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# Towards an $\mathcal{N} = 1$ $SU(3)$-invariant supersymmetric membrane flow in eleven-dimensional supergravity

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ABSTRACT

The $M$-theory lift of $\mathcal{N} = 1$ $G_2$-invariant RG flow via a combinatoric use of the 4-dimensional RG flow and 11-dimensional Einstein–Maxwell equations was found some time ago. The 11-dimensional metric, a warped product of an asymptotically $AdS_4$ space with a squashed and stretched 7-sphere, for $SU(3)$-invariance was found before. In this paper, by choosing the 4-dimensional internal space as $\mathbf{CP}^2$ space, we discover an exact solution of $\mathcal{N} = 1$ $G_2$-invariant flow to the 11-dimensional field equations. By an appropriate coordinate transformation on the three internal coordinates, we also find an 11-dimensional solution of $\mathcal{N} = 1$ $G_2$-invariant flow interpolating from $\mathcal{N} = 8$ $SO(8)$-invariant UV fixed point to $\mathcal{N} = 1$ $G_2$-invariant IR fixed point. In particular, the 11-dimensional metric and 4-forms at the $\mathcal{N} = 1$ $G_2$ fixed point for the second solution will provide some hints for the 11-dimensional lift of whole $\mathcal{N} = 1$ $SU(3)$ RG flow connecting this $\mathcal{N} = 1$ $G_2$ fixed point to $\mathcal{N} = 2$ $SU(3) \times U(1)_R$ fixed point in 4-dimensions.



## 1. Introduction

The low energy limit of $N$ membranes at $\mathbf{C}^4/\mathbf{Z}_k$ singularity is described in the context of 3-dimensional $\mathcal{N} = 6U(N) \times U(N)$ Chern–Simons matter theory with level $k$ [1]. For particular level $k = 1, 2$, the maximal $\mathcal{N} = 8$ supersymmetry is preserved. The matter contents and the superpotential of this theory coincide with the ones in the theory for D3-branes at the conifold in 4-dimensions [2]. The renormalization group (RG) flow of the 3-dimensional theory can be studied from the 4-dimensional $\mathcal{N} = 8$ gauged supergravity via $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence [3]. The holographic $\mathcal{N} = 2$ supersymmetric $\mathrm{SU}(3) \times U(1)_R$-invariant RG flow connecting the maximally supersymmetric $\mathcal{N} = 8\mathrm{SO}(8)$ ultraviolet (UV) point to $\mathcal{N} = 2\mathrm{SU}(3) \times U(1)_R$ infrared (IR) point has been found in [4–6] while the $\mathcal{N} = 1$ supersymmetric $G_2$-invariant RG flow from this maximally supersymmetric $\mathcal{N} = 8\mathrm{SO}(8)$ UV point to $\mathcal{N} = 1G_2$ IR point has been discussed in [5,7]. The former has $\mathrm{SU}(3) \times U(1)_R$ symmetry around IR region including the IR critical point and the latter has $G_2$ symmetry around IR region as well as the IR critical point. The 11-dimensional $M$-theory lifts of these two RG flows have been described in [8,7] by solving the Einstein–Maxwell equations explicitly in 11-dimensions.

The mass deformed $U(2) \times U(2)$ Chern–Simons matter theory with level $k = 1, 2$ preserving the above global $\mathcal{N} = 2\mathrm{SU}(3) \times U(1)_R$ symmetry has been found in [9,10] while the mass deformation for this theory preserving $\mathcal{N} = 1G_2$ symmetry has been found in [11]. Further nonsupersymmetric RG flow equations preserving two $\mathrm{SO}(7)^{\pm}$ symmetries have been studied in [12]. The holographic $\mathcal{N} = 1$ supersymmetric SU(3)-invariant RG flow equations connecting $\mathcal{N} = 1G_2$ point to $\mathcal{N} = 2\mathrm{SU}(3) \times U(1)_R$ point in 4-dimensions have been studied in [13]. Moreover, the other holographic supersymmetric RG flows have been found and further developments on the 4-dimensional gauged supergravity (see also [14,15]) have been made in [16,17].

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The spin-2 Kaluza–Klein modes around a warped product of $AdS_4$ and a seven-ellipsoid having the above global $\mathcal{N} = 1G_2$ symmetry are discussed in [18]. The gauge dual with $\mathcal{N} = 2SU(2) \times SU(2) \times U(1)_R$ symmetry for the 11-dimensional lift of $SU(3) \times U(1)_R$-invariant solution in 4-dimensional supergravity is described in [19] (see also [20]). The 11-dimensional description preserving $\mathcal{N} = 2SU(2) \times U(1) \times U(1)_R$ symmetry is found in [21] and the smaller $\mathcal{N} = 2U(1) \times U(1) \times U(1)_R$ symmetry flow is discussed in [22]. All of these have a common $U(1)_R$ factor corresponding to $\mathcal{N} = 2$ supersymmetry. The symmetries come from the symmetry each 5-dimensional Sasaki–Einstein space has.

When the 11-dimensional theory from the 4-dimensional gauged $\mathcal{N} = 8$ supergravity is constructed, the various 11-dimensional solutions occur even though the flow equations characterized by the 4-dimensional supergravity fields are the same. This is mainly due to the fact that the 11-dimensional metric with common geometric parameters are different from each other. The same flow equations [4,7,5] in 4-dimensions, playing the role of geometric parameters in the internal space, provide different 11-dimensional solutions to the equations of the motion for 11-dimensional supergravity [8,19,21,22]. The invariance of the 11-dimensional metric and 4-forms determines the global symmetry. Sometimes the 4-forms are restricted to the global symmetry the metric possesses and break further into the smaller symmetry group.

The compact 7-dimensional manifold in the compactification of 11-dimensional supergravity can be described by the metric encoded in the vacuum expectation values for 4-dimensional $\mathcal{N} = 8$ gauged supergravity [23]. The SU(3)-singlet space contains various critical points [24] and RG flows (domain walls) along the $AdS_4$ radial coordinate. Based on the nonlinear metric ansatz by [25], the geometric construction of the compact 7-dimensional metric is found and the local frames are obtained by decoding the SU(3)-singlet vacuum expectation values into squashing and stretching parameters of the 7-dimensional manifold [26]. The $M$-theory lift of a supersymmetric RG flow can be done, in general, as follows. We impose the nontrivial $AdS_4$ radial coordinate dependence of vacuum expectation values subject to the 4-dimensional RG flow equations [4,7,5]. Then the geometric parameters in the 7-dimensional metric at certain critical point are controlled by the RG flow equations so that they can be extrapolated from the critical points. Next, we make an appropriate ansatz for the 11-dimensional 4-form field strengths. Finally, the 11-dimensional Einstein–Maxwell bosonic equations [27,28] can be solved by using the RG flow equations [4,7,5] to complete the $M$-theory uplift.

The SU(3)-singlet space with a breaking of the SO(8) gauge group into a group which contains SU(3) can be represented by four real supergravity fields [24]. Although the final goal is to solve the 11-dimensional Einstein–Maxwell equations to complete the 11-dimensional lift of the whole SU(3)-invariant sector (that contains four supergravity fields) including the RG flows [5], we focus on the $G_2$-invariant sector (which is a subset of above SU(3)-invariant sector) in a 4-dimensional viewpoint by constraining the four arbitrary supergravity fields together with the particular condition. There exist a supersymmetric $\mathcal{N} = 1G_2$ critical point and two nonsupersymmetric $SO(7)^{\pm}$ critical points [29,30] in this sector. If we take the different constraints on the supergravity fields, then we are led to the $SU(3) \times U(1)_R$-invariant sector where there are a supersymmetric $\mathcal{N} = 2SU(3) \times U(1)_R$ critical point and a nonsupersymmetric $SU(4)^-$ critical point [31].

The 11-dimensional metric is found by [26] where the compact 7-dimensional metric and warp factor are completely determined in local frames. The two geometric parameters, by constraints, that are nothing but the above supergravity fields, depend on the $AdS_4$ radial coordinate and are subject to the RG flow equations [7,5] in 4-dimensional gauged supergravity. The global coordinates for $\mathbf{S}^7$ appropriate for the base six-sphere $\mathbf{S}^6 \simeq \frac{G_2}{SU(3)}$ preserve the Fubini-Study metric on $\mathbf{CP}^2$ and this describes the ellipsoidally deformed $\mathbf{S}^7$ [25]. On the other hand, by using the relation to the Hopf fibration on $\mathbf{CP}^3 \simeq \frac{SU(4)}{SU(3)\times U(1)}$ explicitly and keeping only the $\mathbf{CP}^2$ space, one can change the remaining three local frames in 7-dimensional manifold via an orthogonal transformation [26]. Each global coordinates depends on each base six-spheres they use.

What is the 11-dimensional lift of holographic $\mathcal{N} = 1$ supersymmetric SU(3)-invariant RG flow [13] connecting from $\mathcal{N} = 1G_2$ critical point to $\mathcal{N} = 2SU(3) \times U(1)_R$ critical point? At each critical point, the 4-forms are known in different backgrounds. Along the supersymmetric RG flow, one expects that there exist the extra 4-forms which should vanish at the critical points. In order to describe the whole RG flow, one needs to have a consistent background. Around the $\mathcal{N} = 2SU(3) \times U(1)_R$ IR fixed point, the use of global coordinates for Hopf fibration on $\mathbf{CP}^3$ was done in [8]. Around the $\mathcal{N} = 1G_2$ IR fixed point, the global coordinates for $\mathbf{S}^7$ appropriate for the base six-sphere $\mathbf{S}^6$ was used in [7]. Therefore, either one should find the $\mathbf{CP}^3$-basis around $\mathcal{N} = 1$ IR fixed point, or one needs to find $\mathbf{S}^6$-basis around $\mathcal{N} = 2$ IR fixed point. Recently, the latter was studied in [32]. Now we are left with the former.

In this paper, we find out an exact solution of $\mathcal{N} = 1G_2$-invariant flow (connecting from the $\mathcal{N} = 8SO(8)$ UV fixed point to $\mathcal{N} = 1G_2$ IR fixed point) to the 11-dimensional Einstein–Maxwell equations where the internal space contains the $\mathbf{CP}^2$ space. By an appropriate coordinate transformation, we also find an 11-dimensional solution of the $\mathcal{N} = 1G_2$-invariant flow (connecting from the $\mathcal{N} = 8SO(8)$ UV fixed point to the $\mathcal{N} = 1G_2$ IR fixed point) in a different background where the rectangular coordinates we use are the same as the ones in [8]. The two solutions share the same geometric parameters satisfying the common RG flow equations [7] in 4-dimensional gauged $\mathcal{N} = 8$ supergravity. At the $\mathcal{N} = 1$ IR critical point, we have found both the metric and 4-forms explicitly. They are completely different from the previous findings in [7] in the sense that the rectangular coordinates with specific angular coordinates are the same as the one in [8] and they are different parametrization from the one in [7]. This finding will give us some hints for the final goal, the 11-dimensional lift of $\mathcal{N} = 1$ supersymmetric SU(3)-invariant RG flow connecting from the $\mathcal{N} = 1G_2$ fixed point to the $\mathcal{N} = 2SU(3) \times U(1)_R$ fixed point.

In Section 2, we introduce the six basis vectors living in the unit round six-sphere $\mathbf{S}^6$ and the normal vector perpendicular to $\mathbf{S}^6$ in the context of octonion description [33]. The three $G_2$-invariant tensors are determined by the components of these seven frames. Then the 4-forms are given by these $G_2$-invariant tensors as well as a geometric superpotential [7]. By applying these 4-forms with varying scalars, the exact solution [7] to the 11-dimensional Einstein–Maxwell equations corresponding to the 11-dimensional lift of the $\mathcal{N} = 1 G_2$-invariant RG flow is reviewed.

By taking the different spherical parametrization where the $\mathbf{CP}^2$ structure is evident, the six basis vectors living in the six-sphere $\mathbf{S}^6$ and the normal vector perpendicular to it are constructed explicitly. The three $G_2$-invariant tensors are also determined by the components of these new seven frames. Assuming that the two supergravity fields satisfy the domain wall solutions [7] as before, we compute the Ricci tensor in this background completely. Surprisingly, the Ricci tensor has the same expression as the one in previous spherical parametrization. This indicates that the 4-form field strengths are constructed similarly by using the above new three $G_2$-invariant tensors and a geometric superpotential introduced in [7]. Eventually, we determine the solution for the 11-dimensional Einstein–Maxwell equations corresponding to the the different 11-dimensional lift of the same $\mathcal{N} = 1 G_2$-invariant RG flow. The 4-forms for both parametrizations depend on the 7-dimensional internal coordinates via each invariant tensors.

In Section 3, by changing only three of them among seven internal coordinates characterized by previous second spherical parametrization, the 11-dimensional metric can be written in terms of these new coordinates which preserve the Fubini-Study metric on $\mathbf{CP}^2$ [26]. The Ricci tensor can be expressed as a linear combination of the Ricci tensor for $G_2$-invariant case and similarly the 4-forms also are given by a linear combination of 4-forms for the $G_2$-invariant flow. Then, we find out a solution for the 11-dimensional Einstein–Maxwell equations corresponding to the 11-dimensional lift of the $\mathcal{N} = 1 G_2$-invariant RG flow connecting from the $\mathcal{N} = 8 SO(8)$ UV fixed point to $\mathcal{N} = 1 G_2$ IR fixed point. Since we focus on the $G_2$-invariant sector, the two geometric parameters satisfy the RG flow equations [7]. In the 11-dimensional point of view, both the metric and 4-forms preserve the $G_2$ symmetry.

In Section 4, we summarize the results of this paper and present some future directions.

In Appendices, we present the detailed expressions for the Ricci tensor, 4-form field strengths, seven frames, three invariant tensors and Maxwell equations.

## 2. An $\mathcal{N} = 1 G_2$-invariant supersymmetric flow in an 11-dimensional theory

We construct three $G_2$-invariant tensors in terms of six-sphere coordinates explicitly in Section 2.1 and find out a $\mathcal{N} = 1 G_2$-invariant solution for 11-dimensional Einstein–Maxwell equations in Section 2.2.

### 2.1. Spherical parametrization I

Let us consider the spherical parametrization describing the unit round five-sphere $\mathbf{S}^5$

$$\begin{aligned}
u^1 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \sin\theta_5,\\
u^2 &= \cos\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \sin\theta_5,\\
u^3 &= \cos\theta_2 \sin\theta_3 \sin\theta_4 \sin\theta_5,\\
u^4 &= \cos\theta_3 \sin\theta_4 \sin\theta_5,\\
u^5 &= \cos\theta_4 \sin\theta_5,\\
u^6 &= \cos\theta_5,
\end{aligned} \tag{2.1}$$

subject to the constraint $\sum_{i=1}^{6}(u^i)^2 = 1$. In order to use the 6-dimensional coordinatization for unit round six-sphere $\mathbf{S}^6$, let us also introduce the two orthogonal unit vectors in $\mathbf{R}^7$ with (2.1)

$$\mathbf{U} = (u^1, u^2, u^3, u^4, u^5, u^6, 0), \qquad \mathbf{V} = (0, 0, 0, 0, 0, 0, 1). \tag{2.2}$$

Then the unit normal vector perpendicular to the six-sphere $\mathbf{S}^6$ can be identified with

$$\hat{\mathbf{n}} = \mathbf{U}\sin\theta_6 + \mathbf{V}\cos\theta_6. \tag{2.3}$$

The unit magnitude of normal vector, $(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = 1$, from (2.2) can be easily checked. The rectangular coordinates parametrizing the six-sphere $\mathbf{S}^6$ inside $\mathbf{R}^7$ are given by $X^i = u^i \sin\theta_6$ $(i = 1, \ldots, 6)$ and $X^7 = \cos\theta_6$ with $\sum_{A=1}^{7}(X^A)^2 = 1$ whose gradient is exactly the unit normal vector introduced in (2.3). This parametrization for $X^A$ can be also obtained from the one corresponding to the unit round seven-sphere $\mathbf{S}^7$ by ignoring seventh coordinate $\theta \equiv y^7$ in [7]. By taking the gradient to the normal vector with respect to the six internal coordinates, a set of basis vectors spanning a tangent space on the six-sphere $\mathbf{S}^6$ is given by [33]

$$\hat{\mathbf{e}}_i = -\frac{\partial\hat{\mathbf{n}}}{\partial\theta_i}, \quad i = 1, \ldots, 6, \tag{2.4}$$

which is orthogonal to the normal vector (2.3). In general, the second fundamental tensor arises in (2.4) for a generic hypersurface. For a round six-sphere, this second fundamental tensor is proportional to the 6-dimensional metric for six-sphere and this leads to a simple relation (2.4) above. The orthonormality for the basis vectors (2.4) can be checked also.

Adding the unit normal vector $\hat{\mathbf{n}}$ as the seventh-frame to the six-sphere $\mathbf{S}^6$-frames $\hat{\mathbf{e}}_i$ provides the tangent frames of Cayley space denoted by $\mathbf{I}^7$ [33].

$$\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_i^A O_A, \qquad \hat{\mathbf{n}} = \hat{\mathbf{n}}^A O_A, \quad A = 1, \ldots, 7, \tag{2.5}$$

where $O_A$ are imaginary octonions satisfying the algebra

$$O_A O_B = -\delta_{AB} + \eta_{ABC} O_C, \tag{2.6}$$

with completely antisymmetric structure constants $\eta_{ABC}$ [34]. The nonzero $\eta_{ABC}$ are given by 1 for the indices $ABC = 123, 516, 624, 435, 471, 673, 572$. In order to obtain the $G_2$-invariant tensors on $\mathbf{S}^6$, one projects the $\mathbf{I}^7$ invariant tensor and its dual to local tangent frames and obtains a 7-dimensional tensor (and its dual). Since the geometry of the $G_2$-invariant $\mathbf{S}^6$ is already built in the 7-dimensional coordinatization, what we have to do is to take the dimensional reduction of 7-dimensional tensor (and its dual) to the 6-dimensional ones. Then the almost complex structure $F_{ij}$, its covariant derivative $T_{ijk}$, and its Hodge dual, $S_{ijk}$ that depend on the internal coordinates $\theta_i$ $(i = 1, 2, \ldots, 6)$, with (2.3)–(2.5), can be summarized by

$$\begin{aligned} F_{ij} &= -\eta_{ABC} \hat{\mathbf{e}}_i^A \hat{\mathbf{e}}_j^B \hat{\mathbf{n}}^C, \\ T_{ijk} &= \eta_{ABC} \hat{\mathbf{e}}_i^A \hat{\mathbf{e}}_j^B \hat{\mathbf{e}}_k^C, \\ S_{ijk} &= \eta_{ABCD} \hat{\mathbf{e}}_i^A \hat{\mathbf{e}}_j^B \hat{\mathbf{e}}_k^C \hat{\mathbf{n}}^D, \end{aligned} \tag{2.7}$$

where the dual tensor [35] $\eta_{ABCD}$ of the structure constant $\eta_{EFG}$ is given by $\eta_{ABCD} = \eta_{ABE}\eta_{CDE} - \delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC}$ explicitly. The nonzero components for the dual tensor can be read off from this relation. Note that $\eta_{ABC}$ is a $\mathbf{I}^7$-invariant tensor while $\eta_{DEFG}$ is its dual. The dimensional reduction of a 7-dimensional tensor leads to $F_{ij}$ and $T_{ijk}$ tensors while the dimensional reduction of its dual provides a $S_{ijk}$ tensor, as shown in (2.7). The last relation of (2.7) can be checked from the definition of $S_{ijk} = F_i^l T_{ljk}$ [33] by using the first two relations of (2.7) together with (2.6). We can introduce the dot product and cross product for arbitrary imaginary octonions [33] and these can be used for the construction of a 7-dimensional tensor and its dual.

By computing the dot product between $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ in $\mathbf{I}^7$, one obtains the 6-dimensional metric on $\mathbf{S}^6$ for the spherical parametrization we start with. That is, as we expect,

$$g_{ij} = \mathrm{diag}(s_2^2 s_3^2 s_4^2 s_5^2 s_6^2,\ s_3^2 s_4^2 s_5^2 s_6^2,\ s_4^2 s_5^2 s_6^2,\ s_5^2 s_6^2,\ s_6^2,\ 1), \qquad s_i \equiv \sin\theta_i. \tag{2.8}$$

Now it is ready to construct the 7-dimensional ellipsoidal metric. Applying the Killing vector to the metric formula [25], one obtains the 7-dimensional inverse metric with undetermined warp factor. Then plugging this inverse metric into the definition of warp factor, the warp factor is fixed. Finally, substituting the warp factor into the inverse metric leads to our 7-dimensional metric. The 7-dimensional warped ellipsoidal metric used in [25,7] can be summarized as

$$\begin{aligned} ds_7^2 &= \sqrt{\Delta a} L^2 \left[ \frac{\xi^2}{a^2}\, d\theta^2 + \sin^2\theta\, d\Omega_6^2 \right], \qquad d\Omega_6^2 = d\theta_6^2 + \sin^2\theta_6\, d\Omega_5^2, \\ d\Omega_5^2 &= d\theta_5^2 + \sin^2\theta_5 \left( d\theta_4^2 + \sin^2\theta_4 \left[ d\theta_3^2 + \sin^2\theta_3 \left( d\theta_2^2 + \sin^2\theta_2\, d\theta_1^2 \right) \right] \right), \end{aligned} \tag{2.9}$$

where the quadratic form $\xi^2$ is given by

$$\xi^2 = a^2 \cos^2\theta + b^2 \sin^2\theta, \tag{2.10}$$

and the warp factor together with (2.10) is also given by

$$\Delta = a^{-1} \xi^{-\frac{4}{3}}. \tag{2.11}$$

The standard 7-dimensional ellipsoidal metric is warped by a factor $\sqrt{\Delta a}$ in (2.9). The 6-dimensional metric $d\Omega_6^2$ is equal to (2.8). The two vacuum expectation values $(a, b)$ in 4-dimensional gauged $\mathcal{N} = 8$ supergravity that appear in the 7-dimensional internal metric (2.9) are functions of the $\mathrm{AdS}_4$ radial coordinate $r \equiv x^4$. The eccentricity for a 7-dimensional ellipsoid is given by $\sqrt{1 - \frac{b^2}{a^2}}$.

The 11-dimensional metric [25,36] by combining the above 7-dimensional metric (2.9) with the 4-dimensional metric together with warp factor (2.11) yields

$$ds_{11}^2 = \Delta^{-1} \left( dr^2 + e^{2A(r)} \eta_{\mu\nu}\, dx^\mu\, dx^\nu \right) + ds_7^2, \tag{2.12}$$

where $r \equiv x^4$ and $\mu, \nu = 1, 2, 3$ with $\eta_{\mu\nu} = \mathrm{diag}(-, +, +)$. The 11-dimensional coordinates with indices $M, N, \ldots$ are decomposed into 4-dimensional spacetime $x^\mu$ and 7-dimensional internal space $y^m$. Denoting the 11-dimensional metric as

$g_{MN}$ with the convention $(-, +, \ldots, +)$ and the antisymmetric tensor fields as $F_{MNPQ}$, the 11-dimensional Einstein–Maxwell equations are given by [27,28]

$$
\begin{aligned}
R_M^N &= \frac{1}{3} F_{MPQR} F^{NPQR} - \frac{1}{36} \delta_M^N F_{PQRS} F^{PQRS}, \\
\nabla_M F^{MNPQ} &= -\frac{1}{576} E \epsilon^{NPQRSTUVWXY} F_{RSTU} F_{VWXY},
\end{aligned}
\tag{2.13}
$$

where the covariant derivative $\nabla_M$ on $F^{MNPQ}$ in (2.13) is given by $E^{-1}\partial_M(EF^{MNPQ})$ together with Elfbein determinant $E \equiv \sqrt{-g_{11}}$. The epsilon tensor $\epsilon_{NPQRSTUVWXY}$ with lower indices is purely numerical. For a given 11-dimensional metric, the nontrivial task is to find the correct 4-forms that satisfy (2.13).

The warped 11-dimensional metric (2.12) with (2.9) generates the Ricci tensor [7] that depends on $(a, b, A)$ and their derivatives with respect to $r$ and $\theta$. Applying the flow equations, all the $r$-derivatives in the Ricci tensor can be replaced with the functions of $(a, b)$. The Ricci tensor is given in Appendix A explicitly after substituting the flow equations [7,5]

$$
\begin{aligned}
\frac{da}{dr} &= -\frac{8}{7L}\left[a^2 \partial_a W + (ab-2)\partial_b W\right] \\
&= -\frac{a^{\frac{3}{2}}\left[a^5 - 88a^2 b + 14a^3 b^2 - 56b^3 + a(80+49b^4)\right]}{2L\sqrt{(a^2+7b^2)^2 - 112(ab-1)}}, \\
\frac{db}{dr} &= -\frac{8}{7L}\left[b^2 \partial_b W + (ab-2)\partial_a W\right] \\
&= \frac{\sqrt{a}\left[96 + 2a^4 - a(176+a^4)b + 100a^2b^2 - 14a^3b^3 + 42b^4 - 49ab^5\right]}{2L\sqrt{(a^2+7b^2)^2 - 112(ab-1)}}, \\
\frac{dA}{dr} &= \frac{2}{L} W = \frac{1}{4L} a^{\frac{3}{2}} \sqrt{(a^2+7b^2)^2 - 112(ab-1)},
\end{aligned}
\tag{2.14}
$$

where the superpotential $W$ in 4-dimensional gauged $\mathcal{N} = 8$ supergravity is given by a function of $(a, b)$:

$$
W = \frac{1}{8} a^{\frac{3}{2}} \sqrt{(a^2+7b^2)^2 - 112(ab-1)}.
\tag{2.15}
$$

There exist only two nonzero off-diagonal components $R_4^5$ and $R_5^4$ as well as nonzero diagonal components. In 4-dimensions, there exist two critical points, $\mathcal{N} = 8\,SO(8)$ critical point at which $(a, b) = (1, 1)$ and $\mathcal{N} = 1\,G_2$ critical point at which $(a, b) = \left(\sqrt{\frac{6\sqrt{3}}{5}}, \sqrt{\frac{2\sqrt{3}}{5}}\right)$. At these two points, $\frac{da}{dr}$ and $\frac{db}{dr}$ vanish because the right hand sides of (2.14) are equal to zero. The criticality can be observed from the fact that the first two right hand sides of (2.14) are written as the derivatives of superpotential $W(a, b)$ with respect to the field $a$ and the field $b$. The superpotential has 1 and $\sqrt{\frac{36\sqrt{2}\,3^{\frac{1}{4}}}{25\sqrt{5}}}$ at two critical values respectively.

What about 4-form field strengths for 11-dimensional solution? By interpreting the 3-form gauge field with membrane indices as a geometric superpotential, which is a generalization of 4-dimensional superpotential (2.15), times volume form and putting the previous $G_2$-invariant tensors (2.7) to various 3-form components, one can construct the most general $G_2$-invariant ansatz. The field strengths are summarized by [7]

$$
\begin{aligned}
&F_{\mu\nu\rho 4} = e^{3A(r)} W_r(r,\theta)\epsilon_{\mu\nu\rho}, \qquad F_{\mu\nu\rho 5} = e^{3A(r)} W_\theta(r,\theta)\epsilon_{\mu\nu\rho}, \\
&F_{mnpq} = 2h_2(r,\theta) \overset{6}{\epsilon}_{mnpqrs} F^{rs}, \qquad F_{5mnp} = \tilde{h}_1(r,\theta) T_{mnp} + \tilde{h}_2(r,\theta) S_{mnp}, \\
&F_{4mnp} = \tilde{h}_3(r,\theta) T_{mnp} + \tilde{h}_4(r,\theta) S_{mnp}, \qquad F_{45mn} = \tilde{h}_5(r,\theta) F_{mn},
\end{aligned}
\tag{2.16}
$$

where the eight coefficient functions $(W_r, W_\theta, \tilde{h}_1, h_2, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4, \tilde{h}_5)$ which depend on both $r$ and $\theta$ are given by [7] and they are also in Appendix B for convenience. The fields $(W_r, W_\theta)$ are related to the geometric superpotential $\widetilde{W}(r, \theta)$ which corresponds to the 3-form gauge field with membrane indices, as mentioned before. Compared with the previous works [25,33] which holds for the critical points only, the mixed field strengths $F_{1235}$, $F_{4mnp}$ and $F_{45mn}$ were new.[1]

[1] One can write down the 2-form and 3-forms in terms of rectangular coordinates $X^i = u^i \sin\theta_6 \sin\theta$ $(i = 1, \ldots, 6)$, $X^7 = \cos\theta_6 \sin\theta$ and $X^8 = \cos\theta$. Since one can express the angles in terms of $X^A$'s as $\cos\theta_i = \frac{X^{i+1}}{\sqrt{\sum_{j=1}^{i+1}(X^j)^2}}$ where $i = 1, \ldots, 6$ and $\cos\theta = X^8$, the rank 2 and rank 3 tensors with angular coordinates can be written in terms of those with $X^A$'s. Let us write the 2-form $F_{mn}\, d\theta^m \wedge d\theta^n$ by using the relation between $(\theta, \theta_i)$ and $X^A$ and changing the differentials $d\theta^m$ in terms of $dX^A$'s. Then it turns out that the 2-form is given by $F^{(2)} = \frac{1}{(\sum_{A=1}^{7}(X^A)^2)^{\frac{3}{2}}} \eta_{ABC}\, X^A dX^B \wedge dX^C$ where $\eta_{ABC}$ is the same as the one in

The details of Einstein equations, the first equation of (2.13), are given in [7] explicitly. The nonzero components of the Maxwell equation, the second equation of (2.13), are characterized by the following free indices

$$(123), \qquad (4np), \qquad (5np), \qquad (mnp), \quad m, n, p = 6, \ldots, 11, \tag{2.17}$$

with the number of components 1, 15 by choosing two out of six, 15 by choosing two out of six and 20 by choosing three out of six respectively. Other remaining components of the Maxwell equation become identically zero. Therefore, there exist 51 nonzero components of the Maxwell equation. The right hand side of Maxwell equations for the (123)-component consists of 35 terms coming from the quadratic 4-forms. For $(4np)$- and $(5np)$-components, the right hand side of Maxwell equations contain only a single term in quadratic 4-forms. The former has $F_{1235}$ while the latter has $F_{1234}$. For the $(mnp)$-components, the two contributions from the quadratic 4-forms arise in the right hand side of Maxwell equations. In this case, both 4-forms $F_{1235}$ and $F_{1234}$ appear. The Elfbein determinant $E = \sqrt{-g_{11}}$ is used and it is

$$E = e^{3A(r)} L^7 a(r) \sin^6\theta \sin\theta_2 \sin^2\theta_3 \sin^3\theta_4 \sin^4\theta_5 \sin^5\theta_6 \left[a^2(r)\cos^2\theta + b^2(r)\sin^2\theta\right]^{\frac{2}{3}},$$

by calculating the determinant of 11-dimensional metric (2.12) with (2.9).

Therefore, the $\mathcal{N} = 1 G_2$-invariant solutions (2.16), with Appendix A where the Ricci tensor is presented and Appendix B where the coefficient functions are listed, satisfy the field equations (2.13) as long as the deformation parameters $(a, b)$ of the 7-ellipsoid (or two supergravity fields in 4-dimensions) and amplitude $A$ develop in the $AdS_4$ radial direction along the $G_2$-invariant flow (2.14). The corresponding gauge dual was constructed in the context of $\mathcal{N} = 1 U(2) \times U(2)$ superconformal Chern–Simons matter theory by adding one mass term for the adjoint $\mathcal{N} = 1$ superfield and the matter multiplet consists of seven flavors transforming in the adjoint together with flavor symmetry (**7** of $G_2$) [11].

### 2.2. Spherical parametrization II

Let us describe the other spherical parametrization describing the unit round five-sphere $\mathbf{S}^5$ whose base is characterized by $\mathbf{CP}^2$ space

$$\begin{aligned}
u^1 + iu^2 &= \sin\theta_4 \cos\left(\frac{\theta_1}{2}\right) e^{\frac{i}{2}(\theta_2+\theta_3)} e^{i\theta_5},\\
u^3 + iu^4 &= \sin\theta_4 \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i}{2}(-\theta_2+\theta_3)} e^{i\theta_5},\\
u^5 + iu^6 &= \cos\theta_4\, e^{i\theta_5}.
\end{aligned} \tag{2.18}$$

The isometry of $\mathbf{S}^5$ is $SU(3) \times U(1)$ where $SU(3)$ acts on three complex coordinates $z^i \equiv u^{2i-1} + iu^{2i}$ $(i = 1, 2, 3)$ and $U(1)$ acts on each $z^i$ as the phase rotations. The vector $u$ spans the $\mathbf{S}^5$ given by Hopf fibration on $\mathbf{CP}^2$ space. This can be described by writing $(du)^2$ as $(du)^2 = ds^2_{FS(2)} + (u, Jdu)^2$ where $ds^2_{FS(2)}$ denotes the Fubini-Study metric on $\mathbf{CP}^2$ and $(u, Jdu)$ is the Hopf fiber on it. The $J$ is the standard Kahler form. One can introduce the two orthogonal unit vectors (2.2) and the unit normal vector (2.3) for the parametrization (2.18) given above. Furthermore, the set of the six basis vectors is given by (2.4) and they as well as the normal vector can be described in terms of imaginary octonions. More explicitly, let us present some of them here

$$\begin{aligned}
\hat{\mathbf{e}}_1^1 &= \frac{1}{2}\cos\left(\frac{\theta_2+\theta_3}{2} + \theta_5\right)\sin\frac{\theta_1}{2}\sin\theta_4\sin\theta_6,\\
\hat{\mathbf{e}}_1^2 &= \frac{1}{2}\sin\frac{\theta_1}{2}\sin\theta_4\sin\left(\frac{\theta_2+\theta_3}{2} + \theta_5\right)\sin\theta_6, \ldots,
\end{aligned}$$

and the other nonvanishing components are given in Appendix C.

The three $G_2$-invariant tensors $\hat{F}_{ij}$, $\hat{T}_{ijk}$ and $\hat{S}_{ijk}$ are given by (2.7) for the choice of (2.18). For example,

$$\begin{aligned}
\hat{F}_{12} = \frac{1}{32}\sin^2\theta_4\sin^2\theta_6[&-4\cos 2\alpha_2\cos\theta_6 + 4\cos 2\alpha_1\cos 2(\alpha_3+\theta_5)\cos\theta_6\\
&+ (8\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_4 - \cos\theta_4(\cos(2\alpha_1-2\alpha_3-3\theta_5)\\
&- \cos(2\alpha_1-2\alpha_3-\theta_5) - \cos(2\alpha_1+2\alpha_3+\theta_5) + \cos(2\alpha_1+2\alpha_3+3\theta_5)\\
&+ 8\cos\alpha_1\cos\theta_5\sin\alpha_1 + 8\cos\alpha_2\sin\alpha_2\sin\theta_5))\sin\theta_6],
\end{aligned}$$

(2.7) and the $X^8$ is the $G_2$-singlet. The $G_2$-invariant structure constant $\eta_{ABC}$ is contracted with the **7**'s of $G_2$ in $X^A$ $(A = 1, \ldots, 7)$ in this way. Similarly the 3-form $S_{mnp}\, d\theta^m \wedge d\theta^n \wedge d\theta^p$ is given by $S^{(3)} = \frac{1}{(\sum_{A=1}^{7}(X^A)^2)^2}\eta_{ABCD}X^A dX^B \wedge dX^c \wedge dX^D$ where $\eta_{ABCD}$ is the same as the one in (2.7) and the 3-form $T^{(3)} = dF^{(2)}$ is given. Then one arrives at $F^{(4)} = dA^{(3)} + h_2\, dS^{(3)} + dr \wedge (\tilde{h}_3\, dF^{(2)} + \tilde{h}_4 S^{(3)}) + \frac{1}{\sqrt{1-(X^8)^2}} dX^8 \wedge (\tilde{h}_1\, dF^{(2)} + \tilde{h}_2 S^{(3)}) - \frac{\tilde{h}_5}{\sqrt{1-(X^8)^2}} dr \wedge F^{(2)} \wedge dX^8$ where the 3-form $A^{(3)} = -e^{3A}\widetilde{W}\, dx^1 \wedge dx^2 \wedge dx^3$.

$$\hat{F}_{13} = \frac{1}{32}\sin^2\theta_4\sin^2\theta_6[-2(\cos 2(\alpha_1-\alpha_2)+\cos 2(\alpha_1+\alpha_2)-2\cos 2(\alpha_3+\theta_5))\cos\theta_6 + (8\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_4+\cos\theta_4(\cos(2\alpha_1-2\alpha_2-\theta_5) - \cos(2\alpha_1+2\alpha_2-\theta_5)-\cos(2\alpha_1-2\alpha_2+\theta_5)+\cos(2\alpha_1+2\alpha_2+\theta_5) + 8\cos\alpha_1\cos\theta_5\sin\alpha_1+4\sin\theta_5\sin 2(\alpha_3+\theta_5)))\sin\theta_6], \ldots.$$

These tensors are given in Appendix D in terms of internal coordinates $\theta_i$ ($i = 1, \ldots, 6$) explicitly. Note that the antisymmetric tensor on $\mathbf{S}^6$, the dual of $T_{ijk}$, has an extra minus sign as follows:

$$\hat{S}_{ijk} = -\hat{F}_i^{\,l}\hat{T}_{ljk}. \tag{2.19}$$

All the identities between three tensors $\hat{F}_{ij}$, $\hat{T}_{ijk}$ and $\hat{S}_{ijk}$ are given in Appendix D.

By computing the dot product between $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ in $\mathbf{I}^7$, one obtains the 6-dimensional metric on $\mathbf{S}^6$. That is,

$$g_{ij} = \begin{pmatrix} \frac{1}{4}s_4^2s_6^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}s_4^2s_6^2 & \frac{1}{4}c_1s_4^2s_6^2 & 0 & \frac{1}{2}c_1s_4^2s_6^2 & 0 \\ 0 & \frac{1}{4}c_1s_4^2s_6^2 & \frac{1}{4}s_4^2s_6^2 & 0 & \frac{1}{2}s_4^2s_6^2 & 0 \\ 0 & 0 & 0 & s_6^2 & 0 & 0 \\ 0 & \frac{1}{2}c_1s_4^2s_6^2 & \frac{1}{2}s_4^2s_6^2 & 0 & s_6^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad c_i \equiv \cos\theta_i, \qquad s_i \equiv \sin\theta_i. \tag{2.20}$$

One sees this 6-dimensional metric from the projection of 7-dimensional metric $g_{mn}^7$ [18] by putting the extra coordinate $\theta$ to $\frac{\pi}{2}$. The rectangular coordinates parametrizing the $\mathbf{S}^7$ inside $\mathbf{R}^8$ with $\mathbf{S}^6 \simeq G_2/\mathrm{SU}(3)$ base are given by introducing the seventh coordinate $\theta$ [26]

$$X^i = u^i\sin\theta_6\sin\theta, \qquad X^7 = \cos\theta, \qquad X^8 = \cos\theta_6\sin\theta, \tag{2.21}$$

with $\sum_{A=1}^8 (X^A)^2 = 1$. As done in previous spherical parametrization, via the nonlinear metric ansatz [25], the warp factor and the 7-dimensional inverse metric are completely determined. Then the 7-dimensional warped ellipsoidal metric in this case can be summarized as [25,26]

$$ds_7^2 = \sqrt{\Delta a}L^2\left[\frac{\xi^2}{a^2}\,d\theta^2 + \sin^2\theta\,d\Omega_6^2\right], \qquad d\Omega_6^2 = d\theta_6^2 + \sin^2\theta_6\left[ds^2_{\mathrm{FS}(2)} + (u, Jdu)^2\right]. \tag{2.22}$$

The quadratic form and warp factor are given by (2.10) and (2.11) respectively. The fact that the vector $(u^1, u^2, u^3, u^4, u^5, u^6)$ in (2.18) spans $\mathbf{S}^5$ with a $\mathbf{CP}^2$ base, compared to the one in (2.9) that looks similar to (2.22) but the 5-dimensional metric inside behaves differently, can be understood by writing $(du)^2$ in terms of the Fubini-Study metric

$$ds^2_{\mathrm{FS}(2)} = d\theta_4^2 + \frac{1}{4}\sin^2\theta_4\left(\sigma_1^2+\sigma_2^2+\cos^2\theta_4\sigma_3^2\right), \tag{2.23}$$

and its Hopf fiber

$$(u, Jdu) = d\theta_5 + \frac{1}{2}\sin^2\theta_4\sigma_3. \tag{2.24}$$

The standard Kahler form $J$ is given by $J_{12} = J_{34} = J_{56} = J_{78} = 1$. One can easily check that the 6-dimensional metric (2.22) is nothing but the metric in (2.20) and the one-forms are given by $\sigma_1 = \cos\theta_3 d\theta_1 + \sin\theta_1\sin\theta_3 d\theta_2$, $\sigma_2 = \sin\theta_3 d\theta_1 - \sin\theta_1\cos\theta_3 d\theta_2$, and $\sigma_3 = d\theta_3 + \cos\theta_1 d\theta_2$, as usual. Recall that the $G_2$ symmetry is the isometry group of the metric. The Killing vector associated to the $G_2$ symmetry is constructed explicitly in [18] and it is shown that the 7-dimensional metric of (2.22) has a vanishing Lie derivative [28] with respect to the Killing vector fields.

The 11-dimensional metric is realized as (2.12) together with (2.22). The nontrivial task is to find the exact solution for 11-dimensional Einstein–Maxwell equations (2.13). Also we assume the nontrivial $\mathrm{AdS}_4$ radial coordinate dependence of vacuum expectation values and they are subject to the 4-dimensional RG flow equations (2.14). The Ricci tensor can be computed from the 11-dimensional metric with (2.12) together with (2.22) directly and they have the same form as the ones in previous spherical parametrization. So they have common feature in the Ricci tensor presented in Appendix A. This indicates that the Ricci tensor is indeed independent of the particular 5-dimensional space. One can express the Ricci tensor in the frame basis rather than in the coordinate basis. The former is exactly the same as the latter except the off-diagonal (4, 5)- and (5, 4)-components.

It is natural to solve the 11-dimensional Einstein–Maxwell equations by assuming that the 4-forms have the same tensorial structure as the ones in (2.16) and these $G_2$-invariant tensors are defined in the present 6-dimensional metric (2.22).

The undetermined coefficient functions depend on the coordinates $(r, \theta)$. One should find out these coefficient functions explicitly by requiring that 4-forms should satisfy (2.13). The field strengths are given by

$$
\begin{aligned}
F_{\mu\nu\rho 4} &= e^{3A(r)} W_r(r,\theta)\epsilon_{\mu\nu\rho}, \qquad F_{\mu\nu\rho 5} = e^{3A(r)} W_\theta(r,\theta)\epsilon_{\mu\nu\rho}, \\
F_{mnpq} &= 2h_2(r,\theta)\overset{6}{\epsilon}_{mnpqrs}\hat{F}^{rs}, \qquad F_{5mnp} = \tilde{h}_1(r,\theta)\hat{T}_{mnp} + \tilde{h}_2(r,\theta)\hat{S}_{mnp}, \\
F_{4mnp} &= \tilde{h}_3(r,\theta)\hat{T}_{mnp} + \tilde{h}_4(r,\theta)\hat{S}_{mnp}, \qquad F_{45mn} = \tilde{h}_5(r,\theta)\hat{F}_{mn},
\end{aligned}
\tag{2.25}
$$

where we denote the three hatted tensors here in order not to confuse with the unhatted ones that appear in (2.16). Applying these field strengths (2.25) to the 11-dimensional Einstein–Maxwell equations, one can determine the unknown coefficient functions, which depend on $r$ and $\theta$, completely. If we compute the right hand side of the Einstein equation, then the linear combination of quadratic coefficient functions appears after using the identities between $G_2$-invariant tensors or calculating the functional expressions component by component explicitly. For the latter case, we do not need to use the identities because we know three $G_2$-invariant tensors according to Appendix D.[2] Except for the (4, 5)- and (5, 4)-components of the Einstein equations, the (1, 1), (4, 4), (5, 5)- and (6, 6)-components depend on eight coefficient functions. The right hand side of the Einstein equations can be summarized by (3.21) of [7] exactly. As we explained before, the left hand side of the Einstein equations, the Ricci tensors, are identical to each other. Therefore, one concludes that the unknown coefficient functions are exactly the same as those in [7] or in Appendix B. Note that the geometric superpotential $\widetilde{W}$ from the field strengths ($W_r = e^{-3A}\partial_r(e^{3A}\widetilde{W})$, $W_\theta = \partial_\theta \widetilde{W}$) was found by requiring that it should coincide with the 4-dimensional superpotential $W$ (2.15) up to a multiplicative constant when the internal coordinate $\theta$ is fixed to some specific value.

The analysis for the checking of Maxwell equations can be done before, based on the paragraph containing (2.17). We have checked that the solutions (2.25) also satisfy the Maxwell equations explicitly. During this computation, the Elfbein determinant $E = \sqrt{-g_{11}}$ from 11-dimensional metric (2.12) and (2.22) is given by

$$
E = \frac{1}{8}\, e^{3A} L^7 a \sin^6\theta \sin\theta_1 \cos\theta_4 \sin^3\theta_4 \sin^5\theta_6 \left[a^2\cos^2\theta + b^2\sin^2\theta\right]^{\frac{2}{3}} .
$$

Therefore, we have established that the solutions (2.25) together with Appendices A, B and D consists of an exact solution to 11-dimensional supergravity by bosonic field equations (2.13), provided that the deformation parameters $(a, b)$ of the 7-dimensional internal space and the domain wall amplitude $A$ develop in the $AdS_4$ radial direction along the $G_2$-invariant RG flow (2.14). Although the 11-dimensional metric and 4-forms are different from those in previous parametrizations, the Ricci tensor (or the quadratic structure of 4-forms appearing on the right hand side of the Einstein equations) are exactly the same as each other.

## 3. Towards an $\mathcal{N} = 1$SU(3)-invariant supersymmetric flow in an 11-dimensional theory

We will use the results of 11-dimensional solutions with $\mathbf{CP}^2$ in the previous subsection. In the context of the round seven-sphere $\mathbf{S}^7$ as a Hopf fibration on $\mathbf{CP}^3$ let us replace the coordinate $\theta_5$ in (2.18) with $\phi + \psi$ where $\psi$ is the coordinate of the Hopf fiber on $\mathbf{S}^7$ as follows:

$$
\begin{aligned}
u^1 + iu^2 &= \sin\theta_4 \cos\left(\frac{\theta_1}{2}\right) e^{\frac{i}{2}(\theta_2+\theta_3)}\, e^{i(\phi+\psi)}, \\
u^3 + iu^4 &= \sin\theta_4 \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i}{2}(-\theta_2+\theta_3)} e^{i(\phi+\psi)}, \\
u^5 + iu^6 &= \cos\theta_4\, e^{i(\phi+\psi)}.
\end{aligned}
\tag{3.1}
$$

As before, the vector $(u^1, u^2, u^3, u^4, u^5, u^6)$ in (3.1) spans the $\mathbf{S}^5$ with $\mathbf{CP}^2$ base and the Fubini-Study metric on $\mathbf{CP}^2$ is given by (2.23). The correct 7-dimensional warped metric, corresponding to (5.15) of [26] is given by

$$
ds_7^2 = \sqrt{\Delta} a L^2 \left[ (dU)^2 + \frac{b^2}{a^2}(dV_1)^2 + (dV_2)^2 \right],
\tag{3.2}
$$

[2] One can easily check that the above rank 2 and rank 3 tensors written in terms of the angle variables can be expressed similarly as in footnote 1. That is, $\hat{F}^{(2)} = \frac{1}{(\sum_{A=1}^{7}(X^A)^2)^{\frac{3}{2}}}\eta_{ABC}X^A dX^B \wedge dX^c$, $\hat{S}^{(3)} = -\frac{1}{(\sum_{A=1}^{7}(X^A)^2)^2}\eta_{ABCD}X^A dX^B \wedge dX^c \wedge dX^D$, and $\hat{T}^{(3)} = d\hat{F}^{(2)}$. Note the extra minus sign in the 3-form $\hat{S}^{(3)}$. As in footnote 1, we interchange the $X^7$ and the $X^8$ with each other. It is not so obvious to express the angular variables in terms of the rectangular coordinates, contrary to footnote 1 and from the beginning we assume the above forms. After inserting the rectangular coordinates written in terms of angular variables (2.18) and (2.21) into them, we have checked that these agree with those in Appendix D. Therefore, the $G_2$-invariance is more obvious in this basis rather than in the angle basis. Note that the $G_2$-invariant tensors $\hat{F}_{mn}$, $\hat{T}_{mnp}$ and $\hat{S}_{mnp}$ in (2.25) can be obtained from those in the previous parametrization via the coordinate transformations between (2.1) and (2.18). In particular, we have checked that the component $F_{46}$ is related to the component $\hat{F}_{46}$ via coordinate transformations.

where the $\mathbf{R}^8$ coordinates $X^A$ are decomposed into

$$\begin{aligned}
\mathbf{U} &= (u^1, u^2, u^3, u^4, u^5, u^6, 0, 0)\cos\mu,\\
\mathbf{V}_1 &= (0, 0, 0, 0, 0, 0, 1, 0)\cos\psi\sin\mu,\\
\mathbf{V}_2 &= (0, 0, 0, 0, 0, 0, 0, 1)\sin\psi\sin\mu.
\end{aligned} \tag{3.3}$$

The deformation parameter $\gamma \equiv \frac{cd}{ab} - 1$ of [26] vanishes for the $G_2$-invariant sector in which the constraints are characterized by $c = a$ and $d = b$. For the $G_2$-invariant sector, the ellipsoidal deformation arises along the $V_1$ direction. Then one can choose the two additional orthonormal frames as $(u, Jdu)$ (2.24) and $dV_1$ respectively. Then the remaining orthonormal frame is determined from (3.2) by completing squares.

The precise relations between the parameters of previous section and those in this section by comparing (2.21) with (3.3) (the other eight coordinates are the same) are given by

$$\begin{aligned}
\cos\theta &= \cos\psi\sin\mu,\\
\theta_5 &= \phi + \psi,\\
\cos\theta_6 &= \frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}.
\end{aligned} \tag{3.4}$$

From these (3.4), one obtains the partial differentiations of old variables $(\theta, \theta_5, \theta_6)$ with respect to the new variables $(\mu, \phi, \psi)$ as follows:

$$\begin{aligned}
&\frac{\partial\theta}{\partial\mu} = -\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}, \qquad \frac{\partial\theta}{\partial\psi} = \frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}},\\
&\frac{\partial\theta_5}{\partial\phi} = 1, \qquad \frac{\partial\theta_5}{\partial\psi} = 1,\\
&\frac{\partial\theta_6}{\partial\mu} = -\frac{\sin\psi}{1-\cos^2\psi\sin^2\mu}, \qquad \frac{\partial\theta_6}{\partial\psi} = -\frac{\sin\mu\cos\mu\cos\psi}{1-\cos^2\psi\sin^2\mu}.
\end{aligned} \tag{3.5}$$

The frames for the 11-dimensional metric are summarized by

$$\begin{aligned}
e^1 &= \Delta^{-\frac{1}{2}} e^A dx^1, \qquad e^2 = \Delta^{-\frac{1}{2}} e^A dx^2, \qquad e^3 = \Delta^{-\frac{1}{2}} e^A dx^3, \qquad e^4 = \Delta^{-\frac{1}{2}} dr,\\
e^5 &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}} \frac{\xi}{a}\sqrt{\frac{1}{1-\cos^2\psi\sin^2\mu}}\left(-\cos\mu\cos\psi\, d\mu + \sin\mu\sin\psi\, d\psi\right),\\
e^6 &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}} \cos\mu \frac{1}{2}\sin\theta_4\sigma_1,\\
e^7 &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}} \cos\mu \frac{1}{2}\sin\theta_4\sigma_2,\\
e^8 &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}} \cos\mu \frac{1}{2}\sin\theta_4\cos\theta_4\sigma_3,\\
e^9 &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}} \cos\mu\, d\theta_4,\\
e^{10} &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}} \cos\mu\left(d\phi + d\psi + \frac{1}{2}\sin^2\theta_4\sigma_3\right),\\
e^{11} &= L\Delta^{\frac{1}{4}} a^{\frac{1}{4}}\sqrt{\frac{1}{1-\cos^2\psi\sin^2\mu}}\left(-\sin\psi\, d\mu - \sin\mu\cos\mu\cos\psi\, d\psi\right).
\end{aligned} \tag{3.6}$$

The $\widetilde{e}^5$, $\widetilde{e}^6$ and $\widetilde{e}^7$ with some sign difference appearing in (5.24) of [26] correspond to $e^{10}$, $e^{11}$ and $e^5$ respectively. The frames $e^6$, $e^7$, $e^8$ and $e^9$ correspond to the Fubini-Study metric on $\mathbf{CP}^2$ whose symmetry group is SU(3). Note that $\psi$ which is the coordinate of the Hopf fiber on $\mathbf{S}^7$ appears in the frame $e^{10}$ and this gives the off-diagonal metric components between the coordinates $(\psi, \theta_2, \theta_3, \phi)$. Also from the structure of $e^5$ and $e^{11}$, the off-diagonal components between $(\mu, \psi)$ occur. The quadratic form (2.10) reads as $\xi^2 = a^2\cos^2\psi\sin^2\mu + b^2(1-\cos^2\psi\sin^2\mu)$ from (3.4). The warp factor is the same as before: (2.11). The one-forms $\sigma_i$ are given as before.[3]

Then how does one find the solution for 11-dimensional Einstein–Maxwell equations for given 11-dimensional background (3.6)? The 6-dimensional space is no longer a unit round six-sphere. Since the metric (2.22) is related to the metric (3.6) by change of variables, one can use the solutions (2.25) and find out the solutions. First of all, the Ricci tensor can be obtained from (3.6) directly or can be determined from the one in the previous section by using the transformation

3 When we take the seven rectangular coordinates except $X^7$ among (3.3), a similar computation as in [18] leads to the fact that the 7-dimensional metric has a vanishing Lie derivative with respect to the Killing vector fields where the generator for $G_2$ is the same as the one in Appendix A of [18]. This implies that the Killing vector is associated to the $G_2$ symmetry.

on the coordinates between the two coordinate systems (3.5) via the tensorial property. In other words, the Ricci tensor is given explicitly by

$$\widetilde{R}_M^N = \left(\frac{\partial z^P}{\partial \widetilde{z}^M}\right)\left(\frac{\partial \widetilde{z}^N}{\partial z^Q}\right) R_P^Q, \tag{3.7}$$

where the 11-dimensional coordinates are given by

$$z^M = (x^1, x^2, x^3, r; \theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6), \qquad \widetilde{z}^M = (x^1, x^2, x^3, r; \mu, \theta_1, \theta_2, \theta_3, \theta_4, \phi, \psi).$$

Eight of them are common and three of them are distinct. The Ricci tensor $R_P^Q$ for the $G_2$-invariant flow is presented in Appendix A. Some of them are given by

$$\widetilde{R}_1^1 = R_1^1, \qquad \widetilde{R}_2^2 = R_2^2\,(=\widetilde{R}_1^1), \qquad \widetilde{R}_3^3 = R_3^3(=\widetilde{R}_1^1), \ldots.$$

There exist off-diagonal components $(4, 10)$, $(4, 11)$, $(5, 10)$, $(5, 11)$, $(11, 4)$, $(11, 5)$, $(11, 10)$, as well as $(4, 5)$- and $(5, 4)$-components. Their full expressions are given in Appendix E.

For the field strengths, one has, by multiplying the transformation matrix,

$$\widetilde{F}_{MNPQ} = \left(\frac{\partial z^R}{\partial \widetilde{z}^M}\right)\left(\frac{\partial z^S}{\partial \widetilde{z}^N}\right)\left(\frac{\partial z^T}{\partial \widetilde{z}^P}\right)\left(\frac{\partial z^U}{\partial \widetilde{z}^Q}\right) F_{RSTU}. \tag{3.8}$$

Then some of the components, using the relations (3.5), are given by

$$\widetilde{F}_{1234} = F_{1234}, \qquad \widetilde{F}_{1235} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{1235}, \ldots.$$

These transformed 4-forms are given in terms of those in $G_2$-invariant flow in the Appendix F and moreover, the transformed 4-forms with upper indices are described. The 4-form $\widetilde{F}_{123\,11}$ is a new object. At the IR critical point where $a = \sqrt{\frac{6\sqrt{3}}{5}}$ and $b = \sqrt{\frac{2\sqrt{3}}{5}}$, due to the fact that the coefficient functions $W_\theta$, $\widetilde{h}_3$, $\widetilde{h}_4$ and $\widetilde{h}_5$ in the Appendix B vanish, the following 4-forms also vanish at this critical point:

$$\widetilde{F}_{1235} = 0 = \widetilde{F}_{123\,11} = \widetilde{F}_{45mn} = \widetilde{F}_{4mnp}.$$

Note that for the $G_2$-invariant flow, the 4-forms $F_{1235}$, $F_{4mnp}$ and $F_{45mn}$ vanish where $m, n, p = 6, \ldots, 11$ at the IR critical point. Once we suppose that the 4-dimensional metric has the domain wall factor $e^{3A(r)}$ which breaks the 4-dimensional conformal invariance, the mixed 4-forms occur along the whole RG flow. Of course, at the UV critical point where $a = 1 = b$, the only nonzero 4-form field is $\widetilde{F}_{1234}$. Some of the transformed 4-forms with upper indices are given by

$$\widetilde{F}^{1234} = F^{1234}, \qquad \widetilde{F}^{1235} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F^{1235}, \ldots.$$

These can be obtained from (3.8) by using the 11-dimensional metric (3.6) or by multiplying the transformation matrices (3.5) into the 4-forms with upper indices of $G_2$-invariant flow. The remaining 4-forms are given explicitly by Appendix F.[4]

How does one check the 11-dimensional solution for the Einstein equation? One shows this by substituting the solutions (3.7) and (3.8) into the first equation of (2.13) with the transformed-Ricci tensor and transformed-4-forms. Or one checks this equality using the solution of the $G_2$-invariant flow in Section 2.2. In the previous section we have shown that (2.25) satisfies the field equations (2.13). Let us go back the transformed-Ricci tensor (3.7) which is written in terms of the Ricci tensor for a $G_2$ invariant flow. Without specifying the explicit form for Ricci tensor $R_P^Q$, one writes down the transformed-Ricci tensor, as in Appendix E. Now one can use the property of the $G_2$-invariant flow: the first equation of (2.13). That is, one replaces the Ricci tensor in terms of the quadratic 4-forms. Then one sees that the transformed-Ricci tensor can be written in terms of the quadratic 4-forms for the $G_2$-invariant flow. Let us return to the right hand side of the Einstein equations. Using (3.8) and (3.5), one can express this in terms of quadratic 4-forms for the $G_2$-invariant flow.

So far, we did not insert the explicit form for the 4-forms $F_{MNPQ}$. One can make the difference between the left hand side and the right hand side of the Einstein equations and see whether this is zero or not. At first sight, some of the components written in terms of quadratic 4-forms in the $G_2$-invariant flow are not exactly vanishing. They contain the terms $F_{4npq}F^{mnpq}$, $F_{5npq}F^{mnpq}$, $F_{mnpq}F^{snpq}$ where $m, n, p, q = 6, \ldots, 11$ and $s = 4, 5, 6, \ldots, 11$. After plugging the explicit solution of

[4] One can also check that the field strengths written in terms of the angle variables can be expressed similarly as in footnotes 1 and 2: $\widetilde{F}^{(4)} = dA^{(3)} + h_2\, d\hat{S}^{(3)} + dr \wedge (\tilde{h}_3\, d\hat{F}^{(2)} + \tilde{h}_4\hat{S}^{(3)}) + \frac{1}{\sqrt{1-(X^8)^2}} dX^8 \wedge (\tilde{h}_1\, d\hat{F}^{(2)} + \tilde{h}_2\hat{S}^{(3)}) - \frac{\tilde{h}_5}{\sqrt{1-(X^8)^2}} dr \wedge \hat{F}^{(2)} \wedge dX^8$ where the 2-form $\hat{F}^{(2)}$ and the 3-forms $A^{(3)}$ and $\hat{S}^{(3)}$ are given in the footnote 2. As in footnote 2, as it is not so obvious to express the angular variables in terms of the rectangular coordinates, we assume the above forms. After inserting the rectangular coordinates written in terms of angular variables (3.1) and (3.3) into them, we have checked this 4-form agrees with those in Appendix F. Therefore, the 4-form explicitly has $G_2$-invariance.

the $G_2$-invariant flow, then all of these are equal to zero identically. One can also check this from the useful identities between three-invariant tensors $\hat{F}_{ij}$, $\hat{T}_{ijk}$ and $\hat{S}_{ijk}$ presented in Appendix D, as done in [7]. Recall that these quadratic 4-forms above correspond to the off-diagonal terms of Einstein equation for $G_2$-invariant flow which vanish identically except the (4, 5)- and (5, 4)-components. Of course, the above extra piece does not possess these nonzero off-diagonal terms, (4, 5)- and (5, 4)-components. Therefore, we have shown the solutions (3.7) and (3.8) indeed satisfy the 11-dimensional Einstein equations.

Let us move on to the Maxwell equations. Let us introduce the notation $\frac{1}{2}E\widetilde{\nabla}_M\widetilde{F}^{MNPQ} \equiv (\widetilde{N}\widetilde{P}\widetilde{Q})$ for simplicity, and present all the nonzero components of the left hand side of the Maxwell equations in terms of the 4-forms in the $G_2$-invariant flow, using the property of the 11-dimensional solution we have found before. For example, the $(\widetilde{1}\widetilde{2}\widetilde{3})$-component of the left hand side of the Maxwell equations reads

$$(\widetilde{1}\widetilde{2}\widetilde{3}) = \sum_{m,n,p,q,r,s=6}^{11} \overset{6}{\epsilon}_{mnpqrs}\left(\frac{1}{3!3!}F_{4mnp}F_{5qrs} - \frac{1}{2!4!}F_{45mn}F_{pqrs}\right).$$

The other nonzero components are given explicitly in Appendix G. The nonzero components of the Maxwell equation, the second equation of (2.13), are characterized by the following indices

$$(\widetilde{1}\widetilde{2}\widetilde{3}), \qquad (\widetilde{4}\widetilde{5}\widetilde{m}), \qquad (\widetilde{4}\widetilde{n}\widetilde{p}), \qquad (\widetilde{5}\widetilde{n}\widetilde{p}), \qquad (\widetilde{m}\widetilde{n}\widetilde{p}), \qquad \widetilde{m}, \widetilde{n}, \widetilde{p} = 6, \ldots, 11,$$

with the number of components 1, 5 by choosing one out of five (the (45 11)-component is equal to zero), 15 by choosing two out of six, 15 by choosing two out of six and 20 by choosing three out of six respectively. The other remaining components of the Maxwell equation become identically zero. Therefore, there exist 56-nonzero-components of the Maxwell equation. Compared with the ones (2.17) in the $G_2$-invariant flow, there are nonzero terms from $(\widetilde{4}\widetilde{5}\widetilde{m})$-components where $\widetilde{m} = 6, \ldots, 10$. The right hand side of the Maxwell equations for the $(\widetilde{1}\widetilde{2}\widetilde{3})$-component above consist of 35 terms coming from the quadratic 4-forms. For the other components, the right hand side contains a single term, two terms or three terms in quadratic 4-forms.

Now it is ready to check the Maxwell equations for the solutions (3.8). For given 4-forms $\widetilde{F}_{MNPQ}$, one can construct the corresponding 4-forms with upper indices $\widetilde{F}^{RSTU}$ by using the 11-dimensional metric (3.6) or by multiplying the transformation matrix with the 4-forms $F^{MNPQ}$ for $G_2$-invariant flow, as was done in (3.8). Then one obtains all these transformed 4-forms with upper indices, as in Appendix E. As done in the previous paragraph, one constructs the left hand side of Maxwell equations in terms of quadratic 4-forms for the $G_2$-invariant flow by using the 11-dimensional field equations. According to the transformation rules, one can express the covariant derivative $\widetilde{\nabla}_M$ and 4-forms $\widetilde{F}^{MNPQ}$ in terms of $\nabla_N$ and $F^{RSTU}$ for the $G_2$-invariant flow together with $(\mu, \psi)$-dependent functions. Then using the Maxwell equations for the $G_2$-invariant flow, one can replace $\nabla_M F^{MNPQ}$ with quadratic 4-forms and arrives at Appendix G. Similarly, let us return to the right hand side of Maxwell equations. Using (3.8) and the 11-dimensional metric, one can express this in terms of quadratic 4-forms in the $G_2$-invariant flow. We also transform the 11-dimensional determinant according to the transformation rules appropriately. It turns out that the difference between the left hand side and the right hand side of Maxwell equations becomes zero identically. Therefore, we have shown that the solution (3.8) indeed satisfies the Maxwell equations correctly. During this check, the Elfbein determinant $E = \sqrt{-g_{11}}$ is used and it is

$$E = -\frac{1}{8}e^{3A}L^7 a \sin\mu \cos^5\mu \sin\theta_1 \cos\theta_4 \sin^3\theta_4 \left[a^2\cos^2\psi \sin^2\mu + b^2(1-\cos^2\psi \sin^2\mu)\right]^{\frac{2}{3}},$$

by computing the determinant of 11-dimensional metric (3.6).

Therefore, we have shown that the solutions (3.8) together with Appendices A, B and D–F consists of an exact solution to the 11-dimensional supergravity by bosonic field equations (2.13), provided that the deformation parameters $(a, b)$ of the 7-dimensional internal space and the domain wall amplitude $A$ develop in the $\mathrm{AdS}_4$ radial direction along the $G_2$-invariant RG flow (2.14) connecting from the $\mathcal{N} = 8\,\mathrm{SO}(8)$ UV fixed point to the $\mathcal{N} = 1\,G_2$ IR fixed point. Compared with the previous solutions for $G_2$-invariant flow, they share the common $\mathbf{CP}^2$ space inside 7-dimensional internal space but three remaining coordinates are different from each other.

## 4. Conclusions and outlook

We have found an exact solution of the $\mathcal{N} = 1\,G_2$-invariant flow (connecting the $\mathcal{N} = 8\,\mathrm{SO}(8)$ UV invariant fixed point to the $\mathcal{N} = 1\,G_2$ IR invariant fixed point) to the 11-dimensional Einstein–Maxwell equations with (2.12) and (2.22). Based on this solution, we also have discovered an exact solution of the $\mathcal{N} = 1\,G_2$-invariant flow connecting above two fixed points to the 11-dimensional Einstein–Maxwell equations with (3.6). Now we are ready to look at the 11-dimensional lift of holographic $\mathcal{N} = 1$ supersymmetric SU(3)-invariant RG flow [13] connecting the $\mathcal{N} = 1\,G_2$ critical point to the $\mathcal{N} = 2\,\mathrm{SU}(3) \times U(1)_R$ critical point. At each critical point, there exists a consistent description for the 11-dimensional metric with $\mathbf{CP}^3$-basis and there exist corresponding 4-forms explicitly and it is an open problem to discover the 11-dimensional solution along the whole $\mathcal{N} = 1$ RG flow.

- What happens when there exist four supergravity fields $(a, b, c, d)$ [26] in which there exists an SU(3)-invariant sector? So far, we have concentrated on the $G_2$-invariant sector where there exist two independent supergravity fields. As explained in the introduction, the 11-dimensional metric is known and it is an open problem to construct the correct

4-forms. In the particular limit, one has 11-dimensional lift [8] of $\mathcal{N}=2$ SU(3) $\times$ $U(1)_R$-invariant flow and for other limit, one obtains the 11-dimensional lift [7] of $\mathcal{N}=1$ $G_2$-invariant flow. At least, the 4-forms should respect the behavior of these two extreme limits. The decoding of the 4-forms written as the SU(3)-singlet vacuum expectation values in [36] will be useful. One of these flows will describe the $\mathcal{N}=1$ SU(3)-invariant RG flow connecting from $\mathcal{N}=8$ SO(8) UV invariant fixed point to $\mathcal{N}=2$ SU(3) $\times$ $U(1)_R$ IR invariant fixed point, by looking at the SU(3) $\times$ $U(1)_R$-invariant sector with two supergravity fields.

- What happens when we replace $\mathbf{CP}^2$ appearing in (2.22) or (3.6) with $\mathbf{CP}^1\times\mathbf{CP}^1$? In 5-dimensional space, this is equivalent to putting $T^{1,1}$ space in (2.22). According to the branching of $G_2$ into SU(2) $\times$ SU(2), one expects that the 11-dimensional solution should preserve SU(2) $\times$ SU(2) symmetry. It would be interesting to find out the correct 4-forms for given 11-dimensional metric for this case: the uplift of $\mathcal{N}=1$ SU(2) $\times$ SU(2)-invariant flow. In order to do this direction, one needs to look at the structure of 4-forms found in this paper, in the frame basis. It is a nontrivial task to find out the corresponding gauge dual, as seen in [19]. Furthermore, the most general 5-dimensional Sasaki–Einstein space can be considered and the global symmetries become smaller than SU(2) $\times$ SU(2) symmetry.
- Any octonion description for the present work? The automorphisms of the Cayley algebra that leaves one of the imaginary octonion units form a subgroup SU(3) of $G_2$ [33,34]. For the split octonion algebra, the automorphism group $G_2$ acts on the basis by an 8-dimensional reducible representation. Two of them are invariant, three split octonions transform like triplets, and three complex conjugate split octonions transform like antitriplets, under the SU(3) subgroup of split $G_2$.

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## Appendix A. The Ricci tensor for $G_2$-invariant flow

The Ricci tensor appearing in (2.13) for $G_2$-invariant flow is given by

$$\begin{aligned}
R_1^1 = & -\frac{1}{24L^2\left[(a^2+7b^2)^2-112(ab-1)\right](a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{8}{3}}}\\
&\times[8a^{14}\cos^4\theta-1120a^{11}b\cos^4\theta+8a^{12}b^2\cos^2\theta(15+13\cos 2\theta)\\
&-112a^9b^3\cos^2\theta(83+63\cos 2\theta)-8a^5b^3(32\cos^2\theta(445+437\cos 2\theta)\\
&+7b^4(617+148\cos 2\theta-177\cos 4\theta))+4a^8b^2(4\cos^2\theta(1139+1031\cos 2\theta)\\
&+7b^4(171+192\cos 2\theta+29\cos 4\theta))+a^{10}(896\cos^4\theta+b^4(941+1172\cos 2\theta+239\cos 4\theta))\\
&+a^6(-128\cos^2\theta(29+27\cos 2\theta)-49b^8(-221-172\cos 2\theta+\cos 4\theta)\\
&+16b^4(4123+4776\cos 2\theta+705\cos 4\theta))-8a^7b(32\cos^2\theta(27+29\cos 2\theta)\\
&+7b^4(564\cos 2\theta+73(7+\cos 4\theta)))-16a^3b(768\cos^2\theta+49b^8(83+\cos 2\theta)\\
&+16b^4(549+697\cos 2\theta))\sin^2\theta+96b^2(-384+616b^4+245b^8)\sin^4\theta\\
&-96ab^3(-1408+1232b^4+343b^8)\sin^4\theta+8a^2\sin^2\theta(4608\cos^2\theta-784b^8(-14+5\cos 2\theta)\\
&+16b^4(37+1209\cos 2\theta)+2401b^{12}\sin^2\theta)+8a^4b^2(4988+7824\cos 2\theta\\
&+b^4(7191+212\cos 2\theta-3287\cos 4\theta)+2868\cos 4\theta+343b^8(9+5\cos 2\theta)\sin^2\theta)]\\
= & R_2^2=R_3^3,\\
R_4^4 = & -\frac{1}{24L^2\left[(a^2+7b^2)^2-112(ab-1)\right](a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{8}{3}}}\\
&\times[8a^{14}\cos^4\theta+8a^{12}b^2\cos^2\theta(15+13\cos 2\theta)-8a^{11}b\cos^2\theta(73+67\cos 2\theta)\\
&-4a^5b^3(7b^4(2425+1340\cos 2\theta-237\cos 4\theta)+16(2461+2268\cos 2\theta-25\cos 4\theta))\\
&-2a^9b^3(3527+4412\cos 2\theta+909\cos 4\theta)+2a^8b^2(9303+11\,692\cos 2\theta+2413\cos 4\theta\\
&+14b^4(171+192\cos 2\theta+29\cos 4\theta))+a^{10}(896\cos^4\theta+b^4(941+1172\cos 2\theta+239\cos 4\theta))\\
&-4a^7b(128\cos^2\theta(51+40\cos 2\theta)+b^4(9281+10\,128\cos 2\theta+1367\cos 4\theta))\\
&+a^6(128\cos^2\theta(55+57\cos 2\theta)-49b^8(-221-172\cos 2\theta+\cos 4\theta)\\
&+8b^4(14\,309+14\,988\cos 2\theta+1671\cos 4\theta))\\
&-8a^3b(24\,576\cos^2\theta+49b^8(349+113\cos 2\theta)+64b^4(942+395\cos 2\theta))\sin^2\theta\\
&+48b^2(1536+3248b^4+1519b^8)\sin^4\theta-48ab^3(5632+7840b^4+1715b^8)\sin^4\theta\\
&+8a^2\sin^2\theta(4608\cos^2\theta+b^8(29\,876-8708\cos 2\theta)+b^4(42\,832-6768\cos 2\theta)\\
&+2401b^{12}\sin^2\theta)-4a^4b^2(8(-2981-1884\cos 2\theta+945\cos 4\theta)\\
&+b^4(-36\,429+524\cos 2\theta+11\,209\cos 4\theta)-686b^8(9+5\cos 2\theta)\sin^2\theta)],
\end{aligned}$$

$$R^5_4 = \frac{1}{8L^3\sqrt{(a^2+7b^2)^2-112(ab-1)}(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{8}{3}}}$$
$$\times [a^{\frac{5}{2}}(-8a^5(80+a^4)\cos^3\theta\sin\theta+4b(a^2(48(4+a^4)-a(192+7a^4)b$$
$$+54a^2b^2-24a^3b^3-60b^4+7ab^5)+b^2(-12+7ab)(-16+7b^4)\sin^2\theta)\sin 2\theta$$
$$-2a^2b(-96-46a^4+6a^5b+50a^2b^2+11a^3b^3+38b^4+2ab(8-21b^4))\sin 4\theta)],$$
$$R^5_5 = \frac{1}{48L^2\left[(a^2+7b^2)^2-112(ab-1)\right](a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{8}{3}}}$$
$$\times [8a^{14}\cos^4\theta-1120a^{11}b\cos^4\theta+8a^{12}b^2\cos^2\theta(15+13\cos 2\theta)-112a^9b^3\cos^2\theta(83$$
$$+63\cos 2\theta)-8a^5b^3(128\cos^2\theta(181+8\cos 2\theta)+7b^4(617+148\cos 2\theta-177\cos 4\theta))$$
$$+4a^8b^2(4\cos^2\theta(1175+1091\cos 2\theta)+7b^4(171+192\cos 2\theta+29\cos 4\theta))$$
$$+a^{10}(16\cos^2\theta(25+31\cos 2\theta)+b^4(941+1172\cos 2\theta+239\cos 4\theta))$$
$$+a^6(1024\cos^2\theta(-7+3\cos 2\theta)-49b^8(-221-172\cos 2\theta+\cos 4\theta)$$
$$+8b^4(8153+8832\cos 2\theta+879\cos 4\theta))-8a^7b(64\cos^2\theta(9+28\cos 2\theta)$$
$$+7b^4(564\cos 2\theta+73(7+\cos 4\theta)))-16a^3b(-1536\cos^2\theta+49b^8(83+\cos 2\theta)$$
$$+32b^4(747+380\cos 2\theta))\sin^2\theta+96b^2(768+1792b^4+245b^8)\sin^4\theta$$
$$-96ab^3(2816+3584b^4+343b^8)\sin^4\theta-4a^2\sin^2\theta(18\,432\cos^2\theta$$
$$+256b^4(-323+15\cos 2\theta)+196b^8(-181+115\cos 2\theta)-4802b^{12}\sin^2\theta)$$
$$-8a^4b^2(64(-116-102\cos 2\theta+15\cos 4\theta)+b^4(-10\,065+1324\cos 2\theta+4625\cos 4\theta)$$
$$-343b^8(9+5\cos 2\theta)\sin^2\theta)],$$
$$R^6_6 = \frac{1}{48L^2\left[(a^2+7b^2)^2-112(ab-1)\right](a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{8}{3}}}$$
$$\times [8a^{14}\cos^4\theta-1120a^{11}b\cos^4\theta+8a^{12}b^2\cos^2\theta(15+13\cos 2\theta)-112a^9b^3\cos^2\theta(83$$
$$+63\cos 2\theta)-8a^5b^3(128\cos^2\theta(100+89\cos 2\theta)+7b^4(617+148\cos 2\theta-177\cos 4\theta))$$
$$+4a^8b^2(4\cos^2\theta(1187+1079\cos 2\theta)+7b^4(171+192\cos 2\theta+29\cos 4\theta))$$
$$+a^{10}(896\cos^4\theta+b^4(941+1172\cos 2\theta+239\cos 4\theta))+a^6(-128\cos^2\theta(17+15\cos 2\theta)$$
$$-49b^8(-221-172\cos 2\theta+\cos 4\theta)+16b^4(3895+4440\cos 2\theta+597\cos 4\theta))$$
$$-8a^7b(64\cos^2\theta(18+19\cos 2\theta)+7b^4(564\cos 2\theta+73(7+\cos 4\theta)))$$
$$-16a^3b(768\cos^2\theta+49b^8(83+\cos 2\theta)+32b^4(216+281\cos 2\theta))\sin^2\theta$$
$$+96b^2(-384+392b^4+245b^8)\sin^4\theta-96ab^3(-1408+896b^4+343b^8)\sin^4\theta$$
$$+8a^2\sin^2\theta(4608\cos^2\theta-112b^8(-92+29\cos 2\theta)+112b^4(-17+147\cos 2\theta)$$
$$+2401b^{12}\sin^2\theta)+8a^4b^2(4076+6480\cos 2\theta+b^4(6927+116\cos 2\theta-2927\cos 4\theta)$$
$$+2436\cos 4\theta+343b^8(9+5\cos 2\theta)\sin^2\theta)]=R^7_7=R^8_8=R^9_9=R^{10}_{10}=R^{11}_{11},$$
$$R^4_5 = \frac{1}{8L\sqrt{a}\sqrt{(a^2+7b^2)^2-112(ab-1)}(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{8}{3}}}$$
$$\times [-2(a^5(80+a^4)-96a^2(4+a^4)b+2a^3(192+7a^4)b^2-12(16+9a^4)b^3$$
$$+16a(7+3a^4)b^4+120a^2b^5-14a^3b^6+84b^7-49ab^8)\sin 2\theta$$
$$-(a-b)(a+b)(a^7-192b-92a^4b+13a^5b^2+8a^2b^3+84b^5+7ab^2(16-7b^4)$$
$$+5a^3(16+7b^4))\sin 4\theta].$$

The Ricci tensor before imposing the RG flow equations (2.14) to two vacuum expectation values $(a, b)$ was given in [7]. The above Ricci tensor holds for both spherical parametrizations in Section 2.

## Appendix B. The coefficient functions for $G_2$ invariant flow

The coefficient functions appearing in (2.16) or (2.25) are given by

$$\tilde{h}_1 = \frac{L^3\sqrt{ab-1}\sin^4\theta\left[-a^4\cos^2\theta+a^2b^2(3+2\cos 2\theta)+3b^4\sin^2\theta\right]}{2a(a^2\cos^2\theta+b^2\sin^2\theta)^2},$$
$$h_2 = \frac{L^3b\sqrt{ab-1}\sin^4\theta}{2(a^2\cos^2\theta+b^2\sin^2\theta)},$$
$$\tilde{h}_2 = \frac{L^3b\sqrt{ab-1}\cos\theta\sin^3\theta\left[a^2(3+\cos 2\theta)+2b^2\sin^2\theta\right]}{2(a^2\cos^2\theta+b^2\sin^2\theta)^2},$$

$$\widetilde{h}_3 = \frac{L^2 a^{\frac{3}{2}} \sqrt{ab-1}\cos\theta\sin^3\theta}{2\sqrt{(a^2+7b^2)^2-112(ab-1)}(a^2\cos^2\theta+b^2\sin^2\theta)^2} \times [-16a^3\cos^2\theta + a^4 b(3+\cos 2\theta) + 2a^2b^3(11+3\cos 2\theta) - 112ab^2\sin^2\theta + 2b(48+7b^4)\sin^2\theta],$$

$$\tilde{h}_4 = \frac{L^2 a^{\frac{1}{2}} \sqrt{ab-1}\sin^4\theta}{4\sqrt{(a^2+7b^2)^2-112(ab-1)}(a^2\cos^2\theta+b^2\sin^2\theta)^2}[2a^6\cos^2\theta - 64a^3 b\cos^2\theta + a^4b^2(1+3\cos 2\theta) + a^2(96\cos^2\theta - 5b^4(9+5\cos 2\theta)) + 128ab^3\sin^2\theta - 6b^2(16+7b^4)\sin^2\theta],$$

$$\tilde{h}_5 = \frac{2L^2\sqrt{ab-1}(7b^3+a^2b-4a)\sin^2\theta}{\sqrt{a}\sqrt{(a^2+7b^2)^2-112(ab-1)}},$$

$$W_r = -\frac{1}{2L}a^2\left[a^5\cos^2\theta + a^2b(ab-2)(4+3\cos 2\theta) + b^3(7ab-12)\sin^2\theta\right],$$

$$W_\theta = -\frac{a^{\frac{3}{2}}\left[48(1-ab)+(a^2-b^2)(a^2+7b^2)\right]}{\sqrt{(a^2+7b^2)^2-112(ab-1)}}\sin\theta\cos\theta.$$

## Appendix C. The seven frames on $\mathbf{S^6}$ with $\mathbf{CP^2}$ space: $G_2$-invariant flow

The seven frames ($\alpha_i \equiv 2\theta_i,\ i = 1, 2, 3$) in Section 2.2 are given by

$$\hat{\mathbf{e}}_1^1 = \frac{1}{2}\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4\sin\theta_6,$$

$$\hat{\mathbf{e}}_1^2 = \frac{1}{2}\sin\alpha_1\sin\theta_4\sin(\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_1^3 = -\frac{1}{2}\cos\alpha_1\cos(-\alpha_2+\alpha_3+\theta_5)\sin\theta_4\sin\theta_6,$$

$$\hat{\mathbf{e}}_1^4 = -\frac{1}{2}\cos\alpha_1\sin\theta_4\sin(-\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_2^1 = \frac{1}{2}\cos\alpha_1\sin\theta_4\sin(\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_2^2 = -\frac{1}{2}\cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sin\theta_4\sin\theta_6,$$

$$\hat{\mathbf{e}}_2^3 = -\frac{1}{2}\sin\alpha_1\sin\theta_4\sin(-\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_2^4 = \frac{1}{2}\cos(-\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4\sin\theta_6,$$

$$\hat{\mathbf{e}}_3^1 = \hat{\mathbf{e}}_2^1, \qquad \hat{\mathbf{e}}_3^2 = \hat{\mathbf{e}}_2^2, \qquad \hat{\mathbf{e}}_3^3 = -\hat{\mathbf{e}}_2^3, \qquad \hat{\mathbf{e}}_3^4 = -\hat{\mathbf{e}}_2^4,$$

$$\hat{\mathbf{e}}_4^1 = -\frac{1}{2}\cos\alpha_1\cos\theta_4\cos(\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_4^2 = -\frac{1}{2}\cos\alpha_1\cos\theta_4\sin(\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_4^3 = -\cos\theta_4\cos(-\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_6,$$

$$\hat{\mathbf{e}}_4^4 = -\cos\theta_4\sin\alpha_1\sin(-\alpha_2+\alpha_3+\theta_5)\sin\theta_6,$$

$$\hat{\mathbf{e}}_4^5 = \cos\theta_5\sin\theta_4\sin\theta_6, \qquad \hat{\mathbf{e}}_4^6 = \sin\theta_4\sin\theta_5\sin\theta_6,$$

$$\hat{\mathbf{e}}_5^1 = 2\hat{\mathbf{e}}_2^1, \qquad \hat{\mathbf{e}}_5^2 = 2\hat{\mathbf{e}}_2^2, \qquad \hat{\mathbf{e}}_5^3 = -2\hat{\mathbf{e}}_2^3, \qquad \hat{\mathbf{e}}_5^4 = -2\hat{\mathbf{e}}_2^4,$$

$$\hat{\mathbf{e}}_5^5 = \cos\theta_4\sin\theta_5\sin\theta_6, \qquad \hat{\mathbf{e}}_5^6 = -\cos\theta_4\cos\theta_5\sin\theta_6,$$

$$\hat{\mathbf{e}}_6^1 = -\cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6\sin\theta_4,$$

$$\hat{\mathbf{e}}_6^2 = -\cos\alpha_1\cos\theta_6\sin\theta_4\sin(\alpha_2+\alpha_3+\theta_5),$$

$$\hat{\mathbf{e}}_6^3 = -\cos(-\alpha_2+\alpha_3+\theta_5)\cos\theta_6\sin\alpha_1\sin\theta_4,$$

$$\hat{\mathbf{e}}_6^4 = -\cos\theta_6\sin\alpha_1\sin\theta_4\sin(-\alpha_2+\alpha_3+\theta_5),$$

$$\hat{\mathbf{e}}_6^5 = -\cos\theta_4\cos\theta_5\cos\theta_6, \qquad \hat{\mathbf{e}}_6^6 = -\cos\theta_4\cos\theta_6\sin\theta_5, \qquad \hat{\mathbf{e}}_6^7 = \sin\theta_6,$$
$$\hat{\mathbf{n}}^1 = -2\hat{\mathbf{e}}_2^2, \qquad \hat{\mathbf{n}}^2 = 2\hat{\mathbf{e}}_2^1, \qquad \hat{\mathbf{n}}^3 = 2\hat{\mathbf{e}}_2^4,$$
$$\hat{\mathbf{n}}^4 = -2\hat{\mathbf{e}}_2^3, \qquad \hat{\mathbf{n}}^5 = -\hat{\mathbf{e}}_5^6, \qquad \hat{\mathbf{n}}^6 = \hat{\mathbf{e}}_5^5, \qquad \hat{\mathbf{n}}^7 = \cos\theta_6.$$

## Appendix D. The $\hat{F}_{ij}$, $\hat{T}_{ijk}$ and $\hat{S}_{ijk}$ tensors on $\mathbf{S}^6$ with $\mathbf{CP}^2$ space: $G_2$-invariant flow

The three invariant tensors appearing in (2.25) are given as follows. The $\hat{F}_{ij}$ tensor appearing in the 4-forms $F_{45mn}$ where $\alpha_i \equiv 2\theta_i$ ($i = 1, 2, 3$) for simplicity is given by

$$\begin{aligned}
\hat{F}_{12} \ = \ & \frac{1}{32}\sin^2\theta_4\sin^2\theta_6[-4\cos 2\alpha_2\cos\theta_6 + 4\cos 2\alpha_1\cos 2(\alpha_3+\theta_5)\cos\theta_6 \\
& + (8\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_4 - \cos\theta_4(\cos(2\alpha_1-2\alpha_3-3\theta_5) \\
& - \cos(2\alpha_1-2\alpha_3-\theta_5) - \cos(2\alpha_1+2\alpha_3+\theta_5) + \cos(2\alpha_1+2\alpha_3+3\theta_5) \\
& + 8\cos\alpha_1\cos\theta_5\sin\alpha_1 + 8\cos\alpha_2\sin\alpha_2\sin\theta_5))\sin\theta_6], \\
\hat{F}_{13} \ = \ & \frac{1}{32}\sin^2\theta_4\sin^2\theta_6[-2(\cos 2(\alpha_1-\alpha_2) + \cos 2(\alpha_1+\alpha_2) - 2\cos 2(\alpha_3+\theta_5))\cos\theta_6 \\
& + (8\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_4 + \cos\theta_4(\cos(2\alpha_1-2\alpha_2-\theta_5) \\
& - \cos(2\alpha_1+2\alpha_2-\theta_5) - \cos(2\alpha_1-2\alpha_2+\theta_5) + \cos(2\alpha_1+2\alpha_2+\theta_5) \\
& + 8\cos\alpha_1\cos\theta_5\sin\alpha_1 + 4\sin\theta_5\sin 2(\alpha_3+\theta_5)))\sin\theta_6], \\
\hat{F}_{14} \ = \ & \frac{1}{4}\sin\theta_4\sin^2\theta_6[\cos\theta_4\cos\theta_6 - \sin 2\alpha_2 + \sin 2(\alpha_3+\theta_5) + 2\cos\theta_6\sin\theta_4 \\
& \times(+\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_5 - \cos\theta_5\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)) \\
& + (\cos 2\alpha_2 - \cos 2(\alpha_3+\theta_5))\sin\theta_5\sin\theta_6], \\
\hat{F}_{15} \ = \ & \frac{1}{8}\sin\theta_4\sin^2\theta_6[-4\cos\theta_4\cos\theta_6\sin\alpha_1\sin\theta_5\sin(\alpha_2+\alpha_3+\theta_5) \\
& + 4\cos^2\theta_4\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_6 + 4\cos^3\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin^2\theta_4\sin\theta_6 \\
& + \sin 2\theta_4(\cos\theta_5\sin 2\alpha_1 + \sin^2\alpha_1\sin\theta_5(\sin 2\alpha_2 + \sin 2(\alpha_3+\theta_5)))\sin\theta_6 \\
& + 2\cos 2\alpha_2\sin^2\alpha_1\sin\theta_4(\cos\theta_6 - \cos\theta_4\cos\theta_5\sin\theta_6) \\
& + 2\cos 2(\alpha_3+\theta_5)\sin^2\alpha_1\sin\theta_4(\cos\theta_6 + \cos\theta_4\cos\theta_5\sin\theta_6) \\
& + 4\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)(-\cos\theta_4\cos\theta_5\cos\theta_6 + \sin^2\alpha_1\sin^2\theta_4\sin\theta_6) \\
& + 2\cos^2\alpha_1(-\cos 2\alpha_2 + \cos 2(\alpha_3+\theta_5)\cos\theta_6\sin\theta_4 \\
& + \sin(\alpha_2+\alpha_3)\sin 2\theta_4\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_6)], \\
\hat{F}_{16} \ = \ & \frac{1}{4}\sin\theta_4\sin\theta_6[-2\cos\alpha_1\cos\theta_4\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_5 \\
& + \sin\theta_4(-\sin 2\alpha_2 + \sin 2(\alpha_3+\theta_5)) + 2\cos\theta_4\cos\theta_5\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)], \\
\hat{F}_{23} \ = \ & \frac{1}{4}\cos\alpha_1\sin\alpha_1\sin^2\theta_4\sin^2\theta_6[\cos\theta_6(\sin 2\alpha_2 + \sin 2(\alpha_3+\theta_5)) \\
& - (2\cos\alpha_1\sin\theta_4\sin(\alpha_2-\alpha_3-\theta_5) + \cos\theta_4(\cos 2\alpha_2 + \cos 2(\alpha_3+\theta_5))\sin\theta_5)\sin\theta_6], \\
\hat{F}_{24} \ = \ & \frac{1}{8}\sin\theta_4\sin^2\theta_6[-2\cos\theta_4\cos 2(\alpha_3+\theta_5)\cos\theta_6\sin 2\alpha_1 \\
& + \cos^2\theta_4(4\cos\theta_5(\sin^2\alpha_1 - \cos 2\alpha_3\sin 2\alpha_1\sin^2\theta_5) \\
& + \sin 2\alpha_1\sin 2\alpha_3(\sin\theta_5 - \sin 3\theta_5))\sin\theta_6 \\
& + 2\sin\theta_4(2\cos\alpha_1\cos\theta_5\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6 - 2\cos\theta_6\sin\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5 \\
& + \sin\theta_4(2\cos\theta_5\sin^2\alpha_1 - \sin 2\alpha_1\sin\theta_5\sin 2(\alpha_3+\theta_5))\sin\theta_6)], \\
\hat{F}_{25} \ = \ & \frac{1}{8}\sin\theta_4\sin^2\theta_6[2\cos\theta_6(2\cos\theta_4(\cos\theta_5\sin\alpha_1\sin(\alpha_2-\alpha_3-\theta_5) \\
& + \cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sin\theta_5) + \sin 2\alpha_1\sin\theta_4(\sin 2\alpha_2 + \sin 2(\alpha_3+\theta_5))) \\
& + (-8\cos^2\alpha_1\sin\alpha_1\sin^2\theta_4\sin(\alpha_2-\alpha_3-\theta_5) + 4\cos\alpha_1\cos^2\theta_4\sin(\alpha_2+\alpha_3+\theta_5) \\
& + \sin 2\theta_4(2\sin^2\alpha_1\sin\theta_5 + \sin 2\alpha_1(-\cos 2\alpha_2\sin\theta_5 + \sin(2\alpha_3+\theta_5))))\sin\theta_6], \\
\hat{F}_{26} \ = \ & \frac{1}{16}\sin\theta_4\sin\theta_6[(2-3\cos 2(\alpha_3+\theta_5))\sin 2\alpha_1\sin\theta_4 \\
& - 2\cos\alpha_1 4\cos\theta_4\cos\theta_5\cos(\alpha_2+\alpha_3+\theta_5) + (2+\cos 2(\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4) \\
& + 8\cos\theta_4\sin\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5],
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{34} &= \frac{1}{4}\sin\theta_4\sin^2\theta_6[\cos\theta_6(\cos^2\alpha_2\cos\theta_4\sin 2\alpha_1-\cos\theta_4\sin 2\alpha_1\sin^2\alpha_2\\
&\quad+2\cos\alpha_1\cos\alpha_2\cos\theta_5\cos(\alpha_3+\theta_5)\sin\theta_4+2\cos(\alpha_3+\theta_5)\sin\alpha_1\sin\alpha_2\sin\theta_4\sin\theta_5\\
&\quad-2\sin\theta_4(\cos\alpha_1\cos\theta_5\sin\alpha_2+\cos\alpha_2\sin\alpha_1\sin\theta_5)\sin(\alpha_3+\theta_5))\\
&\quad+(-2\cos\theta_5\sin^2\alpha_1+\sin 2\alpha_1\sin 2\alpha_2\sin\theta_5)\sin\theta_6],\\
\hat{F}_{35} &= \frac{1}{2}\cos\theta_4\sin\theta_4\sin^2\theta_6[-\sin^2\alpha_1\sin\theta_4\sin\theta_5\sin\theta_6\\
&\quad-\cos\theta_5\sin\alpha_1(\cos\theta_6\sin(\alpha_2-\alpha_3-\theta_5)+\cos\alpha_1\sin 2\alpha_2\sin\theta_4\sin\theta_6)\\
&\quad+\cos\alpha_1(\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6\sin\theta_5+\cos\theta_4\sin(\alpha_2+\alpha_3+\theta_5)\sin\theta_6)],\\
\hat{F}_{36} &= \frac{1}{4}\sin\theta_4\sin\theta_6[\cos^2\alpha_2\sin 2\alpha_1\sin\theta_4-2\cos\alpha_1(\cos\theta_4\cos\theta_5\cos(\alpha_2+\alpha_3+\theta_5)\\
&\quad+\sin\alpha_1\sin^2\alpha_2\sin\theta_4)-2\cos\theta_4\sin\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5],\\
\hat{F}_{45} &= -\frac{1}{4}\sin^2\theta_6[\cos\theta_6(2\cos\alpha_1(-\cos(\alpha_2+\alpha_3)\cos 2\theta_4+\cos(\alpha_2+\alpha_3+2\theta_5))\\
&\quad+2(\cos(\alpha_2-\alpha_3)\cos 2\theta_4+\cos(\alpha_2-\alpha_3-2\theta_5))\sin\alpha_1+\cos 2\alpha_2\sin 2\alpha_1\sin 2\theta_4)\\
&\quad+4(-\cos\theta_5\sin^2\alpha_1\sin\theta_4+\cos\alpha_1(\cos\theta_4\cos(\alpha_2+\alpha_3+\theta_5)\\
&\quad+\sin\alpha_1\sin 2\alpha_2\sin\theta_4\sin\theta_5))\sin\theta_6],\\
\hat{F}_{46} &= \frac{1}{4}\sin\theta_6[\cos(\alpha_1+\alpha_2-\alpha_3)-\cos(\alpha_1-\alpha_2+\alpha_3)-\cos(\alpha_1+\alpha_2-\alpha_3-2\theta_5)\\
&\quad+\cos(\alpha_1-\alpha_2+\alpha_3+2\theta_5)-4\cos\alpha_1\cos\theta_5\sin(\alpha_2+\alpha_3+\theta_5)],\\
\hat{F}_{56} &= \sin\theta_4\sin\theta_6\left[-\cos\alpha_1\cos(\alpha_2+\alpha_3)\cos\theta_4+\cos(\alpha_2-\alpha_3)\cos\theta_4\sin\alpha_1+\frac{1}{2}\cos 2\alpha_2\sin 2\alpha_1\sin\theta_4\right].
\end{aligned}$$

The $\hat{F}^{ij}$ tensor appearing in the 4-forms $F_{mnpq}$ is given by

$$\begin{aligned}
\hat{F}^{12} &= -2\csc\theta_4\csc\theta_6[\cos 2\alpha_2\cot\theta_6\csc\theta_4+\cos\theta_5(\cot\alpha_1\cot\theta_4-\cos(\alpha_2-\alpha_3)\sec\alpha_1)\\
&\quad+(-\cos\alpha_3\sec\alpha_1\sin\alpha_2+\cos\alpha_2(2\cot\theta_4\sin\alpha_2+\sec\alpha_1\sin\alpha_3))\sin\theta_5],\\
\hat{F}^{13} &= 2\csc^2\theta_6[\cos\theta_6(\cos(2\alpha_3+2\theta_5)\csc^2\theta_4+\csc 2\theta_4(\cos(\alpha_1+\alpha_2-\alpha_3)+\cos(\alpha_1-\alpha_2+\alpha_3)\\
&\quad+\cos(\alpha_1+\alpha_2-\alpha_3-2\theta_5)+\cos(\alpha_1-\alpha_2+\alpha_3+2\theta_5)+4\sin\alpha_1\sin\theta_5\sin(\alpha_2+\alpha_3+\theta_5)))\\
&\quad+(\cos\theta_5(\cot\alpha_1\cot\theta_4\csc\theta_4+(\cos 2\alpha_2-\cos(2\alpha_3+2\theta_5))\sec\theta_4)\\
&\quad+\csc\theta_4(\cos(\alpha_2-\alpha_3-\theta_5)\sec\alpha_1-2\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\\
&\quad+\cot\theta_4\sin\theta_5\sin(2\alpha_3+2\theta_5)))\sin\theta_6],\\
\hat{F}^{14} &= \csc\theta_6[2\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\cot\theta_6\sin\theta_5+(\cos 2\alpha_2-\cos(2\alpha_3+2\theta_5))\csc\theta_4\sin\theta_5\\
&\quad+\cot\theta_4\cot\theta_6(-\sin 2\alpha_2+\sin(2\alpha_3+2\theta_5))-2\cos\theta_5\cot\theta_6\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)],\\
\hat{F}^{15} &= \frac{1}{4}\csc\theta_6\sec\theta_4[-\cos(2\alpha_1-2\alpha_2-\theta_5)+\cos(2\alpha_1+2\alpha_2-\theta_5)+\cos(2\alpha_1-2\alpha_2+\theta_5)\\
&\quad-\cos(2\alpha_1+2\alpha_2+\theta_5)-2(\cos(2\alpha_2-\theta_5)+\cos(2\alpha_2+\theta_5)-\cos(2\alpha_3+\theta_5)-\cos(2\alpha_3+3\theta_5)\\
&\quad-4\cos(\alpha_2+\alpha_3+\theta_5)\cot\theta_4\sin\alpha_1+2\cos 2\alpha_1\sin 2\alpha_2\sin\theta_5+\cot\theta_6\csc\theta_4(\cos(\alpha_1+\alpha_2-\alpha_3)\\
&\quad+\cos(\alpha_1-\alpha_2+\alpha_3)+\cos(\alpha_1+\alpha_2-\alpha_3-2\theta_5)+\cos(\alpha_1-\alpha_2+\alpha_3+2\theta_5)\\
&\quad+4\sin\alpha_1\sin\theta_5\sin(\alpha_2+\alpha_3+\theta_5)))],\\
\hat{F}^{16} &= \csc\theta_6[-\sin 2\alpha_2-2\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\cot\theta_4\sin\theta_5\\
&\quad+\sin(2\alpha_3+2\theta_5)+2\cos\theta_5\cot\theta_4\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)],\\
\hat{F}^{23} &= \csc 2\alpha_1\csc^2\theta_6[2\cos\theta_6(-4\csc 2\theta_4(\cos\alpha_1\cos\theta_5\sin(\alpha_2-\alpha_3-\theta_5)\\
&\quad+\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_5)+\csc^2\theta_4(\sin 2\alpha_2+\sin(2\alpha_3+2\theta_5)))\\
&\quad-2(\csc\theta_4(2\cos\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)+\cos 2\alpha_2\cot\theta_4\sin\theta_5)+\sec\theta_4(\cos\theta_5(\cos 2\alpha_1\sin 2\alpha_2\\
&\quad+\sin 2\alpha_3)+(\cos 2\alpha_3+\cos(2\alpha_3+2\theta_5)\csc^2\theta_4+\sin 2\alpha_1)\sin\theta_5)\\
&\quad+2\csc\theta_4\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5))\sin\theta_6],\\
\hat{F}^{24} &= -\csc 2\alpha_1\csc\theta_4\csc^2\theta_6[\cos\theta_4\cos(2\alpha_3+2\theta_5)\cos\theta_6+\cos^2\theta_4\sin\theta_6(\sin\theta_5\sin(2\alpha_3+2\theta_5)\\
&\quad-\cos\theta_5\tan\alpha_1)+\cos 2\alpha_1(\cos^2\alpha_2\cos\theta_4\cos\theta_6-\cos\theta_4\cos\theta_6\sin^2\alpha_2\\
&\quad+\cos(\alpha_3+\theta_5)\cos\theta_6\sec\alpha_1\sin\alpha_2\sin\theta_4\sin\theta_5\\
&\quad+\cos\alpha_2\cos\theta_6\sin\theta_4(\cos\theta_5\cos(\alpha_3+\theta_5)\csc\alpha_1-\sec\alpha_1\sin\theta_5\sin(\alpha_3+\theta_5))\\
&\quad+\sin 2\alpha_2\sin\theta_5\sin\theta_6-\cos\theta_5(\cos\theta_6\csc\alpha_1\sin\alpha_2\sin\theta_4\sin(\alpha_3+\theta_5)+\sin\theta_6\tan\alpha_1))\\
&\quad+\sin\theta_4(\sin\theta_5(\cos\theta_6\sec\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)+\sin\theta_4\sin(2\alpha_3+2\theta_5)\sin\theta_6)\\
&\quad-\cos\theta_5(\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6\csc\alpha_1+\sin\theta_4\sin\theta_6\tan\alpha_1))],
\end{aligned}$$

$$\hat{F}^{26} = -\csc\theta_6[\cos 2\alpha_2 \cot 2\alpha_1 + \csc 2\alpha_1(\cos(2\alpha_3 + 2\theta_5) + 2\cot\theta_4(\cos\theta_5\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1 - \cos\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5))],$$

$$\hat{F}^{25} = \csc 2\alpha_1 \csc^2\theta_6 \sec\theta_4[2\cos\theta_6\csc\theta_4(\cos\alpha_1\cos\theta_5\sin(\alpha_2-\alpha_3-\theta_5) + \cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_5) + (\cos 2\alpha_1\cos\theta_5\sin 2\alpha_2 + (\cos(2\alpha_3+2\theta_5)+\sin 2\alpha_1)\sin\theta_5 + 2\cot\theta_4\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5) + \sin(2\alpha_3+\theta_5))\sin\theta_6],$$

$$\hat{F}^{34} = \csc\theta_6[\cos(2\alpha_3+2\theta_5)\cot 2\alpha_1\cot\theta_4\cot\theta_6 + \cos 2\alpha_2\cot\theta_4\cot\theta_6\csc 2\alpha_1 - 2\cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sec\theta_4 - \cos\theta_5(\csc\theta_4 + 2\cot\theta_6\sin\alpha_1 \times(-\cos\alpha_2\cos(\alpha_3+\theta_5)(-1+\csc 2\alpha_1) + (1+\csc 2\alpha_1)\sin\alpha_2\sin(\alpha_3+\theta_5))) + \sin\theta_5(\csc 2\alpha_1\csc\theta_4\sin 2\alpha_2 + \cot 2\alpha_1\csc\theta_4\sin(2\alpha_3+2\theta_5) + \cot\theta_6(\csc\alpha_1\sin(\alpha_2-\alpha_3-\theta_5) + 2\cos\alpha_1\sin(\alpha_2+\alpha_3+\theta_5))))],$$

$$\hat{F}^{35} = -\frac{1}{2}\csc\theta_6[2\cos\theta_5\sec\theta_4(\csc 2\alpha_1\sin 2\alpha_2 + \cot\theta_6\csc\alpha_1\csc\theta_4\sin(\alpha_2-\alpha_3-\theta_5)) + \cos 2\alpha_1\csc^2\alpha_1\csc\theta_4\sec\alpha_1\sin(\alpha_2+\alpha_3+\theta_5) - \csc^2\alpha_1\csc\theta_4\sec\alpha_1\sec^2\theta_4\sin(\alpha_2+\alpha_3+\theta_5) + \sec\theta_4(2(1+\cos(2\alpha_3+2\theta_5))\cot 2\alpha_1 - \cos(\alpha_2+\alpha_3+\theta_5)\cot\theta_6\csc\theta_4\sec\alpha_1)\sin\theta_5 + 2\cot 2\alpha_1\sin(2\alpha_3+\theta_5) + \csc^2\alpha_1\sec\alpha_1)\sin(\alpha_2+\alpha_3+\theta_5)\tan\theta_4],$$

$$\hat{F}^{36} = \frac{1}{4}\csc\theta_6[\cot 2\alpha_1(-2+4\cos(2\alpha_3+2\theta_5)) + 4\cos\alpha_1\csc 2\alpha_1(2\cos\theta_5\cos(\alpha_2+\alpha_3+\theta_5)\cot\theta_4 + \sin\alpha_1) - 4\cot\theta_4\sec\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5) + 4\csc^2 2\alpha_1(-\cos^2 2\alpha_1 + \csc^2\theta_4)\sec\theta_4 \times(\cos^2\alpha_2\sin 2\alpha_1\sin\theta_4 - 2\cos\alpha_1(\cos\theta_4\cos\theta_5\cos(\alpha_2+\alpha_3+\theta_5) + \sin\alpha_1\sin^2\alpha_2\sin\theta_4) - 2\cos\theta_4\sin\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5)\tan\theta_4 - 4\tan\theta_4(-2\cos\alpha_1\cos(\alpha_2+\alpha_3) + 2\cos(\alpha_2-\alpha_3)\sin\alpha_1 + \cos 2\alpha_2\sin 2\alpha_1\tan\theta_4)],$$

$$\hat{F}^{45} = -\frac{1}{2}\csc\theta_6[2\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_5\cot\theta_6\sin\alpha_1 + \cos\alpha_1((-\cos(\alpha_2+\alpha_3)\cos 2\theta_4 + \cos(\alpha_2+\alpha_3+2\theta_5))\cot\theta_6\sec^2\theta_4 + 2\cos(\alpha_2+\alpha_3+\theta_5)(\sec\theta_4 - \cos\theta_5\cot\theta_6\tan^2\theta_4))],$$

$$\hat{F}^{46} = \frac{1}{4}\csc\theta_6[\cos(\alpha_1+\alpha_2-\alpha_3) - \cos(\alpha_1-\alpha_2+\alpha_3) - \cos(\alpha_1+\alpha_2-\alpha_3-2\theta_5) + \cos(\alpha_1-\alpha_2+\alpha_3+2\theta_5) - 4\cos\alpha_1\cos\theta_5\sin(\alpha_2+\alpha_3+\theta_5)],$$

$$\hat{F}^{56} = \frac{1}{4}\csc\theta_6[-\cos(\alpha_1-\alpha_2-\alpha_3) - \cos(\alpha_1+\alpha_2+\alpha_3) + \cos(\alpha_1-\alpha_2-\alpha_3-2\theta_5) + 4\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_5\sin\alpha_1]\tan\theta_4.$$

One checks the identities

$$\hat{F}^i_k\hat{F}^k_j = -\delta^i_j, \qquad \hat{F}^{[ij}\hat{F}^{kl}\hat{F}^{mn]} = \frac{1}{15}\overset{6}{\epsilon}{}^{ijklmn}.$$

The raising and lowering of the indices are done by the 6-dimensional metric on $\mathbf{S}^6$ (2.20).

The $\hat{T}_{ijk}$ tensor appearing in the 4-forms $F_{4mnp}$ and $F_{5mnp}$ is given by

$$\hat{T}_{123} = \frac{1}{4}\cos\alpha_1\sin^2\alpha_1\sin^3\theta_4\sin(\alpha_2-\alpha_3-\theta_5)\sin^3\theta_6,$$

$$\hat{T}_{124} = \frac{1}{8}\sin^2\theta_4\sin^3\theta_6[2\cos\alpha_1\cos\theta_4\cos(\alpha_2-\alpha_3-\theta_5) + \sin\theta_4(\cos\theta_5\sin 2\alpha_1 + \sin\theta_5(\sin 2\alpha_2 - \cos 2\alpha_1\sin 2(\alpha_3+\theta_5))))],$$

$$\hat{T}_{125} = \frac{1}{8}\sin^2\theta_4\sin^3\theta_6[4\cos\alpha_1\sin^2\alpha_1\sin\theta_4\sin(\alpha_2-\alpha_3-\theta_5) + \cos\theta_4(\sin 2\alpha_1\sin\theta_5 + \cos\theta_5(-\sin 2\alpha_2 + \cos 2\alpha_1\sin 2(\alpha_3+\theta_5))))],$$

$$\hat{T}_{126} = \frac{1}{16}\sin^2\theta_4\sin^2\theta_6[\cos 3\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_6\sin\theta_4 + \cos\alpha_1\cos\theta_6(-4\cos\theta_4\cos\theta_5\sin\alpha_1 + (5-2\cos 2\alpha_1)\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_4) - 2\cos\theta_4\cos\theta_6\sin\theta_5(\sin 2\alpha_2 - \cos 2\alpha_1\sin 2(\alpha_3+\theta_5)) + 2(\cos 2\alpha_2 - \cos 2\alpha_1\cos 2(\alpha_3+\theta_5))\sin\theta_6],$$

$$\hat{T}_{134} = \frac{1}{8}\sin^2\theta_4\sin^3\theta_6[2\cos\alpha_1(\cos\theta_4\cos(\alpha_2-\alpha_3-\theta_5) - \cos\theta_5\sin\alpha_1\sin\theta_4) + \cos^2\alpha_1\sin 2\alpha_2\sin\theta_4\sin\theta_5 - \sin\theta_4\sin\theta_5(\sin^2\alpha_1\sin 2\alpha_2 + \sin 2(\alpha_3+\theta_5))],$$

$$\hat{T}_{135} = \frac{1}{16}\cos\theta_4\sin^2\theta_4\sin^3\theta_6[-\cos(2\alpha_1-\theta_5) + \cos(2\alpha_1+\theta_5) + \cos\theta_5(\sin 2(\alpha_1-\alpha_2) - \sin 2(\alpha_1+\alpha_2) + 2\sin 2(\alpha_3+\theta_5))],$$

$$\hat{T}_{136} = \frac{1}{16}\sin^2\theta_4\sin^2\theta_6[4\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_6\sin\theta_4 + \cos\theta_4\cos\theta_6(2\cos\theta_5\sin 2\alpha_1+\sin\theta_5(\sin 2(\alpha_1-\alpha_2)-\sin 2(\alpha_1+\alpha_2) + 2\sin 2(\alpha_3+\theta_5))) + (\cos 2(\alpha_1-\alpha_2)+\cos 2(\alpha_1+\alpha_2)-2\cos 2(\alpha_3+\theta_5))\sin\theta_6],$$

$$\hat{T}_{145} = \frac{1}{32}\sin\theta_4\sin^3\theta_6[2\cos(2\alpha_2-\theta_5)+\cos(2\alpha_2-2\theta_4-\theta_5)+\cos(2\alpha_2+2\theta_4-\theta_5) + 2\cos(2\alpha_2+\theta_5)+\cos(2\alpha_2-2\theta_4+\theta_5)-2\cos(2\alpha_3-2\theta_4+\theta_5)+\cos(2\alpha_2+2\theta_4+\theta_5) - 2(\cos(2\alpha_3+2\theta_4+\theta_5)+2\cos(2\alpha_3+3\theta_5)+2(\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5) - \cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1)\sin 2\theta_4+2\sin^2\theta_4(-\cos\theta_5\sin 2\alpha_1+\cos 2\alpha_1\sin 2\alpha_2\sin\theta_5))],$$

$$\hat{T}_{146} = \frac{1}{4}\sin\theta_4\sin^2\theta_6[(\cos 2\alpha_2-\cos 2(\alpha_3+\theta_5))\cos\theta_6\sin\theta_5 + (\cos\theta_4(\sin 2\alpha_2-\sin 2(\alpha_3+\theta_5))+2\sin\theta_4(-\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_5 + \cos\theta_5\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)))\sin\theta_6],$$

$$\hat{T}_{156} = \frac{1}{16}\sin\theta_4\sin^2\theta_6[\cos\theta_6(8\cos^2\theta_4\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1 + 8\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin^2\theta_4-\sin 2\theta_4(\cos(2\alpha_2-\theta_5)+\cos(2\alpha_2+\theta_5) - 2(\cos(2\alpha_3+\theta_5)+\cos\theta_5\sin 2\alpha_1-\cos 2\alpha_1\sin 2\alpha_2\sin\theta_5))) + 2((\cos(2\alpha_1-2\alpha_2)+\cos(2\alpha_1+2\alpha_2)-2\cos(2\alpha_3+2\theta_5))\sin\theta_4 + 4\cos\theta_4(\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_5+\sin\alpha_1\sin\theta_5\sin(\alpha_2+\alpha_3+\theta_5)))\sin\theta_6],$$

$$\hat{T}_{234} = \frac{1}{8}\sin 2\alpha_1\sin^2\theta_4\sin^3\theta_6[-2\cos\alpha_1\cos\theta_4\sin(\alpha_2-\alpha_3-\theta_5) + (\cos 2\alpha_2+\cos(2\alpha_3+2\theta_5))\sin\theta_4\sin\theta_5],$$

$$\hat{T}_{235} = -\frac{1}{8}\cos\theta_4\cos\theta_5\sin 2\alpha_1\sin^2\theta_4\sin^3\theta_6\left[\cos 2\alpha_2+\cos(2\alpha_3+2\theta_5)\right],$$

$$\hat{T}_{236} = -\frac{1}{8}\sin 2\alpha_1\sin^2\theta_4\sin^2\theta_6[2\cos\alpha_1\cos\theta_6\sin\theta_4\sin(\alpha_2-\alpha_3-\theta_5) + \cos\theta_4(\cos 2\alpha_2+\cos(2\alpha_3+2\theta_5))\cos\theta_6\sin\theta_5+(\sin 2\alpha_2+\sin(2\alpha_3+2\theta_5))\sin\theta_6],$$

$$\hat{T}_{245} = -\frac{1}{4}\sin\theta_4\sin^3\theta_6[-2\cos^2\alpha_1\sin\alpha_1\sin 2\theta_4\sin(\alpha_2-\alpha_3-\theta_5)+(2\cos^2\theta_4\sin^2\alpha_1 + (\cos 2\alpha_2+\cos(2\alpha_3+2\theta_5))\sin 2\alpha_1\sin^2\theta_4)\sin\theta_5 + \cos\alpha_1(2\cos^2\theta_4\cos\theta_5\sin\alpha_1\sin(2\alpha_3+2\theta_5)-\sin 2\theta_4\sin(\alpha_2+\alpha_3+\theta_5))],$$

$$\hat{T}_{246} = \frac{1}{8}\sin\theta_4\sin^2\theta_6[\cos\theta_6\sin 2\alpha_1\sin 2\alpha_3(\sin\theta_5-\sin 3\theta_5) + 2(\cos\theta_4\cos(2\alpha_3+2\theta_5)\sin 2\alpha_1 + 2\sin\alpha_1\sin\theta_4\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_5)\sin\theta_6+4\cos\theta_5(\cos\theta_6(\sin^2\alpha_1 - \cos 2\alpha_3\sin 2\alpha_1\sin^2\theta_5)-\cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sin\theta_4\sin\theta_6)],$$

$$\hat{T}_{256} = -\frac{1}{4}\sin^2\theta_6[4\cos^2\alpha_1\cos\theta_6\sin\alpha_1\sin^3\theta_4\sin(\alpha_2-\alpha_3-\theta_5) - 2\cos\alpha_1\cos^2\theta_4\cos\theta_6\sin\theta_4\sin(\alpha_2+\alpha_3+\theta_5)-\cos\theta_4\cos\theta_6\sin^2\theta_4(2\sin\alpha_1 \times(-\cos\alpha_1\cos 2\alpha_2+\sin\alpha_1)\sin\theta_5+\sin 2\alpha_1\sin(2\alpha_3+\theta_5)) + \cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sin 2\theta_4\sin\theta_5\sin\theta_6+(\cos\theta_5\sin\alpha_1\sin 2\theta_4\sin(\alpha_2-\alpha_3-\theta_5) + \sin 2\alpha_1\sin^2\theta_4(\sin 2\alpha_2+\sin(2\alpha_3+2\theta_5)))\sin\theta_6],$$

$$\hat{T}_{345} = \frac{1}{4}\sin\theta_4\sin^3\theta_6[2\cos^2\theta_4\sin\alpha_1(\cos\alpha_1\cos\theta_5\sin 2\alpha_2+\sin\alpha_1\sin\theta_5) + \cos\alpha_1\sin 2\theta_4\sin(\alpha_2+\alpha_3+\theta_5)],$$

$$\hat{T}_{346} = \frac{1}{4}\sin\theta_4\sin^2\theta_6[\cos\theta_6\sin 2\alpha_1\sin 2\alpha_2\sin\theta_5-\cos^2\theta_5(2\cos\alpha_1\cos(\alpha_2+\alpha_3) + \cos(\alpha_2-\alpha_3)\sin\alpha_1)\sin\theta_4\sin\theta_6+(\cos\theta_4\sin 2\alpha_1\sin^2\alpha_2 + \sin\alpha_1\sin\theta_4(\cos(\alpha_2-\alpha_3)(1+\sin^2\theta_5)+\cos\alpha_2\sin\alpha_3\sin 2\theta_5) + \cos\alpha_1(-2\cos^2\alpha_2\cos\theta_4\sin\alpha_1+\sin(\alpha_2+\alpha_3)\sin\theta_4\sin 2\theta_5))\sin\theta_6 - 2\cos\theta_5\sin\alpha_1(\cos\theta_6\sin\alpha_1+\cos\alpha_3\sin\alpha_2\sin\theta_4\sin\theta_5\sin\theta_6)],$$

$$\hat{T}_{356} = \frac{1}{4}\sin^2\theta_6[\cos\theta_4\cos\theta_6(-\sin^2\theta_4(\cos\theta_5\sin 2\alpha_1\sin 2\alpha_2+2\sin^2\alpha_1\sin\theta_5) + \cos\alpha_1\sin 2\theta_4\sin(\alpha_2+\alpha_3+\theta_5))+\sin 2\theta_4(\cos\theta_5\sin\alpha_1\sin(\alpha_2-\alpha_3-\theta_5) - \cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sin\theta_5)\sin\theta_6],$$

$$\hat{T}_{456} = \frac{1}{8}\sin^2\theta_6[-4\cos^2\alpha_1\cos\theta_5\cos\theta_6\sin\theta_4 - 4\cos\theta_6\sin 2\alpha_1\sin 2\alpha_2\sin\theta_4\sin\theta_5 + 4\cos(\alpha_2-\alpha_3)\cos^2\theta_5\sin\alpha_1\sin\theta_6 + 2\sin\alpha_1(\cos(\alpha_2-\alpha_3)(-1+2\cos 2\theta_4+\cos 2\theta_5) - 2\cos\alpha_2\sin\alpha_3\sin 2\theta_5)\sin\theta_6 - \cos\alpha_1(8\cos\theta_4\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6 + (-\cos(\alpha_1+2\alpha_2-2\theta_4)+2\cos(\alpha_2+\alpha_3-2\theta_4)+\cos(\alpha_1-2\alpha_2+2\theta_4) + 2\cos(\alpha_2+\alpha_3+2\theta_4)-\cos(\alpha_1-2\alpha_2-2\theta_4)+\cos(\alpha_1+2\alpha_2+2\theta_4) - 4\cos(\alpha_2+\alpha_3+2\theta_5))\sin\theta_6) + 4\cos\theta_5(\cos\theta_6(1+\sin^2\alpha_1)\sin\theta_4 + 2\cos\alpha_3\sin\alpha_1\sin\alpha_2\sin\theta_5\sin\theta_6)].$$

One has also various identities which can be used for checking the 11-dimensional Einstein–Maxwell equations.

$$\hat{T}_{ij}^{m}\hat{F}_{mk} + \hat{T}_{kj}^{m}\hat{F}_{mi} = 0,$$
$$\hat{T}_{ij}^{m}\hat{T}_{mk}^{l} + \hat{T}_{kj}^{m}\hat{T}_{mi}^{l} = -\hat{F}_{ij}\hat{F}_{k}^{l} - \hat{F}_{kj}\hat{F}_{i}^{l} - \delta_i^l\overset{6}{g}_{jk} - \delta_k^l\overset{6}{g}_{ij} + 2\delta_j^l\overset{6}{g}_{ik},$$
$$\overset{6}{\nabla}_k\hat{F}_{ij} = \hat{T}_{ijk},$$
$$\overset{6}{\nabla}_k\hat{T}_{ij}^{l} = -\overset{6}{g}_{ki}\hat{F}_{j}^{l} + \overset{6}{g}_{kj}\hat{F}_{i}^{l} - \delta_k^l\hat{F}_{ij},$$
$$\overset{6}{\nabla}_k\hat{T}_{ij}^{k} = -4\hat{F}_{ij},$$
$$\hat{T}_{imn}\hat{T}^{jmn} = 4\delta_i^j,$$
$$\hat{T}_{ijk}\hat{F}^{jk} = 0,$$
$$\overset{6}{\epsilon}_{ijkmnp}\hat{T}^{mnp}\hat{F}^{jk} = 0,$$
$$\overset{6}{\epsilon}{}^{ijkmnp}\overset{6}{\nabla}_i\hat{T}_{mnp} = 0,$$
$$\overset{6}{\nabla}_{[m}\hat{T}_{npq]} = 0.$$

All these identities can be checked from the scalar product and the vector cross product of two vectors in Cayley space $\mathbf{I}^7$. According to the third equation, by taking the 6-dimensional covariant derivative on the almost complex structure $\hat{F}_{ij}$, one obtains the $\hat{T}_{ijk}$ tensor. In other words, the above expression for $\hat{T}_{ijk}$ can be determined via the explicit form for $\hat{F}_{ij}$. We present them here for convenience. In general, the second fundamental tensor of the hypersurface is different from the metric tensor but for $\mathbf{S}^6$, they are equivalent to each other. We use this property all the time.

The $\hat{S}_{ijk}$ tensor appearing in the 4-forms $F_{4mnp}$ and $F_{5mnp}$ is given by

$$\hat{S}_{123} = \frac{1}{64}\sin 2\alpha_1\sin^3\theta_4\sin^3\theta_6[-8\cos\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6 - \cos\theta_4(\cos(\alpha_1-\alpha_2-\alpha_3)+\cos(\alpha_1+\alpha_2+\alpha_3)+2\cos\alpha_1(\cos(\alpha_2+\alpha_3) - 2\cos(\alpha_2+\alpha_3+2\theta_5)) - 8\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_5\sin\alpha_1)\sin\theta_6],$$
$$\hat{S}_{124} = \frac{1}{8}\sin^2\theta_4\sin^3\theta_6[\cos\theta_6\sin 2\alpha_1\sin\theta_4\sin\theta_5 - 2\cos\theta_4\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5) + 2\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_5\sin\theta_6 + \cos\theta_5(\cos\theta_6\sin\theta_4(\sin 2\alpha_2 + \cos 2\alpha_1\sin(2\alpha_3+2\theta_5)) - 2\cos\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_6)],$$
$$\hat{S}_{125} = \frac{1}{16}\sin^2\theta_4\sin^3\theta_6[2\cos\theta_6(-2\cos\alpha_1\cos\theta_4\cos\theta_5\sin\alpha_1 - 4\cos^2\alpha_1\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4 + \cos\theta_4\sin\theta_5(\sin 2\alpha_2+\cos 2\alpha_1\sin(2\alpha_3+2\theta_5))) + (-2\cos 2\alpha_2\cos^2\theta_4 - 2\cos 2\alpha_1\cos\theta_4(\cos\theta_4\cos(2\alpha_3+2\theta_5) + (\cos\alpha_1\cos(\alpha_2-\alpha_3-2\theta_5) - \cos(\alpha_2+\alpha_3+2\theta_5)\sin\alpha_1)\sin\theta_4) + (\cos\alpha_2\cos\alpha_3(\cos\alpha_1-\sin\alpha_1)^3 + (\cos\alpha_1+\sin\alpha_1)^3\sin\alpha_2\sin\alpha_3)\sin 2\theta_4)\sin\theta_6],$$
$$\hat{S}_{126} = -\frac{1}{8}\sin 2\alpha_1\sin^3\theta_4\sin^2\theta_6[\cot\theta_4(\cos\alpha_2\cos\theta_5\csc\alpha_1\sec\alpha_1\sin\alpha_2 + \sin\theta_5 + \cos\theta_5\cot 2\alpha_1\sin(2\alpha_3+2\theta_5)) + \sec\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)],$$
$$\hat{S}_{134} = \frac{1}{24}\sin^2\theta_4\sin^3\theta_6[3\cos(2\alpha_3+2\theta_5)\cos\theta_6\sin\theta_4\sin\theta_5 + \cos\theta_4\cos\theta_6(\csc\alpha_1 + 4\sin\alpha_1)\sin(\alpha_2+\alpha_3+\theta_5) - (\cos\theta_5(4\cos\alpha_1+\sec\alpha_1)\sin(\alpha_2-\alpha_3-\theta_5) + \cos(\alpha_2+\alpha_3+\theta_5)(\csc\alpha_1+4\sin\alpha_1)\sin\theta_5)\sin\theta_6 + 3\cos\theta_6\sin\theta_4(\sin(2\alpha_3+\theta_5)$$

$$
\begin{aligned}
&+\sin\theta_5\tan\alpha_1)+\cos 2\alpha_1(-\cos\theta_4\cos\theta_6\csc\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)\\
&+\cos\theta_5(3\cos\theta_6\sin 2\alpha_2\sin\theta_4-\sec\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_6)\\
&+\sin\theta_5(\cos(\alpha_2+\alpha_3+\theta_5)\csc\alpha_1\sin\theta_6+3\cos\theta_6\sin\theta_4\tan\alpha_1))],\\
\hat{S}_{135} = &-\frac{1}{32}\sin^3\theta_6[(\cos 2\alpha_1\cos 2\alpha_2+\cos(2\alpha_3+2\theta_5))\sin^2 2\theta_4\sin\theta_6\\
&-4\cos\theta_4\sin^2\theta_4(\cos\theta_5(-\cos\theta_6\sin 2\alpha_1+2\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4\sin\theta_6)\\
&+\sin\theta_5(\cos\theta_6(\cos 2\alpha_1\sin 2\alpha_2+\sin(2\alpha_3+2\theta_5))-2\cos\alpha_1\sin\theta_4\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_6))],\\
\hat{S}_{136} = &\frac{1}{16}\sin^2\theta_4\sin^2\theta_6[-\cos\theta_4(\cos(2\alpha_1-\theta_5)-\cos(2\alpha_1+\theta_5)+2\cos\theta_5(\cos 2\alpha_1\sin 2\alpha_2\\
&+\sin(2\alpha_3+2\theta_5)))+4\sin\alpha_1\sin\theta_4\sin(\alpha_2+\alpha_3+\theta_5)],\\
\hat{S}_{145} = &-\frac{1}{48}\sin\theta_4\sin^3\theta_6[12\cos 2\alpha_2\cos^2\theta_4\cos\theta_6\sin\theta_5+8\cos(2\alpha_3+2\theta_5)\cos\theta_6\sin\theta_5\\
&-4\cos(2\alpha_3+2\theta_5)\cos\theta_6\cot^2 2\alpha_1\sin\theta_5+4\cos(2\alpha_3+2\theta_5)\cos\theta_6\csc^2 2\alpha_1\sin\theta_5\\
&+2\cos 2\alpha_3\cos\theta_6\sin^2\theta_4\sin\theta_5+8\cos\theta_6\sin 2\alpha_1\sin^2\theta_4\sin\theta_5\\
&+\cos\theta_6\csc\alpha_1\sin 2\theta_4\sin(\alpha_2+\alpha_3+\theta_5)+10\cos\theta_6\sin\alpha_1\sin 2\theta_4\sin(\alpha_2+\alpha_3+\theta_5)\\
&+10\cos\theta_6\sin^2\theta_4\sin(2\alpha_3+\theta_5)+6\cos\theta_4\sin 2\alpha_2\sin\theta_6+2\sec\theta_4\sin 2\alpha_2\sin\theta_6\\
&-4\cot^2 2\alpha_1\sec\theta_4\sin 2\alpha_2\sin\theta_6+4\csc^2 2\alpha_1\sec\theta_4\sin 2\alpha_2\sin\theta_6\\
&-4\cos(\alpha_2+\alpha_3+\theta_5)\csc\alpha_1\sin\theta_4\sin\theta_5\sin\theta_6-16\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4\sin\theta_5\sin\theta_6\\
&+6\cos\theta_4\sin(2\alpha_3+2\theta_5)\sin\theta_6+2\sec\theta_4\sin(2\alpha_3+2\theta_5)\sin\theta_6\\
&-4\cot^2 2\alpha_1\sec\theta_4\sin(2\alpha_3+2\theta_5)\sin\theta_6+4\csc^2 2\alpha_1\sec\theta_4\sin(2\alpha_3+2\theta_5)\sin\theta_6\\
&+2\cos\theta_5\sin\theta_4(\cos\theta_6\sin 2\alpha_3\sin\theta_4-3(2\cos\alpha_1+\sec\alpha_1)\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_6)\\
&+4\cos\theta_6\sin^2\theta_4\sin\theta_5\tan\alpha_1+\cos 2\alpha_1(4\cos(\alpha_2+\alpha_3+\theta_5)\csc\alpha_1\sin\theta_4\sin\theta_5\sin\theta_6\\
&+6\cos\theta_5\sin\theta_4(2\cos\theta_6\sin 2\alpha_2\sin\theta_4-\sec\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\sin\theta_6)\\
&+\cos\theta_6(-\csc\alpha_1\sin 2\theta_4\sin(\alpha_2+\alpha_3+\theta_5)+4\sin^2\theta_4\sin\theta_5\tan\alpha_1))\\
&+\cos\theta_6\csc^2\alpha_1\sec\alpha_1\sin(\alpha_2-\alpha_3-\theta_5)\tan\theta_4-6\sin 2\alpha_2\sin\theta_4\sin\theta_6\tan\theta_4\\
&-6\sin\theta_4\sin(2\alpha_3+2\theta_5)\sin\theta_6\tan\theta_4\\
&+2\cos\alpha_1\cos\theta_6\sin(\alpha_2-\alpha_3-\theta_5)(5\sin 2\theta_4-2(\cot^2 2\alpha_1+\sin^2\theta_4)\tan\theta_4)],\\
\hat{S}_{146} = &\frac{1}{4}\cos\theta_5\sin\theta_4\sin^2\theta_6\,[\cos 2\alpha_2+\cos(2\alpha_3+2\theta_5)]\,,\\
\hat{S}_{156} = &\frac{1}{8}\sin\theta_4\sin^2\theta_6[-4\cos^2\alpha_1\cos\alpha_2\cos\theta_4\cos\theta_5\sin\alpha_2\sin\theta_4\\
&+\sin 2\theta_4(\cos\theta_5(\sin^2\alpha_1\sin 2\alpha_2-2\cos\alpha_3\sin\alpha_3)+(\cos 2\alpha_2-\cos 2\alpha_3)\sin\theta_5)\\
&-4\cos\alpha_1\cos\theta_4(\cos\theta_4\sin(\alpha_2-\alpha_3-\theta_5)+\sin\alpha_1\sin\theta_4\sin\theta_5)+4\sin\alpha_1\sin^2\theta_4\sin(\alpha_2+\alpha_3+\theta_5)],\\
\hat{S}_{234} = &\frac{1}{8}\sin 2\alpha_1\sin^2\theta_4\sin^3\theta_6[-2\cos\theta_4\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6\sin\alpha_1\\
&+\cos\theta_5(\cos 2\alpha_2-\cos(2\alpha_3+2\theta_5))\cos\theta_6\sin\theta_4-2(\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_5\\
&+\sin\alpha_1\sin\theta_5\sin(\alpha_2+\alpha_3+\theta_5))\sin\theta_6],\\
\hat{S}_{235} = &\frac{1}{8}\cos\theta_4\sin 2\alpha_1\sin^2\theta_4\sin^3\theta_6[(\cos 2\alpha_2-\cos(2\alpha_3+2\theta_5))\cos\theta_6\sin\theta_5\\
&+(\cos\theta_4(\sin 2\alpha_2-\sin(2\alpha_3+2\theta_5))+2\sin\theta_4(-\cos\alpha_1\cos(\alpha_2-\alpha_3-\theta_5)\sin\theta_5\\
&+\cos\theta_5\sin\alpha_1\sin(\alpha_2+\alpha_3+\theta_5)))\sin\theta_6],\\
\hat{S}_{236} = &-\frac{1}{8}\sin 2\alpha_1\sin^2\theta_4\sin^2\theta_6[\cos\theta_4\cos\theta_5(\cos 2\alpha_2-\cos(2\alpha_3+2\theta_5))\\
&+2\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin\theta_4],\\
\hat{S}_{245} = &\frac{1}{8}\sin\theta_4\sin^3\theta_6[-\cos\theta_6(2\cos(\alpha_2-\alpha_3-\theta_5)\sin\alpha_1\sin 2\theta_4\\
&+\sin 2\alpha_1(2\cos 2\alpha_2\cos\theta_5\sin^2\theta_4+\sin 2\alpha_3\sin 3\theta_5))\\
&+\cos 2\alpha_3\sin 2\alpha_1((-\cos 2\theta_4\cos\theta_5+\cos 3\theta_5)\cos\theta_6+2\cos\theta_4\cos 2\theta_5\sin\theta_6)\\
&-4\cos^2\alpha_1\cos\theta_5(\cos^2\theta_4\cos\theta_6-2\sin\alpha_1\sin(\alpha_2-\alpha_3)\sin\theta_4\sin\theta_5\sin\theta_6)\\
&+2\cos\alpha_1(2\cos(\alpha_2+\alpha_3+\theta_5)\cos\theta_6\sin^2\alpha_1\sin 2\theta_4\\
&+2\cos(\alpha_2-\alpha_3)\cos^2\theta_5\sin 2\alpha_1\sin\theta_4\sin\theta_6+\sin\alpha_1\sin\theta_5(\cos 2\theta_4\cos\theta_6\sin 2\alpha_3\\
&+4(-\cos\theta_4\cos\theta_5\sin 2\alpha_3+\sin\alpha_1\sin\theta_4\sin(\alpha_2+\alpha_3+\theta_5))\sin\theta_6))],
\end{aligned}
$$

$$\hat{S}_{246} = \frac{1}{2}\cos\alpha_1 \sin\theta_4 \sin^2\theta_6\left[-\cos\alpha_1\sin\theta_5 + \cos\theta_5\sin\alpha_1\sin(2\alpha_3+2\theta_5)\right],$$
$$\hat{S}_{256} = \frac{1}{8}\sin 2\alpha_1 \sin\theta_4\sin^2\theta_6[-2\cos^2\theta_4\cos(\alpha_2-\alpha_3-\theta_5)\sec\alpha_1 - 4\cos(\alpha_2+\alpha_3+\theta_5)\sin\alpha_1\sin^2\theta_4$$
$$+(\cos(2\alpha_3+\theta_5)+\cos\theta_5(-\cos 2\alpha_2+\cot\alpha_1))\sin 2\theta_4],$$
$$\hat{S}_{345} = \frac{1}{4}\cos\theta_4\sin 2\alpha_1\sin\theta_4\sin^3\theta_6[\cos(\alpha_2-\alpha_3-\theta_5)\cos\theta_6\sec\alpha_1\sin\theta_4$$
$$-\cos\theta_4\cos\theta_6(\cos\theta_5\cot\alpha_1+\sin 2\alpha_2\sin\theta_5)+\cos 2\alpha_2\sin\theta_6],$$
$$\hat{S}_{346} = \frac{1}{4}\sin 2\alpha_1\sin\theta_4\sin^2\theta_6\left[\cos\theta_5\sin 2\alpha_2 - \cot\alpha_1\sin\theta_5\right],$$
$$\hat{S}_{356} = \frac{1}{4}\cos\theta_4\sin 2\alpha_1\sin^2\theta_4\sin^2\theta_6[\cos\theta_5\cot\alpha_1+\cos(\alpha_2-\alpha_3-\theta_5)\cot\theta_4\sec\alpha_1+\sin 2\alpha_2\sin\theta_5],$$
$$\hat{S}_{456} = \frac{1}{2}\sin^2\theta_6[2\cos\alpha_2\cos\theta_4\cos\theta_5\sin\alpha_1\sin\alpha_3+\sin\theta_4(-\cos\theta_5\sin 2\alpha_1\sin 2\alpha_2+2\cos^2\alpha_1\sin\theta_5)$$
$$+2\cos\theta_4\sin\alpha_1(-\cos\alpha_3\sin(\alpha_2-\theta_5)+\sin\alpha_2\sin\alpha_3\sin\theta_5)].$$

Let us also present the other identities which contain the above $\hat{S}_{ijk}$ tensor.

$$\hat{S}_{imn}\hat{S}^{jmn} = 4\delta_i^j,$$
$$\hat{S}_{imn}\hat{T}^{jmn} = -\frac{1}{2}\overset{6}{g}_{ik}\overset{6}{\epsilon}{}^{jkmnpq}\hat{F}_{mn}\hat{F}_{pq} = 4\hat{F}_i^j,$$
$$\hat{S}_{ijk}\hat{F}^{jk} = 0,$$
$$\overset{6}{\nabla}_i\hat{S}^{ijk} = 0,$$
$$\hat{T}_{imn}\hat{S}^{jmn} + \hat{S}_{imn}\hat{T}^{jmn} = 0.$$

There is no $G_2$-invariant vector. As mentioned, there exists a relation (2.19) which shows how the above tensor $\hat{S}_{ijk}$ can be obtained from the previous two tensors.

## Appendix E. The Ricci tensor

The Ricci tensor appearing in (3.7) is given explicitly by

$$\widetilde{R}_1^1 = R_1^1, \qquad \widetilde{R}_2^2 = R_2^2(=\widetilde{R}_1^1), \qquad \widetilde{R}_3^3 = R_3^3(=\widetilde{R}_1^1),$$
$$\widetilde{R}_4^4 = R_4^4,$$
$$\widetilde{R}_4^5 = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]R_4^5,$$
$$\widetilde{R}_4^{10} = -\left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]R_4^5,$$
$$\widetilde{R}_4^{11} = \left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]R_4^5(=-\widetilde{R}_4^{10}),$$
$$\widetilde{R}_5^4 = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]R_5^4,$$
$$\widetilde{R}_5^5 = \left[\frac{\cos^2\mu\cos^2\psi}{1-\cos^2\psi\sin^2\mu}\right]R_5^5 + \left[\frac{\sin^2\psi}{1-\cos^2\psi\sin^2\mu}\right]R_{11}^{11},$$
$$\widetilde{R}_5^{10} = \left[\frac{\cot\mu\sin\psi\cos\psi}{1-\cos^2\psi\sin^2\mu}\right]R_5^5 - \left[\frac{\cot\mu\sin\psi\cos\psi}{1-\cos^2\psi\sin^2\mu}\right]R_{11}^{11},$$
$$\widetilde{R}_5^{11} = -\left[\frac{\cot\mu\sin\psi\cos\psi}{1-\cos^2\psi\sin^2\mu}\right]R_5^5 + \left[\frac{\cot\mu\sin\psi\cos\psi}{1-\cos^2\psi\sin^2\mu}\right]R_{11}^{11}(=-\widetilde{R}_5^{10}),$$
$$\widetilde{R}_6^6 = R_6^6, \qquad \widetilde{R}_7^7 = R_7^7(=\widetilde{R}_6^6), \qquad \widetilde{R}_8^8 = R_8^8(=\widetilde{R}_6^6),$$
$$\widetilde{R}_9^9 = R_9^9(=\widetilde{R}_6^6), \qquad \widetilde{R}_{10}^{10} = R_{10}^{10}(=\widetilde{R}_6^6),$$

$$\widetilde{R}^4_{11} = \left[\frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] R^4_5,$$
$$\widetilde{R}^5_{11} = -\left[\frac{\sin\mu\cos\mu\sin\psi\cos\psi}{1-\cos^2\psi\sin^2\mu}\right] R^5_5 + \left[\frac{\sin\mu\cos\mu\sin\psi\cos\psi}{1-\cos^2\psi\sin^2\mu}\right] R^{11}_{11},$$
$$\widetilde{R}^{10}_{11} = -\left[\frac{\sin^2\psi}{1-\cos^2\psi\sin^2\mu}\right] R^5_5 + R^{10}_{10} - \left[\frac{\cos^2\mu\cos^2\psi}{1-\cos^2\psi\sin^2\mu}\right] R^{11}_{11},$$
$$\widetilde{R}^{11}_{11} = \left[\frac{\sin^2\psi}{1-\cos^2\psi\sin^2\mu}\right] R^5_5 + \left[\frac{\cos^2\mu\cos^2\psi}{1-\cos^2\psi\sin^2\mu}\right] R^{11}_{11}.$$

Here the Ricci tensor $R^N_M$ on the right hand side above is for the $G_2$-invariant case we presented in Appendix A and the old variable $\theta$ should be replaced by the new variables $(\mu, \psi)$ through (3.4). Note that there are off-diagonal components (4, 10), (4, 11), (5, 10), (5, 11), (11, 4), (11, 5), and (11, 10) while there exist only (4, 5) and (5, 4) components in the $G_2$-invariant flow. One also obtains these components from the 11-dimensional metric (3.6) directly.

## Appendix F. The 4-forms

The 4-forms appearing in (3.8) are given by

$$\widetilde{F}_{1234} = F_{1234}, \qquad \widetilde{F}_{1235} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{1235},$$
$$\widetilde{F}_{123\,11} = \left[\frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{1235},$$
$$\widetilde{F}_{45mn} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{45mn} - \left[\frac{\sin\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{4mn\,11}, \quad (m, n = 6, \ldots, 10),$$
$$\begin{aligned}\widetilde{F}_{45m\,11} \;=\; & -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{45m\,10} + \left[\frac{\sin\mu}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{45m\,11} \\ & - \left[\frac{\sin\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{4m\,10\,11}, \quad (m = 6, \ldots, 9),\end{aligned}$$
$$\widetilde{F}_{45\,10\,11} = \left[\frac{\sin\mu}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{45\,10\,11}, \qquad \widetilde{F}_{4mnp} = F_{4mnp}, \quad (m, n, p = 6, \ldots, 10),$$
$$\widetilde{F}_{4mn\,11} = \left[\frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{45mn} + F_{4mn\,10} - \left[\frac{\sin\mu\cos\mu\cos\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{4mn\,11}, \quad (m, n = 6, \ldots, 9),$$
$$\widetilde{F}_{4m\,10\,11} = \left[\frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{45m\,10} - \left[\frac{\sin\mu\cos\mu\cos\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{4m\,10\,11}, \quad (m = 6, \ldots, 9),$$
$$\widetilde{F}_{5mnp} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{5mnp} + \left[\frac{\sin\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{mnp\,11}, \quad (m, n, p = 6, \ldots, 10),$$
$$\begin{aligned}\widetilde{F}_{5mn\,11} \;=\; & -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{5mn\,10} + \left[\frac{\sin\mu}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{5mn\,11} \\ & + \left[\frac{\sin\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{mn10\,11}, \quad (m, n = 6, \ldots, 9),\end{aligned}$$
$$\widetilde{F}_{5m\,10\,11} = \left[\frac{\sin\mu}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{5m\,10\,11}, \quad (m = 6, \ldots, 9),$$
$$\widetilde{F}_{mnpq} = F_{mnpq}, \quad (m, n, p, q = 6, \ldots, 10),$$
$$\widetilde{F}_{mnp\,11} = -\left[\frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right] F_{5mnp} + F_{mnp\,10} - \left[\frac{\sin\mu\cos\mu\cos\psi}{1-\cos^2\psi\sin^2\mu}\right] F_{mnp\,11}, \quad (m, n, p = 6, \ldots, 9),$$

$$\widetilde{F}_{mn10\,11} = -\left[\frac{\sin\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F_{5mn\,10} - \left[\frac{\sin\mu\cos\mu\cos\psi}{1-\cos^2\psi\sin^2\mu}\right]F_{mn10\,11}, \quad (m,n=6,\ldots,9).$$

Here the 4-forms $F_{MNPQ}$ on the right hand side are for the $G_2$-invariant case and are given by (2.25) together with Appendices B and D. Note that the old variables $(\theta, \theta_5, \theta_6)$ should be replaced by the new variables $(\mu, \phi, \psi)$ through (3.4). The 4-form $\widetilde{F}_{123\,11}$ is new.

The 4-forms with upper indices are given by

$$\widetilde{F}^{1234} = F^{1234}, \qquad \widetilde{F}^{1235} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{1235},$$

$$\widetilde{F}^{123\,10} = -\left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{1235}, \qquad \widetilde{F}^{123\,11} = \left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{1235},$$

$$\widetilde{F}^{45mn} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{45mn} - [\sin\psi]\,F^{4mn\,11}, \quad (m,n=6,\ldots,9),$$

$$\begin{aligned}\widetilde{F}^{45m\,10} &= -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{45m\,10} - \left[\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}\right]F^{45m\,11}\\ &\quad - [\sin\psi]\,F^{4m\,10\,11}, \quad (m=6,\ldots,9),\end{aligned}$$

$$\widetilde{F}^{45m\,11} = \left[\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}\right]F^{45m\,11}, \quad (m=6,\ldots,10),$$

$$\widetilde{F}^{4mnp} = F^{4mnp}, \quad (m,n,p=6,\ldots,9),$$

$$\widetilde{F}^{4mn\,10} = F^{4mn\,10} + [\cos\psi\cot\mu]\,F^{4mn\,11} - \left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{45mn}, \quad (m,n=6,\ldots,9),$$

$$\widetilde{F}^{4mn\,11} = \left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{45mn} - [\cos\psi\cot\mu]\,F^{4mn\,11}, \quad (m,n=6,\ldots,10),$$

$$\widetilde{F}^{5mnp} = -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{5mnp} + [\sin\psi]\,F^{mnp\,11}, \quad (m,n,p=6,\ldots,9),$$

$$\begin{aligned}\widetilde{F}^{5mn\,10} &= -\left[\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{5mn\,10} - \left[\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}\right]F^{5mn\,11}\\ &\quad + [\sin\psi]\,F^{mn\,10\,11}, \quad (m,n=6,\ldots,9),\end{aligned}$$

$$\widetilde{F}^{5mn\,11} = \left[\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}\right]F^{5mn\,11}, \quad (m,n=6,\ldots,10), \qquad \widetilde{F}^{6789} = F^{6789},$$

$$\widetilde{F}^{mnp\,10} = F^{mnp\,10} + [\cos\psi\cot\mu]\,F^{mnp\,11} + \left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{5mnp}, \quad (m,n,p=6,\ldots,9),$$

$$\widetilde{F}^{mnp\,11} = -\left[\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}\right]F^{5mnp} - [\cos\psi\cot\mu]\,F^{mnp\,11}, \quad (m,n,p=6,\ldots,10).$$

The 4-forms $\widetilde{F}^{123\,10}$ and $\widetilde{F}^{123\,11}$ are new, compared to the $G_2$-invariant flow.

## Appendix G. The left hand side of the Maxwell equation

The Maxwell equations can be summarized as follows. Let us introduce the notation

$$\frac{1}{2}E\widetilde{\nabla}_M\widetilde{F}^{MNPQ} \equiv (NPQ),$$

where we ignore the tilde in $(NPQ)$ for simplicity and present all the nonzero components of left hand side of Maxwell equations in terms of the 4-forms in $G_2$-invariant case

$$\begin{aligned}(123) = & -F_{49\,10\,11}F_{5678} + F_{48\,10\,11}F_{5679} - F_{489\,11}F_{567\,10} + F_{489\,10}F_{567\,11} - F_{47\,10\,11}F_{5689} \\ & + F_{479\,11}F_{568\,10} - F_{479\,10}F_{568\,11} - F_{478\,11}F_{569\,10} + F_{478\,10}F_{569\,11} - F_{4789}F_{56\,10\,11} \\ & + F_{46\,10\,11}F_{5789} - F_{469\,11}F_{578\,10} + F_{469\,10}F_{578\,11} + F_{468\,11}F_{579\,10} - F_{468\,10}F_{579\,11} \\ & + F_{4689}F_{57\,10\,11} - F_{467\,11}F_{589\,10} + F_{467\,10}F_{589\,11} - F_{4679}F_{58\,10\,11} + F_{4678}F_{59\,10\,11} \\ & - F_{45\,10\,11}F_{6789} + F_{459\,11}F_{678\,10} - F_{459\,10}F_{678\,11} - F_{458\,11}F_{679\,10} + F_{458\,10}F_{679\,11} \\ & - F_{4589}F_{67\,10\,11} + F_{457\,11}F_{689\,10} - F_{457\,10}F_{689\,11} + F_{4579}F_{68\,10\,11} - F_{4578}F_{69\,10\,11} \\ & - F_{456\,11}F_{789\,10} + F_{456\,10}F_{789\,11} - F_{4569}F_{78\,10\,11} + F_{4568}F_{79\,10\,11} - F_{4567}F_{89\,10\,11},\end{aligned}$$

$$(456) = \sin\psi F_{1235}F_{789\,10}, \qquad (457) = -\sin\psi F_{1235}F_{689\,10},$$

$$(458) = \sin\psi F_{1235}F_{679\,10}, \qquad (459) = -\sin\psi F_{1235}F_{678\,10},$$

$$(45\,10) = \sin\psi F_{1235}F_{6789}, \qquad (467) = F_{1235}F_{89\,10\,11},$$

$$(468) = -F_{1235}F_{79\,10\,11}, \qquad (469) = F_{1235}F_{78\,10\,11},$$

$$(46\,10) = F_{1235}(\cos\psi\cot\mu F_{789\,10} - F_{789\,11}),$$

$$(46\,11) = -\cos\psi\cot\mu F_{1235}F_{789\,10}, \qquad (478) = F_{1235}F_{69\,10\,11},$$

$$(479) = -F_{1235}F_{68\,10\,11}, \qquad (47\,10) = F_{1235}(-\cos\psi\cot\mu F_{689\,10} + F_{689\,11}),$$

$$(47\,11) = \cos\psi\cot\mu F_{1235}F_{689\,10}, \qquad (489) = F_{1235}F_{67\,10\,11},$$

$$(48\,10) = F_{1235}(\cos\psi\cot\mu F_{679\,10} - F_{679\,11}), \qquad (48\,11) = -\cos\psi\cot\mu F_{1235}F_{679\,10},$$

$$(49\,10) = F_{1235}(-\cos\psi\cot\mu F_{678\,10} + F_{678\,11}),$$

$$(49\,11) = \cos\psi\cot\mu F_{1235}F_{678\,10}, \qquad (4\,10\,11) = -\cos\psi\cot\mu F_{1235}F_{6789},$$

$$(567) = \sin\psi(-F_{1235}F_{489\,10} + F_{1234}F_{589\,10}) + \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{89\,10\,11},$$

$$(568) = \sin\psi(F_{1235}F_{479\,10} - F_{1234}F_{579\,10}) - \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{79\,10\,11},$$

$$(569) = \sin\psi(-F_{1235}F_{478\,10} + F_{1234}F_{578\,10}) + \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{78\,10\,11},$$

$$\begin{aligned}(56\,10) = & -\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{789\,11} + \csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{789\,10} \\ & + \sin\psi(F_{1235}F_{4789} - F_{1234}F_{5789}),\end{aligned}$$

$$(56\,11) = -\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{789\,10},$$

$$(578) = \sin\psi(-F_{1235}F_{469\,10} + F_{1234}F_{569\,10}) + \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{69\,10\,11},$$

$$(579) = \sin\psi(F_{1235}F_{468\,10} - F_{1234}F_{568\,10}) - \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{68\,10\,11},$$

$$\begin{aligned}(57\,10) = & \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{689\,11} - \csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{689\,10} \\ & + \sin\psi(-F_{1235}F_{4689} + F_{1234}F_{5689}),\end{aligned}$$

$$(57\,11) = \csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{689\,10},$$

$$(589) = \sin\psi(-F_{1235}F_{467\,10} + F_{1234}F_{567\,10}) + \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{67\,10\,11},$$

$$\begin{aligned}(58\,10) = & -\frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{679\,11} + \csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{679\,10} \\ & + \sin\psi(F_{1235}F_{4679} - F_{1234}F_{5679}),\end{aligned}$$

$$(58\,11) = -\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{679\,10},$$

$$\begin{aligned}(59\,10) = & \frac{\cos\mu\cos\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{678\,11} - \csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{678\,10} \\ & + \sin\psi(-F_{1235}F_{4678} + F_{1234}F_{5678}),\end{aligned}$$

$$(59\,11) = \csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}F_{1234}F_{678\,10},$$

$$
\begin{aligned}
(5\,10\,11) &= -\csc\mu\sqrt{1-\cos^2\psi\sin^2\mu}\,F_{1234}F_{6789},\\
(678) &= -(F_{1235}F_{49\,10\,11}-F_{1234}F_{59\,10\,11}), \qquad (679) = (F_{1235}F_{48\,10\,11}-F_{1234}F_{58\,10\,11}),\\
(67\,10) &= -F_{1235}F_{489\,11}+\cos\psi\cot\mu(F_{1235}F_{489\,10}-F_{1234}F_{589\,10})\\
&\quad +F_{1234}F_{589\,11}+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{89\,10\,11},\\
(67\,11) &= \cos\psi\cot\mu(-F_{1235}F_{489\,10}+F_{1234}F_{589\,10})-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{89\,10\,11},\\
(689) &= -(F_{1235}F_{47\,10\,11}-F_{1234}F_{57\,10\,11}),\\
(68\,10) &= F_{1235}F_{479\,11}-\cos\psi\cot\mu(F_{1235}F_{479\,10}-F_{1234}F_{579\,10})\\
&\quad -F_{1234}F_{579\,11}-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{79\,10\,11},\\
(68\,11) &= \cos\psi\cot\mu(F_{1235}F_{479\,10}-F_{1234}F_{579\,10})+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{79\,10\,11},\\
(69\,10) &= -F_{1235}F_{478\,11}+\cos\psi\cot\mu(F_{1235}F_{478\,10}-F_{1234}F_{578\,10})\\
&\quad +F_{1234}F_{578\,11}+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{78\,10\,11},\\
(69\,11) &= \cos\psi\cot\mu(-F_{1235}F_{478\,10}+F_{1234}F_{578\,10})-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{78\,10\,11},\\
(6\,10\,11) &= \cos\psi\cot\mu(F_{1235}F_{4789}-F_{1234}F_{5789})+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{789\,11},\\
(789) &= (F_{1235}F_{46\,10\,11}-F_{1234}F_{56\,10\,11}),\\
(78\,10) &= -F_{1235}F_{469\,11}+\cos\psi\cot\mu(F_{1235}F_{469\,10}-F_{1234}F_{569\,10})\\
&\quad +F_{1234}F_{569\,11}+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{69\,10\,11},\\
(78\,11) &= \cos\psi\cot\mu(-F_{1235}F_{469\,10}+F_{1234}F_{569\,10})-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{69\,10\,11},\\
(79\,10) &= F_{1235}F_{468\,11}-\cos\psi\cot\mu(F_{1235}F_{468\,10}-F_{1234}F_{568\,10})\\
&\quad -F_{1234}F_{568\,11}-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{68\,10\,11},\\
(79\,11) &= \cos\psi\cot\mu(F_{1235}F_{468\,10}-F_{1234}F_{568\,10})+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{68\,10\,11},\\
(7\,10\,11) &= \cos\psi\cot\mu(-F_{1235}F_{4689}+F_{1234}F_{5689})-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{689\,11},\\
(89\,10) &= -F_{1235}F_{467\,11}+\cos\psi\cot\mu(F_{1235}F_{467\,10}-F_{1234}F_{567\,10})\\
&\quad +F_{1234}F_{567\,11}+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{67\,10\,11},\\
(89\,11) &= \cos\psi\cot\mu(-F_{1235}F_{467\,10}+F_{1234}F_{567\,10})-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{67\,10\,11},\\
(8\,10\,11) &= \cos\psi\cot\mu(F_{1235}F_{4679}-F_{1234}F_{5679})+\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{679\,11},\\
(9\,10\,11) &= \cos\psi\cot\mu(-F_{1235}F_{4678}+F_{1234}F_{5678})-\frac{\csc\mu\sin\psi}{\sqrt{1-\cos^2\psi\sin^2\mu}}F_{1234}F_{678\,11}.
\end{aligned}
$$

Note that there are nonzero components $(45m)$ where $m = 6, \ldots, 10$ while these are vanishing for $G_2$-invariant flow.

## References

[1] O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, $N = 6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals, J. High Energy Phys. 0810 (2008) 091. arXiv:0806.1218 [hep-th].

[2] I.R. Klebanov, E. Witten, Superconformal field theory on threebranes at a Calabi–Yau singularity, Nuclear Phys. B 536 (1998) 199. arXiv:hep-th/9807080.
[3] J.M. Maldacena, The large $N$ limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231. [Int. J. Theor. Phys., 38, (1999), 1113] arXiv:hep-th/9711200;
E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253. arXiv:hep-th/9802150;
S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B 428 (1998) 105. arXiv:hep-th/9802109.
[4] C. Ahn, J. Paeng, Three-dimensional SCFTs, supersymmetric domain wall and renormalization group flow, Nuclear Phys. B 595 (2001) 119. arXiv:hep-th/0008065.
[5] C. Ahn, K. Woo, Supersymmetric domain wall and RG flow from 4-dimensional gauged $N = 8$ supergravity, Nuclear Phys. B 599 (2001) 83. arXiv:hep-th/0011121.
[6] C. Ahn, S.J. Rey, More CFTs and RG flows from deforming M2/M5-brane horizon, Nuclear Phys. B 572 (2000) 188. arXiv:hep-th/9911199.
[7] C. Ahn, T. Itoh, An $N = 1$ supersymmetric G(2)-invariant flow in $M$-theory, Nuclear Phys. B 627 (2002) 45. arXiv:hep-th/0112010.
[8] R. Corrado, K. Pilch, N.P. Warner, An $N = 2$ supersymmetric membrane flow, Nuclear Phys. B 629 (2002) 74. arXiv:hep-th/0107220.
[9] C. Ahn, Holographic supergravity dual to three dimensional $N = 2$ gauge theory, J. High Energy Phys. 0808 (2008) 083. arXiv:0806.1420 [hep-th].
[10] M. Benna, I. Klebanov, T. Klose, M. Smedback, Superconformal Chern–Simons theories and $AdS_4/CFT_3$ correspondence, J. High Energy Phys. 0809 (2008) 072. arXiv:0806.1519 [hep-th];
I. Klebanov, T. Klose, A. Murugan, $AdS_4/CFT_3$—squashed, stretched and warped, J. High Energy Phys. 0903 (2009) 140. arXiv:0809.3773 [hep-th];
I.R. Klebanov, S.S. Pufu, F.D. Rocha, The squashed, stretched, and warped gets perturbed, J. High Energy Phys. 0906 (2009) 019. arXiv:0904.1009 [hep-th].
[11] C. Ahn, Towards holographic gravity dual of $N = 1$ superconformal Chern–Simons gauge theory, J. High Energy Phys. 0807 (2008) 101. arXiv:0806.4807 [hep-th].
[12] C. Ahn, Comments on holographic gravity dual of $N = 6$ superconformal Chern–Simons gauge theory, J. High Energy Phys. 0903 (2009) 107. arXiv:0812.4363 [hep-th].
[13] N. Bobev, N. Halmagyi, K. Pilch, N.P. Warner, Holographic, $N = 1$ supersymmetric RG flows on M2 branes, J. High Energy Phys. 0909 (2009) 043. arXiv:0901.2736 [hep-th].
[14] C.N. Gowdigere, N.P. Warner, Flowing with eight supersymmetries in $M$-theory and $F$-theory, J. High Energy Phys. 0312 (2003) 048. arXiv:hep-th/0212190.
[15] C.N. Pope, N.P. Warner, A dielectric flow solution with maximal supersymmetry, J. High Energy Phys. 0404 (2004) 011. arXiv:hep-th/0304132.
[16] C. Ahn, K. Woo, $N = 8$ gauged supergravity theory and $N = 6$ superconformal Chern–Simons matter theory, arXiv:0901.4666 [hep-th];
C. Ahn, K. Woo, Are there any new vacua of gauged $N = 8$ supergravity in four dimensions?, arXiv:0904.2105 [hep-th].
[17] C. Ahn, The gauge dual of gauged $N = 8$ supergravity theory, arXiv:0905.3943 [hep-th].
[18] C. Ahn, K. Woo, Perturbing around a warped product of $AdS_4$ and seven-ellipsoid, J. High Energy Phys. 0908 (2009) 065. arXiv:0907.0969 [hep-th].
[19] C. Ahn, K. Woo, The gauge dual of a warped product of $AdS_4$ and a squashed and stretched seven-manifold, Classical Quantum Gravity 27 (2010) 035009. arXiv:0908.2546 [hep-th].
[20] S. Franco, I.R. Klebanov, D. Rodriguez-Gomez, M2-branes on orbifolds of the cone over $Q^{1,1,1}$, J. High Energy Phys. 0908 (2009) 033. arXiv:0903.3231 [hep-th].
[21] C. Ahn, New $N = 2$ supersymmetric membrane flow in eleven-dimensional supergravity, J. High Energy Phys. 0911 (2009) 022. arXiv:0909.3745 [hep-th].
[22] C. Ahn, The eleven-dimensional uplift of four-dimensional supersymmetric RG flow, arXiv:0910.3533 [hep-th].
[23] B. de Wit, H. Nicolai, $N = 8$ supergravity, Nuclear Phys. B 208 (1982) 323;
B. de Wit, H. Nicolai, $N = 8$ supergravity with local SO(8) X SU(8) invariance, Phys. Lett. B 108 (1982) 285.
[24] N.P. Warner, Some new extrema of the scalar potential of gauged $N = 8$ supergravity, Phys. Lett. B 128 (1983) 169;
N.P. Warner, Some properties of the scalar potential in gauged supergravity theories, Nuclear Phys. B 231 (1984) 250.
[25] B. de Wit, H. Nicolai, N.P. Warner, The embedding of gauged $N = 8$ supergravity into $D = 11$ supergravity, Nuclear Phys. B 255 (1985) 29.
[26] C. Ahn, T. Itoh, The 11-dimensional metric for AdS/CFT RG flows with common SU(3) invariance, Nuclear Phys. B 646 (2002) 257. arXiv:hep-th/0208137.
[27] E. Cremmer, B. Julia, J. Scherk, Supergravity theory in 11 dimensions, Phys. Lett. B 76 (1978) 409.
[28] M.J. Duff, B.E.W. Nilsson, C.N. Pope, Kaluza–Klein supergravity, Phys. Rept. 130 (1986) 1.
[29] B. de Wit, H. Nicolai, A new SO(7) invariant solution of $D = 11$ supergravity, Phys. Lett. B 148 (1984) 60.
[30] F. Englert, Spontaneous compactification of eleven-dimensional supergravity, Phys. Lett. B 119 (1982) 339.
[31] C.N. Pope, N.P. Warner, An SU(4) invariant compactification of $D = 11$ supergravity on a stretched seven sphere, Phys. Lett. B 150 (1985) 352.
[32] C. Ahn, K. Woo, Holographic $N = 1$ supersymmetric membrane flows, arXiv:1004.0619 [hep-th].
[33] M. Gunaydin, N.P. Warner, The G2 invariant compactifications in eleven-dimensional supergravity, Nuclear Phys. B 248 (1984) 685.
[34] M. Gunaydin, F. Gursey, Quark structure and octonions, J. Math. Phys. 14 (1973) 1651.
[35] A.K. Waldron, G.C. Joshi, Gauging the octonion algebra, arXiv:hep-th/9211123.
[36] B. de Wit, H. Nicolai, The consistency of the S**7 truncation in $D = 11$ supergravity, Nuclear Phys. B 281 (1987) 211.