



# M theory fivebrane and confining phase of $N = 1$ $SO(N_c)$ gauge theories

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## Abstract

The moduli space of vacua for the confining phase of  $N = 1$   $SO(N_c)$  supersymmetric gauge theories in four dimensions is analyzed by studying the M theory fivebrane. The type IIA brane configuration consists of a single NS5 brane, *multiple copies* of NS'5 branes, D4 branes between them, and D6 branes intersecting D4 branes. We construct M theory fivebrane configuration corresponding to the superpotential perturbation where the power of adjoint field is connected to the number of NS'5 branes. At a singular point in the moduli space where mutually local dyons become massless, the quadratic degeneracy of the  $N = 2$   $SO(N_c)$  hyperelliptic curve determines whether this singular point gives a  $N = 1$  vacua or not. The comparison of the meson vevs in M theory fivebrane configuration with field theory results turns out that the effective superpotential by the “integrating in” method with enhanced gauge group  $SU(2)$  is exact. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In the last years we have seen how string/M theory can be used to study nonperturbative dynamics of low energy supersymmetric gauge theories in various dimensions. One of the main motivations is to understand the D(irichet) brane dynamics where the gauge theory is realized on the worldvolume of D branes.

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This work was pioneered by Hanany and Witten [1] where the mirror symmetry of  $N = 4$  gauge theory in three dimensions was described by changing the location of the Neveu–Schwarz(NS)5 brane in spacetime (see, for example, [2,3]). As one changes the relative orientation of the two NS5 branes [4] while keeping their common four spacetime dimensions intact, the  $N = 2$  supersymmetry is broken to  $N = 1$  [5,6]. Using this configuration they [5] described and checked a stringy derivation of Seiberg’s duality for  $N = 1$  supersymmetric gauge theory with  $SU(N_c)$  gauge group with  $N_f$  flavors in the fundamental representation which was conjectured some time ago in [7]. This result was generalized to brane configurations with orientifolds which give  $N = 1$  supersymmetric theories with gauge group  $SO(N_c)$  or  $Sp(N_c)$  [6,8] (see also [9–11] for this approach and [12–16] for an equivalent geometrical approach).

The branes in type IIA/IIB string theory were considered to be rigid without any bendings. As the branes are intersecting each other, a singularity occurs. In order to avoid that kind of singularities, a nice explanation was found by reinterpreting brane configuration in string theory from the point of view of M theory by Witten [17]. Then both the D4 branes and NS5 branes used in type IIA string theory originate from the fivebrane of M theory (the former is an M theory fivebrane wrapped over  $S^1$  and the latter is the one on  $\mathbf{R}^{10} \times S^1$ ). That is, D4 brane’s worldvolume projects to a five-dimensional manifold in  $\mathbf{R}^{10}$  and NS5 brane’s worldvolume is located at a point in  $S^1$  and fills a six dimensional manifold in  $\mathbf{R}^{10}$ . In order to insert D6 branes one has to use a multiple Taub–NUT space [18] whose metric is complete and smooth. Therefore, the singularities are removed in 11 dimensions where the picture becomes smooth, the D4 branes and NS5 branes become the unique M theory fivebrane and the D6 branes are the Kaluza–Klein monopoles. The property of  $N = 2$  supersymmetry in four dimensions requires that the worldvolume of M theory fivebrane is  $\mathbf{R}^{1,3} \times \Sigma$  where  $\Sigma$  is uniquely identified with the curves [19–24] that appear in the solutions to Coulomb branch of the field theory. Further generalizations of this configuration with orientifolds were studied in [25,26]. The original work [17] was appropriated for understanding the moduli space for  $N = 2$  supersymmetric gauge theories. In [27,28] (see also [29–31]), this was seen in M theory, by considering the possible deformation of the curve  $\Sigma$ .

The exact low energy description of  $N = 2$  supersymmetric  $SU(N_c)$  gauge theories with  $N_f$  flavors in four dimensions have been found in [28]. They obtained the information regarding the Affleck–Dine–Seiberg superpotential [32] for  $N_f < N_c$ , in M theory approach. It has  $N_c$  branches corresponding to  $N_c$  D4 branes and there exist two asymptotic regions corresponding to two NS5 branes. The M theory fivebrane [27] is described by the curve

$$t^2 - 2C_{N_c}(v, u_k)t + A_{N=2}^{2N_c-N_f} \prod_{i=1}^{N_f} (v + m_i) = 0, \quad (1.1)$$

where  $v = x^4 + ix^5$ ,  $t = \exp(-(x^6 + ix^{10})/R)$  where  $x^{10}$  is the 11th coordinate of M theory compactified on a circle of radius  $R$ ,  $C_{N_c}(v, u_k)$  is a degree  $N_c$  polynomial in  $v$  with coefficients depending on the moduli  $u_k$  and  $m_i (i = 1, 2, \dots, N_f)$  is the mass of quark. By rotating the  $N = 2$  configuration (which implies to add a mass term  $\mu_2 \text{Tr}(\Phi^2)$  where  $\Phi$  is the adjoint field) an  $N = 1$  configuration is obtained. The asymptotic conditions are

changed and the M theory fivebrane is described now in an  $(v, t, w)$  space where  $w = x^8 + ix^9$ .

This approach has been developed further and used to study the moduli space of vacua of confining phase of  $N = 1$  supersymmetric gauge theories in four dimensions [31]. In terms of brane configuration of IIA string theory, this was achieved by taking multiples of NS'5 branes rather than a single NS'5 brane. In field theory, this is done by generalizing to the case of the superpotential  $\Delta W = \sum_{k=2}^{N_c} \mu_k \text{Tr}(\Phi^k)$ . This perturbation lifts the nonsingular locus of the  $N = 2$  Coulomb branch while at singular locus there exist massless monopoles that can condense due to the perturbation.

In the present work we extend the results of [31] to  $N = 1$  supersymmetric theories with gauge groups  $SO(2N_c)$  and  $SO(2N_c + 1)$  and also generalize our previous work [33] which dealt with a single NS'5 brane in the sense that we are considering *multiple copies* of NS'5 branes. We will describe how the field theory analysis obtained in the low energy superpotential gives rise to the geometrical structure in  $(v, t, w)$  space. For more than one massless dyon (what we mean by dyon is a state charged electrically or magnetically or both), a mismatch is found between field theory results which have been studied in [34,35] and brane configuration results for the exact result  $W_\Delta = 0$ . As in  $SU(N_c)$  case, this implies that the minimal form for the effective superpotential obtained by “integrating in” is not exact [36], in general, for several massless dyons.

This paper is organized as follows. In Section 2, we summarize some results concerning the  $N = 2$  moduli space of vacua for  $SO(N_c)$  supersymmetric gauge theory. By adding tree level superpotential perturbation  $\Delta W$  to  $N = 2$  superpotential, we can analyze the  $N = 1$  field theory. In Section 3, we describe the M theory fivebrane configuration corresponding to  $N = 1$  theory with superpotential perturbation  $\Delta W$ . In Section 4, we calculate meson vacuum expectation values (vevs) and the result is in complete agreement with field theory results discussed in Section 2, for one massless dyon. Finally, in Section 5, we come to the conclusions and the outlook in the future directions.

Note that these techniques of intersecting branes in string/M theory have been used to obtain much information about supersymmetric gauge theories with different gauge groups and in various dimensions [37–64].

## 2. Field theory analysis

Let us first review some field theory results already obtained in [34,35,65–68]. We claim no originality for most of the results presented in this section except the description of the singular point in the moduli space where mutually local dyons become massless and the computation of the generating function for the parameter  $\mu_{2k}$  (see, for example, (2.23) and (2.27)).

### 2.1. $N = 2$ theory

Let us consider  $N = 2$  supersymmetric  $SO(N_c)$  gauge theory with matter in the  $N_c$ -dimensional representation of  $SO(N_c)$ . In terms of  $N = 1$  superfields,  $N = 2$  vector

multiplet consists of a field strength chiral multiplet  $W_\alpha^{ab}$  and a scalar chiral multiplet  $\Phi_{ab}$ , both in the adjoint representation of the gauge group  $SO(N_c)$ . The quark hypermultiplets are made of a chiral multiplet  $Q_a^i$  which couples to the Yang–Mills fields where  $i = 1, \dots, 2N_f$  are flavor indices and  $a = 1, \dots, N_c$  are color indices. The  $N = 2$  superpotential takes the form

$$W = \sqrt{2} Q_a^i \Phi_{ab} Q_b^j J_{ij} + \sqrt{2} m_{ij} Q_a^i Q_a^j, \quad (2.1)$$

where  $J_{ij}$  is the symplectic metric  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1}_{N_f \times N_f}$  used to raise and lower  $SO(N_c)$  flavor indices ( $\mathbf{1}_{N_f \times N_f}$  is the  $N_f \times N_f$  identity matrix) and  $m_{ij}$  is a quark mass matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \dots, m_{N_f})$ . Classically, the global symmetries are the flavor symmetry  $Sp(2N_f)$  when there are no quark masses, in addition to  $U(1)_R \times SU(2)_R$  chiral R-symmetry. The theory is asymptotically free for the region  $N_f < N_c - 2$  and generates dynamically a strong coupling scale  $\Lambda_{N=2}$  where we denote the  $N = 2$  theory by indicating it in the subscript of  $\Lambda$ . The instanton factor is proportional to  $\Lambda_{N=2}^{2N_c-4-2N_f}$ . Then the  $U(1)_R$  symmetry is anomalous and is broken down to a discrete  $\mathbf{Z}_{2N_c-4-2N_f}$  symmetry by instantons.

The  $[N_c/2]$  complex dimensional moduli space of vacua contains the Coulomb and Higgs branches (we denote  $[N_c/2]$  by the value of integer part of  $N_c/2$ ). The Coulomb branch is parameterized by the gauge invariant order parameters

$$u_{2k} = \langle \text{Tr}(\phi^{2k}) \rangle, \quad k = 1, \dots, [N_c/2], \quad (2.2)$$

where  $\phi$  is the scalar field in  $N = 2$  chiral multiplet. Up to a gauge transformation  $\phi$  can be skew diagonalized to a complex matrix,  $\langle \phi \rangle = \text{diag}(A_1, \dots, A_{[N_c/2]})$  where  $A_i = \begin{pmatrix} 0 & -ia_i \\ ia_i & 0 \end{pmatrix}$ . At a generic point the vevs of  $\phi$  breaks the  $SO(N_c)$  gauge symmetry to  $U(1)^{[N_c/2]}$  and the dynamics of the theory is that of an Abelian Coulomb phase. The Wilsonian effective Lagrangian in the low energy can be made of the multiplets of  $A_i$  and  $W_i$  where  $i = 1, 2, \dots, [N_c/2]$ . If  $k$   $a_i$ 's are equal and nonzero then there exists an enhanced  $SU(k)$  gauge symmetry. On the other hand when they are also zero, there is an enhanced  $SO(2k)$  or  $SO(2k+1)$  depending on whether  $N_c$  is even or odd. The property of  $N = 2$  supersymmetry implies that there are no perturbative corrections beyond one loop and there exist nonperturbative instanton corrections.

The quantum moduli space is described by a family of genus  $2N_c - 1$  hyperelliptic spectral curves (we will use the subscript “even” for the quantity corresponding to  $SO(2N_c)$ , “odd” for  $SO(2N_c + 1)$ ; otherwise they have common expression) [66,67,69] with associated meromorphic one forms,

$$y_{\text{even}}^2 = C_{2N_c}^2(v^2) - \Lambda_{N=2}^{4N_c-4-2N_f} v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2) \quad \text{for } SO(2N_c),$$

$$y_{\text{odd}}^2 = C_{2N_c}^2(v^2) - \Lambda_{N=2}^{4N_c-2-2N_f} v^2 \prod_{i=1}^{N_f} (v^2 - m_i^2) \quad \text{for } SO(2N_c + 1), \quad (2.3)$$

where  $C_{2N_c}(v^2)$  is a degree  $2N_c$  polynomial in  $v$  with coefficients depending on the moduli  $u_{2k}$  appearing in (2.2) and  $m_i$  ( $i = 1, 2, \dots, N_f$ ) is the mass of quark, that is, nonzero quark mass matrix element. Note that the polynomial  $C_{2N_c}(v^2)$  is an even function of  $v$  which will be identified with a complex coordinate  $(x^4, x^5)$  directions in spacetime in next section and is given by

$$C_{2N_c}(v^2) = v^{2N_c} + \sum_{i=1}^{N_c} s_{2i} v^{2(N_c-i)} = \prod_{i=1}^{N_c} (v^2 - a_i^2), \quad (2.4)$$

where  $s_{2k}$  and  $u_{2k}$  are related each other by so-called Newton's formula

$$2ks_{2k} + \sum_{i=1}^k s_{2k-2i} u_{2i} = 0, \quad k = 1, 2, \dots, N_c, \quad (2.5)$$

with  $s_0 = 1$ . Recall that the symmetric polynomial  $s_{2k}$  in  $a_i^2$  is

$$s_{2k} = (-1)^k \sum_{i_1 < \dots < i_k} a_{i_1}^2 \dots a_{i_k}^2 \quad (2.6)$$

at the classical level. From this recurrence relation, we obtain

$$\frac{\partial s_{2j}}{\partial u_{2k}} = -\frac{1}{2k} s_{2(j-k)} \quad \text{for } j \geq k, \quad (2.7)$$

which will be used later. When  $2r$  branch points of (2.3) coincide, the Riemann surface degenerates as we vary the moduli, giving a singularity in the effective action and there exists an unbroken  $SO(2r)$  or  $SO(2r+1)$  enhanced gauge symmetry. On the submanifold with all but  $(N_c - r)/2$  of the  $a_i$  being zero (when  $N_c - r$  is even), the curve becomes

$$y_{\text{even}}^2 = v^{4r} (C_{2(N_c-r)}^2(v^2) - \Lambda_{N=2}^{4N_c-4-2N_f} v^{2N_f-4r+4}), \quad (2.8)$$

$$y_{\text{odd}}^2 = v^{4r+2} (C_{2N_c-2r-1}^2(v^2) - \Lambda_{N=2}^{4N_c-2-2N_f} v^{2N_f-4r}) \quad (2.9)$$

for massless matter. By absorbing the factor  $v^{4r} (v^{4r+2})$  into the new variable  $\tilde{y}_{\text{even}} (\tilde{y}_{\text{odd}})$  we will study the property of singular point in the moduli space.

## 2.2. Breaking $N = 2$ to $N = 1$

We are interested in a microscopic  $N = 1$  theory mainly in a phase with a single confined photon coupled to the light dyon hypermultiplet while the photons for the rest are free. By taking a tree level superpotential perturbation  $\Delta W$  of [35] made out of the Casimirs of the adjoint fields in the vector multiplets to the  $N = 2$  superpotential (2.1), the  $N = 2$  supersymmetry can be broken to  $N = 1$  supersymmetry. That is,

$$W = \sqrt{2} Q_a^i \Phi_{ab} Q_b^j J_{ij} + \sqrt{2} m_{ij} Q_a^i Q_a^j + \Delta W, \quad (2.10)$$

where (our  $\mu_{2k}$  is the same as their  $g_{2k}/2k$  in [35])  $\Phi$  is the adjoint  $N = 1$  superfields in the  $N = 2$  vector multiplet and

$$\begin{aligned}\Delta W_{\text{even}} &= \sum_{k=1}^{N_c-2} \mu_{2k} \text{Tr}(\Phi^{2k}) + \mu_{2(N_c-1)} s_{2(N_c-1)} + \lambda \text{Pf} \Phi, \\ \Delta W_{\text{odd}} &= \sum_{k=1}^{N_c-1} \mu_{2k} \text{Tr}(\Phi^{2k}) + \mu_{2N_c} s_{2N_c}.\end{aligned}\quad (2.11)$$

Here there exists an extra invariant quantity  $\text{Pf} \Phi = (1/2^{N_c} N_c!) \epsilon_{i_1 j_1 \dots i_{N_c} j_{N_c}} \Phi^{i_1 j_1} \dots \Phi^{i_{N_c} j_{N_c}}$  when  $N_c$  is even while  $\text{Pf} \Phi$  vanishes for odd  $N_c$ . Note that the  $\mu_{2(N_c-1)}$  term is not associated with  $u_{2(N_c-1)}$  but  $s_{2(N_c-1)}$  which is proportional to the sum of  $u_{2(N_c-1)}$  and the polynomials of other  $u_{2k}$  ( $k < N_c - 1$ ) according to (2.5) (similar argument for odd  $N_c$ ). Then microscopic  $N = 1$   $SO(N_c)$  gauge theory is obtained from  $N = 2$   $SO(N_c)$  Yang–Mills theory perturbed by  $\Delta W$ .

### 2.2.1. Pure Yang–Mills theory

Let us first study  $N = 1$  pure  $SO(N_c)$  Yang–Mills theory with tree level superpotential (2.11). Near the singular points where monopole singlets charged under  $U(1)$  factors become massless, the macroscopic superpotential of the theory is given by

$$W_{\text{even}} = \sqrt{2} \sum_{i=1}^{N_c-1} M_i A_i M_i + \sum_{k=1}^{N_c-2} \mu_{2k} U_{2k} + \mu_{2(N_c-1)} S_{2(N_c-1)} + \lambda U \quad (2.12)$$

and

$$W_{\text{odd}} = \sqrt{2} \sum_{i=1}^{N_c-1} M_i A_i M_i + \sum_{k=1}^{N_c-1} \mu_{2k} U_{2k} + \mu_{2N_c} S_{2N_c}. \quad (2.13)$$

We denote by  $A_i$  the  $N = 2$  chiral superfield of  $(N_c - 1)$   $N = 2$   $U(1)$  gauge multiplets, by  $M_i$  those of  $N = 2$  dyon hypermultiplets, by  $U_{2k}$  the chiral superfields corresponding to  $\text{Tr}(\Phi^{2k})$ , by  $S_{2k}$  the chiral superfields which are related to  $U_{2k}$  through (2.5), and by  $U$  the one corresponding to  $\text{Pf} \Phi$ , in the low energy theory. The vevs of the lowest components of  $A_i$ ,  $M_i$ ,  $U_{2k}$ ,  $S_{2k}$ ,  $U$  are written as  $a_i$ ,  $m_{i,\text{dy}}$ ,  $u_{2k}$ ,  $s_{2k}$ ,  $u$ , respectively. Recall that  $N = 2$  configuration is invariant under the group  $U(1)_R$  and  $SU(2)_R$  corresponding to the chiral R-symmetry of the field theory we mentioned last subsection. However, in  $N = 1$  theory  $SU(2)_R$  is broken to  $U(1)_J$ . In order to the theory to be consistent we should specify the charges of  $U(1)_R \times U(1)_J$  of the fields and parameters as follows:

	$U(1)_R$	$U(1)_J$
$A_i$	2	0
$M_i M_i$	0	2
$\mu_{2k}$	$4 - 4k$	4
$U_{2k}$	$-2 + 4k$	$-2$
$S_{2k}$	$-2 + 4k$	$-2$
$\Lambda_{N=2}$	2	0

(2.14)

The equations of motion obtained by varying the superpotential with respect to each field read

$$\begin{aligned}
 -\frac{\mu_{2k}}{\sqrt{2}} &= \sum_{i=1}^{N_c-1} \frac{\partial a_i}{\partial u_{2k}} m_{i,\text{dy}}^2, \quad k = 1, \dots, N_c - 2, \\
 -\frac{\mu_{2(N_c-1)}}{\sqrt{2}} &= \sum_{i=1}^{N_c-1} \frac{\partial a_i}{\partial s_{2(N_c-1)}} m_{i,\text{dy}}^2, \\
 -\frac{\lambda}{\sqrt{2}} &= \sum_{i=1}^{N_c-1} \frac{\partial a_i}{\partial u} m_{i,\text{dy}}^2
 \end{aligned}
 \tag{2.15}$$

and

$$a_i m_{i,\text{dy}} = 0, \quad i = 1, \dots, N_c - 1. \tag{2.16}$$

At a generic point in the moduli space, no massless fields appear ( $a_i \neq 0$  for  $i = 1, \dots, N_c - 1$ ) which implies  $m_{i,\text{dy}} = 0$  by (2.16). Thus  $\mu_{2k}, \mu_{2(N_c-1)}$  and  $\lambda$  vanish according to (2.15). Then we obtain the moduli space of vacua of  $N = 2$  theory.

On the other hand, we consider a singular point in the moduli space where  $l$  mutually local dyons are massless. (We can choose local coordinates so that the quantum discriminant factorizes into linearly independent factors. This implies that all branches intersect transversely.) This means that  $l$  one cycles shrink to zero. Then the curve (2.3) of genus  $2N_c - 1$  degenerates to a curve of genus  $2N_c - 2l - 1$ . The right-hand side of (2.3) becomes, for  $SO(2N_c)$ ,

$$y_{\text{even}}^2 = C_{2N_c}^2(v^2, u_{2k}) - \Lambda_{N=2}^{4(N_c-1)} v^4 = \prod_{i=1}^l (v^2 - p_i^2)^2 \prod_{j=1}^{2N_c-2l} (v^2 - q_j^2) \tag{2.17}$$

with  $p_i$  and  $q_j$  distinct. A point in the  $N = 2$  moduli space of vacua is characterized by  $p_i$  and  $q_j$ . The degeneracy of this curve is checked by explicitly evaluating both  $y_{\text{even}}^2$  and  $\partial y_{\text{even}}^2 / \partial v^2$  at the point  $v = \pm p_i$ , obtaining thus a zero. Since  $a_i = 0$  for  $i = 1, \dots, l$  and  $a_i \neq 0$  for  $i = l + 1, \dots, N_c - 1$ , (2.16) leads to

$$m_{i,\text{dy}} = 0, \quad i = l + 1, \dots, N_c - 1, \tag{2.18}$$

while  $m_{i, \text{dy}}$  ( $i = 1, \dots, l$ ) are not constrained. We will see how the vevs  $m_{i, \text{dy}}$  originate from the information about  $N = 2$  moduli space of vacua which is encoded in the values of  $p_i$  and  $q_j$ . We assume that the matrix  $\partial a_i / \partial u_{2k}$  is nondegenerate and a complex  $2N_c - 2l - 1$ -dimensional moduli space of  $N = 1$  vacua remains after perturbation. In order to calculate  $\partial a_i / \partial u_{2k}$ , which appears in Eq. (2.15), we need the relation between  $\partial a_i / \partial s_{2k}$  and the period integral on a basis of holomorphic one forms on the curve (we thank A. Hanany for communicating us the misprint of the power of  $v$  in the original hep-th version; the correct expression appeared in the published version of [67]),

$$\frac{\partial a_i}{\partial s_{2k}} = \oint_{\alpha_i} \frac{v^{2(N_c-k)} dv}{y}. \quad (2.19)$$

By plugging the expression of  $y$  of (2.17) into (2.19) and by integrating along one cycles around  $v = \pm p_i$  ( $i = 1, \dots, l$ ), we get

$$\frac{\partial a_i}{\partial s_{2k}} = \frac{p_i^{2(N_c-k)}}{\prod_{j \neq i}^l (p_i^2 - p_j^2) \prod_t^{2N_c-2l} (p_i^2 - q_t^2)^{1/2}}, \quad (2.20)$$

since the  $l$  one cycles shrink to zero. Through Eqs. (2.7), (2.15) and (2.20), we arrive at the following relation between the parameters  $\mu_{2k}$  and the dyon vevs  $m_{i, \text{dy}}^2$

$$\frac{-\mu_{2k}}{\sqrt{2}} = \sum_{i=1}^l \sum_{j=1}^{N_c} \frac{-1}{2k} s_{2(j-k)} p_i^{2(N_c-j)} \frac{m_{i, \text{dy}}^2}{\prod_{s \neq i}^l (p_i^2 - p_s^2) \prod_t^{2N_c-2l} (p_i^2 - q_t^2)^{1/2}}. \quad (2.21)$$

By using the definition

$$\omega_i = \frac{\sqrt{2} m_{i, \text{dy}}^2}{\prod_{s \neq i}^l (p_i^2 - p_s^2) \prod_t^{2N_c-2l} (p_i^2 - q_t^2)^{1/2}}, \quad (2.22)$$

which will be useful for comparison with brane configuration, we can express the generating function for the  $\mu_{2k}$ ,  $\sum_{k=1}^{N_c-1} 2k \mu_{2k} v^{2(k-1)}$ , in terms of  $\omega_i$  as follows:

$$\begin{aligned} \sum_{k=1}^{N_c-1} 2k \mu_{2k} v^{2(k-1)} &= \sum_{k=1}^{N_c-1} \sum_{i=1}^l \sum_{j=1}^{N_c} v^{2(k-1)} s_{2(j-k)} p_i^{2(N_c-j)} \omega_i \\ &= \sum_{k=-\infty}^{N_c-1} \sum_{i=1}^l \sum_{j=1}^{N_c} v^{2(k-1)} s_{2(j-k)} p_i^{2(N_c-j)} \omega_i + \mathcal{O}(v^{-4}) \\ &= \sum_{i=1}^l \frac{C_{2N_c}(v^2)}{v^2(v^2 - p_i^2)} \omega_i + \mathcal{O}(v^{-4}). \end{aligned} \quad (2.23)$$

In principle, we can find the parameter  $\mu_{2k}$  by reading the right-hand side of (2.23). This result determines whether a point in the  $N = 2$  moduli space of vacua classified by the set of  $p_i, q_j$  in (2.17) remains as an  $N = 1$  vacuum after the perturbation, if given a set of



perturbation parameters  $\mu_{2k}$ ,  $\mu_{2(N_c-1)}$  and  $\lambda$ , and gives the dyon vevs  $m_{i,\text{dy}}^2$ . We will see in Section 3 that this corresponds to one of the boundary conditions on a complex coordinate in  $(x^8, x^9)$  directions as  $v$  (which is realized as a complex coordinate in  $(x^4, x^5)$  directions in string/M theory point of view) goes to infinity. In order to make the comparison with the brane picture, it is very useful to define the polynomial  $H_{\text{even}}(v^2)$  of degree  $2l - 4$  by

$$\sum_{i=1}^l \frac{\omega_i}{v^2(v^2 - p_i^2)} = \frac{2H_{\text{even}}(v^2)}{\prod_{i=1}^l (v^2 - p_i^2)}. \quad (2.24)$$

At a given point  $p_i$  and  $q_j$  in the  $N = 2$  moduli space of vacua,  $H_{\text{even}}(v^2)$  determines the dyon vevs, that is,

$$m_{i,\text{dy}}^2 = \sqrt{2} p_i^2 H_{\text{even}}(p_i^2) \prod_m (p_i^2 - q_m^2)^{1/2}, \quad (2.25)$$

which will be described in terms of the geometric brane picture in next section. Therefore, all the vevs of dyons  $m_{i,\text{dy}}$  ( $i = 1, \dots, l$ ) are found:  $m_{i,\text{dy}}$  ( $i = l + 1, \dots, N_c - 1$ ) vanishes according to (2.18).

Similarly we can proceed (we will describe  $SO(2N_c + 1)$  case very briefly and simply write down the main results throughout this paper because the arguments for  $SO(2N_c)$  case go through in the same way.) the case of  $SO(2N_c + 1)$  by starting from

$$C_{2N_c}^2(v^2, u_{2k}) - A_{N=2}^{4N_c-2} v^2 = \prod_{i=1}^l (v^2 - p_i^2)^2 \prod_{j=1}^{2N_c-2l} (v^2 - q_j^2) \quad (2.26)$$

going near a singular point where we get

$$\begin{aligned} \sum_{k=1}^{N_c} 2k \mu_{2k} v^{2(k-1)} &= \sum_{k=1}^{N_c} \sum_{i=1}^l \sum_{j=1}^{N_c} v^{2(k-1)} s_{2(j-k)} p_i^{2(N_c-j)} \omega_i \\ &= \sum_{i=1}^l \frac{C_{2N_c}(v^2)}{(v^2 - p_i^2)} \omega_i + \mathcal{O}(v^{-2}). \end{aligned} \quad (2.27)$$

We will see how this generating function arises when we consider the brane approach which is the main subject in Section 3. Also we define the polynomial  $H_{\text{odd}}(v^2)$  of degree  $2l - 2$  by

$$\sum_{i=1}^l \frac{\omega_i}{(v^2 - p_i^2)} = \frac{2H_{\text{odd}}(v^2)}{\prod_{i=1}^l (v^2 - p_i^2)}, \quad (2.28)$$

which will be of use later. At a given point  $p_i$  and  $q_m$ ,  $H_{\text{odd}}(v^2)$  determines the dyon vevs

$$m_{i,\text{dy}}^2 = \sqrt{2} H_{\text{odd}}(p_i^2) \prod_m (p_i^2 - q_m^2)^{1/2}. \quad (2.29)$$

Of course, the equations of motion leads to  $m_{i,\text{dy}} = 0$  for  $i = l + 1, \dots, N_c - 1$ .

### 2.2.2. Yang–Mills theory with massless matter

The theory with  $N_f$  flavors is similar to the pure case we have discussed. When some of branch points of (2.3) collide as we change the moduli, the Riemann surface degenerates and gives a singularity in the effective theory corresponding to an additional massless field. The gauge group is  $SO(r) \times U(1)^{(N_c-r)/2}$  where  $N_c - r$  is even. At very special points in the moduli space, there are  $(N_c - r)/2$  hypermultiplets charged under these  $U(1)$ 's becoming simultaneously massless. The superpotential at these points is

$$W_{\text{even}} = \sqrt{2} \sum_{i=1}^{N_c-r-1} M_i A_i M_i + \sum_{k=1}^{N_c-2} \mu_{2k} U_{2k} + \mu_{2(N_c-1)} S_{2(N_c-1)} + \lambda U \quad (2.30)$$

and

$$W_{\text{odd}} = \sqrt{2} \sum_{i=1}^{N_c-r-1} M_i A_i M_i + \sum_{k=1}^{N_c-1} \mu_{2k} U_{2k} + \mu_{2N_c} S_{2N_c}. \quad (2.31)$$

As we did in the pure case, the equations of motion can be written as

$$\begin{aligned} -\frac{\mu_{2k}}{\sqrt{2}} - \sum_{j=N_c-r}^{N_c-2} \frac{\mu_{2j}}{\sqrt{2}} \frac{\partial u_{2j}}{\partial u_{2k}} &= \sum_{i=1}^{N_c-r-1} \frac{\partial a_i}{\partial u_{2k}} m_{i,\text{dy}}^2, \quad k = 1, \dots, N_c - r - 2, \\ -\frac{\mu_{2(N_c-1)}}{\sqrt{2}} &= \sum_{i=1}^{N_c-r-1} \frac{\partial a_i}{\partial s_{2(N_c-1)}} m_{i,\text{dy}}^2, \\ -\frac{\lambda}{\sqrt{2}} &= \sum_{i=1}^{N_c-r-1} \frac{\partial a_i}{\partial u} m_{i,\text{dy}}^2, \end{aligned} \quad (2.32)$$

and

$$a_i m_{i,\text{dy}} = 0, \quad i = 1, \dots, N_c - r - 1. \quad (2.33)$$

Notice that the extra term in the left-hand side of (2.33) comes from the fact that  $U_{2k}$  for  $k > N_c - r - 1$  are dependent on  $U_{2k}$  for  $k \leq N_c - r$ . At a generic point in the moduli space, no massless fields  $m_{i,\text{dy}}$  appear ( $a_i \neq 0$  for  $i = 1, \dots, N_c - r - 1$ ) which implies  $m_{i,\text{dy}} = 0$  by (2.33). Then we get the moduli space of vacua of  $N = 2$  theory since all the parameters are zero which gives  $\Delta W = 0$ .

In order to reduce this case to the one analogous to the pure Yang–Mills case where mutually local dyons are massless, we define  $\tilde{y}_{\text{even}}^2 = y_{\text{even}}^2/v^{4r}$  to get the  $2r$  branch points of Riemann surface: the curve (2.3) of genus  $2N_c - 2r - 1$  degenerates to a curve of genus  $2N_c - 2r - 2l - 1$ . Then

$$\tilde{y}_{\text{even}}^2 = C_{2(N_c-r)}^2(v^2) - \Lambda_{N=2}^{2(2N_c-2-N_f)} v^{2N_f-4r+4} \quad (2.34)$$

or

$$\begin{aligned}\tilde{y}_{\text{even}}^2 &= C_{2(N_c-r)}^2(v^2) - \Lambda_{N=2}^{4(N_c-r)-(2N_f-4r+4)} v^{2N_f-4r+4} \\ &= \prod_{i=1}^l (v^2 - p_i^2)^2 \prod_{j=1}^{2(N_c-r-l)} (v^2 - q_j^2)\end{aligned}\quad (2.35)$$

with all  $p_i, q_j$  distinct. Similarly, by redefinition of  $\tilde{y}_{\text{odd}}^2 = y_{\text{odd}}^2/v^{4r+2}$  we obtain

$$\begin{aligned}\tilde{y}_{\text{odd}}^2 &= C_{(2N_c-2r-1)}^2(v^2) - \Lambda_{N=2}^{2(2N_c-2r-1)-(2N_f-4r)} v^{2N_f-4r} \\ &= \prod_{i=1}^l (v^2 - p_i^2)^2 \prod_{j=1}^{2(N_c-r-l)} (v^2 - q_j^2).\end{aligned}\quad (2.36)$$

Now Eq. (2.33) implies that

$$m_{i, \text{dy}} = 0, \quad i = l+1, \dots, N_c - r - 1, \quad (2.37)$$

while  $m_{i, \text{dy}}$  for  $i = 1, \dots, l$  are not constrained since  $a_i = 0$  for  $i = 1, \dots, l$  and  $a_i \neq 0$  for  $i = l+1, \dots, N_c - r - 1$ . By analogy with the pure Yang–Mills case, we use again the relation:

$$\frac{\partial a_i}{\partial s_{2k}} = \oint_{\alpha_i} \frac{v^{2(N_c-k)} dv}{\tilde{y}_{\text{even}}} = \oint_{\alpha_i} \frac{v^{2(N_c-r-k)} dv}{y_{\text{even}}}. \quad (2.38)$$

Then, after making the steps between (2.19) and (2.23), it turns out the generating function for  $\mu_{2k}$  is given by

$$\sum_{k=1}^{N_c-1} 2k \mu_{2k} v^{2(k-1)} = \sum_{i=1}^l \frac{C_{2(N_c-r)}(v^2)}{v^2(v^2 - p_i^2)} \omega_i + \mathcal{O}(v^{-4}), \quad (2.39)$$

where the relation between the function  $H(v^2)$ ,  $\omega_i$  and the dyon vevs  $m_{i, \text{dy}}^2$  is the same as in pure Yang–Mills case. Also we get similar result for  $SO(2N_c + 1)$ ,

$$\sum_{k=1}^{N_c} 2k \mu_{2k} v^{2(k-1)} = \sum_{i=1}^l \frac{C_{2N_c-2r-1}(v^2)}{(v^2 - p_i^2)} \omega_i + \mathcal{O}(v^{-2}). \quad (2.40)$$

### 2.3. The meson vevs

Let us discuss the vevs of the meson field along the singular locus of the Coulomb branch. This is due to the nonperturbative effects of  $N = 1$  theory and obviously was zero before the perturbation (2.11). We will see the property of exactness in field theory analysis in the context of M theory fivebrane in Section 4. Equivalently, the exactness of superpotential for any values of the parameters is to assume  $W_\Delta = 0$ .

### 2.3.1. $SO(2N_c)$ case

We will follow the method presented in [70]. Let us consider the vacuum where one massless dyon exists with unbroken  $SU(2) \times U(1)^{N_c-1}$  where

$$\Phi_{\text{even}}^{\text{cl}} = \sigma_2 \otimes \text{diag}(a_1, a_1, a_2, \dots, a_{N_c-1}), \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.41)$$

and the chiral multiplet  $Q = 0$ . These eigenvalues of  $\Phi$  can be obtained by differentiating the superpotential (2.10) with respect to  $\Phi$  and setting the chiral multiplet  $Q = 0$ .

$$\begin{aligned} W'(\Phi) &= \sum_{i=1}^{N_c-2} 2i\mu_{2i}(\Phi^{2i-1})_{ij} + \mu_{2(N_c-1)} \frac{\partial s_{2(N_c-1)}}{\partial \Phi} \\ &\quad - \frac{\lambda}{2^{N_c}(N_c-1)!} \epsilon_{ijk_1l_1 \dots k_{N_c}l_{N_c}} \Phi^{k_1l_1} \Phi^{k_2l_2} \dots \Phi^{k_{N_c}l_{N_c}} = 0. \end{aligned} \quad (2.42)$$

The vacua with classical  $SU(2) \times U(1)^{N_c-1}$  group are those with two eigenvalues equal to  $a_1$  and the rest given by  $a_2, a_3, \dots, a_{N_c-1}$ . It is known from [34,35] that, if using  $s_{2(N_c-1)}$  in the superpotential perturbation rather than  $u_{2(N_c-1)}$  the degenerate eigenvalue of  $\Phi$  is obtained to be

$$a_1^2 = \frac{(N_c-2)\mu_{2(N_c-2)}}{(N_c-1)\mu_{2(N_c-1)}}. \quad (2.43)$$

We will see in Section 3 that in the context of string/M theory, the asymptotic behavior of a complex coordinate in  $(x^8, x^9)$  directions for large  $v$  determines this degenerate eigenvalue by using the condition for generating function of  $\mu_{2k}$  (2.23). Recall that  $\mu_{2(N_c-1)}s_{2(N_c-1)}$  term in (2.11) is used rather than  $\mu_{2(N_c-1)}u_{2(N_c-1)}$  to get this result. The scale matching condition between the high energy  $SO(2N_c)$  scale  $\Lambda_{N=2}$  and the low energy  $SU(2)$  scale  $\Lambda_{SU(2), N_f}$  is related by the following relation

$$\Lambda_{SU(2), N_f}^{6-2N_f} = (2(N_c-1)\mu_{2(N_c-1)})^2 \Lambda_{N=2}^{4(N_c-1)-2N_f}. \quad (2.44)$$

After integrating out  $SU(2)$  quarks we obtain the scale matching between  $\Lambda_{N=2}$  and  $\Lambda_{SU(2)}$  for pure  $N=1$   $SU(2)$  gauge theory. That is,

$$\Lambda_{SU(2)}^6 = (2(N_c-1)\mu_{2(N_c-1)})^2 \Lambda_{N=2}^{4(N_c-1)-2N_f} \det(a_1^2 - m^2), \quad (2.45)$$

where matrix  $a_1^2$  means  $i\sigma \otimes a_1^2$  and quark mass matrix  $m$  being  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \dots, m_{N_f})$ . Then the full exact low energy effective superpotential is given by

$$\begin{aligned} W_L &= W_{\text{cl}} \pm 2\Lambda_{SU(2)}^3 \\ &= \sum_{k=1}^{N_c-2} \mu_{2k} \text{Tr}(\Phi_{\text{cl}}^{2k}) + \mu_{2(N_c-1)} s_{2(N_c-1)}^{\text{cl}} + \lambda \text{Pf} \Phi_{\text{cl}} \pm 2\Lambda_{SU(2)}^3, \end{aligned} \quad (2.46)$$

where  $W_{\text{cl}}$  is the superpotential evaluated in the classical  $SU(2) \times U(1)^{N_c-1}$  vacua and the last term is generated by gaugino condensation in the low energy  $SU(2)$  theory (the sign reflects the vacuum degeneracy). In terms of the original  $N = 2$  scale, it is written as

$$W_{\text{even}}(\mu_{2k}, \lambda, m) = \sum_{k=1}^{N_c-2} \mu_{2k} \text{Tr}(\Phi_{\text{cl}}^{2k}) + \lambda \text{Pf} \Phi_{\text{cl}} + \mu_{2(N_c-1)} s_{2(N_c-1)}^{\text{cl}} \pm 4(N_c - 1) \mu_{2(N_c-1)} \Lambda_{N=2}^{2(N_c-1)-N_f} \det(a_1^2 - m^2)^{1/2}. \quad (2.47)$$

Therefore, one can obtain the vevs of meson  $M_i = Q_a^i Q_a^i$  by taking the mass matrix  $m$  to be  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \dots, m_{N_f})$  which gives

$$M_i = \frac{\partial W_{\text{even}}}{\partial m_i^2} = \pm \frac{4 \Lambda_{N=2}^{2(N_c-1)-N_f}}{\sqrt{2}(a_1^2 - m_i^2)} (N_c - 1) \mu_{2(N_c-1)} \det(a_1^2 - m^2)^{1/2}, \quad (2.48)$$

where  $a_1$  is given by (2.43). It is easy to see that the vacua of gauge invariant order parameters which are obtained from  $W_L$  parametrize the singularities of the curve (2.3) and reproduce the  $N = 2$  curve (2.3). We will see in Section 3 that the finite value of a complex coordinate in  $(x^8, x^9)$  directions corresponds to the above vevs of meson when  $v \rightarrow \pm m_i$  and the other complex coordinate in  $(x^6, x^{10})$  directions vanishes.

### 2.3.2. $SO(2N_c + 1)$ case

Let us go now to the  $SO(2N_c + 1)$  group for which there is no contribution from  $\text{Pf} \Phi$  and consider again the case of one massless dyon, i.e., the case of unbroken  $SU(2) \times U(1)^{N_c-1}$  vacua with:

$$\Phi_{\text{odd}}^{\text{cl}} = \sigma_2 \otimes \text{diag}(a_1, a_1, a_2, \dots, a_{N_c-1}, 0), \quad (2.49)$$

which can be determined by differentiating the superpotential with respect to  $\Phi$ ,

$$W'(\Phi) = \sum_{i=1}^{N_c-1} 2i \mu_{2i} (\Phi^{2i-1})_{ij} + \mu_{2N_c} \frac{\partial s_{2N_c}}{\partial \Phi} = 0. \quad (2.50)$$

Again we use the result of [34,35] where the degenerate eigenvalue of  $\Phi$  was obtained to be

$$a_1^2 = \frac{(N_c - 1) \mu_{2(N_c-1)}}{N_c \mu_{2N_c}}. \quad (2.51)$$

We will use this value when we discuss the property of the function of a complex coordinate in  $(x^8, x^9)$  directions in Section 3. The scale matching between the high energy  $SO(2N_c + 1)$  scale and the low energy  $SU(2)$  scale is

$$\Lambda_{SU(2), N_f}^{6-2N_f} = (2N_c \mu_{2N_c}) (2(N_c - 1) \mu_{2(N_c-1)}) \Lambda_{N=2}^{2(2N_c-1)-N_f} \quad (2.52)$$

and after integrating out  $SU(2)$  quarks it leads to for pure  $N = 1$   $SU(2)$  gauge theory

$$\Lambda_{SU(2), N_f}^6 = (2N_c \mu_{2N_c}) (2(N_c - 1) \mu_{2(N_c-1)}) \Lambda_{N=2}^{2(2N_c-1)-N_f} \det(a_1^2 - m^2) \quad (2.53)$$

as in  $SO(2N_c)$  case. The matrices  $a_1$  and  $m$  are the same as those in even  $N_c$  case. As a result the low energy effective superpotential is given by

$$W_L = W_{\text{cl}} \pm 2\Lambda_{SU(2)}^3 = \sum_{k=1}^{N_c-1} \mu_{2k} \text{Tr}(\Phi_{\text{cl}}^{2k}) + \mu_{2N_c} s_{2N_c}^{\text{cl}} \pm 2\Lambda_{SU(2)}^3, \quad (2.54)$$

where again the sign reflects the vacuum degeneracy. The quadratic degeneracy of the curve (2.3) is confirmed by the vevs of gauge invariants obtained by  $W_L$ . In terms of original  $N = 2$  scale it is written as

$$W_{\text{odd}}(\mu_{2k}, m) = \sum_{k=1}^{N_c-1} \mu_{2k} \text{Tr}(\Phi_{\text{cl}}^{2k}) + \mu_{2N_c} s_{2N_c}^{\text{cl}} \\ \pm 4\sqrt{N_c(N_c-1)\mu_{2N_c}\mu_{2(N_c-1)}} \Lambda_{N=2}^{2N_c-1-N_f} \det(a_1^2 - m^2)^{1/2}. \quad (2.55)$$

Finally, one gets the vevs of meson  $M_i = Q_a^i Q_a^i$  by using the corresponding mass matrix  $m$ ,

$$M_i = \frac{\partial W_{\text{odd}}}{\partial m_i^2} \\ = \pm \frac{4\Lambda_{N=2}^{2N_c-1-N_f}}{\sqrt{2}(a_1^2 - m_i^2)} \sqrt{N_c(N_c-1)\mu_{2N_c}\mu_{2(N_c-1)}} \det(a_1^2 - m^2)^{1/2}, \quad (2.56)$$

where  $a_1$  given by (2.51). We will learn how this vevs of meson will occur and relate to the asymptotic location in  $(x^8, x^9)$  directions of semiinfinite D4 branes in brane geometry.

#### 2.4. Several massless dyons

We discuss now the case of several massless dyons. The basic procedure in this direction already appeared in [71] for the  $SU(N_c)$  case. Let us start with the  $SO(2N_c)$  case. The classical moduli space is given then by

$$\Phi_{\text{even}}^{\text{cl}} = \sigma_2 \otimes \text{diag}(a_1^{r_1}, \dots, a_k^{r_k}), \quad (2.57)$$

where the eigenvalue  $a_1$  occurs  $r_1$  times, the eigenvalue  $a_2$  does  $r_2$  times and so on. When  $r_1 = 2$  and  $r_i = 1$  ( $i > 1$ ), this will lead to the case of one dyon (2.41). The unbroken group is identical to the one of  $SU$  case, i.e.,  $SU(r_1) \times \dots \times SU(r_k) \times U(1)^{k-1}$ . In (2.57), we consider that all of the  $a_i$ 's are nonzero. If some of them are zero, the unbroken gauge group will be a product of several  $SO$  groups with several  $SU$  groups.

The procedure to obtain the meson vevs is already clear from the arguments of one massless dyon. That is, we decompose  $\Phi = \Phi_{\text{cl}} + \delta\Phi$  and some of the  $\delta\Phi$ 's commute with  $\Phi_{\text{cl}}$  and are integrated first out. Then one obtains just a product of  $SU(r_i)$  groups, each one with adjoints. Some of the vector particles are massive after the symmetry breaking and are to be integrated out. After that we integrate out the adjoint fields in each  $SU(r_i)$  so we go

from  $N = 2$  to  $N = 1$  theory. In the  $N = 1$  theory we integrate out the quarks. The final formula for the scale of the pure  $N = 1$   $SU(r_i)$  theory is:

$$\Lambda_i^{3r_i} = \det(a_i^2 - m^2) \phi(a_i)^{r_i} \prod_{j \neq i} (a_j^2 - a_i^2)^{r_i - 2r_j} \Lambda_{N=2}^{4(N_c-1)-2N_f} \quad (2.58)$$

for some polynomial  $\phi(a_i)$  and the low-energy effective superpotential is given by

$$W_{\text{even}} = \sum_{k=1}^{N_c-2} \mu_{2k} \text{Tr}(\Phi_{\text{cl}}^{2k}) + \mu_{2(N_c-1)} s_{2(N_c-1)}^{\text{cl}} + \lambda \text{Pf } \Phi_{\text{cl}} \pm \sum_{i=1}^k j_i v_i \Lambda_i^3, \quad (2.59)$$

where  $v_i$  is an  $r_i$ th root of unity and after differentiating with respect to  $m_j^2$  we obtain

$$\begin{aligned} \sqrt{2} M_j = \pm \sum_i \frac{v_i \phi(a_i)}{a_i^2 - m_j^2} \det(a_i^2 - m^2)^{1/r_i} \\ \times \prod_{j \neq i} (a_j^2 - a_i^2)^{1-2r_j/r_i} \Lambda_{N=2}^{(4(N_c-1)-2N_f)/r_i}. \end{aligned} \quad (2.60)$$

In the case of  $SO(2N_c + 1)$  gauge group, the classical moduli space is taken to be

$$\Phi_{\text{odd}}^{\text{cl}} = \sigma_2 \otimes \text{diag}(a_1^{r_1}, \dots, a_k^{r_k}, 0). \quad (2.61)$$

Then we obtain the meson vevs by using the same procedure as in the even case and we integrate out all the massive fields. The only difference as compared with the even case (2.60) is the power of  $\Lambda_{N=2}$  which becomes  $(2(2N_c - 1) - 2N_f)/r_i$ . We will find for the unbroken gauge group  $SU(2)$  that there exists an agreement between the meson vevs result in this section and the one which will be discussed in Section 4. However, we will find for the unbroken gauge group  $SU(r)$ ,  $r > 2$ , that there is a disagreement between these two approaches.

### 3. Brane configuration from M theory

In this section we study the theory with the superpotential perturbation  $\Delta W$  (2.11) by analyzing M theory fivebranes. Let us first describe them in the type IIA brane configuration.

Following [5], the brane configuration in  $N = 2$  theory consists of three kinds of branes: the two parallel NS5 branes extend in the directions  $(x^0, x^1, x^2, x^3, x^4, x^5)$ , the D4 branes are stretched between two NS5 branes and extend over  $(x^0, x^1, x^2, x^3)$  and are finite in the direction of  $x^6$ , and the D6 branes extend in the directions  $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$ . In order to study orthogonal gauge groups, we will consider an O4 orientifold which is parallel to the D4 branes in order to keep the supersymmetry and is not of finite extent in  $x^6$  direction. The D4 branes is the only brane which is not intersected by this O4 orientifold. The orientifold gives a spacetime reflection as  $(x^4, x^5, x^7, x^8, x^9) \rightarrow (-x^4, -x^5, -x^7, -x^8, -x^9)$ , in addition to the gauging of worldsheet parity  $\Omega$ . The fixed points of the spacetime symmetry define this O4 planes. Each object which does not lie at the fixed points (i.e., over the

orientifold plane), must have its mirror image. Thus NS5 branes have a mirror in  $(x^4, x^5)$  directions and D6 branes have a mirror in  $(x^7, x^8, x^9)$  directions.

For  $SO(2N_c)$  gauge group, each D4 brane at  $v = x^4 + ix^5$  has its mirror image at  $-v$ :  $N_c$  D4 branes and its mirror  $N_c$  ones. Similarly, for  $SO(2N_c + 1)$  gauge group, there exist an extra single D4 brane which lies over the O4 orientifold being frozen at  $v = 0$  because it does not contain its mirror image, as well as  $N_c$  D4 branes and their  $N_c$  mirror branes. Another important ingredient of O4 orientifold is its charge which is related to the sign of  $\Omega^2$ . When the D4 brane carries one unit of this charge, the charge of the O4 orientifold is  $\mp 1$ , for  $\Omega^2 = \pm 1$  in the D4 brane sector. We are considering the four-dimensional  $N = 2$  supersymmetric gauge theory on D4 brane's worldvolume,  $(x^0, x^1, x^2, x^3)$  directions. The Higgs branch of the theory can be described as the D4 branes broken the D6 brane, suspended them and being allowed to move on the directions  $(x^7, x^8, x^9)$ . The dimension of the Higgs moduli space is found by counting all possible breakings of D4 branes into D6 branes.

In order to realize the  $N = 1$  theory with a perturbation (2.11) we can think of a single NS5 brane and *multiple copies* of NS'5 branes which are orthogonal to a NS5 brane with worldvolume,  $(x^0, x^1, x^2, x^3, x^8, x^9)$  and between them there exist D4 branes intersecting D6 branes. The number of NS'5 branes is  $N_c - 2$  for  $SO(2N_c)$  and  $N_c - 1$  for  $SO(2N_c + 1)$  by identifying the power of adjoint field appearing in the superpotential (2.11). The brane description for  $N = 1$  theory with a superpotential (2.10) where  $\mu_{2(N_c-1)} = \mu_{2N_c} = \lambda = 0$  has been studied in the paper [6] in type IIA brane configuration. In this case, all the couplings,  $\mu_{2k}$  can be regarded as tending uniformly to infinity. On the other hand, we will see in M theory configuration there will be no such restrictions.

### 3.1. M theory fivebrane configuration

#### 3.1.1. $SO(2N_c)$ case

Let us describe how the above brane configuration is embedded in M theory in terms of a single M theory fivebrane whose worldvolume is  $\mathbf{R}^{1,3} \times \Sigma$  where  $\Sigma$  is identified with Seiberg–Witten curves [66,67,69] that determine the solutions to Coulomb branch of the field theory. As usual, we write  $s = (x^6 + ix^{10})/R$ ,  $t = e^{-s}$  where  $x^{10}$  is the 11th coordinate of M theory which is compactified on a circle of radius  $R$ . Then the curve  $\Sigma$ , describing  $N = 2$   $SO(N_c)$  gauge theory with  $N_f$  flavors and even  $N_c$ , is given [25] by an equation in  $(v, t)$  space

$$t^2 - \frac{2C_{2N_c}(v^2, u_{2k})}{v^2}t + \Lambda_{N=2}^{4N_c-4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0. \quad (3.1)$$

Here  $C_{2N_c}(v^2, u_{2k})$  is a degree  $2N_c$  polynomial in  $v$  with only even degree of terms and the coefficients depending on the moduli  $u_{2k}$ , and  $m_i$  is the mass of quark. It is easy to check that this description is the same as (2.3) under the identification

$$v^2 t = y + C_{2N_c}(v^2, u_{2k}). \quad (3.2)$$



By adding (2.11) which corresponds to the adjoint chiral multiplet, the  $N = 2$  supersymmetry will be broken to  $N = 1$ . To describe the corresponding brane configuration in M theory, let us introduce a complex coordinate

$$w = x^8 + ix^9. \quad (3.3)$$

To match the superpotential perturbation  $\Delta W_{\text{even}}$  (2.11), we propose the following boundary conditions for  $SO(2N_c)$ :

$$\begin{aligned} w^2 &\rightarrow \sum_{k=2}^{N_c-1} 2k\mu_{2k}v^{2(k-1)} \quad \text{as } v \rightarrow \infty, & t &\sim \Lambda_{N=2}^{2(2N_c-2-N_f)} v^{2N_f-2N_c+2}, \\ w &\rightarrow 0 \quad \text{as } v \rightarrow \infty, & t &\sim v^{2N_c-2}. \end{aligned} \quad (3.4)$$

After deformation,  $SU(2)_{7,8,9}$  is broken to  $U(1)_{8,9}$  if  $\mu_{2k}$  has the charges  $(4 - 4k, 4)$  under  $U(1)_{4,5} \times U(1)_{8,9}$ . The charges of coordinates and parameters are given by

	$U(1)_{4,5}$	$U(1)_{8,9}$
$v$	2	0
$w$	0	2
$t$	$4(N_c - 1)$	0
$\mu_{2k}$	$4 - 4k$	4
$\Lambda_{N=2}$	2	0

(3.5)

where  $U(1)_{4,5} = U(1)_R$  and  $U(1)_{8,9} = U(1)_I$  we have mentioned last section. If we consider now only the value  $k = 2$ , this reduces to the case of [33] and one obtains in (3.4) that  $w^2 \sim \mu_4 v^2$  as  $v \rightarrow \infty$  which is the same as the relation  $w \rightarrow \mu v$  obtained in [33] if we identify  $\mu_4$  with  $\mu^2$ . This identification comes also from the  $U(1)_{4,5}$  and  $U(1)_{8,9}$  charges of  $\mu$  and  $\mu_4$ . So the reduction to the case of a single NS5 brane is found.

After perturbation, only the singular part of the  $N = 2$  Coulomb branch with  $l$  or more mutually local massless dyons remains in the moduli space of vacua. The corresponding brane configuration is possible only when the curve  $\Sigma$  degenerates to a curve of genus less than  $2N_c - 2l - 1$ . Let us construct the M theory fivebrane configuration corresponding to the correct boundary conditions and assume the condition that  $w^2$  is a rational function of  $v^2$  and  $t$ . Our result is really similar to the case of  $SU(N_c)$  [31] and we will follow their notations. We write  $w^2$  as follows:

$$w^2(t, v^2) = \frac{a(v^2)t + b(v^2)}{c(v^2)t + d(v^2)}, \quad (3.6)$$

where  $a, b, c, d$  are arbitrary polynomials of  $v^2$  and  $t$  satisfies Eq. (3.1). Now we can calculate the following two quantities using the two solutions of  $t$ , denoted by  $t_+$  and  $t_-$  which satisfy the equation for  $t$ , (3.1):

$$w^2(t_+(v^2), v^2) + w^2(t_-(v^2), v^2) = \frac{2acG + 2adC + 2bcC + 2bd}{c^2G + 2cdC + d^2} \quad (3.7)$$

and

$$w^2(t_+(v^2), v^2) - w^2(t_-(v^2), v^2) = \frac{2(ad - bc)S\sqrt{T}}{v^2(c^2G + 2cdC + d^2)}, \quad (3.8)$$

where there is a relation between

$$C \equiv C_{2N_c}(v^2, u_{2k})/v^2 \quad \text{and} \quad G \equiv A_{N=2}^{4N_c-4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2), \quad (3.9)$$

implying that

$$C^2 - G(v^2) \equiv \frac{S^2(v^2)T(v^2)}{v^4}, \quad (3.10)$$

where

$$S(v^2) = \prod_{i=1}^l (v^2 - p_i^2), \quad T(v^2) = \prod_{j=1}^{2N_c-2l} (v^2 - q_j^2) \quad (3.11)$$

with all  $p_i, q_j$ 's different. Remember that  $N = 2$  moduli space of vacua is determined by these  $p_i$  and  $q_j$ . Since  $w^2$  has no poles for finite value of  $v^2$ ,  $w^2(t_+(v^2), v^2) \pm w^2(t_-(v^2), v^2)$  also does not have poles which leads to arbitrary polynomials  $H(v^2)$  and  $N(v^2)$  given by

$$\frac{acG + adC + bcC + bd}{c^2G + 2cdC + d^2} = N, \quad (3.12)$$

$$\frac{(ad - bc)S}{v^2(c^2G + 2cdC + d^2)} = H. \quad (3.13)$$

It will turn out that the function  $H(v^2)$  is exactly the same as the one (2.24) or (2.28) defined in field theory analysis. By making a shift of  $a \rightarrow a + Nc, b \rightarrow b + Nd$  due to the arbitrariness of the polynomials  $a$  and  $b$  the following relations come out:

$$\begin{aligned} w^2 &= N + \frac{a(v^2)t + b(v^2)}{c(v^2)t + d(v^2)}, \\ 0 &= acG + adC + bcC + bd, \\ H &= \frac{(ad - bc)S}{v^2(c^2G + 2cdC + d^2)}. \end{aligned} \quad (3.14)$$

The second equation implies

$$a(cG + dC) + b(d + cC) = 0, \quad (3.15)$$

which can be written as

$$cG + dC = -be, \quad d + cC = ae \quad (3.16)$$

for arbitrary rational function  $e$ . Plugging the values of  $c$  and  $d$  into the (3.14), it turns out  $e = S/(Hv^2)$ . By combining all the information for  $b$  and  $d$ , we get the most general rational function  $w^2$  which has no poles for finite value of  $v^2$  is

$$w^2 = N + \frac{at + cHST/v^2 - aC}{ct - cC + aS/(Hv^2)}, \quad (3.17)$$

where  $N, a, c, H$  are arbitrary polynomials. As we choose two  $w^2$ 's, each of them possessing different polynomials  $a$  and  $c$  and subtract them, the numerator of it will be proportional to  $t^2 - 2Ct + G$  which vanishes according to (3.1). This means  $w^2$  does not depend on  $a$  and  $c$ . Therefore, when  $c = 0$ , the form of  $w^2$  is very simple. That is,

$$w^2 = N + \frac{v^2 H}{S}(t - C_{2N_c}/v^2). \quad (3.18)$$

This result will be used throughout the remaining part of this paper. Now we want to impose the boundary conditions on  $w^2$  from the most general solution (3.18). From the previous relation, by recognizing  $T^{1/2} = v^2(t - C_{2N_c}/v^2)/S$ ,

$$w^2(t_{\pm}(v^2), v^2) = N \pm H\sqrt{T} \quad (3.19)$$

and from the boundary condition  $w \rightarrow 0$  for  $v \rightarrow \infty$ ,  $t = t_-(v) \sim v^{2N_c-2}$  it is easy to see the value of  $N(v^2)$ ,

$$N(v^2) = [H(v^2)\sqrt{T(v^2)}]_+, \quad (3.20)$$

where  $[H(v^2)\sqrt{T(v^2)}]_+$  means only nonnegative power of  $v^2$  when we expand around  $v = \infty$ . Next, by applying the other boundary condition  $v \rightarrow \infty$ ,  $t \sim \Lambda_{N=2}^{2(2N_c-2-N_f)} v^{2N_f-2N_c+2}$ , we obtain

$$w^2 = [2H(v^2)\sqrt{T(v^2)}]_+ + \mathcal{O}(v^{-2}). \quad (3.21)$$

By noting that  $w^2$  satisfies the following equation:

$$w^4 - 2Nw^2 + N^2 - TH^2 = 0 \quad (3.22)$$

and restricting the form of  $N, T$  and  $H$  like as  $N \sim c_1 v^2 + c_2$ ,  $T \sim c_3 v^6 + c_4 v^4 + c_5 v^2 + c_6$ ,  $H \sim c_7/v^2$ , it leads to

$$w^4 + (c_8 + c_9 v^2)w^2 + c_{10} = 0 \quad (3.23)$$

for some constants  $c_i$  ( $i = 1, \dots, 10$ ). Then we can solve for  $v^2$  in terms of  $w^2$  to reproduce the result of [33]. As all the couplings  $\mu_{2k}$  are becoming very large,  $H(v^2)$  and  $N(v^2)$  go to infinity. From (3.22)  $N^2 - TH^2$  goes to zero as we take the limit of  $\Lambda_{N=2} \rightarrow 0$ . This

tells us that  $w^2$  becomes  $(N^2 - TH^2)/2N$  and as  $N(v^2)$  goes to zero,  $w^2 \rightarrow \infty$  showing the findings in [6].

The brane configuration was constructed only at the singular point in the  $N = 2$  moduli space of vacua where  $(v, t)$ -plane curves are degenerate to curves of genus  $2N_c - 2l - 1$  given in (2.17). The general solution for  $w^2$  is

$$w^2 = N(v^2) + v^2 H(v^2) \frac{t - C_{2N_c}(v^2)/v^2}{\prod_{i=1}^l (v^2 - p_i^2)}, \quad (3.24)$$

where  $H(v^2)$  and  $N(v^2)$  are arbitrary polynomials of  $v^2$ . The boundary condition determines  $N(v^2)$  as follows:

$$N(v^2) = \left[ H(v^2) \prod_{j=1}^{2N_c-2l} (v^2 - q_j^2)^{1/2} \right]_+. \quad (3.25)$$

The other boundary condition shows that  $w^2$  behaves as  $w^2 \rightarrow \sum_{k=1}^{N_c-1} 2k\mu_{2k}v^{2(k-1)}$  from (3.4). Then by expanding  $w^2$  in powers of  $v^2$  we can identify  $H(v^2)$  with parameter  $\mu_{2k}$ . Using  $T^{1/2} = v^2(t - C/v^2)/S$  and  $t = 2C/v^2 + \dots$  from (3.1) we get

$$\begin{aligned} w^2 &= 2v^2 H(v^2) \frac{C_{2N_c}(v^2)/v^2}{\prod_{i=1}^l (v^2 - p_i^2)} + \mathcal{O}(v^{-2}) \\ &= \sum_{i=1}^l \frac{C_{2N_c}(v^2)\omega_i}{v^2(v^2 - p_i^2)} = \sum_{k=1}^{N_c-1} 2k\mu_{2k}v^{2(k-1)}, \end{aligned} \quad (3.26)$$

where we used the definition of  $H_{\text{even}}$  in (2.24) and the generating function of  $\mu_{2k}$  in (2.23). From this result one can find the explicit form of  $H(v^2)$  in terms of  $\mu_{2k}$  by comparing both sides in the above relation. This is an explanation for field theory results of (2.23) and (2.24) which determine the  $N = 1$  moduli space of vacua after the perturbation, from the point of view of M theory fivebrane. It reproduces the equations which determine the vevs of massless dyons along the singular locus. The dyon vevs  $m_{i, \text{dy}}^2$  given by (2.25)

$$m_{i, \text{dy}}^2 = \sqrt{2} p_i^2 H_{\text{even}}(p_i^2) \sqrt{T(p_i^2)}, \quad (3.27)$$

are nothing but the difference between the two finite values of  $v^2 w^2$ . This can be seen by taking  $v = \pm p_i$  in (3.19) and (3.20). The  $N = 2$  curve of (3.1) and (2.17) contains double points at  $v = \pm p_i$  and  $t = C_{2N_c}(p_i^2)$ . The perturbation  $\Delta W$  of (2.11) splits these into separate points in  $(v, t, w)$  space and the difference in  $v^2 w^2$  between these points becomes the dyon vevs. This is a geometric interpretation of dyon vevs in M theory brane configuration.

### 3.1.2. $SO(2N_c + 1)$ case

The curve  $\Sigma$  for  $N = 2$   $SO(2N_c + 1)$  gauge theory with  $N_f$  flavors reads in  $(v, t)$  space

$$t^2 - \frac{2C_{2N_c}(v^2, u_{2k})}{v} t + \Lambda_{N=2}^{4N_c-2-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0, \quad (3.28)$$

where  $t$  is related to  $y$  by

$$vt = y + C_{2N_c}(v^2, u_{2k}). \quad (3.29)$$

The configuration of M theory fivebrane corresponding to type IIA brane configuration has the following boundary conditions:

$$\begin{aligned} w^2 &\rightarrow \sum_{k=1}^{N_c} 2k\mu_{2k}v^{2(k-1)} \quad \text{as } v \rightarrow \infty, & t &\sim \Lambda_{N=2}^{2(2N_c-1-N_f)} v^{2N_f-2N_c+1}, \\ w &\rightarrow 0 \quad \text{as } v \rightarrow \infty, & t &\sim v^{2N_c-1}. \end{aligned} \quad (3.30)$$

After doing the similar procedure as in  $SO(2N_c)$  case, we arrive at the final expression

$$\begin{aligned} w^2 &= 2vH(v^2) \frac{C_{2N_c}(v^2)/v}{\prod_{i=1}^l (v^2 - p_i^2)} + \mathcal{O}(v^{-2}) \\ &= \sum_{i=1}^l \frac{C_{2N_c}(v^2)\omega_i}{(v^2 - p_i^2)} = \sum_{k=1}^{N_c} 2k\mu_{2k}v^{2(k-1)}, \end{aligned} \quad (3.31)$$

where we used the definition of  $H_{\text{odd}}$  in (2.28) and the generating function of  $\mu_{2k}$  in (2.27). One can find the explicit form of  $H(v^2)$  in terms of  $\mu_{2k}$  by comparing both sides.

### 3.2. Yang–Mills theory with massless matter

We have seen that  $(v, \tilde{t} = \tilde{y}/v^2 + C_{2(N_c-r)}(v^2, u_{2k})/v^2)$  curve (2.3) of genus  $2N_c - 2r - 1$  degenerates to a curve of genus  $2N_c - 2r - 2l - 1$  by redefining  $\tilde{y}_{\text{even}}^2 = y_{\text{even}}^2/v^{4r}$  to get the  $2r$  branches of the curve. Now it is straightforward to get the most general form of the solution by looking at Eq. (3.18),

$$w^2 = N(v^2) + v^2 H(v^2) \frac{\tilde{t} - C_{2(N_c-r)}(v^2)/v^2}{\prod_{i=1}^l (v^2 - p_i^2)}, \quad (3.32)$$

where  $H(v^2)$  and  $N(v^2)$  are arbitrary polynomials of  $v^2$ . Once again the boundary condition  $w \rightarrow 0$  as  $v \rightarrow \infty$  and  $\tilde{t} \sim \Lambda_{N=2}^{2(2N_c-2-r)} v^{-2N_c+2r+2}$  gives the form of  $N(v^2)$

$$N(v^2) = \left[ H(v^2) \prod_{j=1}^{2(N_c-r-l)} (v^2 - q_j^2)^{1/2} \right]_+. \quad (3.33)$$

The other boundary condition  $w^2 \rightarrow \sum_{k=1}^{N_c-1} 2k\mu_{2k}v^{2(k-1)}$  as  $v \rightarrow \infty$ ,  $\tilde{t} \sim v^{2(N_c-r-1)}$  and the relation  $\tilde{t} = 2C_{2(N_c-r)} + \dots$  yield to

$$\begin{aligned} w_{\text{even}}^2 &= 2v^2 H(v^2) \frac{C_{2(N_c-r)}(v^2)/v^2}{\prod_{i=1}^l (v^2 - p_i^2)} + \mathcal{O}(v^{-2}) \\ &= \sum_{i=1}^l \frac{C_{2(N_c-r)}(v^2)}{v^2(v^2 - p_i^2)} \omega_i = \sum_{k=1}^{N_c-1} 2k\mu_{2k}v^{2(k-1)}, \end{aligned} \quad (3.34)$$

which is precisely in agreement with Eq. (2.39) which determines the relation between the function  $H(v^2)$ ,  $\omega_i$  and dyon vevs  $m_{i,\text{dy}}^2$  after perturbation. Also we will see the  $SO(2N_c+1)$  result analogous to the  $SO(2N_c)$ . For most general deformation of the brane, it is

$$w^2 = N(v^2) + vH(v^2) \frac{\tilde{t} - C_{(2N_c-2r-1)}(v^2)/v}{\prod_{i=1}^l (v^2 - p_i^2)} \quad (3.35)$$

and

$$N(v^2) = \left[ H(v^2) \prod_{j=1}^{2(N_c-r-l)} (v^2 - q_j^2)^{1/2} \right]_+ \quad (3.36)$$

By applying the second boundary condition, we arrive at

$$\begin{aligned} w_{\text{odd}}^2 &= 2vH(v^2) \frac{C_{(2N_c-2r-1)}(v^2)/v}{\prod_{i=1}^l (v^2 - p_i^2)} + \mathcal{O}(v^{-2}) \\ &= \sum_{i=1}^l \frac{C_{2N_c-2r-1}(v^2)}{(v^2 - p_i^2)} \omega_i = \sum_{k=1}^{N_c} 2k\mu_{2k} v^{2(k-1)}, \end{aligned} \quad (3.37)$$

which is again the same as Eq. (2.40).

#### 4. Brane configuration and field theory

In this section we continue to study for the meson vevs from the singularity structure of  $N = 2$  Riemann surface. The vevs of meson will depend on the moduli structure of  $N = 2$  Coulomb branch (see, for example, (4.8)). Also, the finite values of  $w^2$  can be determined fully by using the property of boundary conditions of  $w^2$  when  $v$  goes to be very large. We will illustrate some examples which studied before in field theory analysis that was limited for the case of a small number of  $N_c$  because it becomes very difficult to find the vacua from the quantum discriminant when  $N_c$  is very large.

##### 4.1. $SO(2N_c)$ case

Let us consider the case of finite  $w^2$  at  $t = 0$ ,  $v = \pm m_i$  and we want to compare with the meson vevs we have studied in (2.48). At a point where there exists a single massless dyon (in other words, by putting  $l = 1$  into (2.17)) and recalling the definition of  $T(v^2)$ , we have for Yang–Mills with matter

$$C_{2N_c}^2(v^2) - \Lambda_{N=2}^{4N_c-4-2N_f} v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2) = (v^2 - p_1^2)^2 T(v^2) \quad (4.1)$$

and the function  $w^2$  according to (3.19) and (3.20) reads

$$w^2 = \left[ \frac{h}{v^2} \sqrt{T(v^2)} \right]_+ \pm \frac{h}{v^2} \sqrt{T(v^2)}, \quad (4.2)$$

where in this case  $l = 1$  means that the polynomial  $v^2 H(v^2)$  has the degree of zero and we denote it by a constant  $h$ . From (4.1) we see for  $N_f < 2N_c - 2$

$$\frac{\sqrt{T(v^2)}}{v^2} = \frac{C_{2N_c}(v^2)}{v^2(v^2 - p_1^2)} + \mathcal{O}(v^{-4}) \quad (4.3)$$

and we decompose  $C_{2N_c}$  as

$$\frac{C_{2N_c}(v^2)}{v^2} = \frac{C_{2N_c}(p_1^2)}{p_1^2} + (v^2 - p_1^2) \tilde{C}_{2N_c-4}(v^2) \quad (4.4)$$

for some polynomial  $v^2 \tilde{C}_{2N_c-4}(v^2)$  of degree  $2N_c - 2$  which can be determined completely. This means that the coefficients of  $\tilde{C}_{2N_c-4}(v^2)$  can be fixed from the explicit form of the polynomial  $C_{2N_c}(v^2)$ . Through (4.3) and (4.4) the part with nonnegative powers of  $v^2$  in  $\sqrt{T(v^2)}/v^2$  becomes  $\tilde{C}_{2N_c-4}(v^2)$  as follows:

$$\frac{\sqrt{T(v^2)}}{v^2} = \tilde{C}_{2N_c-4}(v^2) + \mathcal{O}(v^{-2}) \rightarrow \left[ \frac{\sqrt{T(v^2)}}{v^2} \right]_+ = \tilde{C}_{2N_c-4}(v^2). \quad (4.5)$$

Thus as  $v \rightarrow \pm m_i$  the finite value of  $w^2$ , denoted by  $w_i^2$ , can be written

$$w_i^2 = w^2(v^2 \rightarrow m_i^2) = h \tilde{C}_{2N_c-4}(m_i^2) \pm \frac{h}{m_i^2} \sqrt{T(m_i^2)}. \quad (4.6)$$

From (4.1), the relation  $\sqrt{T(m_i^2)}/m_i^2 = C_{2N_c}(m_i^2)/m_i^2(m_i^2 - p_1^2) + \mathcal{O}(m_i^{-4})$  holds and the decomposition of (4.4) yields the following relation:

$$\frac{\sqrt{T(m_i^2)}}{m_i^2} = \frac{C_{2N_c}(p_1^2)}{p_1^2(m_i^2 - p_1^2)} + \tilde{C}_{2N_c-4}(m_i^2). \quad (4.7)$$

By plugging this value into (4.6) and taking the minus sign which corresponds to  $t \rightarrow 0$ , we end up with

$$w_i^2 = \frac{h}{p_1^2} \frac{C_{2N_c}(p_1^2)}{(p_1^2 - m_i^2)}. \quad (4.8)$$

In order to find  $C_{2N_c}(p_1^2)$  we evaluate it from (4.1) at  $v^2 = p_1^2$  to arrive at

$$w_i^2 = h \Lambda_{N=2}^{2N_c-2-N_f} \frac{\det(p_1^2 - m^2)^{1/2}}{(p_1^2 - m_i^2)}. \quad (4.9)$$

In the above expression we need to know the values of  $h$  and  $p_1$ . On the other hand the boundary condition for  $w^2$  for large  $v$  leads to

$$w^2 \sim 2 \frac{h}{v^2} \frac{C_{2N_c}(v^2)}{v^2 - p_1^2} \sim 2h v^{2(N_c-2)} + 2hp_1^2 v^{2(N_c-3)} + \dots, \quad (4.10)$$

which should be equal to  $\sum_{k=1}^{N_c-1} 2k\mu_{2k} v^{2(k-1)}$ . Then we can read off the values of  $h$  and  $p_1$  by comparing both sides term by term:

$$2h = 2(N_c - 1)\mu_{2(N_c-1)}, \quad p_1^2 = \frac{(N_c - 2)\mu_{2(N_c-2)}}{(N_c - 1)\mu_{2(N_c-1)}}. \quad (4.11)$$

Finally, the finite value for  $w^2$  can be written as

$$w_i^2 = (N_c - 1)\mu_{2(N_c-1)} A_{N=2}^{2N_c-2-N_f} \frac{\det(a_1^2 - m^2)^{1/2}}{(a_1^2 - m_i^2)}, \quad (4.12)$$

which is exactly, up to constant, the same expression for meson vevs (2.48) obtained from field theory analysis in the low energy superpotential (2.46). This illustrates the fact that at vacua with enhanced gauge group  $SU(2)$  the effective superpotential by “integrating in” method with the assumption of  $W_\Delta = 0$  is really exact.

#### 4.1.1. Example 1: $SO(6)$ with one flavor

We would like to demonstrate the above descriptions by taking the specific models. The  $N = 2$  theory in this model is described by the curve  $\Sigma$ :

$$t^2 - \frac{2C_6(v^2, u_{2k})}{v^2} t + A_{N=2}^6(v^2 - m_1^2) = 0, \quad (4.13)$$

where the polynomial  $C_6(v^2)$  is given as (2.4)

$$C_6(v^2, u_{2k}) = v^6 + s_2 v^4 + s_4 v^2 + s_6 \quad (4.14)$$

in terms of  $s$  or

$$C_6(v^2, u_{2k}) = v^6 - \frac{u_2}{2} v^4 - \left( \frac{u_4}{4} - \frac{u_2^2}{8} \right) v^2 - \frac{1}{6} \left( u_6 - \frac{3u_4 u_2}{4} + \frac{u_2^3}{8} \right) \quad (4.15)$$

in terms of  $u$ . When one dyon becomes massless the locus in the moduli space becomes

$$C_6^2(v^2, u_{2k}) - A_{N=2}^6 v^4 (v^2 - m_1^2) = (v^2 - p_1^2)^2 T(v^2). \quad (4.16)$$

By putting  $v^2 = p_1^2$  in Eq. (4.16) we get one relation and by differentiating with respect to  $v^2$  and evaluating the derived equation at  $v^2 = p_1^2$  we obtain a second relation between  $s_2, s_4$  and  $s_6$ . So one of those vevs will remain undetermined. This is because for only one massless dyon there exist only terms with  $v^4$  so after taking two derivatives, in the right-hand side of (4.16) we would have only terms that cannot cancel at  $v^2 = p_1^2$ . For several massless dyons the power of  $v^2$  in the right-hand side of (4.16) would be bigger than  $v^4$



and we could take two derivatives with respect to  $v^2$ . Now we express  $s_2$  and  $s_4$  in terms of  $s_6$  whose classical vev vanishes

$$\begin{aligned}s_2 &= -2p_1^2 + \frac{s_6}{p_1^4} \pm \frac{\Lambda_{N=2}^3}{2b}, \\s_4 &= p_1^4 - \frac{2s_6}{p_1^2} \pm \frac{\Lambda_{N=2}^3}{2b} \sqrt{2m_1^2 - p_1^2},\end{aligned}\quad (4.17)$$

where  $b = \sqrt{p_1^2 - m_1^2}$ . These vevs of gauge invariant variables are exactly the same as those obtained in [35] from the low energy effective superpotential, for  $N_c = 2$ ,  $\mu_2 \text{Tr}(\Phi_{\text{cl}}^2) + \mu_4 s_4^{\text{cl}} + \lambda \text{Pf} \Phi_{\text{cl}} \pm 2\Lambda_{SU(2)}^3$  where the classical vacua of  $\Phi_{\text{cl}} = \sigma_2 \otimes \text{diag}(a_1, a_1, a_2)$  breaks  $SO(6)$  into  $SU(2) \times U(1) \times U(1)$ . For the case of pure Yang–Mills theory, it is known from [72] that only six points remain as mutually local dyons for  $SO(8)$  gauge theory while in  $SO(6)$  four points give the correct  $N = 1$  vacua. By using relation (4.4), it is easy to find the value of  $N(v^2)$  and from (4.16) one obtains  $T(v^2)$ :

$$\begin{aligned}N &= h \left( v^2 + (p_1^2 + s_2) - \frac{s_6}{p_1^2 v^2} \right), \\N^2 - h^2 T &= h^2 \left[ -2p_1^4 - 2p_1^2 s_2 - 2s_4 - \frac{2s_6}{p_1^2} + \frac{1}{v^2} \left( -4p_1^6 + \Lambda_{N=2}^6 - 6p_1^4 s_2 \right. \right. \\&\quad \left. \left. - 2p_1^2 s_2^2 - 4p_1^2 s_4 - 2s_2 s_4 - 4s_6 - \frac{2s_2 s_6}{p_1^2} \right) \right. \\&\quad \left. + \frac{1}{v^4} \left( -5p_1^8 + \Lambda_{N=2}^6 (2p_1^2 - m_1^2) - 8p_1^6 s_2 - 3p_1^4 s_2^2 \right. \right. \\&\quad \left. \left. - 6p_1^4 s_4 - 4p_1^2 s_2 s_4 - s_4^2 - 4p_1^2 s_6 - 2s_2 s_6 + \frac{s_6^2}{p_1^4} \right) \right].\end{aligned}\quad (4.18)$$

Also we get the dyon vevs from the relation  $m_{1,\text{dy}}^4 = 2h^2 T(p_1^2)$ ,

$$\begin{aligned}m_{1,\text{dy}}^4 &= 2h^2 (15p_1^8 - \Lambda_{N=2}^6 (3p_1^2 - m_1^2) + 20p_1^6 s_2 + 6p_1^4 s_2^2 \\&\quad + 12p_1^4 s_4 + 6p_1^2 s_2 s_4 + s_4^2 + 6p_1^2 s_6 + 2s_2 s_6).\end{aligned}\quad (4.19)$$

The boundary condition for  $w^2$  gives the relation between  $\mu_4$  and  $h$ ,

$$h = 2\mu_4. \quad (4.20)$$

By explicitly calculating  $C_6(p_1^2)$ , we get the meson vevs,

$$w_1^2 = h \frac{\Lambda_{N=2}^3 b}{p_1^2 - m_1^2}. \quad (4.21)$$

#### 4.1.2. Example 2: $SO(6)$ with two flavors

Let us consider the previous example with two flavors which is described by the curve:

$$t^2 - \frac{2C_6(v^2, u_{2k})}{v^2}t + \Lambda_{N=2}^4(v^2 - m_1^2)(v^2 - m_2^2) = 0, \quad (4.22)$$

where  $m_1, m_2$  are the masses for the two flavors and  $C_6$  is the same as before. When one dyon becomes massless the locus in the moduli space becomes as in (4.16):

$$C_6^2 - \Lambda_{N=2}^4 v^4 (v^2 - m_1^2)(v^2 - m_2^2) = (v^2 - p_1^2)^2 T(v^2). \quad (4.23)$$

From this equation we again obtain the values for  $s_{2k}$ . As in Example 1, for one massless dyon we have only two equations for three variables  $s_2, s_4$  and  $s_6$  and we can solve  $s_2, s_4$  in terms of  $s_6$ . Our results are

$$\begin{aligned} s_2 &= -2p_1^2 + \frac{s_6}{p_1^4} \pm \frac{\Lambda_{N=2}^2}{2B} (2p_1^2 - m_1^2 - m_2^2), \\ s_4 &= p_1^4 - \frac{2s_6}{p_1^2} \pm \frac{\Lambda_{N=2}^2}{2B} (p_1^2 m_1^2 + p_1^2 m_2^2 - 2m_1^2 m_2^2), \end{aligned} \quad (4.24)$$

where  $B = \pm \sqrt{(p_1^2 - m_1^2)(p_1^2 - m_2^2)}$  and we take only the plus sign. Remark that  $B$  has dimension 2 while  $b$  has dimension 1. The value for  $N(v^2)$  remains the same as in Eq. (4.18) given different  $s_2$  and  $s_4$  as in (4.24) and from the explicit form of  $T(v^2)$  we obtain the following relation:

$$\begin{aligned} N^2 - h^2 T &= h^2 \left[ -2p_1^4 + \Lambda_{N=2}^4 - 2p_1^2 s_2 - 2s_4 - \frac{2s_6}{p_1^2} \right. \\ &\quad + \frac{1}{v^2} \left( -4p_1^6 + \Lambda_{N=2}^4 (2p_1^2 - m_1^2 - m_2^2) - 6p_1^4 s_2 \right. \\ &\quad \left. \left. - 2p_1^2 s_2^2 - 4p_1^2 s_4 - 2s_2 s_4 - 4s_6 - \frac{2s_2 s_6}{p_1^2} \right) \right. \\ &\quad + \frac{1}{v^4} \left( -5p_1^8 + \Lambda_{N=2}^4 (3p_1^4 - 2p_1^2 m_1^2 - 2p_1^2 m_2^2 + m_1^2 m_2^2) - 8p_1^6 s_2 \right. \\ &\quad \left. \left. - 3p_1^4 s_2^2 - 6p_1^4 s_4 - 4p_1^2 s_2 s_4 - s_4^2 - 4p_1^2 s_6 - 2s_2 s_6 + \frac{s_6^2}{p_1^4} \right) \right] \end{aligned} \quad (4.25)$$

and the dyon vevs are given by

$$\begin{aligned} m_{i, \text{dy}}^4 &= 2h^2 (15p_1^8 + \Lambda_{N=2}^4 (-6p_1^4 + 3p_1^2 m_1^2 + 3p_1^2 m_2^2 - m_1^2 m_2^2) + 20p_1^6 s_2 \\ &\quad + 6p_1^4 s_2^2 + 12p_1^4 s_4 + 6p_1^2 s_2 s_4 + s_4^2 + 6p_1^2 s_6 + 2s_2 s_6). \end{aligned} \quad (4.26)$$

Finally for meson vevs, we get

$$w_i^2 = h \frac{\Lambda^2 B}{p_1^2 - m_i^2}. \quad (4.27)$$

#### 4.2. $SO(2N_c + 1)$ case

At the locus where there is one dyon which becomes massless, we have

$$C_{2N_c}^2(v^2) - \Lambda_{N=2}^{4N_c-2-2N_f} v^2 \prod_{i=1}^{N_f} (v^2 - m_i^2) = (v^2 - p_1^2)^2 T(v^2). \quad (4.28)$$

The function of  $w^2$  is

$$w^2 = [H\sqrt{T(v^2)}]_+ \pm H\sqrt{T(v^2)}, \quad (4.29)$$

where  $H$  is a constant. For the finite values of  $w^2$  as  $t \rightarrow 0$  and  $v \rightarrow \pm m_i$  we find

$$w_i^2 = H \frac{C_{2N_c}(p_1^2)}{(p_1^2 - m_i^2)}. \quad (4.30)$$

By inserting  $v^2 = p_1^2$  into (4.28) and writing  $C_{2N_c}(p_1^2)$  as  $\Lambda_{N=2}^{2N_c-1-N_f} p_1 \prod_{i=1}^{N_f} (p_1^2 - m_i^2)^{1/2}$ , we obtain

$$w_i^2 = H \Lambda_{N=2}^{2N_c-1-N_f} p_1 \frac{\det(p_1^2 - m^2)^{1/2}}{(p_1^2 - m_i^2)}. \quad (4.31)$$

The asymptotic behavior of  $w^2$  for large  $v$  leads to

$$w^2 \sim 2H \frac{C_{2N_c}(v^2)}{v^2 - p_1^2} \sim 2H v^{2(N_c-1)} + 2H p_1^2 v^{2(N_c-2)} + \dots, \quad (4.32)$$

which is the same as  $\sum_{k=1}^{N_c} 2k\mu_{2k} v^{2(k-1)}$ . From this relation we get

$$2H = 2N_c \mu_{2N_c}, \quad p_1^2 = \frac{(N_c - 1)\mu_{2(N_c-1)}}{N_c \mu_{2N_c}}. \quad (4.33)$$

Therefore we find  $w_i^2$  completely

$$w_i^2 = 2\sqrt{N_c(N_c - 1)\mu_{2N_c}\mu_{2(N_c-1)}} \Lambda_{N=2}^{2N_c-1-N_f} \frac{\det(a_1^2 - m^2)^{1/2}}{(a_1^2 - m_i^2)}, \quad (4.34)$$

which exactly coincides with the meson vevs (2.56) from field theory results in the low energy superpotential (2.54) in Section 2.

##### 4.2.1. Example 3: $SO(5)$ with one flavor

As we did in the case of  $SO(6)$ , the  $N = 2$  theory of this model is given by the curve

$$t^2 - \frac{2C_4(v^2, u_{2k})}{v} t + \Lambda_{N=2}^4 (v^2 - m_1^2) = 0, \quad (4.35)$$

where  $C_4(v^2, u_{2k}) = v^4 + s_2 v^2 + s_4$  in terms of  $s_{2k}$  or  $C_4(v^2) = v^4 - \frac{1}{2} u_2 v^2 - (\frac{1}{4} u_4 - \frac{1}{8} u_2^2)$  in terms of  $u_{2k}$ .

When one dyon becomes massless, the locus in the moduli space becomes

$$\left(v^4 - \frac{u_2}{2}v^2 - \left(\frac{u_4}{4} - \frac{u_2^2}{8}\right)\right)^2 - \Lambda_{N=2}^4 v^2(v^2 - m_1^2) = (v^2 - p_1^2)^2 T(v^2). \quad (4.36)$$

By inserting  $v^2 = p_1^2$  in Eq. (4.36) and its derivative with respect to  $v^2$  we obtain two equations for  $u_2, u_4$  which are solved to give

$$\begin{aligned} u_2 &= 4p_1^2 \pm \Lambda_{N=2}^2 \left(\frac{p_1}{b} + \frac{b}{p_1}\right), \\ u_4 &= 4p_1^4 \pm \Lambda_{N=2}^2 \left(4p_1^2 \left(\frac{p_1}{b} + \frac{b}{p_1}\right) - \frac{2p_1 m_1^2}{b}\right) + \frac{\Lambda_{N=2}^4}{2} \left(\frac{p_1}{b} + \frac{b}{p_1}\right)^2, \end{aligned} \quad (4.37)$$

where  $b = \pm\sqrt{p_1^2 - m^2}$  and we take only the plus sign for  $b$ . It is easy to see that these vevs of gauge invariant variables are exactly coincident with those of [35,73] obtained from the low energy effective superpotential, for  $N_c = 2$ ,  $\mu_2 \text{Tr}(\Phi_{\text{cl}}^2) + \mu_4 s_4^{\text{cl}} \pm 2\Lambda_{SU(2)}^3$  where the classical vacua of  $\Phi_{\text{cl}} = \sigma_2 \otimes \text{diag}(a_1, a_1, 0)$  breaks  $SO(5)$  into  $SU(2) \times U(1)$ . Note that the pure case has been discussed in [74] where the three intersection points correspond to a pair of mutually local dyons becoming massless. We want to see the dyon vev and the finite value for  $w^2$ . The sum of two solutions of  $w^2$  satisfying (3.22) and product of them can be summarized as

$$\begin{aligned} N &= H \left(v^2 + p_1^2 - \frac{u_2}{2}\right), \\ N^2 - H^2 T &= H^2 (-2p_1^4 + \Lambda_{N=2}^4 - 2s_4 + p_1^2 u_2). \end{aligned} \quad (4.38)$$

The dyon vevs are obtained from the explicit form of  $T(p_1^2)$  as follows:

$$m_{1,\text{dy}}^4 = 2H^2 T(p_1^2) = 2H^2 \left(-2p_1 b \Lambda_{N=2}^2 + \frac{m_1^4}{4p_1^2 b^2} \Lambda_{N=2}^4\right). \quad (4.39)$$

Also we have the parameter  $\mu_4$

$$H = 2\mu_4 \quad (4.40)$$

and the meson vevs  $w_1^2$

$$w_1^2 = H \frac{\Lambda_{N=2}^2 p_1 b}{p_1^2 - m_1^2}. \quad (4.41)$$

#### 4.3. Several massless dyons

We discuss the even case  $SO(2N_c)$ , the odd case  $SO(2N_c + 1)$  going exactly the same. If there are  $l$  massless dyons, the curve can be factorized as in (2.17):

$$C_{2N_c}^2(v^2) - \Lambda_{N=2}^{4N_c-4-2N_f} v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2) = \prod_{i=1}^l (v^2 - p_i^2)^2 T(v^2) \quad (4.42)$$

and  $w^2$  is given by

$$w^2 = \left[ \frac{h(v^2)}{v^2} \sqrt{T(v^2)} \right]_+ \pm \frac{h(v^2)}{v^2} \sqrt{T(v^2)}, \quad (4.43)$$

where now  $h(v^2)$  is a polynomial of degree  $2l - 2$  in  $v$ . The degree of  $h$  comes from the degree of  $w^2(v^2)$  which is  $2N_c - 4$  from (3.4) and the degree of  $T(v^2)$  is  $4N_c - 4l$  giving the degree  $2l - 2$  for  $h(v^2)$ . As in the case with one massless dyon, for  $N_f < 2N_c - 2$  we can write

$$\frac{h(v) \sqrt{T(v^2)}}{v^2} = \frac{h(v) C_{2N_c}(v^2)}{v^2 \prod_{i=1}^l (v^2 - p_i^2)} + \mathcal{O}(v^{-4}) \quad (4.44)$$

and we decompose

$$\frac{h(v) C_{2N_c}(v^2)}{v^2} = G_1(p_i^2) + G_2(v^2) \prod_{i=1}^l (v^2 - p_i^2). \quad (4.45)$$

We obtain  $w_i^2 = w^2(v^2 \rightarrow m_i^2) = G_1(m_i^2) / \prod_{j=1}^l (m_i^2 - p_j^2)$ . The value for  $G_1(p_i^2)$  is

$$G_1(p_i^2) = h(p_i) \sqrt{\prod_{j=1}^{N_f} (p_i^2 - m_j^2)} \Lambda_{N=2}^{2N_c-2-N_f}. \quad (4.46)$$

Then again the discussion goes the same way as in [31] and is similar to the one involving only one massless dyon. We determined the coefficients of the polynomial  $G_1(m_i^2)$  and we plugged back into the expression for  $w_i^2$  to obtain

$$w_j^2 = \sum_{i=1}^l \frac{h(p_i) \det(p_i^2 - m^2)^{1/2}}{\prod_{k \neq i} (p_k^2 - p_i^2)(p_i^2 - m_j^2)} \Lambda_{N=2}^{2N_c-2-N_f}. \quad (4.47)$$

Because the unbroken gauge group is the same as the one obtained in the  $SU$  case, formulas (2.60) and (4.47) agree only for  $r_1 = \dots = r_l = 2$  by identifying  $a_i$  with  $p_i$  and  $\phi$  with  $h$ . So in  $SO(2N_c)$  case as one of the  $r_i$ 's is greater than 2, we have a disagreement between the field theory and brane configuration result. This implies that  $W_\Delta \neq 0$  in the “integrating in” method. It would be very interesting how to obtain the singular submanifolds of the  $N = 2$  Coulomb branch the low energy effective superpotential parametrizes.

## 5. Conclusions

In the present work we have considered  $N = 2$  supersymmetric gauge theories with gauge groups  $SO(N_c)$  by using field theory approach. By adding a general superpotential (2.11) corresponding to the relative orientation between two NS5 branes, we obtained the description of the resulting  $N = 1$  gauge theory in both field theory and string/M theory. The nonsingular locus of the  $N = 2$  Coulomb branch is lifted while the only singular points

remain, where massless monopoles condense by perturbation. We explicitly calculated the monopole vevs, in field theory by studying a point in the  $N = 2$  moduli space of vacua and in M theory by exploiting the M theory fivebrane configuration. We have obtained the same contradiction as in the case of  $SU(N_c)$  groups given by the low energy effective superpotential obtained by the “integrated in” method. In other words, this is zero for the enhanced gauge group  $SU(2)$  but is different from zero for  $SU(r)$  with  $r > 2$ . We have also given the examples of  $SO(5)$  and  $SO(6)$  gauge groups in order to illustrate our methods.

As in the case of the simplest mass superpotential studied previously [33], we did not obtain any information about the particles at singular points, i.e., about their exact electric and magnetic charges. This remains an interesting direction to pursue both in field theory and in M theory. Also, it is extremely important to obtain the corrections to  $W_\Delta$  in order to see a complete match between field theory and M theory. It would be related to calculate the superpotential using the fivebrane configuration. These two directions are very important and deserve further study.

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