $\Gamma^P \circ \pi : P \times_{G_0} Q \longrightarrow \pi^*(CP)$ with this bundle map. Note also that the prescription corresponds precisely to that given in equation (65) at the level of horizontal bundles.

3.3. Second order jet groupoids and induced actions

In this subsection, we apply the general procedure developed in Ref. [2] of "differentiating" actions of Lie groupoids on fiber bundles once more, namely, to the natural actions of the jet groupoid JG of the gauge groupoid $G = (P \times P)/G_0$ on the jet bundle JP and the connection bundle CP of the principal bundle P itself, to obtain natural induced actions [6]

$$
\begin{array}{rccc}
\Phi_{J(JP)}: & J(JG) \times_M J(JP) & \longrightarrow & J(JP) \\
& (u'_{u_g}, u'_{u_p}) & \longmapsto & u'_{u_g} \cdot u'_{u_p}
\end{array}
\tag{71}
$$

and

$$
\begin{array}{rccc}
\Phi_{J(CP)}: & J(JG) \times_M J(CP) & \longrightarrow & J(CP) \\
& (u'_{u_g}, u_{[w_p]}) & \longmapsto & u'_{u_g} \cdot u_{[w_p]}
\end{array}
\tag{72}
$$

derived from the actions Φ_{JP} in equation (53) and Φ_{CP} in equation (68) by applying the general formula in equation (16) of the previous section. Explicitly, we have

$$
u'_{u_g} \cdot u'_{u_p} = T_{(u_g,u_p)}\Phi_{JP} \circ (u'_{u_g}, u'_{u_p}) \circ \pi^{\mathrm{fr}}_{J(JG)}(u'_{u_g})^{-1},
\tag{73}
$$

and

$$
u'_{u_g} \cdot u'_{[w_p]} = T_{(u_g,[w_p])}\Phi_{CP} \circ (u'_{u_g}, u'_{[w_p]}) \circ \pi^{\mathrm{fr}}_{J(JG)}(u'_{u_g})^{-1},
\tag{74}
$$

---

[6] In this subsection, we often write $g = [p', p]$ for points in the gauge groupoid $G = (P \times P)/G_0$.

respectively. These actions admit restrictions to several subgroupoids and subbundles, among which the following will become important to us at some point or another: the natural induced actions

$$
\begin{aligned}
\Phi_{\bar{J}^2 P}: \quad \bar{J}^2 G \times_M \bar{J}^2 P &\longrightarrow \bar{J}^2 P \\
(u'_{u_g}, u'_{u_p}) &\longmapsto u'_{u_g} \cdot u'_{u_p}
\end{aligned}
\tag{75}
$$

of the semiholonomous second order jet groupoid $\bar{J}^2 G$ of G and

$$
\begin{aligned}
\Phi_{\bar{J}^2 P}: \quad J^2 G \times_M \bar{J}^2 P &\longrightarrow \bar{J}^2 P \\
(u'_{u_g}, u'_{u_p}) &\longmapsto u'_{u_g} \cdot u'_{u_p}
\end{aligned}
\tag{76}
$$

of the second order jet groupoid $J^2 G$ of G on the semiholonomous second order jet bundle $\bar{J}^2 P$ of P, as well as the action

$$
\begin{aligned}
\Phi_{J^2 P}: \quad J^2 G \times_M J^2 P &\longrightarrow J^2 P \\
(u'_{u_g}, u'_{u_p}) &\longmapsto u'_{u_g} \cdot u'_{u_p}
\end{aligned}
\tag{77}
$$

of the second order jet groupoid $J^2 G$ of G on the second order jet bundle $J^2 P$ of P, all defined by the same formula,

$$
u'_{u_g} \cdot u'_{u_p} = T_{(u_g, u_p)} \Phi_{JP} \circ (u'_{u_g}, u'_{u_p}) \circ \pi^{\mathrm{fr}}_{JG}(u_g)^{-1},
\tag{78}
$$

and similarly, the natural induced actions

$$
\begin{aligned}
\Phi_{J(CP)}: \quad \bar{J}^2 G \times_M J(CP) &\longrightarrow J(CP) \\
(u'_{u_g}, u_{[w_p]}) &\longmapsto u'_{u_g} \cdot u_{[w_p]}
\end{aligned}
\tag{79}
$$

of the semiholonomous second order jet groupoid $\bar{J}^2 G$ of G and

$$
\begin{aligned}
\Phi_{J(CP)}: \quad J^2 G \times_M J(CP) &\longrightarrow J(CP) \\
(u'_{u_g}, u_{[w_p]}) &\longmapsto u'_{u_g} \cdot u_{[w_p]}
\end{aligned}
\tag{80}
$$

of the second order jet groupoid $J^2 G$ of G on the jet bundle $J(CP)$ of the connection bundle CP of P, defined by

$$
u'_{u_g} \cdot u_{[w_p]} = T_{(u_g, [w_p])} \Phi_{CP} \circ (u'_{u_g}, u_{[w_p]}) \circ \pi^{\mathrm{fr}}_{JG}(u_g)^{-1}.
\tag{81}
$$

As noted in the discussion preceding Proposition 2 in the previous section, the simplification in the last term on the rhs of equations (78) and (81), as compared to equations (73) and (74), comes from the assumption that u'_{u_g} is semiholonomous, and the definition of the actions in equations (75) and (77) relies on the fact that when u'_{u_g} and u'_{u_p} are both semiholonomous or both holonomous, then so is $u'_{u_g} \cdot u'_{u_p}$.

A more profound understanding of the situation can be obtained by extending the "magical square" for connection bundles in equation (69) to the corresponding jet bundles, considering the commutative diagram

$$
\begin{array}{ccc}
J(JP) & \xrightarrow{\ J\rho_C\ } & J(CP) \\
{\scriptstyle \pi_{J(JP)}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{J(CP)}} \\
JP & \xrightarrow{\ \rho_C\ } & CP \\
{\scriptstyle \pi_{JP}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{CP}} \\
P & \xrightarrow{\ \rho\ } & M
\end{array}
\tag{82}
$$

and noting that $J\rho_C$, although no longer an isomorphism on each fiber (it is still onto but has a kernel), is $J(JG)$-equivariant. Even more importantly, by restricting to the semiholonomous second order jet bundle of P, we arrive at a "magical square" for jet bundles of connection bundles, i.e., the commutative diagram

$$
\begin{array}{ccc}
\bar{J}^2 P & \xrightarrow{\ J\rho_C\ } & J(CP) \\
{\scriptstyle \pi_{\bar{J}^2 P}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{J(CP)}} \\
JP & \xrightarrow{\ \rho_C\ } & CP \\
{\scriptstyle \pi_{JP}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{CP}} \\
P & \xrightarrow{\ \rho\ } & M
\end{array}
\tag{83}
$$

in which all three horizontal projections define principal G_0-bundles while the vertical projections provide affine bundles such that ρ_C and $J\rho_C$ are both isomorphisms on each fiber, ρ_C is JG-equivariant and $J\rho_C$ is $\bar{J}^2 G$-equivariant.

To prove these statements, let us pick a point $p \in P$ with $\rho(p) = x$ and a jet $w_p \in J_p P$ and take tangent maps to the commutative diagram in equation (69) to obtain the commutative diagram

$$
\begin{array}{ccc}
T_{w_p}(JP) & \xrightarrow{\;T_{w_p}\rho_C\;} & T_{[w_p]}(CP) \\[2pt]
{\scriptstyle T_{w_p}\pi_{JP}}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{[w_p]}\pi_{CP}} \\[2pt]
T_p P & \xrightarrow[\;T_p\rho\;]{} & T_x M
\end{array}
\tag{84}
$$

Since ρ_C is a submersion and hence its tangent maps are surjective, this means that the tangent spaces $T_{[w_p]}(CP)$ of the orbit space CP can be realized as quotient spaces, namely, the linear maps

$$
T_{w_p}\rho_C : T_{w_p}(JP) \;\longrightarrow\; T_{[w_p]}(CP)
\tag{85}
$$

induce isomorphisms

$$
T_{[w_p]}(CP) \;\cong\; T_{w_p}(JP)/\ker T_{w_p}\rho_C \,,
\tag{86}
$$

and this leads to an analogous realization of the jet spaces $J_{[w_p]}(CP)$ of the orbit space CP as quotient spaces, namely, the affine maps

$$
J_{w_p}\rho_C : J_{w_p}(JP) \;\longrightarrow\; J_{[w_p]}(CP)
\tag{87}
$$

defined by

$$
J_{w_p}\rho_C(u'_{w_p}) = T_{w_p}\rho_C \circ u'_{w_p}
\tag{88}
$$

induce isomorphisms

$$
J_{[w_p]}(CP) \;\cong\; J_{w_p}(JP) \,/\, L(T_x M, \ker T_{w_p}\rho_C) \,.
\tag{89}
$$

Now using the JG-equivariance of ρ_C, which means that $\Phi_{CP} \circ (\mathrm{id}_{JG} \times_M \rho_C) = \rho_C \circ \Phi_{JP}$, we can prove the $J(JG)$-equivariance of $J\rho_C$. To this end, let us also pick a point $g = [p', p] \in G$ and a jet $u_g \in J_g G$, together with iterated jets $u'_{u_g} \in J_{u_g}(JG)$ and $u'_{w_p} \in J_{w_p}(JP)$, and calculate

$$
\begin{aligned}
& u'_{u_g} \cdot J_{w_p}\rho_C(u'_{w_p}) \\[4pt]
&= T_{(u_g,[w_p])}\Phi_{CP} \circ \big(u'_{u_g}, T_{w_p}\rho_C \circ u'_{w_p}\big) \circ \pi^{\mathrm{fr}}_{J(JG)}(u'_{u_g})^{-1} \\[4pt]
&= T_{(u_g,[w_p])}\Phi_{CP} \circ T_{(u_g,w_p)}(\mathrm{id}_{JG} \times_M \rho_C) \circ \big(u'_{u_g}, u'_{w_p}\big) \circ \pi^{\mathrm{fr}}_{J(JG)}(u'_{u_g})^{-1} \\[4pt]
&= T_{u_g \cdot w_p}\rho_C \circ T_{(u_g,w_p)}\Phi_{JP} \circ \big(u'_{u_g}, u'_{w_p}\big) \circ \pi^{\mathrm{fr}}_{J(JG)}(u'_{u_g})^{-1} \\[4pt]
&= J_{u_g \cdot w_p}\rho_C \big(u'_{u_g} \cdot u'_{u_p}\big) \,.
\end{aligned}
$$

But here we can actually do better if we replace iterated jets by semiholonomous second order jets because that will eliminate the need of passing to a quotient and convert the commutative diagram in equation (82) to the one in equation (83). To show this, we first note that, as before,

$$
\ker T_{w_p}\rho_C = \{ (X_0)_{JP}(w_p) \,|\, X_0 \in \mathfrak{g}_0 \} \;\cong\; \mathfrak{g}_0 \;\cong\; V_p P \,,
\tag{90}
$$

where $(X_0)_{JP}$ denotes the fundamental vector field on JP associated to a generator $X_0 \in \mathfrak{g}_0$ via the pertinent action of G_0, defined by the appropriate analogue of equation (51) above. Here, we shall need a more explicit form of this isomorphism between the spaces $\ker T_{w_p}\rho_C$ and $V_p P$: it is simply the restriction

$$
T_{w_p}\pi_{JP} : \ker T_{w_p}\rho_C \;\xrightarrow{\;\cong\;}\; V_p P
\tag{91}
$$

of the linear map

$$
T_{w_p}\pi_{JP} : T_{w_p}(JP) \;\longrightarrow\; T_p P
\tag{92}
$$

that appears in the definition of semiholonomous second order jets. (Indeed, the right action of G_0 on JP being induced from that on P, the tangent map $T_{w_p}\pi_{JP}$ will of course take any fundamental vector field $(X_0)_{JP}$ at w_p to the corresponding fundamental vector field $(X_0)_P$ at p). This in turn implies that the restriction of the (affine) map in equation (87) to the (affine) subspace $\bar{J}^2_{w_p}P$ of the (affine) space $J_{w_p}(JP)$ will establish an isomorphism

$$J_{w_p}\rho_C : \bar{J}^2_{w_p}P \;\overset{\cong}{\longrightarrow}\; J_{[w_p]}(CP) \tag{93}$$

so we can replace equation (89) by the much simpler equation

$$J_{[w_p]}(CP) \cong \bar{J}^2_{w_p}P . \tag{94}$$

To prove this statement, we have to show that the affine map in equation (87), when restricted to the affine subspace $\bar{J}^2_{w_p}P$, (a) becomes injective and (b) remains surjective. For (a), assume we are given two semiholonomous second order jets $u'^1_{w_p}, u'^2_{w_p} \in \bar{J}^2_{w_p}P$ which under $J_{w_p}\rho_C$ have the same image; then their difference is a linear map from T_xM to $T_{w_p}(JP)$ satisfying two conditions, namely that its composition with $T_{w_p}\rho_C$ is zero, so it takes value in $\ker T_{w_p}\rho_C$, and that its composition with $T_{w_p}\pi_{JP}$ is also zero, since $u'^1_{w_p}$ and $u'^2_{w_p}$ are both semiholonomous. But this implies that it must itself be zero since according to equation (91), $T_{w_p}\pi_{JP}$ is injective on $\ker T_{w_p}\rho_C$. For (b), assume we are given a general iterated jet $u'_{w_p} \in J_{w_p}(JP)$ and consider the difference $T_{w_p}\pi_{JP} \circ u'_{w_p} - w_p$, which is a linear map from T_xM to V_pP, so that according to equation (91), there is a unique linear map $\vec{u}'_{w_p}$ from T_xM to $\ker T_{w_p}\rho_C \subset T_{w_p}(JP)$ satisfying $T_{w_p}\pi_{JP} \circ u'_{w_p} - w_p = T_{w_p}\pi_{JP} \circ \vec{u}'_{w_p}$. But this implies that the difference $\bar{u}'_{w_p} = u'_{w_p} - \vec{u}'_{w_p}$ is a semiholonomous second order jet, $\bar{u}'_{w_p} \in \bar{J}^2_{w_p}P$, which under $J_{w_p}\rho_C$ has the same image as the original iterated jet $u'_{w_p} \in J_{w_p}(JP)$.

4. Minimal coupling and Utiyama's theorem

In the context of the formalism adopted in the previous section, the minimal coupling prescription and the curvature map can be viewed as stemming from bundle maps

$$D : CP \times_M J(P \times_{G_0} Q) \;\longrightarrow\; \vec{J}(P \times_{G_0} Q) , \tag{95}$$

and

$$F : J(CP) \;\longrightarrow\; \textstyle\bigwedge^2 T^*M \otimes (P \times_{G_0} \mathfrak{g}_0) , \tag{96}$$

over M, which have already appeared in Ref. [4] (see the diagrams in equations (52) and (57) there). What we want to show here is that, and in precisely what sense, these bundle maps are equivariant under the action not only of the pertinent Lie group bundles but also of the pertinent Lie groupoids. To this end, it turns out to be convenient to "lift" all bundles to the space appearing in the upper left hand corner of the appropriate "magical square", that is, the space $P \times Q$ in the first case (see equation (45)) and the space JP in the second case (see equation (69)), where these bundle maps take a much simpler form.

4.1. Minimal coupling

To deal with the minimal coupling prescription, we observe that the bundle map D in equation (95) fits into the following commutative diagram

$$
\begin{array}{ccc}
(JP \times Q) \times_{P \times Q} J(P \times Q) & \overset{D}{\longrightarrow} & \vec{J}(P \times Q) \\
{\scriptstyle (\rho_C \circ \mathrm{pr}_1, J\rho_Q)} \Big\downarrow & & \Big\downarrow {\scriptstyle \vec{J}\rho_Q} \\
CP \times_M J(P \times_{G_0} Q) & \underset{D}{\longrightarrow} & \vec{J}(P \times_{G_0} Q)
\end{array}
\tag{97}
$$

where the bundles in the top row are over $P \times Q$ while those in the bottom row are over M. (Here, we have identified the pull-back of JP by the projection from $P \times Q$ to P with the cartesian product $JP \times Q$). In fact, it is convenient to expand this to a commutative diagram

$$
\begin{array}{ccc}
(JP \times Q) \times_{P \times Q} J(P \times Q) & \xrightarrow{\ D\ } & \vec{J}(P \times Q) \\
\Big\downarrow & & \Big\downarrow \\
\pi^*(CP) \times_{P \times_{G_0} Q} J(P \times_{G_0} Q) & \xrightarrow{\ D\ } & \vec{J}(P \times_{G_0} Q) \\
\Big\downarrow & & \Big\downarrow \\
CP \times_M J(P \times_{G_0} Q) & \xrightarrow{\ D\ } & \vec{J}(P \times_{G_0} Q)
\end{array}
\tag{98}
$$

where the bundles in the middle row are over the quotient space $P \times_{G_0} Q$, i.e., the total space of the corresponding associated bundle. (Here, we omit the labels on the vertical maps, which are either the same as in the previous diagram or else are obvious.) Then the bundle map D in the middle row is the composition of the difference map already introduced at the beginning of this paper (see equation (3)) and the canonical bundle map of equation (70) in the first factor, up to a sign that can be taken care of by switching the two factors. Continuing to use the same notation as in Section 3.1, we see that this corresponds to the bundle map D in the bottom row being given in terms of that in the top row according to

$$
D_x\big([w_p], J_{(p,q)}\rho_Q(u_p, u_q)\big) = \vec{J}_{(p,q)}\rho_Q\big(D_{(p,q)}\big(w_p, (u_p, u_q)\big)\big),
\tag{99}
$$

whereas the latter is simply defined by

$$
D_{(p,q)}\big(w_p, (u_p, u_q)\big) = (u_p - w_p, u_q).
\tag{100}
$$

(This follows from equations (59)–(62) together with the same equations with J replaced by $\vec{J}$). To show that D_x is well defined, note first that if we replace the point p in P by another point in P in the same fiber over x, which is of the form $p \cdot g_0$ for some (unique) $g_0 \in G$, then we must replace w_p by $w_p \cdot g_0 = T_p R_{g_0} \circ w_p$ and similarly u_p by $u_p \cdot g_0 = T_p R_{g_0} \circ u_p$, as well as u_q by $u_q \cdot g_0 = T_q L_{g_0^{-1}} \circ u_q$, so as to guarantee that $J_{(p,q)}\rho_Q(u_p, u_q)$ remains unaltered:

$$
J_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q\big(u_p \cdot g_0, u_q \cdot g_0\big) = T_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q \circ T_{(p,q)}\big(R_{g_0} \times L_{g_0^{-1}}\big) \circ (u_p, u_q)
$$

$$
= T_{(p,q)}\big(\rho_Q \circ \big(R_{g_0} \times L_{g_0^{-1}}\big)\big) \circ (u_p, u_q) = T_{(p,q)}\rho_Q \circ (u_p, u_q)
$$

$$
= J_{(p,q)}\rho_Q(u_p, u_q).
$$

But then $\vec{J}_{(p,q)}\rho_Q(u_p - w_p, u_q)$ will remain unaltered as well:

$$
\vec{J}_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q\big((u_p - w_p) \cdot g_0, u_q \cdot g_0\big) = T_{(p \cdot g_0, g_0^{-1} \cdot q)}\rho_Q \circ T_{(p,q)}\big(R_{g_0} \times L_{g_0^{-1}}\big) \circ (u_p - w_p, u_q)
$$

$$
= T_{(p,q)}\big(\rho_Q \circ \big(R_{g_0} \times L_{g_0^{-1}}\big)\big) \circ (u_p - w_p, u_q) = T_{(p,q)}\rho_Q \circ (u_p - w_p, u_q)
$$

$$
= \vec{J}_{(p,q)}\rho_Q(u_p - w_p, u_q).
$$

Moreover, even if we leave p fixed, we may still modify the second component in the argument of $D_{(p,q)}$, i.e., the pair $(u_p, u_q) \in J_p P \times L(T_x M, T_q Q)$, without changing its image under $J_{(p,q)}\rho_Q$, namely, by adding a pair $(\vec{u}_p, \vec{u}_q) \in L(T_x M, \ker T_{(p,q)}\rho_Q)$. But then since $w_p \in J_p P$ remains unaltered, the expression $(u_p - w_p, u_q) \in \vec{J}_p P \times L(T_x M, T_q Q)$ will be modified in the same way and, in particular, without changing its image under $\vec{J}_{(p,q)}\rho_Q$.

Now we are ready to formulate the first main theorem in this paper, which extends the left part of the commutative diagram in equation (52) of Ref. [4], as follows.

Theorem 1. *The minimal coupling map D in equation (95) is equivariant under the actions of the pertinent Lie groupoids, i.e., the diagram*

$$
\begin{array}{ccc}
JG \times_M (CP \times_M J(P \times_{G_0} Q)) & \longrightarrow & CP \times_M J(P \times_{G_0} Q) \\
{\scriptstyle (\pi_{JG}^{\mathrm{fr}} \times \pi_{JG}) \times_M D}\Big\downarrow & & \Big\downarrow {\scriptstyle D} \\
(GL(TM) \times_M G) \times_M \vec{J}(P \times_{G_0} Q) & \longrightarrow & \vec{J}(P \times_{G_0} Q)
\end{array}
\tag{101}
$$

commutes.

Proof. This follows immediately from equivariance of $J\rho_Q$ under JG (which as we have seen implies equivariance of the canonical bundle map in equation (70) under JG) and equivariance of $\vec{J}\rho_Q$ under $GL(TM) \times_M G$ (which can be shown in

precisely the same way), in combination with Proposition 1, to prove that the bundle maps D in the top and middle rows of the diagram in equation (98) are equivariant in the same sense, the former obviously being equivariant under the right action of G_0 as well. □

To complete the discussion, let us specify in what sense the map D in equation (95) captures the essence of the minimal coupling prescription. Abbreviating $P \times_{G_0} Q$ to E, assume that $\Gamma : M \longrightarrow CP$ is a section of CP representing a principal connection in P, $\Gamma^E : E \longrightarrow JE$ is the section of JE (as a bundle over E) representing the resulting associated connection in E, obtained by push-forward with the canonical bundle map in equation (70), $\varphi : M \longrightarrow E$ is a section of E and $\partial\varphi : M \longrightarrow JE$ is its derivative (also denoted by $j\varphi$ and called its jet prolongation); then $D \circ (\Gamma, \partial\varphi) : M \longrightarrow \bar{J}E$ is indeed the covariant derivative of φ with respect to that connection, because it is elementary to see that equation (99) combined with equation (100) will boil down to the formula in equation (2).

4.2. Utiyama's theorem

To deal with the curvature map, we observe that the bundle map F of equation (96) fits into the following commutative diagram

$$
\begin{array}{ccc}
\bar{J}^2 P & \xrightarrow{\ F\ } & \pi_{JP}^*\Big(\rho^*(\bigwedge^2 T^*M) \otimes VP\Big) \\
{\scriptstyle J\rho_C}\Big\downarrow & & \Big\downarrow \\
J(CP) & \xrightarrow{\ F\ } & \bigwedge^2 T^*M \otimes (P \times_{G_0} \mathfrak{g}_0)
\end{array}
\tag{102}
$$

where the bundles in the upper row are over JP while those in the lower row are over M. (Here, we have identified the vertical bundle VP of P with the trivial vector bundle $P \times \mathfrak{g}_0$ over P; then the second tensor factor in the vertical map on the rhs of this diagram is just the map $\rho_{\mathfrak{g}_0}$ in the "magical square" of equation (45) for the adjoint bundle $P \times_{G_0} \mathfrak{g}_0$, pulled back to JP). Again, it is convenient to expand this to a commutative diagram

$$
\begin{array}{ccc}
\bar{J}^2 P & \xrightarrow{\ F\ } & \pi_{JP}^*\Big(\rho^*(\bigwedge^2 T^*M) \otimes VP\Big) \\
\Big\downarrow & & \Big\downarrow \\
\bar{J}^2 P & \xrightarrow{\ F\ } & \rho^*(\bigwedge^2 T^*M) \otimes VP \\
\Big\downarrow & & \Big\downarrow \\
J(CP) & \xrightarrow{\ F\ } & \bigwedge^2 T^*M \otimes (P \times_{G_0} \mathfrak{g}_0)
\end{array}
\tag{103}
$$

where the bundles in the middle row are over the total space P of the principal bundle. (And again, we omit the labels on the vertical maps, which are either the same as in the previous diagram or else are obvious.) Then the bundle map F in the middle row is the alternator or antisymmetrizer already introduced at the beginning of this paper (see equation (8)). Continuing to use the same notation as in Section 3.2, we see that this corresponds to the bundle map F in the bottom row being given in terms of that in the top row according to

$$
F_x\big(J_{w_p}\rho_C(u'_{w_p})\big)(v_1, v_2) = \rho_{\mathfrak{g}_0}\big(F_{w_p}(u'_{w_p})(v_1, v_2)\big),
\tag{104}
$$

for $v_1, v_2 \in T_x M$, whereas the latter, as we recall from Section 2, is explicitly defined as follows: given a semiholonomous second order jet $u'_{w_p} \in \bar{J}^2_{w_p} P$, we arbitrarily choose some holonomous second order jet $u'^0_{w_p} \in J^2_{w_p} P$ (this choice will ultimately drop out under the antisymmetrization) to form the difference $u'_{w_p} - u'^0_{w_p}$, which is a linear map from $T_x M$ to the vertical space $V^{\mathrm{jt}}_{w_p}(JP)$ of JP with respect to the jet target projection π_{JP}; then we can apply the canonical isomorphism

$$
V^{\mathrm{jt}}_{w_p}(JP) = \ker T_{w_p}\pi_{JP} = T_{w_p}(J_p P) \cong \bar{J}_p P = L(T_x M, V_p P)
\tag{105}
$$

to identify it with a linear map from $T_x M$ to $L(T_x M, V_p P)$, that is, with an element of $L^2(T_x M, V_p P)$, and obtain $F_{w_p}(u'_{w_p}) \in L^2_a(T_x M, V_p P)$ by antisymmetrizing in the usual sense. The last step then consists in applying the additional canonical isomorphism

$$
V_p P \cong \mathfrak{g}_0.
\tag{106}
$$

To show that F_x is well defined, note that if we replace the point p in P by another point in P in the same fiber over x, which is of the form $p \cdot g_0$ for some (unique) $g_0 \in G$, then we must replace w_p by $w_p \cdot g_0 = T_p R^P_{g_0} \circ w_p$, u'_{w_p} by $u'_{w_p} \cdot g_0 =$

$T_{w_p} R^{JP}_{g_0} \circ u'_{w_p}$ and similarly $u'^0_{w_p}$ by $u'^0_{w_p} \cdot g_0 = T_{w_p} R^{JP}_{g_0} \circ u'^0_{w_p}$, where $R^P_{g_0}$ and $R^{JP}_{g_0}$ denote right translation by g_0 in P and in JP, respectively, so as to guarantee that $J_{w_p} \rho_C(u'_{w_p})$ remains unaltered:

$$J_{w_p \cdot g_0} \rho_C(u'_{w_p} \cdot g_0) = T_{w_p \cdot g_0} \rho_C \circ T_{w_p} R^{JP}_{g_0} \circ u'_{w_p} = T_{w_p}(\rho_C \circ R^{JP}_{g_0}) \circ u'_{w_p}$$
$$= T_{w_p} \rho_C \circ u'_{w_p} = J_{w_p} \rho_C(u'_{w_p}).$$

But then

$$u'_{w_p} \cdot g_0 - u'^0_{w_p} \cdot g_0 = T_{w_p} R^{JP}_{g_0} \circ u'_{w_p} - T_{w_p} R^{JP}_{g_0} \circ u'^0_{w_p},$$

so that applying the isomorphism in equation (105), we get

$$\left(u'_{w_p} - u'^0_{w_p}\right) \cdot g_0 = T_p R^P_{g_0} \circ \left(u'_{w_p} - u'^0_{w_p}\right),$$

and applying the additional isomorphism in equation (106), we get

$$\left(u'_{w_p} - u'^0_{w_p}\right) \cdot g_0 = \mathrm{Ad}(g_0^{-1}) \circ \left(u'_{w_p} - u'^0_{w_p}\right),$$

implying that

$$[\, p \cdot g_0 \,, \left(u'_{w_p} - u'^0_{w_p}\right) \cdot g_0\,] = [\, p \,, u'_{w_p} - u'^0_{w_p}\,].$$

(To justify this conclusion, note that the linear isomorphism $T_{w_p} R^{JP}_{g_0} : T_{w_p}(JP) \longrightarrow T_{w_p \cdot g_0}(JP)$, when restricted to the vertical space of JP with respect to the jet target projection π_{JP}, reduces to the tangent map $T_{w_p} R^{JP}_{g_0,p} : T_{w_p}(J_p P) \longrightarrow T_{w_p \cdot g_0}(J_{p \cdot g_0} P)$ of the restricted right translation $R^{JP}_{g_0,p} : J_p P \longrightarrow J_{p \cdot g_0} P$ by g_0. But this is an affine map between affine spaces, so under the isomorphism in equation (105), its tangent map at each point becomes the corresponding difference map, which is a linear map $\vec{R}^{JP}_{g_0,p} : \vec{J}_p P \longrightarrow \vec{J}_{p \cdot g_0} P$, and that is just composition with $T_p R^P_{g_0} : V_p P \longrightarrow V_{p \cdot g_0} P$. Finally, it is well known that under the isomorphism in equation (106), this becomes $\mathrm{Ad}(g_0^{-1}) : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$.)

Now we are ready to formulate the second main theorem in this paper, which extends the left part of the commutative diagram in equation (57) of Ref. [4], as follows.

Theorem 2. *The curvature map F in equation (96) is equivariant under the actions of the pertinent Lie groupoids, i.e., the diagram*

$$
\begin{array}{ccc}
J^2 G \times_M J(CP) & \longrightarrow & J(CP) \\
{\scriptstyle((\pi^{\mathrm{fr}}_{JG} \times \pi_{JG}) \circ \pi_{J^2 G}, F)} \Big\downarrow & & \Big\downarrow {\scriptstyle F} \\
(GL(TM) \times_M G) \times_M \left(\bigwedge^2 T^* M \otimes (P \times_{G_0} \mathfrak{g}_0)\right) & \longrightarrow & \bigwedge^2 T^* M \otimes (P \times_{G_0} \mathfrak{g}_0)
\end{array}
\tag{107}
$$

commutes.

Proof. This follows immediately from Proposition 2, together with the fact that, as shown in Section 3.1, the canonical isomorphism $VP \cong P \times \mathfrak{g}_0$ and the projection $\rho_{\mathfrak{g}_0} : P \times \mathfrak{g}_0 \longrightarrow P \times_{G_0} \mathfrak{g}_0$ are both G-equivariant. □ □

To complete the discussion, let us specify in what sense the map F in equation (96) captures the essence of the prescription for defining the curvature of a principal connection. Assume that $\Gamma : M \longrightarrow CP$ is a section of CP representing a principal connection in P and $\partial \Gamma : M \longrightarrow J(CP)$ is its derivative (also denoted by $j\Gamma$ and called its jet prolongation); then $F \circ \partial \Gamma : M \longrightarrow \bigwedge^2 T^* M \otimes (P \times_{G_0} \mathfrak{g}_0)$ is a 2-form on M with values in the adjoint bundle $P \times_{G_0} \mathfrak{g}_0$ which is precisely the curvature form of that connection, because it is elementary to see that equation (104) will boil down to the formula in equation (9).

5. Conclusions and outlook

The equivariance statements formulated in the two theorems in this paper are very general, in that this equivariance holds for the full jet groupoid JG of the gauge groupoid G, in the case of Theorem 1, and for the full second order jet groupoid $J^2 G$ of the gauge groupoid G, in the case of Theorem 2. But this does of course not mean that a concrete field theoretical model will have such a huge amount of symmetry – quite to the contrary! Any such model will be subject to restrictions on what are its allowed symmetries coming from the dynamics, which is governed, say, by its Lagrangian: such a Lagrangian will typically be invariant not under the pertinent jet groupoid but rather only under a certain Lie subgroupoid thereof. The generic situation here, which prevails for all standard Lagrangians in gauge theories, is that M

comes equipped with some metric g and the aforementioned Lie subgroupoid will be the inverse image of the corresponding orthonormal frame groupoid $O(TM, g) \subset GL(TM)$ under the "frame" projection from the pertinent jet groupoid to the linear frame groupoid $GL(TM)$ of M. Thus what the two theorems in the previous section really prove is that there are no other restrictions, so this is in fact the correct Lie groupoid for hosting the symmetries of any such theory, and remarkably, it is large enough to accommodate not only its gauge symmetries but also its space-time symmetries, including isometries as well as orthonormal frame transformations, unifying them all within a single mathematical object. Finally, the formalism can also be adapted to handle symmetry breaking, as has been discussed in Ref. [7] (even though only at the level of Lie group bundles and not of full Lie groupoids, which is however enough to deal with that subject).

With this picture in mind, we hope to have demonstrated, in the two papers of this series, that Lie groupoids provide a much wider and more flexible mathematical framework than Lie groups for describing symmetries in physics, and in some cases such as that of gauge theories, we would venture to say they provide the "right" one. What remains to be seen is how this approach will evolve when one tries to extend it from classical to quantum field theories.

Acknowledgements

References

[1] D. Bleecker, Gauge Theory and Variational Principles, Addison-Wesley, Reading, 1981.
[2] B.T. Costa, M. Forger, L.H.P. Pêgas, Lie groupoids in classical field theory I: Noether's theorem, J. Geom. Phys. 131 (2018) 220–245.
[3] L. Fatibene, M. Francaviglia, Natural and Gauge Natural Formalism for Classical Field Theories, Kluwer, Boston, 2003.
[4] M. Forger, B.L. Soares, Local symmetries in gauge theories in a finite-dimensional setting, J. Geom. Phys. 62 (2012) 1925–1938.
[5] M. Göckeler, T. Schücker, Differential Geometry, Gauge Theories and Gravity, Cambridge University Press, Cambridge, 1987.
[6] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge, 1973.
[7] M. Keyl, About the geometric structure of symmetry breaking, J. Math. Phys. 32 (1991) 1065–1071.
[8] K.C.H. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, Cambridge University Press, Cambridge, 2005.
[9] D.J. Saunders, The Geometry of Jet Bundles, Cambridge University Press, Cambridge, 1989.