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Deformations of compact coassociative 4-folds with boundary

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ABSTRACT

Let M be a 7-manifold with a G_2-structure induced by a closed 'positive' differential 3-form. We study deformations of a compact coassociative 4-submanifold $N \subset M$ with non-empty boundary ∂N contained in a fixed, codimension 1 submanifold S of M with a compatible Hermitian symplectic structure. We show that 'small' coassociative deformations of N with special Lagrangian boundary in S form a smooth moduli space of finite dimension not greater than the Betti number $b^1(\partial N)$. It is also shown that N is 'stable' under small deformations of the closed G_2 3-form on the ambient 7-manifold M. The results can be compared to those for minimal Lagrangian submanifolds of Calabi–Yau manifolds proved in [A. Butscher, Deformations of minimal Lagrangian submanifolds with boundary, Proc. Amer. Math. Soc. 131 (2002), 1953–1964]. Some examples are also discussed.

1. Introduction

Coassociative 4-folds are a particular class of 4-dimensional submanifolds which may be defined in a 7-dimensional manifold M endowed with a 'G_2 form' φ. The latter is a differential 3-form which is invariant at each point under the action of the exceptional Lie group G_2. This 3-form induces a G_2-*structure* on M and, consequently, a Riemannian metric and orientation. If the form φ is coclosed then every coassociative 4-fold in M is calibrated and hence minimal. See Section 2 for precise definitions and a summary of the relevant theory. Coassociative submanifolds were introduced by Harvey and Lawson [8] who also gave SU(2)-invariant examples of these submanifolds in Euclidean $\mathbb{R}^7$. Examples of compact coassociative submanifolds of compact 7-manifolds with holonomy G_2 were given by Joyce [11] and later by the first author [15].

McLean [17] showed that, when the G_2-structure 3-form is closed, the deformations of a compact coassociative 4-fold without boundary are *unobstructed* and the moduli space of local deformations is a finite-dimensional smooth manifold.

There is some analogy between coassociative submanifolds of G_2-manifolds and special Lagrangian submanifolds of Calabi–Yau manifolds. Both are calibrated minimal submanifolds and may be equivalently defined by the vanishing of appropriate differential forms on the ambient manifold. Compact closed special Lagrangian submanifolds also have an unobstructed deformation theory and finite-dimensional moduli space, a result due again to McLean [17]. Butscher [3]

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extended this result to compact minimal Lagrangian submanifolds with boundary and showed that these have a finite-dimensional moduli space of deformations if the boundary lies in an appropriately chosen symplectic submanifold which he called a *scaffold*.

On a 7-manifold endowed with a G_2-structure, there is also a distinguished class of 3-dimensional submanifolds known as associative 3-folds. Recently, compact associative 3-folds with boundary in a coassociative 4-fold were studied in [5]. Here the deformation theory is obstructed, but the expected dimension of the moduli space is given in terms of the boundary of the associative 3-fold.

In this article, we study the deformations of compact coassociative 4-folds N with boundary in a particular fixed 6-dimensional submanifold $S \subset M$ which, by analogy with [3], we also call a scaffold (Definition 3.5). The condition on S is that it has a Hermitian symplectic structure compatible with the SU(3)-structure it inherits from M. We also require that the normal vectors to S at ∂N are tangent to N. The culmination of the research presented here is the following two theorems.

Theorem 1.1. *Suppose that M is a 7-manifold with a G_2-structure given by a closed 3-form. The moduli space of compact coassociative local deformations of N in M with boundary ∂N in a scaffold S is a finite-dimensional smooth manifold of dimension not greater than $b^1(\partial N)$.*

Theorem 1.2. *Let $\varphi(t)$ be a smooth 1-parameter family of closed 3-forms defining G_2-structures on M. Suppose that a compact submanifold $N \subset M$ with boundary is coassociative with respect to the G_2-structure of $\varphi(0)$ and the boundary ∂N is contained in a scaffold S.*

If $\varphi(t)|_N$ and the normal part of $\varphi(t)|_N$ on ∂N are exact for all t then N can be extended to a smooth family $N(t)$ for small $|t|$, with $N(0) = N$, such that $N(t)$ is coassociative with respect to $\varphi(t)$ and the boundary of $N(t)$ is in S.

The principal ingredient in the proof of Theorems 1.1 and 1.2 is the construction of an appropriate boundary value problem with Fredholm properties. For geometric reasons, the boundary value problem for coassociative 4-folds with boundary *cannot* be of a standard Dirichlet or Neumann type (see Remark on Section 3.2). Our study of coassociative deformations leads to a boundary value problem of second-order and altogether quite different from that for the minimal Lagrangians in [3], which is Neumann first-order.

We set-up the infinitesimal deformation problem for coassociative submanifolds with boundary in a scaffold in Section 4, where we also study the Fredholm properties and give a version of the Tubular Neighbourhood Theorem which is adapted to our needs. Then, in Section 4, we define a 'deformation map' and apply the Implicit Function Theorem to it in order to prove Theorems 1.1 and 1.2. We also briefly discuss some applications of the deformation theory in Section 4.4.

Note. Submanifolds are taken to be embedded, for convenience, since the results hold for immersed submanifolds by simple modification of the arguments given. Smooth functions (and, more generally, sections of vector bundles) on N are understood as 'smooth up to the boundary', so at each point of ∂N these have one-sided partial derivatives of any order in the inward-pointing normal direction.

2. Coassociative 4-folds

The key to defining coassociative 4-folds lies with the introduction of a distinguished 3-form on the Euclidean space $\mathbb{R}^7$.

Definition 2.1. Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \cdots \wedge dx_k$. Define a 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ by:

$$\varphi_0 = d\mathbf{x}_{123} + d\mathbf{x}_{145} + d\mathbf{x}_{167} + d\mathbf{x}_{246} - d\mathbf{x}_{257} - d\mathbf{x}_{347} - d\mathbf{x}_{356}. \tag{1}$$

The Hodge dual of φ_0 is a 4-form given by:

$$*\varphi_0 = d\mathbf{x}_{4567} + d\mathbf{x}_{2367} + d\mathbf{x}_{2345} + d\mathbf{x}_{1357} - d\mathbf{x}_{1346} - d\mathbf{x}_{1256} - d\mathbf{x}_{1247}.$$

The usual basis of $\mathbb{R}^7$ may be identified with the standard basis of the cross-product algebra of pure imaginary octonions. Then

$$\varphi_0(\mathbf{x}, \mathbf{y}, \mathbf{z}) = g_0(\mathbf{x} \times \mathbf{y}, \mathbf{z}) \quad \text{for any } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^7, \tag{2}$$

where g_0 denotes the Euclidean metric.

The subgroup of $GL(7, \mathbb{R})$ preserving the cross-product is the Lie group G_2, which is also a subgroup of SO(7), so G_2 is the stabilizer of φ_0 in the action of $GL(7, \mathbb{R})$. In light of this property, φ_0 is sometimes called a 'G_2 3-form' on $\mathbb{R}^7$; our choice of expression (1) for φ_0 follows that of [12, Definition 11.1.1].

Remark. We note, for later use, that the stabilizer of a non-zero vector $\mathbf{e} \in \mathbb{R}^7$ in the action of G_2 is a maximal subgroup of G_2 isomorphic to SU(3). Thus SU(3) is the stabilizer of a pair (ω, Υ), for a 2-form $\omega = (\mathbf{e} \lrcorner \varphi_0)|_{\mathbf{e}^\perp}$ and a 3-form $\Upsilon = \varphi_0|_{\mathbf{e}^\perp}$, in the action of $GL(6, \mathbb{R})$ on the orthogonal complement $\mathbf{e}^\perp \cong \mathbb{R}^6$ (cf. [10]).

Definition 2.2. A 4-dimensional submanifold P of $\mathbb{R}^7$ is *coassociative* if and only if $\varphi_0|_P \equiv 0$.

The condition $\varphi_0|_P \equiv 0$ forces $*\varphi_0$ to be a non-vanishing 4-form on P. Thus $*\varphi_0|_P$ induces a canonical *orientation* on P.

Definition 2.2 is equivalent to the definition used in *calibrated geometry* [7,12]. That is, a coassociative 4-fold P is a submanifold calibrated by $*\varphi_0$, which means that $*\varphi_0|_P$ is the volume form for the Riemannian metric induced by g_0 on P with the canonical orientation [12, Proposition 12.1.4]. Every coassociative submanifold of $\mathbb{R}^7$ is a *minimal* submanifold [8, Theorem II.4.2] and, moreover, volume minimizing [7, Theorem 7.5].

So that we may describe coassociative submanifolds of more general 7-manifolds, we make two definitions following [2, p. 7].

Definition 2.3. Let M be an oriented 7-manifold. A 3-form φ on M is *positive* if $\varphi(x) = \iota_x^*(\varphi_0)$ for all $x \in M$ for some orientation preserving linear isomorphism $\iota_x : T_xM \to \mathbb{R}^7$, where φ_0 is given in Eq. (1). Denote the subbundle of positive 3-forms on M by $\Lambda^3_+T^*M \subset \Lambda^3T^*M$ and the fibre of this subbundle over $x \in M$ by $\Lambda^3_+T^*_xM$. We write $\Omega^3_+(M)$ for the space of all (smooth) positive 3-forms on M.

For each $x \in M$, $\Lambda^3_+T^*_xM$ is the image of the open $\mathrm{GL}_+(7,\mathbb{R})$-orbit of φ_0 in $\Lambda^3(\mathbb{R}^7)^*$ under ι_x^* given in the above definition. It follows that $\Lambda^3_+T^*M$ is an open subbundle of Λ^3T^*M.

Since a positive 3-form φ is identified at each point in M with the 3-form φ_0 stabilized by G_2, it determines a G_2-structure on M. We shall sometimes simply say that $\varphi \in \Omega^3_+(M)$ *is* a G_2-structure on the oriented 7-manifold M.

Furthermore, as $\mathrm{G}_2 \subset \mathrm{SO}(7)$, we can uniquely associate to each $\varphi \in \Omega^3_+(M)$ a Riemannian metric $g = g(\varphi)$ and the Hodge dual 4-form $*\varphi$ relative to the Hodge star of g. The triple $(\varphi, *\varphi, g)$ corresponds to $(\varphi_0, *\varphi_0, g_0)$ at each point.

Definition 2.4 (*[12, pp. 228, 264]*). Let M be an oriented 7-manifold. We call a G_2-structure $\varphi \in \Omega^3_+(M)$ *torsion-free* if φ is closed and coclosed with respect to the induced metric $g(\varphi)$.

An *almost* G_2*-manifold* (M, φ) is a 7-manifold endowed with a G_2-structure φ such that $d\varphi = 0$. If a G_2-structure φ is torsion-free then (M, φ) is called a G_2*-manifold*.

Remark. By [19, Lemma 11.5], the holonomy of a metric g on M is contained in G_2 if and only if $g = g(\varphi)$ for some torsion-free G_2-structure φ on M. Manifolds with a closed G_2 3-form are important in the constructions of examples of compact irreducible G_2-manifolds in [11,14].

We are now able to give a more general version of Definition 2.2.

Definition 2.5. Let M be an oriented 7-manifold and $\varphi \in \Omega^3_+(M)$ a G_2-structure on M. A 4-dimensional submanifold P of M is *coassociative* if and only if $\varphi|_P \equiv 0$.

The deformation theory of compact coassociative submanifolds was studied by McLean [17]. His results, summarized in Theorem 2.7 below, were stated for G_2-manifolds but it was later observed in [6] that the proof does not use the coclosed condition on the G_2 3-form φ.

By way of preparation, we note a standard consequence of the proof of the Tubular Neighbourhood Theorem in [16, Chapter IV, Theorem 9].

Proposition 2.6. *Let P be a closed submanifold of a Riemannian manifold M. There exist an open subset V_P of the normal bundle $\nu_M(P)$ of P in M, containing the zero section, and a tubular neighbourhood T_P of P in M, such that the exponential map $\exp_M|_{V_P} : V_P \to T_P$ is a diffeomorphism onto T_P.*

The local deformations of P are understood as submanifolds of the form $\exp_{\mathbf{v}}(P)$, where $\mathbf{v}$ is a C^1-section of the normal bundle $\nu_M(P)$ and $\mathbf{v}$ is assumed sufficiently small in the sup-norm. We shall call sections of $\nu_M(P)$ the *normal vector fields* at P.

Now suppose that (M, φ) is an almost G_2-manifold and a submanifold $P \subset M$ is coassociative. Then the local deformations of P may equivalently be given by self-dual 2-forms on P using an isometry of vector bundles (cf. [17, Proposition 4.2])

$$\jmath_P : \mathbf{v} \in \nu_M(P) \to (\mathbf{v} \lrcorner \varphi)|_P \in \Lambda^2_+T^*P; \tag{3}$$

where $\Lambda^2_+T^*P$ denotes the bundle of self-dual 2-forms. The map

$$F : \alpha \in \Omega^2_+(P) \to \exp^*_{\mathbf{v}}(\varphi) \in \Omega^3(P), \quad \mathbf{v} = \jmath_P^{-1}(\alpha), \tag{4}$$

is defined for 'small' α, and $F(\alpha) = 0$ precisely if $\exp_{\mathbf{v}}(P)$ is a coassociative deformation.

Theorem 2.7 (*Cf. [17, Theorem 4.5], [13, Theorem 2.5]*). *Let (M, φ) be an almost G_2-manifold and $P \subset M$ a coassociative submanifold (not necessarily closed).*

(a) *Then for each $\alpha \in \Omega^2_+(P)$, one has $dF|_0(\alpha) = d\alpha$ and the 3-form $F(\alpha)$ (if defined) is exact.*
(b) *If, in addition, P is compact and without boundary then every closed self-dual 2-form α on P arises as $\alpha = \jmath_P(\mathbf{v})$, for some normal vector field $\mathbf{v}$ tangent to a smooth 1-parameter family of coassociative submanifolds containing P. Thus, in this case, the space of local coassociative deformations of P is a smooth manifold parameterized by the space $\mathcal{H}^2_+(P)$ of closed self-dual 2-forms on P.*

Remark. Self-dual 2-forms on a compact manifold without boundary are closed precisely if they are harmonic. By Hodge theory, the dimension of $\mathcal{H}^2_+(P)$ is therefore equal to the dimension $b^2_+(P)$ of a maximal positive subspace for the intersection form on P. It is thus a topological quantity.

Finally, there is a useful extension of Theorem 2.7 in the situation where the G_2-structure is allowed to vary. This result is stated in [12, Theorem 12.3.6] and can be proved using the techniques of [17, Section 4].

Theorem 2.8. *Let $\varphi(t) \in \Omega^3_+(M)$, $t \in \mathbb{R}$, be a smooth path of closed G_2-structure forms on M. Suppose that P is a compact submanifold of M without boundary such that $\varphi(0)|_P = 0$ and the form $\varphi(t)|_P$ is exact for each t.*

Then there is an $\varepsilon > 0$ and, for each $|t| < \varepsilon$, a section $\mathbf{v}(t)$ of $\nu_M(P)$ smoothly depending on t, such that $\mathbf{v}(0) = 0$ and $\varphi(t)$ vanishes on the submanifold $P(t) = \exp_{\mathbf{v}(t)}(P)$. Thus $P(t)$ is a coassociative 4-fold in $(M, \varphi(t))$. Here the normal bundle $\nu_M(P)$ and the exponential map are understood with respect to the metric induced by $\varphi(0)$.

Roughly speaking, this result says that compact coassociative 4-folds are 'stable' under small variations of the ambient closed G_2-structure.

3. The infinitesimal deformation problem

Throughout this section, (M, φ) is an almost G_2-manifold, $g = g(\varphi)$ is the metric induced by φ and N is a compact coassociative submanifold of M.

3.1. SU(3)-structures on 6-dimensional submanifolds

In order to explain and motivate our choice of the boundary condition in the next subsection, we recall some results on SU(3)-structures and Calabi–Yau geometry and their relation to G_2 geometry.

To begin, suppose that S is an orientable 6-dimensional submanifold of M. Then the normal bundle of S is trivial and, by the Tubular Neighbourhood Theorem (Proposition 2.6), there exists a neighbourhood T_S of S which is diffeomorphic to $S \times \{-\varepsilon_S < s < \varepsilon_S\}$, for some $\varepsilon_S > 0$, so that $S = \{s = 0\}$ and $\mathbf{n}_S = \frac{\partial}{\partial s}$ is a unit vector field on T_S, with $\mathbf{n}_S|_S$ orthogonal to S. We shall sometimes call s the normal coordinate near S. We can write

$$\varphi|_{T_S} = \omega_s \wedge \mathrm{d}s + \Upsilon_s \tag{5}$$

for some 1-parameter family of 2-forms and 3-forms ω_s and Υ_s on S.

It is not difficult to see, from the remark following Definition 2.1, that the forms $\omega_0 = (\mathbf{n}_S \lrcorner \varphi)|_S$ and $\Upsilon_0 = \varphi|_S$ together induce an SU(3)-structure (in general, with torsion) on S. In particular, S is oriented by

$$\frac{\omega_0^3}{3!} = \frac{1}{4}\Upsilon_0 \wedge *_6\, \Upsilon_0$$

and has an induced almost complex structure I, compatible with the orientation, which may be given by $I(\mathbf{u}) = \mathbf{n}_S \times \mathbf{u}$ for all $\mathbf{u} \in T_xS \subset T_xM$. Here we denoted by $*_6$ the Hodge star on the 6-manifold S with respect to the induced metric from M, and we used the cross-product on T_xM given by a G_2-invariant identification with the standard G_2 3-form on $\mathbb{R}^7$ as in Definition 2.3.

The metric induced on S is Hermitian with respect to I and its fundamental $(1, 1)$-form is ω_0. The non-vanishing complex 3-form $\Omega_0 = *_6\, \Upsilon_0 - i\Upsilon_0$ has type $(3, 0)$ relative to I. We also note that, as ds is a unit 1-form, the pointwise model Eq. (1) for φ yields the relation

$$\omega_0^3 = \frac{3\mathrm{i}}{4}\Omega_0 \wedge \bar{\Omega}_0 \tag{6}$$

at each point in S.

Denote by d_S the exterior derivative on S.

Lemma 3.1. *Let S be an oriented 6-dimensional submanifold of an almost G_2-manifold (M, φ) and suppose that $d_S\,\omega_0 = 0$, where $\omega_0 \in \Omega^2(S)$ is defined in Eq. (5). Let $N \subset M$ be a coassociative submanifold intersecting S in a 3-dimensional submanifold L such that $\mathbf{n}_S|_L$ is tangent to N. Then $\omega_0|_L = 0$ and $\Upsilon_0|_L = 0$.*

Proof. The last assertion is clear as $\Upsilon_0|_L = \varphi|_L$, $L \subset N$ and $\varphi|_N = 0$. As

$$0 = d\varphi|_{T_S} = \left(d_S\omega_s - \frac{\partial}{\partial s}\Upsilon_s\right) \wedge \mathrm{d}s + d_S\Upsilon_s$$

we find that $\frac{\partial}{\partial s}\Upsilon_s|_{s=0} = 0$. The submanifolds S and N intersect transversely, so ds never vanishes on a neighbourhood of L in N. Restricting Eq. (5) to this neighbourhood we obtain $\omega_0|_L = 0$ as $\mathbf{n}_S$ is a normal vector field at S. □

The property $\omega_0|_L = 0$ means that L is a Lagrangian submanifold of a symplectic manifold (S, ω_0). To explain the role of the additional condition $\Upsilon_0|_L = \varphi|_L = 0$, we begin with the following.

Definition 3.2. An orientable 6-dimensional submanifold S is a *symplectic submanifold* of the almost G_2-manifold (M, φ) if $d_S\omega_0 = 0$, where $\omega_0 = (\mathbf{n}_S \lrcorner \varphi)|_S$ and s is the normal coordinate near S.

A 3-dimensional submanifold $L \subset S$ of a symplectic submanifold $S \subset M$ is said to be *special Lagrangian* if $\omega_0|_L = 0$ and $\varphi|_L = 0$.

It is not difficult to see, using the pointwise model Eq. (1), that every special Lagrangian submanifold of S is oriented by a nowhere-vanishing 3-form $*_6 \Upsilon_0 = (\mathbf{n}_S \lrcorner * \varphi)|_L$.

Definition 3.2 extends the concept of special Lagrangian submanifolds usually found in the literature to a more general class of ambient manifolds, similar to [20]. To clarify this generalisation we note the following.

Proposition 3.3. *Let S be a symplectic submanifold of an almost G_2-manifold. The almost complex structure I on S is integrable if and only if $d_S *_6 \Upsilon_0 = 0$. Furthermore, in this case the Kähler metric defined by ω_0 is Ricci-flat.*

Proof. As $d\varphi = 0$ on M, we obtain from Eq. (5) that $d_S \Upsilon_0 = 0$. Thus, $d_S \Omega_0 = 0$ if and only if $d_S *_6 \Upsilon_0 = 0$. The proposition now follows by the arguments in [9, Section 2]; we omit the details. □

When I is integrable the nowhere-vanishing (3, 0)-form is automatically holomorphic. A Kähler metric is Ricci-flat if and only if its restricted holonomy group is contained in SU(3). A complex 3-fold (S, ω_0, Ω_0) endowed with the Ricci-flat Kähler metric and a holomorphic (3, 0)-form satisfying (6) is sometimes called a *Calabi–Yau 3-fold*. Thus, a symplectic submanifold of an almost G_2-manifold is a natural generalisation of a Calabi–Yau 3-fold by weakening the condition on the complex structure.

Deformations of compact closed special Lagrangian submanifolds in Calabi–Yau manifolds are unobstructed: there is a theorem of a similar type to Theorem 2.7 and due again to McLean [17, Theorem 3.6]. Salur [20] showed that the integrability of the complex structure on the Calabi–Yau manifold is unnecessary for the deformation theory result to hold. Thus, the deformation theory remains valid for symplectic submanifolds S of an almost G_2-manifold (M, φ).

The next result provides motivation for our choice of boundary conditions for coassociative submanifolds later.

Theorem 3.4 (*Cf.* [20]). *Let (S, ω_0) be a symplectic submanifold of an almost G_2-manifold and let $L \subset S$ be a compact special Lagrangian submanifold without boundary. Then*

(a) *$\jmath_L : \mathbf{v} \mapsto (\mathbf{v} \lrcorner \omega_0)|_L$ defines a vector bundle isometry between the normal bundle of L in S, $\nu_S(L)$, and T^*L.*

(b) *The normal vector fields at L defining infinitesimal special Lagrangian deformations correspond, via $\jmath_L$, to closed and coclosed 1-forms on L. Conversely, every closed and coclosed 1-form on L corresponds via $\jmath_L$ to a normal vector field $\mathbf{v}$ at L which is tangent to a path of special Lagrangian deformations of L.*

Proof. This is immediate from the proof of the main theorem in [20]. □

3.2. Boundary conditions

From now on we assume that a compact coassociative submanifold $N \subset M$ has non-empty boundary ∂N. To achieve the Fredholm property of our deformation problem for a coassociative submanifold N with boundary, we need to impose a condition that the boundary is confined to move in a suitable submanifold of M. We call this submanifold a scaffold, borrowing the terminology of [3, Definition 1], where the deformations of minimal Lagrangians with boundary are studied.

Definition 3.5. We say that an orientable 6-dimensional submanifold S of M is a *scaffold* for N if

(a) $\partial N \subset S$, $\mathbf{n} \in \nu_M(S)|_{\partial N}$ and

(b) S is a symplectic submanifold of (M, φ).

The condition (a) implies that $\partial N \subset S$ is special Lagrangian, by Lemma 3.1. Then, by Theorem 3.4, (b) ensures that special Lagrangian deformations of ∂N in S are unobstructed with a smooth moduli space. Thus it is suggestive to consider coassociative deformations of N which remain 'orthogonal to S', i.e. $\mathbf{n}_S$ is tangent to the deformation.

We shall always assume that $\mathbf{n} = \mathbf{n}_S|_{\partial N} \in C^\infty(TN|_{\partial N})$ is a unit inward-pointing normal vector field on N at ∂N. Respectively, $s_\mathbf{n} = s|_N$ is a normal coordinate near the boundary of N. That is, a map

$$(x, s_\mathbf{n}) \mapsto \exp_N(s_\mathbf{n}\mathbf{n}(x)) \tag{7}$$

is defined for all $(x, s_\mathbf{n}) \in \partial N \times [0, \epsilon_N)$ and gives a diffeomorphism of $\partial N \times [0, \epsilon_N)$ onto a collar neighbourhood $C_{\partial N}$ of ∂N in N. In particular, $\mathbf{n} = \frac{\partial}{\partial s_\mathbf{n}}$.

Recall from Theorem 2.7 that infinitesimal deformations of N are given by closed self-dual 2-forms α on N. On the collar neighbourhood $C_{\partial N} = T_S \cap N$ of ∂N in N, we can write a self-dual 2-form α as

$$\alpha|_{C_{\partial N}} = \xi_s \wedge \mathrm{d}s + *_s \xi_s \tag{8}$$

for a 1-parameter family of 1-forms ξ_s on ∂N, where $*_s$ denotes the Hodge star on the submanifold $(\partial N) \times \{s\} \subset C_{\partial N}$ corresponding to a fixed value of s.

For a self-dual 2-form α on N, we easily calculate, using Eq. (8) and restricting to $C_{\partial N}$, that $d\alpha = 0$ implies that

$$d_{\partial N}\xi_s + \frac{\partial}{\partial s}(*_s \xi_s) = 0 \quad \text{and} \quad d_{\partial N} *_s \xi_s = 0. \tag{9}$$

The infinitesimal special Lagrangian deformations correspond to closed and coclosed 1-forms by Theorem 3.4. Therefore, $d\alpha = 0$ leads to infinitesimal special Lagrangian deformations of the boundary if and only if

$$\frac{\partial}{\partial s}(*_s\,\xi_s)|_{s=0} = 0. \tag{10}$$

Notice that

$$d_{\partial N}(\mathbf{n}\lrcorner\alpha) = -d_{\partial N}\xi_0. \tag{11}$$

Hence, the equations

$$d\alpha = 0 \quad \text{on } N \quad \text{and} \quad d_{\partial N}(\mathbf{n}\lrcorner\alpha) = 0 \quad \text{on } \partial N, \tag{12}$$

by Eq. (9), are equivalent to $d\alpha = 0$ with condition (10).

We can describe the boundary condition corresponding to (10) for normal vector fields $\mathbf{v} = \jmath_N^{-1}(\alpha)$ at N directly as follows. Recall that, in a neighbourhood T_S of S, $\varphi|_{T_S} = \omega_s \wedge ds + \Upsilon_s$ with ω_0 closed as S is a scaffold, hence $\frac{\partial \Upsilon_s}{\partial s}|_{s=0} = 0$ as $d\varphi = 0$. The equation (10) that α satisfies is equivalent to the condition $\frac{\partial}{\partial s}(\mathbf{v}_s\lrcorner\Upsilon_s)|_{s=0} = 0$, whence

$$\left.\frac{\partial \mathbf{v}_s}{\partial s}\right|_{s=0} \lrcorner\Upsilon_0 = 0 \quad \text{on } \partial N \tag{13}$$

where we have expressed $\mathbf{v}$ on the collar neighbourhood of ∂N as a 1-parameter family of vector fields on ∂N.

Remark. For a general 2-form $\tilde{\alpha} = \alpha_\tau + \alpha_\nu \wedge ds$, the familiar Dirichlet and Neumann boundary conditions are given by, respectively, $\alpha_\tau = 0$ and $\alpha_\nu = 0$. For a self-dual α, the two conditions are equivalent and force α and the corresponding normal vector field $\jmath_N^{-1}(\alpha)$ to vanish at each point of ∂N. However, if $d\alpha = 0$ and α vanishes on the boundary then $\alpha = 0$ by [4, Lemma 2]. This may be understood as an extension of [8, Theorem IV.4.3], which states that there is a locally unique coassociative submanifold containing any real analytic 3-dimensional submanifold upon which φ vanishes.

For a coassociative submanifold without boundary, a self-dual 2-form is closed if and only if it is harmonic, i.e. satisfies $\Delta\alpha = d_+^* d\alpha = 0$. When there is a non-empty boundary a harmonic self-dual 2-form need not in general be closed. The following lemma is a direct corollary of [21, Proposition 3.4.5] proved by integration by parts.

Lemma 3.6. *For $\alpha \in \Omega_+^2(N)$, $d\alpha = 0$ if and only if*

$$d_+^* d\alpha = 0 \quad \text{on } N \quad \text{and} \quad \mathbf{n}\lrcorner d\alpha = 0 \quad \text{on } \partial N, \tag{14}$$

where $d_+^* = \frac{1}{2}(d^* + *d^*) : \Omega^3(N) \to \Omega_+^2(N)$ is the L^2-adjoint of $d|_{\Omega_+^2}$.

Our next proposition relates the infinitesimal coassociative deformations of N to solutions of a Fredholm linear boundary value problem.

Proposition 3.7. *For $p > 1$ and $k \geq 1$, the map*

$$\alpha \in L_{k+1}^p\Omega_+^2(N) \to \left(d_+^* d\alpha,\; d_{\partial N}(\mathbf{n}\lrcorner\alpha),\; d_{\partial N}(\alpha|_{\partial N})\right) \in L_{k-1}^p\Omega_+^2(N) \oplus L_{k-\frac{1}{p}}^p\left(d\Omega^1(\partial N) \oplus d\Omega^2(\partial N)\right), \tag{15}$$

is surjective. The kernel of (15) consists of smooth forms and has dimension $b^1(\partial N)$. In particular, the map (15) is Fredholm.

Proof. By [21, Theorem 3.4.10], for each $\eta \in \Omega^2(N)$, the solution α of $\Delta\alpha = \eta$ on N exists and is uniquely determined by its tangential $\alpha|_{\partial N} = \xi_\tau \in \Omega^2(\partial N)$ and normal $\mathbf{n}\lrcorner\alpha = \xi_\nu \in \Omega^1(\partial N)$ components at the boundary. Now if $\eta \in \Omega_+^2(N)$, $\xi_\tau = *_{\partial N}\,\xi_\nu$ and α is a solution of the latter boundary value problem, then so is $*\alpha$, so α is self-dual by uniqueness. Thus $\alpha \in \Omega_+^2(N)$ is uniquely determined in this case by $\Delta\alpha$ and $\alpha|_{\partial N}$. As the manifold ∂N is compact and without boundary, it now follows from the Hodge theory on ∂N that the values of $d_{\partial N}(\mathbf{n}\lrcorner\alpha)$ and $d_{\partial N}(\alpha|_{\partial N})$ can be prescribed independently and the operator (15) is surjective. These values are both zero precisely when $\mathbf{n}\lrcorner\alpha$ is a harmonic 1-form on ∂N, which gives the dimension $b^1(\partial N)$ of the kernel of (15).

We see from [21, Theorem 3.4.10], that solutions to $\Delta\alpha = 0$ with $\alpha = \psi$ on ∂N are smooth if ψ is smooth. A similar result holds if the first derivatives of $\alpha = \psi$ on ∂N are smooth, so the kernel of (15) consists of smooth forms. □

In light of the work in this section, we shall be interested in the following subspace of self-dual 2-forms:

$$\Omega_+^2(N)_{\mathrm{bc}} = \{\alpha \in \Omega_+^2(N) :\; \mathbf{n}\lrcorner d\alpha = 0 \text{ and } d_{\partial N}(\mathbf{n}\lrcorner\alpha) = 0 \text{ on } \partial N\}. \tag{16}$$

Our next result follows immediately from Lemma 3.6 and Proposition 3.7.

Corollary 3.8. *The image of $L^p_{k+1}\Omega^2_+(N)_{\rm bc}$ under the map (15) is a closed subspace*

$$V^p_{k-1} \subseteq L^p_{k-1}\Omega^2_+(N) \oplus L^p_{k-\frac{1}{p}}\left(d\Omega^1(\partial N) \oplus d\Omega^2(\partial N)\right). \tag{17}$$

The kernel of (15) intersects $L^p_{k+1}\Omega^2_+(N)_{\rm bc}$ in the subspace of (smooth) closed forms in $\Omega^2_+(N)_{\rm bc}$,

$$(\mathcal{H}^2_+)_{\rm bc} = \{\alpha \in \Omega^2_+(N) \,:\, d\alpha = 0 \text{ on } N \text{ and } d_{\partial N}(\mathbf{n}\lrcorner\alpha) = 0 \text{ on } \partial N\}, \tag{18}$$

and $\dim(\mathcal{H}^2_+)_{\rm bc} \le b^1(\partial N)$.

An example when strict inequality $\dim(\mathcal{H}^2_+)_{\rm bc} < b^1(\partial N)$ occurs is given in Section 4.4.

4. Coassociative local deformations

In this section, like in Section 3, (M, φ) is an almost G_2-manifold and $N \subset M$ is a compact coassociative submanifold with boundary in a scaffold S.

We shall now define a version of a *deformation map G* whose linearization gives the linear problem set up in the previous section. The role of G for the study of deformations of N with boundary in S is similar to the role of F in Eq. (4) for the closed coassociative submanifolds. However, our deformation map modifies (4) in two ways. First, we use the exponential mapping $\widehat{\exp}$ of a metric $\hat{g}$ defined in Section 4.1 which in general is not the metric $g(\varphi)$ induced by the G_2-structure φ. Second, our non-linear differential operator G is of second-order, with the derivative at zero given by (15) restricted to $\Omega^2_+(N)_{\rm bc}$, defined in Eq. (16). An application of the Implicit Function Theorem to G will show that the space of local coassociative deformations of N is smooth and has finite dimension equal to that of the space of closed forms $(\mathcal{H}^2_+)_{\rm bc}$ in $\Omega^2_+(N)_{\rm bc}$.

4.1. Adapted tubular neighbourhoods

We wish to parameterize nearby deformations of N with boundary in the scaffold S by normal vector fields (or self-dual 2-forms on N) via an exponential map. In general, we cannot use, as in Section 2, exponential deformations of N given by $g(\varphi)$, since the scaffold may not be preserved under these deformations. Therefore, we shall define on M a modified metric whose related exponential map does preserve the scaffold; that is, the scaffold is totally geodesic with respect to the new metric. A similar approach was previously used in [3] for minimal Lagrangian submanifolds with boundary.

We first describe the local structure of the almost G_2-manifold M near S, applying a tubular neighbourhood argument (cf. Proposition 2.6).

Lemma 4.1. *There exist $\epsilon_S > 0$, an open neighbourhood T_S of S in M and a diffeomorphism $\eta_S : S \times (-\epsilon_S, \epsilon_S) \to T_S$, such that $\eta_S(x, 0) = x$, and $d\eta_S|_{(x,0)} : T_xS \times \mathbb{R} \to T_xM$ satisfies $d\eta_S|_{(x,0)}(\mathbf{0}, 1) = \mathbf{n}_S(x)$ for all $x \in S$, where $\mathbf{n}_S$ is the unit normal to S in M as in Section 3.1. Furthermore, η_S can be chosen so that*

$$\eta_S(x, s_{\mathbf{n}}) = \exp_N(s_{\mathbf{n}}\mathbf{n}(x))$$

for all $x \in \partial N$ and $s_{\mathbf{n}} \in [0, \epsilon_S)$.

Proof. Recall that $\nu_M(S)$ is trivialised by $\mathbf{n}_S$. Therefore, from the proof of the Tubular Neighbourhood Theorem in [16], we obtain $\epsilon > 0$, an open neighbourhood T of S in M and a diffeomorphism $\eta : S \times (-\epsilon, \epsilon) \to T$ given by

$$\eta(x, s_{\mathbf{n}}) = \exp_M(s_{\mathbf{n}}\mathbf{n}_S(x)).$$

The map η satisfies all but the last of the required conditions, as $\exp_M(s_{\mathbf{n}}\mathbf{n})$ need not agree with $\exp_N(s_{\mathbf{n}}\mathbf{n})$, even if $0 < \varepsilon_S \le \varepsilon_N$ (with ε_N as in the beginning of Section 3). However, since $T_{(x,0)}\eta^{-1}(N) = T_{(x,0)}\partial N \oplus \mathbb{R}\mathbf{n}$, a standard inverse mapping argument shows that, by composing η with a diffeomorphism ψ of $S \times (-\epsilon, \epsilon)$ such that $d\psi|_{(x,0)} = \mathrm{id}$, and choosing a sufficiently small $\varepsilon_S > 0$, we obtain $\eta_S : S \times (-\varepsilon_S, \varepsilon_S) \to T_S$ having all the required properties. □

We can now construct the new metric.

Proposition 4.2 (*Cf. [3, Proposition 6]*). *There is a metric $\hat{g}$ on M, which equals g outside of T_S, such that S is totally geodesic with respect to $\hat{g}$.*

Proof. First recall Lemma 4.1 and define a metric h on $S \times (-\epsilon_S, \epsilon_S)$ by

$$h = \eta_S^*(g|_S) + \mathrm{d}s_{\mathbf{n}} \otimes \mathrm{d}s_{\mathbf{n}}.$$

Let $\chi : M \to [0, 1]$ be a smooth function such that $\chi = 0$ outside T_S and $\chi = 1$ in some tubular neighbourhood of S contained in T_S. We then define $\hat{g}$ on M by

$$\hat{g} = \chi(\eta_S^{-1})^*(h) + (1 - \chi)g.$$

As in the proof of [3, Proposition 6], we see that S is totally geodesic with respect to $\hat{g}$. □

We shall need a variant of the isomorphism (3) for the normal bundle of N with respect to $\hat{g}$.

Proposition 4.3. *Let $\hat{\nu}_M(N)$ denote the normal bundle of N relative to the metric $\hat{g}$ given by Proposition 4.2. The map $\jmath_N : \hat{\nu}_M(N) \to (\Lambda^2_+)_g T^*N$ given by $\jmath_N(\mathbf{v}) = (\mathbf{v} \lrcorner \varphi)|_{TN}$ is an isomorphism. Moreover, $\hat{\nu}_M(N)|_{\partial N} \subseteq TS|_{\partial N}$.*

Proof. Recall from Eq. (3) that $\jmath_N : \nu_M(N) \to (\Lambda^2_+)_g T^*N$ is an isomorphism. Since $TM|_N = TN \oplus \nu_M(N) = TN \oplus \hat{\nu}_M(N)$, $\nu_M(N) \cong \hat{\nu}_M(N)$. Moreover, the fibres of $\hat{\nu}_M(N)$ are transverse to those of TN and $\jmath_N$ maps $TM|_N$ to $(\Lambda^2_+)_g T^*N$ since φ vanishes on TN. So $\jmath_N$ defines an isomorphism between the vector bundles $\hat{\nu}_M(N)$ and $(\Lambda^2_+)_g T^*N$.

The final claim follows because $\hat{g}(x)$ coincides with $g(x)$ at each $x \in \partial N$. □

As a consequence of Propositions 2.6, 4.2 and 4.3 we obtain a version of the Tubular Neighbourhood Theorem which is adapted to local deformations of N with boundary in S. Denote the exponential map on M with respect to $\hat{g}$ by $\widehat{\exp}_M$. However, we emphasise that the self-dual forms on N are always taken with respect to the metric $g = g(\varphi)$. For an open subset $\mathcal{W}$ of a vector bundle W on N, we define a subset of the smooth sections $\Gamma(W)$ on W by

$$\Gamma(\mathcal{W}) = \{w \in \Gamma(W) : w(N) \subset \mathcal{W}\}.$$

We also make similar definitions for subsets of Banach spaces of sections when the Banach spaces consist of continuous sections.

Proposition 4.4. *There exist an open subset $\mathcal{V}_N$ of $\hat{\nu}_M(N)$, containing the zero section, and a 7-dimensional submanifold $\mathcal{T}_N$ of M with boundary, containing N, such that $\widehat{\exp}_M : \mathcal{V}_N \to \mathcal{T}_N$ is a diffeomorphism such that if $\mathbf{v} \in \Gamma(\mathcal{V}_N)$, then $\widehat{\exp}_M(\mathbf{v}(x)) \in S$ for all $x \in \partial N$.*

Respectively, $\mathcal{U}_N = \jmath_N(\mathcal{V}_N)$ is an open neighbourhood of the zero section in $\Lambda^2_+ T^*N$ and $\delta_N = \widehat{\exp}_M \circ \jmath_N^{-1} : \mathcal{U}_N \to \mathcal{T}_N$ is a diffeomorphism such that, if $\alpha \in \Gamma(\mathcal{U}_N)$, then $\delta_N(\alpha(x)) \in S$ for all $x \in \partial N$, so $N_\alpha := \delta_N(\alpha(N)) \subset \mathcal{T}_N$ is a compact 4-dimensional submanifold of M with boundary $\partial N_\alpha \subset S$.

4.2. The deformation map

Definition 4.5. Let $\jmath_N$, $\widehat{\exp}$, $\mathcal{V}_N$ and $\mathcal{U}_N$ be as defined in Section 4.1. Denote

$$\hat{F} : \alpha \in \Gamma(\mathcal{U}_N) \to \widehat{\exp}_{\mathbf{v}}^* \left(\varphi|_{N_\alpha}\right) \in \Omega^3(N), \tag{19}$$

where $\mathbf{v} = \jmath_N^{-1}(\alpha) \in \Gamma(\mathcal{V}_N)$. We shall call the second-order non-linear differential operator

$$G = d_+^* \circ \hat{F} : \Gamma(\mathcal{U}_N) \to \Omega^2_+(N) \tag{20}$$

the *deformation map* for N.

Notice that the argument in [17, p. 731] proving Theorem 2.7(a) does not depend on the choice of metric for the exponential map, so we obtain the same result for $\hat{F}$ given in Eq. (19).

Lemma 4.6.

$$d\hat{F}|_0(\alpha) = d\alpha \quad \textit{and} \quad dG|_0(\alpha) = d_+^* d\alpha \tag{21}$$

for all $\alpha \in \Gamma(\mathcal{U}_N)$.

Next we impose boundary conditions on the deformations of the special Lagrangian submanifold ∂N in S. By Theorem 3.4, there exists an isomorphism $\jmath_{\partial N}$ between $\hat{\nu}_S(\partial N)$ and $T^*\partial N$ (noting that we can use the normal bundle with respect to the metric $\hat{g}$). From the neighbourhoods given in Proposition 4.4, one deduces that there exist open neighbourhoods $\mathcal{V}_{\partial N}$ and $\mathcal{U}_{\partial N}$ of the zero sections in $\hat{\nu}_S(\partial N)$ and $T^*\partial N$ respectively, with $\mathcal{U}_{\partial N} = \jmath_{\partial N}(\mathcal{V}_{\partial N})$, and a tubular neighbourhood $\mathcal{T}_{\partial N}$ of ∂N in S such that $\widehat{\exp}_S : \mathcal{V}_{\partial N} \to \mathcal{T}_{\partial N}$ is a diffeomorphism. Define

$$\hat{F}_{\partial N} : \beta \in \Gamma(\mathcal{U}_{\partial N}) \to \widehat{\exp}_{\mathbf{v}}^*(\mathbf{n}_S \lrcorner \varphi|_{\partial N_\beta}) \in d\Omega^1(\partial N), \tag{22}$$

where $\mathbf{v} = \jmath_{\partial N}^{-1}(\beta)$ and $\partial N_\beta = \widehat{\exp}_{\mathbf{v}}(\partial N)$. The kernel of $\hat{F}_{\partial N}$ characterises the Lagrangian (but not necessarily special Lagrangian) local deformations of ∂N in S. The fact that $\hat{F}_{\partial N}$ maps into the space of forms claimed is a consequence of arguments in [17, Section 3] since the form $\mathbf{n}_S \lrcorner \varphi$ is exact near ∂N as it is closed and vanishes on ∂N. Define

$$\Gamma(\mathcal{U}_N)_{\mathrm{bc}} = \{\alpha \in \Gamma(\mathcal{U}_N) : \hat{F}_{\partial N}(\mathbf{n} \lrcorner \alpha) = 0 \text{ and } \mathbf{n} \lrcorner \hat{F}(\alpha) = 0 \text{ on } \partial N\}. \tag{23}$$

It follows from the deformation theory for ∂N in S and the work in Section 3.2 that, by taking completion in the appropriate Sobolev norm, $L^p_{k+1}(\mathcal{U}_N)_{\mathrm{bc}}$ becomes a Banach submanifold of $L^p_{k+1}\Omega^2_+(N)$ and its tangent space at $\alpha = 0$ is $L^p_{k+1}\Omega^2_+(N)_{\mathrm{bc}}$ defined in Eq. (16).

The next result shows that we can define coassociative local deformations with boundary in S using a second-order differential operator.

Proposition 4.7. *For $\alpha \in \Gamma(\mathcal{U}_N)_{\mathrm{bc}}$, $G(\alpha) = 0$ if and only if N_α is coassociative. For any coassociative deformation $\alpha \in \Gamma(\mathcal{U}_N)_{\mathrm{bc}}$ defined by $G(\alpha) = 0$, the local deformation $\partial N_{\mathbf{n} \lrcorner \alpha} \subset S$ is special Lagrangian.*

Proof. It is clear that N_α is coassociative precisely if $\hat{F}(\alpha) = 0$. Therefore, we suppose that $G(\alpha) = 0$ and show that then $\hat{F}(\alpha) = 0$.

We know that the 3-form $\hat{F}(\alpha)$ is exact on N by the work in [17, Section 4] since φ is exact near N as it is closed and vanishes on N. As the last condition in Eq. (23) asserts the vanishing of the normal component $\mathbf{n}\lrcorner\hat{F}(\alpha)$ at the boundary of N, the integration by parts argument applies to show that $\hat{F}(\alpha) = 0$.

For the last part of the Proposition, recall that the metric $\hat{g}$ constructed in the proof of Proposition 4.2 is a product metric near S, independent of the normal coordinate s, and the exponential map for $\hat{g}$ has the same s-invariant property. It can be checked that $\widehat{\exp}^*_{\mathbf{v}|_{\partial N}}(\varphi|_{\partial N_{\mathbf{n}\lrcorner\alpha}}) = \hat{F}(\alpha)|_{\partial N}$ (in the notation of Eq. (22)). The latter vanishes since $\hat{F}(\alpha) = 0$, thus the Lagrangian deformation $\mathbf{n}\lrcorner\alpha$ is in fact special Lagrangian. □

To apply the Banach space version of the Implicit Function Theorem we note the following, by application of [1, Theorem 2.2.15] or [13, Section 2.2].

Proposition 4.8. *The map G given in Eq. (20) extends to a smooth map of Sobolev spaces* $G : L^p_{k+1}(\mathcal{U}_N)_{\mathrm{bc}} \to L^p_{k-1}\Omega^2_+(N)$, *for any* $p > 4$, $k \geq 2$.

Remark. The conditions $p > 4$ and $k \geq 2$ ensure that the map G of Sobolev spaces in Proposition 4.8 is well-defined since $L^p_{k+1} \hookrightarrow C^2$ in four dimensions, by the Sobolev Embedding Theorem.

Recall the space V^p_{k-1} defined in Eq. (17). Let Π denote the L^2-orthogonal projection $L^p_{k-1}\Omega^2_+(N) \oplus L^p_{k-\frac{1}{p}}(d\Omega^1(\partial N) \oplus d\Omega^2(\partial N)) \to V^p_{k-1}$ and define

$$\widetilde{G}(\alpha) = \Pi \circ (G(\alpha), \hat{F}_{\partial N}(\mathbf{n}\lrcorner\alpha), \mathbf{n}\lrcorner\hat{F}(\alpha)), \tag{24}$$

where $\alpha \in L^p_{k+1}(\mathcal{U}_N)$ and $\hat{F}_{\partial N}$ is given in Eq. (22). As $L^p_k\Omega^2_+(N)_{\mathrm{bc}}$ is the tangent space to $L^p_{k+1}(\mathcal{U}_N)_{\mathrm{bc}}$ at $\alpha = 0$ and G is smooth, we find, by reducing the neighbourhood $\mathcal{U}_N$ if necessary, that $\widetilde{G}(\alpha) = 0$ if and only if $\alpha \in L^p_{k+1}(\mathcal{U}_N)_{\mathrm{bc}}$ and $G(\alpha) = 0$.

Proposition 4.9. *Let* $p > 4$ *and* $k \geq 2$. *If* $\alpha \in L^p_{k+1}(\mathcal{U}_N)_{\mathrm{bc}}$ *and* $\widetilde{G}(\alpha) = 0$ *then* α *is smooth.*

Proof. We can apply to $\widetilde{G}$ the general elliptic regularity result [18, Theorem 6.8.2], which implies that C^2 solutions to a (non-linear) second-order elliptic equation (with suitable boundary conditions) are smooth. □

Our first main theorem, Theorem 1.1, now follows from this technical result.

Theorem 4.10. *Let* $\varphi \in \Omega^3_+(M)$ *be a closed positive 3-form on a 7-manifold M. Recall the map G given in Eq. (20) and the space* $\Gamma(\mathcal{U}_N)_{\mathrm{bc}}$ *defined in Eq. (23). Let N be a compact coassociative submanifold of* (M, φ) *with non-empty boundary* $\partial N \subset S$, *where S is a scaffold for N. For* $p > 4$, $k \geq 2$, *an* L^p_{k+1}*-neighbourhood of zero in the space*

$$\mathcal{M}(N, S) = \{\alpha \in \Gamma(\mathcal{U}_N)_{\mathrm{bc}} \ : \ G(\alpha) = 0\}$$

of coassociative local deformations of N is a smooth manifold parameterized by the finite-dimensional vector space $(\mathcal{H}^2_+)_{\mathrm{bc}}$ *given in Corollary 3.8.*

Proof. For $p > 4$ and $k \geq 2$, let $W = L^p_{k+1}(\mathcal{U}_N)_{\mathrm{bc}}$ and $X = L^p_{k+1}\Omega^2_+(N)_{\mathrm{bc}}$. We see that X is a Banach space and an open neighbourhood of zero in X parameterizes an open neighbourhood of zero in the Banach submanifold $W \subset L^p_{k+1}\Omega^2_+(N)$. Further, $\widetilde{G}$, given in Eq. (24), satisfies $\widetilde{G}(W) \subset V = V^p_{k-1}$, $\widetilde{G}(0) = 0$ and, by construction, $d\widetilde{G}|_0 : X \to V$ is the surjective linear operator (15).

Therefore, we can apply the Implicit Function Theorem to deduce that, as $\widetilde{G}$ is a smooth map, the kernel of $\widetilde{G}$ in W near zero is a manifold smoothly parameterized by a neighbourhood of zero in the kernel $(\mathcal{H}^2_+)_{\mathrm{bc}}$ of $d\widetilde{G}|_0$ in X. By Proposition 4.9, elements of the kernel of $\widetilde{G}$ near zero are smooth, so by further reducing, if necessary, the neighbourhood $\mathcal{U}_N$ of the zero section we obtain, noting also the comments after Proposition 4.8, that $\widetilde{G}^{-1}(0)$ in W near zero is exactly $\mathcal{M}(N, S)$. □

4.3. Varying the G_2-structure

In this subsection we prove our second main result, Theorem 1.2, which shows that coassociative submanifolds with boundary are 'stable' under small perturbations of the G_2-structure on the ambient 7-manifold.

Theorem 4.11. *Let* $\varphi(t) \in \Omega^3_+(M)$, $t \in \mathbb{R}$, *be a smooth path of closed positive 3-forms on a 7-manifold M. Let N be a compact coassociative submanifold of* $(M, \varphi(0))$ *with non-empty boundary* $\partial N \subset S$, *where S is a scaffold for N. Suppose that* $\varphi(t)|_N$ *is exact on N and* $\mathbf{n}\lrcorner(\varphi(t)|_N)$ *is exact on* ∂N *for each t.*

There is an $\varepsilon > 0$ *and, for each* $|t| < \varepsilon$, *a normal vector field* $\mathbf{v}(t) \in \Gamma(\hat{\nu}_M(N))$ *smoothly depending on t and such that*

$$\mathbf{v}(0) = 0, \quad \partial N(t) \subset S \quad \text{and} \quad \varphi(t) \text{ vanishes on } N(t) = \widehat{\exp}_{\mathbf{v}(t)}(N).$$

Here the normal bundle $\hat{\nu}_M(N)$ *and the exponential map are taken with respect to the metric* $\hat{g}$ *given by Proposition 4.2 applied to* $g(\varphi(0))$.

Remarks. Observe that S need not be a scaffold for $N(t)$ relative to the G_2-structure $\varphi(t)$ when $t \neq 0$. Furthermore, the normal vectors $\mathbf{v}(t)$ can be chosen to satisfy the boundary condition (13).

Proof. Let $\jmath_N : \hat{\nu}_M(N) \to (\Lambda^2_+)_{g_0} T^*N$ be the isomorphism given by Proposition 4.3 and let $p > 4$ and $k \geq 2$. Let $W_0 = L^p_{k+1}(\mathcal{U}_N)_0$ denote a Banach submanifold of $L^p_{k+1}(\mathcal{U}_N)_{\rm bc}$ such that the tangent space of W_0 at the zero form is the L^2-orthogonal complement X_0 to $(\mathcal{H}^2_+)_{\rm bc}$ in $L^p_{k+1}\Omega^2_+(N)_{\rm bc}$, in the metric $g_0 = g(\varphi(0))$. Let $\Gamma(\mathcal{U}_N)_0$ be the subset of smooth sections in W_0. We use a 'parametric' version of the deformation map (19) which we still denote by $\hat{F}$,

$$\hat{F} : (t, \alpha) \in \mathbb{R} \times \Gamma(\mathcal{U}_N) \to \widehat{\exp}^*_{\mathbf{v}}\left(\varphi(t)|_{N_\alpha}\right) \in \Omega^3(N),$$

where $\mathbf{v} = \jmath_N^{-1}(\alpha)$. Further, define 'parametric' versions of the maps $G = d^*_+ \circ \hat{F}$ (where d^*_+ is calculated using g_0) and $\hat{F}_{\partial N}$, given in Eqs. (20) and (22), by replacing φ with $\varphi(t)$ throughout, so these maps now take an additional argument t. Notice that, if

$$\widehat{G}(t, \alpha) = (G(t, \alpha), \hat{F}_{\partial N}(t, \mathbf{n} \lrcorner \alpha), \mathbf{n} \lrcorner \hat{F}(t, \alpha)),$$

then its partial derivative $d_2\widehat{G}|_{(0,0)}$ acts on $(t, \alpha) \in \mathbb{R} \oplus L^p_{k+1}\Omega^2_+(N)$ as $(t, \alpha) \to L(\alpha)$, where L is the surjective linear operator (15). Thus, the image of $d_2\widehat{G}|_{(0,0)}$ restricted to $\mathbb{R} \times X_0$ is the Banach space $V_0 = V^p_{k-1}$ given in Eq. (17). Let Π be the L^2-orthogonal projection to V_0 defined using the metric g_0 and let $\widetilde{G} = \Pi \circ \widehat{G}$.

Since $\varphi(t)|_N$ and its normal part on ∂N are exact for each t, a simple adaptation of the argument in Proposition 4.7 shows that, for $\alpha \in \Gamma(\mathcal{U}_N)_0$, $\widetilde{G}(t, \alpha) = 0$ if and only if N_α is coassociative relative to $\varphi(t)$. Moreover, $\widetilde{G}(\mathbb{R} \times W_0) \subset V_0$, $\widetilde{G}(0, 0) = 0$ and $d_2\widetilde{G}|_{(0,0)} : \mathbb{R} \oplus X_0 \to V_0$ is an isomorphism.

By the Implicit Function Theorem for $\widetilde{G}(t, \alpha)$ and an analogous regularity result to Proposition 4.9 (valid for t small as $\widetilde{G}(0, \alpha)$ is elliptic and ellipticity is an open condition), there is a smooth map h defined on a neighbourhood E_0 of zero in $\mathbb{R} \times (\mathcal{H}^2_+)_{\rm bc}$, taking values in X_0, such that $h(0, 0) = 0$ and

$$\widetilde{G}(t, \alpha_0 + h(t, \alpha_0)) = 0, \quad (t, \alpha_0) \in E_0,$$

are all the zeros of $\widetilde{G}$ near $(0, 0)$. The required $\mathbf{v}(t)$, for small $|t|$, may be taken to be $\jmath_N^{-1}(h(t, 0))$. $\square$

4.4. Examples

We now give some simple examples for this deformation theory.

Example 4.12 (G_2-*manifolds with Symplectic Boundary*). Suppose (M, φ) is an (almost) G_2-manifold with boundary ∂M. Then ∂M has trivial normal bundle and it receives an induced SU(3)-structure from M. If $\mathbf{n}_{\partial M} \lrcorner \varphi|_{\partial M}$ is a closed form on ∂M, where $\mathbf{n}_{\partial M}$ is the unit normal vector field at ∂M, then ∂M is a scaffold for coassociative submanifolds of $N \subset M$ with boundary in ∂M and with $\mathbf{n}_{\partial M}$ tangent to N. Our deformation theory results apply to this situation.

Example 4.13 (G_2-*manifolds with Nearly Kähler Boundary*). Perhaps the most obvious G_2-manifold with boundary to study is the unit ball B in $\mathbb{R}^7$. The boundary of B is the nearly Kähler 6-sphere $\mathcal{S}^6$. Suppose N is a compact coassociative submanifold of B with boundary in $\mathcal{S}^6$. Then ∂N is a *Lagrangian* (also called *totally real*) submanifold of $\mathcal{S}^6$: the non-degenerate, but not closed, 2-form on $\mathcal{S}^6$ vanishes on ∂N. Current work in progress of the second author shows that, for any nearly Kähler 6-manifold, the deformation theory of a Lagrangian submanifold is expected to be *obstructed* up to rigid motion. Therefore, one could not hope, in general, for a smooth moduli space of deformations of N in the unit ball B in $\mathbb{R}^7$. This negative result extends to any G_2-manifold with nearly Kähler boundary, or any coassociative 4-fold with boundary in a nearly Kähler 'scaffold'. This gives another motivation for our definition of a scaffold.

Example 4.14 (*Product* G_2-*manifolds 1*). A Kähler complex 3-fold (S, ω) is called *almost Calabi–Yau* if it admits a nowhere vanishing holomorphic (3, 0)-form Ω. Then $M = S \times \mathcal{S}^1$ is an almost G_2-manifold with G_2-structure $\omega \wedge d\theta + \mathrm{Re}\,\Omega$, where θ is a coordinate on $\mathcal{S}^1$. Let $N = L \times \mathcal{S}^1 \subset M$ be a compact coassociative 4-fold. Then L is special Lagrangian in S. We can think of N as an embedding of a manifold $L \times [0, 1]$ whose two boundary components, $L \times \{0\}$ and $L \times \{1\}$, are mapped to L in S. It is not difficult to see that $S \times$ pt is a scaffold for N. Theorem 4.10 gives us that N has a smooth moduli space of coassociative deformations $\mathcal{M}(N, S)$ with dimension $\leq 2b^1(L)$.

Let $\alpha \in \Omega^2_+(N)$. Then $\alpha = \xi_\theta \wedge d\theta + *_L \xi_\theta$, for some path of 1-forms ξ_θ on L. It follows from [21, Theorem 3.4.10] that a harmonic self-dual 2-form on N is uniquely determined by its values ξ_0, ξ_1 on the boundary. The subspace of harmonic $\alpha \in \Omega^2_+(N)$ such that ξ_0 and ξ_1 are harmonic on L has dimension $2b^1(L)$ and corresponds precisely to the paths $\xi_\theta = (1 - \theta)\xi_0 + \theta\xi_1$. On the other hand, $\alpha \in (\mathcal{H}^2_+)_{\rm bc}$ if and only if α is harmonic and $\frac{\partial \xi_\theta}{\partial \theta} = 0$, so $\xi_0 = \xi_1$. Thus $\dim(\mathcal{H}^2_+)_{\rm bc} = b^1(L) < b^1\left((L \times \{0\}) \sqcup (L \times \{1\})\right)$ in this example.

This can also be seen geometrically. If the deformations of the aforementioned two boundary components coincide in $S \times$ pt then, by taking a product with $\mathcal{S}^1$, we obtain a coassociative deformation of $N = L \times \mathcal{S}^1$ defining a point in $\mathcal{M}(N, S)$. On the other hand, if a coassociative deformation $\tilde{N}$ of N is such that the deformations $\tilde{L}_0$ and $\tilde{L}_1$ of $L \times \{0\}$ and $L \times \{1\}$ are special Lagrangian but *distinct*, then $\tilde{N}$ and $\tilde{L}_0 \times \mathcal{S}^1$ are two distinct coassociative 4-folds intersecting in a real analytic

3-fold on which φ vanishes, which contradicts [8, Theorem IV.4.3]. Therefore, $\mathcal{M}(N, S)$ is identified with special Lagrangian deformations of L in the almost Calabi–Yau manifold S. It is well known that these deformations have a smooth moduli space of dimension $b^1(L)$ [12, Theorem 8.4.5].

Moreover, suppose that we have a smooth path of closed positive 3-forms $\varphi(t)$ on M with $\varphi(0) = \varphi$. Suppose that $\varphi(t)|_N$ is exact and the normal part of $\varphi(t)|_N$ on ∂N is exact. Theorem 4.11 says that N extends to a smooth family $N(t)$ of compact 4-folds with boundary in S such that $N(t)$ is coassociative in $(M, \varphi(t))$.

Now, we can write

$$\varphi(t) = \omega(t) \wedge \mathrm{d}\theta + \Upsilon(t)$$

with $\omega(0) = \omega$ and $\Upsilon(0) = \Upsilon$. The conditions on $\varphi(t)$ are equivalent to the exactness of $\omega(t)|_L$ and $\Upsilon(t)|_L$ on L, together with the fact that $\omega(t)$ and $\Upsilon(t)$ define an (almost) Calabi–Yau structure on S. These are precisely the necessary and sufficient conditions, by [12, Theorem 8.4.7], for L to be extended to a smooth family $L(t)$ of compact 3-folds in S such that $L(t)$ is special Lagrangian in S with respect to $(\omega(t), \Upsilon(t))$. This applies to embeddings of $N(t) = L(t) \times \mathcal{S}^1$ and to the more general embeddings of $N(t) = L(t) \times [0, 1]$ with images of $L(t) \times \{0\}$ and $L(t) \times \{1\}$ in $S \times \mathrm{pt}$.

Finally we relate our work to the theory presented in [3].

Example 4.15 (*Product* G_2*-manifolds 2*). Suppose $M = S \times \mathcal{S}^1$ is as in the previous example and $N = L \times \mathcal{S}^1$ is a coassociative 4-fold in M. However, now suppose that the special Lagrangian 3-fold L has boundary ∂L in a (real) 4-dimensional scaffold W in the almost Calabi–Yau manifold S in the sense of [3]. Thus N has boundary $\partial N = \partial L \times \mathcal{S}^1$ in $W \times \mathcal{S}^1$. The 5-dimensional submanifold $W \times \mathcal{S}^1$ is obviously not a scaffold in the sense of Definition 3.5, so our deformation theory does not apply. However the result of Butscher's work [3] is that the deformation theory of L as a *minimal Lagrangian* in S with boundary in W is unobstructed and the dimension of the moduli space is $b^1(L)$.

Motivated by this example the authors considered the possibility of a 5-dimensional 'scaffold' and derived the conditions that it would have to satisfy if it were to be a generalisation of Butscher's scaffold. Unfortunately, the resulting deformation problem turned out not to be elliptic and, moreover, that the deformation theory of coassociative submanifolds in such a 5-dimensional 'scaffold' would be obstructed in general. We can see this problem as follows.

The deformation theory we need for a coassociative N with boundary in $W \times \mathcal{S}^1$ is for special Lagrangians with boundary in W. These are deformations of minimal Lagrangians with 'fixed phase' which means the vanishing of $\mathrm{Im}(e^{i\lambda}\Omega)$ for some $\lambda \in \mathbb{R}$ fixed once and for all. However, the deformation theory of special Lagrangians with boundary in W is obstructed. Therefore, even in this simplest case of a 5-dimensional 'scaffold' we do not get a smooth moduli space of deformations.

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