



Generalized holomorphic structures



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ABSTRACT

We define the notion of generalized holomorphic principal bundles and establish that their associated vector bundles of holomorphic representations are generalized holomorphic vector bundles defined by M. Gualtieri. Motivated by our definition, several examples of generalized holomorphic structures are constructed. A reduction theorem of generalized holomorphic structures is also included.

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1. Introduction

In generalized complex (GC for short) geometry initiated by N. Hitchin [1] and further developed by M. Gualtieri [2, 3] and others, generalized holomorphic (GH for short) structures are the analogue of holomorphic structures in classical complex geometry. They are special examples of Lie algebroid modules, and include some already known geometric objects, e.g. holomorphic Poisson modules.

In general, unlike its complex-geometric counterpart, it is not easy to construct nontrivial GH structures. The existing examples in the literature are flat bundles over symplectic manifolds, co-Higgs bundles over complex manifolds [4–7], Poisson modules over holomorphic Poisson manifolds [8,9]. In [9], Hitchin also adapted a differential geometric version of the Serre construction in algebraic geometry to produce rank-2 GH vector bundles over a compact connected GC 4-manifold whose type change locus has a nondegenerate component.

In [10], from a viewpoint of deformations of GC structures, the author investigated some aspects of GH vector bundles. This paper is then a continuation of that one, but from a different viewpoint: we extend the notion of holomorphic principal bundles to the generalized setting [Definition 4.4](#). In complex geometry, there are three equivalent ways to define a holomorphic principal bundle: by a 1-cocycle of holomorphic transition functions valued in a complex Lie group, by an equivariant complex structure in the total space and a holomorphic projection, or by a complex distribution in the total space. However, the third way is the one we choose to generalize—it seems that no suitable notions of GH functions and GH maps in the literature can be used to generalize the notion of holomorphic principal bundles in the other two ways. So we define the notion of GH principal bundles in terms of (generalized) distributions. Another ingredient to define this notion is the reduction theory of Courant algebroids and Dirac structures [11]. It is adapted to apply to a generalized distribution in the total space which is not maximal—the reduced generalized distribution is precisely the GC structure in the base manifold.

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After defining the notion of GH principal bundles, we establish the relation between GH principal bundles and GH vector bundles—the associated vector bundle of a GH principal bundle and a holomorphic representation of the structure group is a GH vector bundle canonically [Theorem 4.13](#). This is the analogue of the well-known relation between holomorphic principal bundles and holomorphic vector bundles. Therefore, our notion of GH principal bundles provides another possible way to construct GH structures.

The paper is organized as follows. In [Section 2](#), we recall the basic and necessary knowledge of GC geometry and the reduction theory of Courant algebroids and Dirac structures. In [Section 3](#), we briefly investigate some properties of GH vector bundles and show that co-Higgs bundles are the basic local ingredient of such bundles, at least around a regular point [Propositions 3.1](#) and [3.2](#). Motivated by this observation, and starting with a co-Higgs bundle, we construct an example of GH vector bundle, which is not a co-Higgs bundle. In [Section 4](#), we define the notion of GH principal bundles and explore some examples in this context, e.g. co-Higgs principal bundles and Poisson principal bundles (they are the counterparts of co-Higgs vector bundles and Poisson modules). The relation between GH principal bundles and GH vector bundles is also studied there [Proposition 4.12](#) and [Theorem 4.13](#). We also show the particularity of GH principal \mathbb{C}^* -bundles, i.e. the total space of a GH principal \mathbb{C}^* -bundle acquires a canonical GC structure [Theorem 4.14](#). In [Section 5](#), under certain compatibility conditions, we prove that a GH principal bundle over a manifold with symmetries descends to another GH principal bundle over the quotient [Theorem 5.3](#). An illustrative example is also given.

2. Background of generalized geometry

We recall some preliminary material of GC geometry, of which the basic references are [\[2,3,8,11\]](#). In the paper, M is a smooth connected $2m$ -manifold and all Lie groups involved are connected.

Generalized geometry is the geometry related to the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, or more generally, an exact Courant algebroid E over M . We follow [\[11\]](#) for the axioms defining a Courant bracket $[\cdot, \cdot]_E$ and all Courant algebroids in the paper refer to exact ones.

Given E , one can always find an isotropic right splitting $s : TM \rightarrow E$, which has a curvature form $H \in \Omega_{cl}^3(M)$ defined by

$$H(X, Y, Z) = 2([s(X), s(Y)]_E, s(Z)), X, Y, Z \in \Gamma(TM).$$

Via the bundle isomorphism $s + \frac{1}{2}\pi^* : TM \oplus T^*M \rightarrow E$, the Courant algebroid structure can be transported onto $\mathbb{T}M$. Then the pairing $\langle \cdot, \cdot \rangle$ on E is the natural one, i.e. $\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X))$, and the bracket is

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y \iota_X H,$$

called the H -twisted Courant bracket. Different splittings are related by B-field transforms, i.e. $e^B(X + \xi) = X + \xi + \iota_X B$, where B is a 2-form.

An isotropic subbundle $\mathfrak{A} \subset E$ is called a generalized distribution and called integrable if it is involutive w.r.t. the Courant bracket. An integrable maximal generalized distribution \mathfrak{D} is called a Dirac structure. These notions can all be extended to the complexified case and what interest us here are those complex Dirac structures called GC structures, i.e. complex Dirac structures L such that $L \oplus \bar{L} = E_{\mathbb{C}}$.¹

Two extremal GC structures are symplectic and complex structures. Let $E = \mathbb{T}M$ with $H = 0$. If ω is a symplectic structure, then $L = \{X - i\omega(X)|X \in T_{\mathbb{C}}M\}$; if J is a complex structure, then $L = T_{0,1} \oplus T_{1,0}^*$. A more complicated example is a holomorphic Poisson manifold. Let β be a holomorphic Poisson structure on a complex manifold (M, J) . Then $L = \{X + \xi + \beta(\xi)|X + \xi \in T_{0,1} \oplus T_{1,0}^*\}$.

Local GC geometry can already be nontrivial: the dimension of $\ker \pi|_L$, called the type, may vary along some subset of M ; If it does not change around a point x , x is called regular. Around such an x , up to diffeomorphism and B-field transform, L is precisely the product of a symplectic structure and a complex structure (the transverse complex structure) [\[3\]](#).

If a right splitting is chosen, then $E \cong (TM, H)$, and the bundle \mathcal{S} of forms can be viewed as the spin bundle of $\mathbb{T}M$; in particular, a GC structure L is characterized by a line bundle $l \subset \mathcal{S}_{\mathbb{C}}$ (the canonical line bundle): L is precisely the annihilator of l under the Clifford action $(X + \xi) \cdot \eta = \iota_X \eta + \xi \wedge \eta$, and integrability of L means, for any $\eta \in \Gamma(l)$, there exists $A \in \Gamma(\mathbb{T}_{\mathbb{C}}M)$ such that

$$d_H \eta := d\eta - H \wedge \eta = A \cdot \eta.$$

Via the pairing and the bracket, a differential operator $d_L : \Gamma(\wedge^k \bar{L}) \rightarrow \Gamma(\wedge^{k+1} \bar{L})$ can be defined: for $\sigma \in \Gamma(\wedge^k \bar{L})$, $X_i \in \Gamma(L)$,

$$d_L \sigma(X_0, \dots, X_k) = \sum_i (-1)^i \pi(X_i) \sigma(X_0, \dots, \hat{X}_i, \dots, X_k) + \sum_{i < j} (-1)^{i+j} \sigma([X_i, X_j]_E, X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k). \quad (2.1)$$

Since L is involutive, $d_L^2 = 0$. The analogue of holomorphic structures in complex geometry is defined as follows.

¹ We denote the complexification of a real space (or bundle) R by $R_{\mathbb{C}}$.

Definition 2.1 ([8]). Let L be a GC structure in E and V a complex vector bundle over M . An L -connection D in V is a differential operator $D : \Gamma(V) \rightarrow \Gamma(\bar{L} \otimes V)$ satisfying

$$D(fs) = d_L f \otimes s + fDs, \quad s \in \Gamma(V), f \in C^\infty(M).$$

If D is flat, i.e. $D^2 = 0$, D is called a GH structure and V a GH vector bundle.

Example 2.2. The canonical line bundle l of L is GH, with D given rise by d_H . This is the example motivating the above definition in [2].

Over a symplectic manifold, a GH vector bundle is precisely a flat one. More interesting are the following:

Example 2.3. If (M, J) is a complex manifold, then a GH structure in V is precisely a holomorphic structure in V coming together with a $\Phi \in H^0(T_{1,0} \otimes \text{Hom}(V))$ (Higgs field) such that $\Phi \wedge \Phi = 0$. The pair (V, Φ) is called a co-Higgs bundle. Even in this somewhat trivial case, the relevant geometry turns out to be interesting. See [4–6] for a detailed investigation mainly about stable co-Higgs bundles of rank 2 over \mathbb{P}^1 and \mathbb{P}^2 .

Example 2.4. On a holomorphic Poisson manifold (M, J, β) , a GH vector bundle is precisely a holomorphic Poisson module, as was pointed out by Gualtieri [8]. Some constructions of Poisson modules were investigated by Hitchin [9].

In the ordinary way, the dual bundle of a GH vector bundle, and the tensor product of two GH vector bundles are GH again.

We still need some knowledge concerning the reduction theory of Courant algebroids and Dirac structures developed in [11]. Let G be a real Lie group and \mathfrak{g} its Lie algebra.

Definition 2.5. A Courant algebra over \mathfrak{g} is a vector space \mathfrak{a} equipped with a bilinear bracket $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathfrak{a}$ and a map $\pi_0 : \mathfrak{a} \rightarrow \mathfrak{g}$, which satisfy

- (i) $[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]]$,
- (ii) $\pi_0([a_1, a_2]) = [\pi_0(a_1), \pi_0(a_2)]$.

Courant algebra \mathfrak{a} over \mathfrak{g} is called exact if π_0 is surjective and $\ker \pi_0$ is abelian; in particular, $\Gamma(E)$ is an exact Courant algebra over $\Gamma(TM)$. Exact Courant algebras are introduced to formulate the following notion of extended actions.

Definition 2.6. Let G act on M on the right, with infinitesimal action $\varphi_0 : \mathfrak{g} \rightarrow \Gamma(TM)$. An extension of this action to E is an exact Courant algebra \mathfrak{a} over \mathfrak{g} together with a Courant algebra anti-morphism² $\varphi : \mathfrak{a} \rightarrow \Gamma(E)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \pi_0 & \longrightarrow & \mathfrak{a} & \xrightarrow{\pi_0} & \mathfrak{g} \longrightarrow 0 \\ & & & & \downarrow \varphi & & \downarrow \varphi_0 \\ & & & & \Gamma(E) & \xrightarrow{\pi} & \Gamma(TM) \end{array}$$

which is such that $\ker \pi_0 \subset \mathfrak{a}$ acts trivially, and the induced action of $\mathfrak{g} = \mathfrak{a}/\ker \pi_0$ on $\Gamma(E)$ integrates to a G -action on the total space of E .

What of most interest to us is the following relatively simple case:

Definition 2.7. If \mathfrak{a} in Definition 2.6 is \mathfrak{g} , and $\pi_0 = Id$, then φ is called a trivially extended G -action.

In this paper, unless otherwise stated, all group actions on manifolds are on the right and free, and on E all actions are trivially extended. Note that $\varphi(\mathfrak{g})$ generates a subbundle $K \subset E$. By K^\perp denote the annihilator of K in E w.r.t. the pairing. If $K \subset K^\perp$, φ is called isotropic. By [11, Theorem 3.3], we have

Lemma 2.8. If G acts on M freely and properly, and φ is an isotropic trivially extended G -action on E , then E descends to E_r , a Courant algebroid over the quotient manifold M/G .

More precisely, in the above situation, $\frac{K^\perp}{K}$ is a vector bundle carrying a G -action and $E_r = \frac{K^\perp}{K}/G$. The pairing in E_r is induced from that in E restricted on K^\perp . The Courant bracket on $\Gamma(E_r)$ is derived from that on G -invariant sections of $K^\perp \subset E$.

In this formalism, a G -invariant complex Dirac structure $\mathfrak{D} \subset E_{\mathbb{C}}$ descends to another one $\mathfrak{D}_{red} \subset E_{r\mathbb{C}}$, provided a certain smoothness condition holds, e.g. $\mathfrak{D} \cap K_{\mathbb{C}}$ has constant rank, see [11, Theorem 4.2]. As a vector space, \mathfrak{D}_{red} at the point $[xG]$ is isomorphic to $\frac{\mathfrak{D}_x \cap K_{x\mathbb{C}}^\perp + K_{x\mathbb{C}}}{K_{x\mathbb{C}}}$. Besides, it is of basic interest to mention that this reduction procedure actually applies to general integrable generalized distributions, not necessarily only to Dirac structures.

² That is $\varphi([\cdot, \cdot]) = -[\varphi(\cdot), \varphi(\cdot)]_{\mathfrak{c}}$. If the action is on the left, φ should be a Courant algebra morphism.

Lemma 2.9. *Let the assumptions of Lemma 2.8 hold. If \mathfrak{A} is a G -invariant integrable generalized distribution such that both $\mathfrak{A} \cap K_{\mathbb{C}}$ and $\mathfrak{A} \cap K_{\mathbb{C}}^{\perp}$ are of constant rank, then \mathfrak{A} descends to an integrable generalized distribution $\mathfrak{A}_r \subset E_{r\mathbb{C}}$.*

Proof. Let

$$\mathfrak{A}_r := \frac{\mathfrak{A} \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}}{K_{\mathbb{C}}} / G \cong \frac{\mathfrak{A} \cap K_{\mathbb{C}}^{\perp}}{\mathfrak{A} \cap K_{\mathbb{C}}} / G.$$

As for integrability of \mathfrak{A}_r , the argument in the proof of [11, Theorem 4.2] still works here, since it does not depend on whether the generalized distribution is maximal or not. \square

3. Some properties of GH structures

To motivate the next section and also for its own right, we briefly investigate some properties of GH structures. In this section, the underlying Courant algebroid E and GC structure L are fixed.

Let V be a GH vector bundle over M , and s a local frame in V . Then $Ds = s\theta$, with θ being the connection form. Generalized holomorphicity implies the Maurer–Cartan equation $d_L\theta + \theta \wedge \theta = 0$. At a point $x \in M$, if $\pi(\theta)$ is tangent to the symplectic leaf through x , D is said adapted there. This notion is actually independent of s .

In [10, Theorem 6.4], it was proved that if D is adapted around a regular point, then generalized holomorphicity means precisely one can always find GH local frames s there, i.e. $Ds = 0$. Therefore if L is regular and D is adapted everywhere, then V is flat in symplectic directions and holomorphic w.r.t. the transverse complex structure.

Assume L is regular everywhere and the type > 0 . Then $\ker \pi|_L$, denoted by F , is itself a vector bundle. If z^{α} are transverse holomorphic coordinates of M , then F is locally generated by dz^{α} .

Proposition 3.1. *Let V be GH over (M, L) , with L being regular everywhere. Then canonically there is a $\Phi \in \Gamma(F^* \otimes \text{Hom}(V))$ such that $\Phi \wedge \Phi = 0$.*

Proof. Let $S := \pi(\bar{L})/(\pi(L) \cap \pi(\bar{L}))$ and denote the image of $X \in \bar{L}$ under the composition of π and the quotient map by $[X]$. $[\cdot]$ can be extended to $\wedge^* \bar{L}$. Note that $S \cong F^*$.

Let s be a local frame in V , with θ being its connection form. If $s' = sA$ is another frame, then its connection form θ' is related to θ by

$$\theta' = A^{-1}\theta A + A^{-1}d_L A.$$

Noting that $[d_L A] = 0$, we get

$$[\theta'] = A^{-1}[\theta]A,$$

implying that $[\theta]$ is a global section Φ of $S \otimes \text{Hom}(V)$. Since $[d_L\theta] = 0$, $d_L\theta + \theta \wedge \theta = 0$ implies that $[\theta] \wedge [\theta] = 0$, which is precisely $\Phi \wedge \Phi = 0$. \square

We show that a general GH vector bundle is, to some extent, a combination of flat bundles and co-Higgs bundles, at least around a regular point.

Proposition 3.2. *Up to diffeomorphism, B-field transform and suitable choice of local frames, around a regular point a GH vector bundle corresponds to a co-Higgs bundle over the space of symplectic leaves.*

Proof. Since the problem is local, we assume that M is the product of a symplectic manifold and a complex manifold, with specified coordinates p_k, q_k and z^{α} respectively. Then \bar{L} is freely generated by

$$\{\partial_{p_k} - idq_k, \partial_{q_k} + idp_k, d\bar{z}^{\alpha}, \partial_{z^{\alpha}}\}.$$

If a frame s is chosen, the connection form θ decomposes into $\theta_1 + \theta_2$, where θ_2 only involves $\partial_{z^{\alpha}}$ -components such that $[\theta_2] = \Phi$. The Maurer–Cartan equation takes the following form:

$$d_L\theta_1 + \theta_1 \wedge \theta_1 = 0, \quad d_L\theta_2 + \theta_1 \wedge \theta_2 + \theta_2 \wedge \theta_1 = 0, \quad \theta_2 \wedge \theta_2 = 0. \quad (3.1)$$

The same argument as in the proof of [10, Theorem 6.4] also applies to the above first equation, i.e. we can choose another frame s' such that $\theta_1 = 0$. In this frame, the rest two equations become

$$d_L\theta_2 = 0, \quad \theta_2 \wedge \theta_2 = 0. \quad (3.2)$$

The former one implies that the components of θ_2 do not depend on p_k, q_k and depend on z^{α} holomorphically. Thus (3.2) means that (V, θ_2) can be viewed as a co-Higgs bundle over the space of symplectic leaves. Note that the complex structure in the latter space comes from the transverse one associated to L . \square

F has a canonical adapted GH structure D^0 , i.e. the one defined by $D^0(dz^{\alpha}) = 0$. Hence $F^* \otimes \text{End}(V)$ has a GH structure induced from D^0 and D . We still denote this GH structure by D .

Corollary 3.3. The section $\Phi \in \Gamma(F^* \otimes \text{Hom}(V))$ in Proposition 3.1 is GH, i.e. $D\Phi = 0$.

Proof. We continue to use the notation in the proof of Proposition 3.2. If $\theta_2 = \tilde{\theta}_\alpha \partial_{z^\alpha}$, then $\Phi(dz^\alpha) = \tilde{\theta}_\alpha$. In the local frame of $\text{Hom}(V)$ induced from s and its dual,

$$D\tilde{\theta}_\alpha = d_L \tilde{\theta}_\alpha + [\theta, \tilde{\theta}_\alpha].$$

Note that $d_L \tilde{\theta}_\alpha + [\theta_1, \tilde{\theta}_\alpha] = 0$ by the second equation of (3.1) and $[\theta_2, \tilde{\theta}_\alpha] = 0$ by the third. Therefore, $D\tilde{\theta}_\alpha = 0$ and the claim follows. \square

For a co-Higgs bundle (V, ϕ) , the above Φ is precisely the Higgs field ϕ . In this sense, we call Φ the Higgs field associated to D . In general, the subset of regular points is open and dense, and Φ is well-defined in this open set.

As was noted by Cavalcanti [12, Chapter 2], (M, L) may be a symplectic fibration over another GC base B ; in particular, B can be a complex manifold. From Proposition 3.2, it is reasonable to expect that, if M is such a GC manifold, then co-Higgs bundles over B may be the starting point to construct GH vector bundles over M .

Example 3.4. Let $B = \mathbb{P}^1$, $M = \mathbb{P}(\mathcal{O}(n) \oplus 1)$ (the Hirzebruch surface F_n), and $p : M \rightarrow B$ be the projection. Let $E = \mathbb{T}M$ with $H = 0$. We can specify $\mathcal{O}(n) = T_{1,0}^{n/2}$, which then has a hermitian structure induced from the Fubini–Study metric. 1 can be viewed as a unitary frame of the trivial line bundle. Hence the tautological line bundle l over M is holomorphic and has a natural hermitian structure. Let $i\omega$ be the curvature of the Chern connection in l . ω , restricted on each fibre of M , is a multiple of the standard Kähler form on \mathbb{P}^1 . Therefore M is a symplectic fibration over B and can be viewed as a GC 4-manifold: if (U, z) is the euclidean coordinate chart of B , then a local pure spinor of this GC structure L_λ is $e^{i\lambda\omega} p^*(dz)$ ($\lambda \neq 0$ is a real constant). L is of type 1.

Let (V, Φ) be a co-Higgs bundle over B . The holomorphic structure in V can naturally be pulled back, inducing an adapted GH structure D in $p^*(V)$ w.r.t. L_λ . We want to find a $\tilde{\Phi} \in \Gamma(\bar{L}_\lambda \otimes \text{Hom}(p^*(V)))$, solving the equation

$$D\tilde{\Phi} + \tilde{\Phi} \wedge \tilde{\Phi} = 0 \quad (3.3)$$

subject to the condition $p_*(\pi(\tilde{\Phi})) = \Phi$. Once this $\tilde{\Phi}$ is given, $D + \tilde{\Phi}$ will be a new GH structure in $p^*(V)$.

To be more concrete, let $V = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$ such that $n_1 - n_2 > 2$. A general Higgs field is of the following form [5]

$$\Phi = \begin{pmatrix} p_1 \partial_z & p_2 \partial_z \otimes (\partial_z)^{\frac{n_1-n_2}{2}} \\ 0 & p_3 \partial_z \end{pmatrix},$$

where p_1, p_2, p_3 are polynomials in z , of degree 2, $2 + n_1 - n_2$ and 2 respectively.

$(\partial_z)^{\frac{n}{2}}$ is a local frame of $\mathcal{O}(n)$ over U , trivializing $p^{-1}(U) \subset M$ away from the section at infinity. Let (z, u) be the corresponding coordinates. Note that L_λ is of the form $e^{-i\lambda\omega} L_0$, with L_0 being a Dirac structure generated by $\{\partial_u, \partial_{\bar{u}}, dz, \partial_{\bar{z}}\}$. $e^{-i\lambda\omega}$ is actually an isomorphism between the two Lie algebroids L_λ and L_0 . So, to solve Eq. (3.3), we can identify L_0^* with L_λ , and the Lie algebroid derivative associated to L_0 with d_{L_λ} . Here we just give the final result: let $\tilde{\Phi}$ be the matrix differing from Φ only by that ∂_z in $p_i \partial_z$ is replaced by

$$\partial_z + i\lambda\omega(\partial_z, \cdot) - \frac{\omega(\partial_z, \partial_{\bar{u}})}{\omega(\partial_u, \partial_{\bar{u}})} [\partial_u + i\lambda\omega(\partial_u, \cdot)] - 2i\lambda \left[\frac{|\omega(\partial_z, \partial_{\bar{u}})|^2}{\omega(\partial_u, \partial_{\bar{u}})} + \omega(\partial_z, \partial_{\bar{z}}) \right] d\bar{z}.$$

$\tilde{\Phi}$ extends to a global section of $\bar{L}_\lambda^* \otimes \text{Hom}(V)$. It satisfies $d_{L_\lambda} \tilde{\Phi} = 0$ and $\tilde{\Phi} \wedge \tilde{\Phi} = 0$. Hence Eq. (3.3) is satisfied.

To conclude this section, we briefly discuss the situation around a singular point. By the local structure theorems of [13,14], around such a point, L is equivalent to the product of a holomorphic Poisson structure and a regular GC structure. Note that a GH vector bundle over a holomorphic Poisson manifold is precisely a holomorphic Poisson module. With these facts in mind, we can say that flat bundles, co-Higgs bundles and holomorphic Poisson modules are the only possible local ingredients of a GH vector bundle.

4. GH principal bundles

In this section, we define a notion of GH structures in the context of (generalized) principal bundles and explore its relation with GH vector bundles. Some examples of GH structures are constructed.

Let G be a real Lie group of finite dimension and \mathfrak{g} its Lie algebra.

Definition 4.1. A generalized principal G -bundle over M is a triple $(\mathbf{P}, \mathbf{E}, \varphi)$ such that

- (i) $p : \mathbf{P} \rightarrow M$ is an ordinary principal G -bundle,
- (ii) \mathbf{E} is a Courant algebroid over \mathbf{P} and φ is an isotropic trivially extended G -action on \mathbf{E} .

By Lemma 2.8, \mathbf{E} descends to a Courant algebroid E over M . By π_1 and π_2 denote the anchor maps of \mathbf{E} and E respectively. Note that if G acts on the right on a manifold N freely and properly, and φ is an isotropic trivially extended G -action on a Courant algebroid \mathbf{E} over N , then (N, \mathbf{E}, φ) is naturally a generalized principal G -bundle over N/G . In this way, several examples of generalized principal bundles were already included in [11]. If the underlying \mathbf{E} , φ are clear, we will just refer to \mathbf{P} as a generalized principal G -bundle.

Example 4.2. Let $p : \mathbf{P} \rightarrow M$ be an ordinary principal G -bundle over M and H a real closed 3-form on M . Let \mathbf{E} be $\mathbb{T}\mathbf{P}$ equipped with the $p^*(H)$ -twisted Courant bracket. G acts on \mathbf{E} in the ordinary manner, i.e. $g \cdot (X, \xi) = (g_*(X), g^{*-1}(\xi))$. In this way, \mathbf{P} becomes a generalized principal G -bundle. Note that the reduced Courant algebroid is precisely $\mathbb{T}M$ equipped with the H -twisted Courant bracket.

For a general generalized principal G -bundle, due to [15, Lemma 2.7], at least locally, it is safe to assume that G acts on \mathbf{E} in the manner described in the above example.

Example 4.3. Let $(\mathbf{P}, \mathbf{E}, \varphi)$ be a generalized principal G -bundle over M . If $G_1 \subset G$ is a subgroup with Lie algebra \mathfrak{g}_1 and $\iota : \mathbf{Q} \rightarrow \mathbf{P}$ is a reduction of the structure group from G to G_1 , then since \mathbf{Q} is a submanifold of \mathbf{P} , \mathbf{E} can be pulled back to generate a Courant algebroid $\mathbf{E}_Q := \frac{\pi^{-1}(TQ)}{\text{Ann}TQ}$ over \mathbf{Q} (see [11, Lemma 3.7]). By restriction of $\varphi(\mathfrak{g}_1)$ to \mathbf{Q} , φ induces an isotropic trivially extended G_1 -action $\tilde{\varphi}$ on \mathbf{E}_Q . Then $(\mathbf{Q}, \mathbf{E}_Q, \tilde{\varphi})$ is a generalized principal G_1 -bundle over M . It can be checked that the reduced Courant algebroid of \mathbf{E}_Q is the same as that of \mathbf{E} .

Let G be a complex Lie group, \mathfrak{g} its real Lie algebra. Note that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_h \oplus \mathfrak{g}_a$, i.e. the sum of the $(1, 0)$ - and $(0, 1)$ -parts. If $(\mathbf{P}, \mathbf{E}, \varphi)$ is a generalized principal G -bundle, then $\varphi(\mathfrak{g}_a)$, $\varphi(\mathfrak{g}_h)$ generate K_a , K_h , both subbundles of $K_{\mathbb{C}}$. Motivated by some observations in [10], we introduce the following concept.

Definition 4.4. Let $(\mathbf{P}, \mathbf{E}, \varphi)$ be a generalized principal G -bundle over M and $\mathfrak{A} \subset \mathbf{E}_{\mathbb{C}}$ a G -invariant generalized distribution such that

- (i) $K_a \subset \mathfrak{A} \subset K_{\mathbb{C}}^{\perp}$,
- (ii) $\mathfrak{A} \oplus \overline{\mathfrak{A}} = K_{\mathbb{C}}^{\perp}$,
- (iii) \mathfrak{A} descends to a GC structure L in E .

Then \mathfrak{A} is called an almost GH structure w.r.t. L . If furthermore \mathfrak{A} is integrable, it is called a GH structure.

Remark. (1) For a G -invariant generalized distribution \mathfrak{A} satisfying the above conditions (i), (ii), since $\mathfrak{A} \cap K_{\mathbb{C}} = K_a$ and $\mathfrak{A} \cap K_{\mathbb{C}}^{\perp} = \mathfrak{A}$, it always descends to a generalized distribution \mathfrak{A}_{red} such that $\mathfrak{A}_{red} \oplus \overline{\mathfrak{A}}_{red} = E_{\mathbb{C}}$. If furthermore \mathfrak{A} is integrable, so is \mathfrak{A}_{red} by Lemma 2.9. Therefore if an integrable generalized distribution \mathfrak{A} satisfies (i) and (ii), then (iii) follows automatically.

(2) In other words, a GH structure is a G -invariant complex structure \mathcal{J} in K^{\perp} , which restricted to K is $-J$ (J is the canonical complex structure on G) and whose i -eigenbundle is involutive and descends to a GC structure on M . We call \mathcal{J} the canonical complex structure associated with \mathfrak{A} .

If a GH structure \mathfrak{A} is specified on a generalized principal G -bundle $(\mathbf{P}, \mathbf{E}, \varphi)$, we will simply call \mathfrak{A} or \mathbf{P} a GH principal G -bundle when no ambiguity arises. The following proposition shows how Higgs fields turn up in the context of GH principal bundles. Let (M, L) be a regular GC manifold, and F be the bundle in Section 3.

Proposition 4.5. Let \mathfrak{A} be an almost GH structure in a generalized principal G -bundle $(\mathbf{P}, \mathbf{E}, \varphi)$ over (M, L) . Then canonically there is an equivariant map $\Phi : p^*(F) \rightarrow \mathfrak{g}_h$, i.e. $\Phi_{g^{*-1}(\xi)}(pg) = Ad_{g^{-1}}\Phi_{\xi}(p)$ for $p \in \mathbf{P}$, $g \in G$ and $\xi \in p^*(F)|_p$. Furthermore, if \mathfrak{A} is integrable, then $[\Phi, \Phi] = 0$.

Proof. Let $\mathfrak{Z} := \{a \in \mathfrak{A} | \pi_1(a) \in \pi_1(K_h)\}$. \mathfrak{Z} is actually a subbundle of \mathfrak{A} and $p^*(F) \cong \frac{\mathfrak{Z} + K_h}{K_h} \cong \mathfrak{Z}$. Then \mathfrak{Z} is the image of a bundle map $\phi : p^*(F) \rightarrow K_h$. It is easy to check \mathfrak{Z} is G -invariant, so ϕ is induced from an equivariant map $\Phi : p^*(F) \rightarrow \mathfrak{g}_h$.

Let z^{α} be holomorphic coordinates w.r.t. the transverse complex structure in M . Then $p^*(dz^{\alpha}) + \phi(p^*(dz^{\alpha}))$ are local sections of \mathfrak{Z} . Integrability of \mathfrak{A} implies that

$$[p^*(dz^{\alpha}) + \phi(p^*(dz^{\alpha})), p^*(dz^{\beta}) + \phi(p^*(dz^{\beta}))]_{\mathbb{C}} = [\phi(p^*(dz^{\alpha})), \phi(p^*(dz^{\beta}))]_{\mathbb{C}}$$

must vanish. By the equivariance of ϕ , this fact implies that $[\Phi, \Phi] = 0$. \square

Example 4.6. The frame bundle of a GH vector bundle. Given a GH vector bundle V of rank $r > 1$, let $p : F(V) \rightarrow M$ be the frame bundle of V . Choose a splitting for the underlying E , and let H be the curvature. In the manner of Example 4.2, $F(V)$ becomes a generalized principal $GL(r, \mathbb{C})$ -bundle. If s is a local frame over an open set $W \subset M$ (small enough), then $Ds_{\alpha} = \theta_{\alpha}^{\beta}s_{\beta}$, where θ_{α}^{β} is the connection form. By s , $F(V)$ restricted on W looks like $W \times GL(r, \mathbb{C})$. Let $\{a_i\}$ be a frame of L over W and w_{α}^{β} be the standard coordinates of $GL(r, \mathbb{C})$. Let \mathfrak{A} be the generalized distribution locally generated by

$$\{\partial_{\bar{w}}^{\alpha}, a_i - \theta_{\alpha}^{\tau}(a_i)w_{\beta}^{\alpha}\partial_{w_{\beta}^{\tau}}, \alpha, \beta = 1, \dots, r, i = 1, \dots, 2m\}.$$

\mathfrak{A} is actually globally defined: let η be a pure spinor of L over W . \mathfrak{A} is precisely the annihilator of the bundle of spinors generated by

$$\{(dw_\beta^\alpha + \theta_\gamma^\alpha w_\beta^\gamma) \cdot p^*(\eta), \alpha, \beta = 1, \dots, r\},$$

while it is easy to check the latter bundle is globally well-defined. \mathfrak{A} is integrable due to the equation $d_L\theta + \theta \wedge \theta = 0$. Hence $F(V)$ is a GH principal $GL(r, \mathbb{C})$ -bundle.

Example 4.7. Let $p: \mathbf{P} \rightarrow M$ be a principal G -bundle over a symplectic manifold M and \mathfrak{A} a GH structure in \mathbf{P} . Then $\pi_1(\mathfrak{A}) \cap \pi_1(\tilde{\mathfrak{A}}) = R_{\mathbb{C}}$ for some real distribution R of rank $2m$ in \mathbf{P} . R is actually a horizontal distribution and generalized holomorphicity implies that it is integrable. Therefore \mathbf{P} is a flat principal G -bundle.

Example 4.8 (Co-Higgs Principal G -Bundles). Let $p: \mathbf{P} \rightarrow M$ be a principal G -bundle over complex manifold (M, J) . Then \mathbf{P} is a generalized principal G -bundle as in Example 4.2 with $H = 0$. Let \mathfrak{A} be a GH structure and π the projection from \mathbf{E} to $T^*\mathbf{P}$. Then $\ker \pi|_{\mathfrak{A}}$ is an ordinary integrable complex distribution, determining a complex structure in the total space of \mathbf{P} , or more precisely, a holomorphic structure. If this holomorphic structure is fixed, then the remainder datum to specify \mathfrak{A} is an equivariant holomorphic map $\Phi: p^*(T_{1,0}^*M) \rightarrow \mathfrak{g}_h$ such that $[\Phi, \Phi] = 0$. We call such a pair (\mathbf{P}, Φ) a co-Higgs principal G -bundle and Φ the associated Higgs field.

With the above interpretation, we can construct some concrete co-Higgs principal G -bundles.

Example 4.9. Let \mathbf{P} be the space of 3×2 complex matrices of rank 2, the total space of a holomorphic principal $GL(2, \mathbb{C})$ -bundle over \mathbb{P}^2 . We shall construct some Higgs fields w.r.t. the natural holomorphic structure.

Let $U \subset \mathbf{P}$ be the subset of matrices whose first two rows are linearly independent. U can be parameterized by a pair (z, g) , where $z = (z_1, z_2)$ are the euclidean coordinates of \mathbb{P}^2 and $g = \begin{pmatrix} z_3 & z_4 \\ z_5 & z_6 \end{pmatrix} \in GL(2, \mathbb{C})$. Then by the equivariance of Φ (e is the identity of $GL(2, \mathbb{C})$),

$$\Phi(z, g) = \text{Ad}_{g^{-1}}(\Phi(z, e)).$$

So we only need to specify $\Phi(z, e) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$, where $\phi_{ij} = \phi_{ij}^1 \partial_{z_1} + \phi_{ij}^2 \partial_{z_2}$, $i, j = 1, 2$, with ϕ_{ij}^k all being holomorphic functions in z_1, z_2 . In other words, we should find two commuting 2×2 matrices ϕ^1, ϕ^2 . There are many choices, e.g.

$$\phi^1 = \begin{pmatrix} f_1 & 1 \\ 0 & f_2 \end{pmatrix}, \quad \phi^2 = \begin{pmatrix} f_3 & f_4 \\ 0 & f_3 - f_4(f_1 - f_2) \end{pmatrix},$$

where f_i are all constant. It is left to the interested reader to verify that the corresponding Φ extends to a global Higgs field.

Example 4.10 (Poisson Principal G -Bundle). Let $p: \mathbf{P} \rightarrow M$ be a principal G -bundle over a holomorphic Poisson manifold (M, J, β) . \mathbf{P} is a generalized principal G -bundle as in Example 4.2 with $H = 0$. Let \mathfrak{A} be a GH structure and π the projection from \mathbf{E} to $T^*\mathbf{P}$. Then $\ker \pi|_{\mathfrak{A}}$ determines a holomorphic structure in \mathbf{P} . If this holomorphic structure is fixed, then the extra datum to specify \mathfrak{A} is an equivariant holomorphic bundle map $\Phi: p^*(T_{1,0}^*M) \rightarrow T_{1,0}\mathbf{P}$, satisfying the following equations

$$p_*(\Phi_{df}) = \beta(df), \quad [\Phi_{df}, \Phi_{dg}] = \Phi_{d\Phi_{df}g}, \quad (4.1)$$

where f, g are locally defined holomorphic functions on M and df, dg are interpreted as local sections of $p^*(T_{1,0}^*M)$. We call such a pair (\mathbf{P}, Φ) a Poisson principal G -bundle. Note that if $\beta = 0$, then the image of Φ is included in K_h (K_h can be identified with $\mathbf{P} \times \mathfrak{g}_h$) and we come back to the special case of co-Higgs principal bundles.

To be more concrete, we construct a Poisson principal $GL(2, \mathbb{C})$ -bundle over \mathbb{P}^2 .

Example 4.11. We continue to use the main notation of Example 4.9. Since a Poisson structure over \mathbb{P}^2 lies in $H^0(\mathbb{P}^2, \mathcal{O}(3))$, we simply take $\beta = c \partial_{z_1} \wedge \partial_{z_2}$ for a constant c . In $p^{-1}(U)$, let

$$\Phi_{dz_1} := c \partial_{z_2} + \phi_1, \quad \Phi_{dz_2} := -c \partial_{z_1} + \phi_2,$$

where ϕ_1, ϕ_2 are $GL(2, \mathbb{C})$ -equivariant holomorphic $\mathfrak{gl}(2, \mathbb{C})$ -valued functions in z, g . As in Example 4.9, we only need to specify $\tilde{\phi}_i := \phi_i(z, e)$. That $(\mathbf{P}|_U, \Phi)$ is a Poisson principal $GL(2, \mathbb{C})$ -bundle is equivalent to that $\tilde{\phi}_i$ satisfy

$$c \frac{\partial \tilde{\phi}_1}{\partial z_1} + c \frac{\partial \tilde{\phi}_2}{\partial z_2} - [\tilde{\phi}_1, \tilde{\phi}_2] = 0.$$

There are many choices, e.g.

$$\tilde{\phi}_1 = \begin{pmatrix} cf_1 & c \\ 0 & cf_2 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} cf_3 & cf_4 \\ 0 & cf_3 - cf_4(f_1 - f_2) \end{pmatrix}$$

where f_i are all constant. Again we leave it to the interested reader to check that the corresponding Φ extends to a global solution of Eq. (4.1).

We now show how GH principal G -bundles are related to GH vector bundles, justifying the introduction of [Definition 4.4](#).

Let $\rho : G \rightarrow \text{Aut}(U)$ be a holomorphic representation of G in a complex vector space U of finite dimension and \mathfrak{A} an almost GH structure in a generalized principal G -bundle $(\mathbf{P}, \mathbf{E}, \varphi)$ over (M, L) . Then there is a canonical L -connection in the associated vector bundle $V := \mathbf{P} \times_{\rho} U$. Recall that $\Gamma(V)$ can be identified with $\Gamma(\mathbf{P}, U)^G$, the space of G -equivariant functions from \mathbf{P} to U . For $s \in \Gamma(V)$, by \tilde{s} denote the corresponding element in $\Gamma(\mathbf{P}, U)^G$. One can define the L -connection D as follows: let $x \in M$ and W be a small neighbourhood of x . For $A \in \Gamma(L)$, choose $\hat{A} \in \Gamma(\mathfrak{A}|_{p^{-1}(W)})^G$, which descends to $A|_W$. Then

$$(D_A s)(x) := [(q, (\pi_1(\hat{A})\tilde{s})(q))], \quad (4.2)$$

where q is any point in $p^{-1}(x)$. We call \hat{A} a lift of $A|_W$.

Proposition 4.12. D defined in Eq. (4.2) is an L -connection in V .

Proof. (i) The definition is independent of the choice of \hat{A} . This is because the difference of two different lifts lies in $\Gamma(K_a|_W)^G$ and contributes zero since ρ is a holomorphic representation.

(ii) $\pi_1(\hat{A})\tilde{s}$ is G -equivariant. Note that by construction, $\pi_1(\hat{A})$ is G -invariant. Let $\gamma(t)$ be a curve through $\gamma(0) = qg$ at which the tangent vector is $\pi_1(\hat{A})(qg)$. Then³

$$\begin{aligned} (\pi_1(\hat{A})\tilde{s})(qg) &= \lim_{t \rightarrow 0} \frac{\tilde{s}(\gamma(t)) - \tilde{s}(qg)}{t} \\ &= \rho(g)^{-1} \lim_{t \rightarrow 0} \frac{\tilde{s}(\gamma(t)g^{-1}) - \tilde{s}(q)}{t} \\ &= \rho(g)^{-1} (\pi_1(\hat{A})\tilde{s})(q), \end{aligned}$$

where we have used the fact that $\tilde{s}(qg) = \rho(g)^{-1}\tilde{s}(q)$.

(iii) D is indeed an L -connection: let $f \in C^\infty(M)$. Then note that

$$\pi_1(\hat{A})\tilde{f}s = \pi_1(\hat{A})(p^*(f)\tilde{s}) = p^*(\pi_2(A)f)\tilde{s} + p^*(f)\pi_1(\hat{A})\tilde{s},$$

where we have used the fact that \hat{A} descends to A . The conclusion then follows. \square

Theorem 4.13. Given a GH principal G -bundle \mathbf{P} over a GC manifold (M, L) and a holomorphic representation (ρ, U) of G , then canonically the associated vector bundle $V := \mathbf{P} \times_{\rho} U$ is a GH vector bundle.

Proof. We need to check that D defined by Eq. (4.2) is a GH structure in V . Since the problem is local in essence, we can assume that \mathbf{P} is of the form $M \times G$ and that the extended G -action is the one described in [Example 4.2](#).

Due to the product structure, one can embed $\Gamma(\mathbb{T}M)$ in $\Gamma(\mathbb{T}\mathbf{P})$. The lift of $A \in \Gamma(L)$ can be specified by

$$\hat{A} = A - \theta^i(A)\tilde{e}_i,$$

where $\tilde{e}_i = \varphi(e_i)$ with $\{e_i\}$ being a basis of \mathfrak{g}_h , and $\{\theta^i\}$ is a set of sections of $p^*(\tilde{L})$ over $M \times G$ (in the notation $\theta^i(A)$, A should be interpreted as a section of $p^*(L)$ by pull-back). The G -invariance of \hat{A} implies that the \mathfrak{g}_h -valued function $\theta^i(A)e_i$ should be equivariant and holomorphic in the G -orbit directions.

Let $A, B \in \Gamma(L)$, and \hat{A}, \hat{B} be their corresponding lifts. We only need to prove for the lift $\widehat{[A, B]}_c$ of $[A, B]_c \in \Gamma(L)$,

$$\pi_1(\hat{A})\pi_1(\hat{B})\tilde{s} - \pi_1(\hat{B})\pi_1(\hat{A})\tilde{s} - \pi_1(\widehat{[A, B]}_c)\tilde{s} = 0, \quad s \in \Gamma(V)$$

which is equivalent to

$$\pi_1([\hat{A}, \hat{B}] - \widehat{[A, B]}_c) = 0 \quad \text{mod } \pi_1(\Gamma(K_a)^G),$$

where, to distinguish the two Courant brackets, we have omitted the subscript c of the bracket on \mathbf{E} . It is easy to check that

$$[\theta^i(A)\tilde{e}_i, \theta^j(B)\tilde{e}_j] = -\theta^i(A)\theta^j(B)\widetilde{[e_i, e_j]}.$$

Therefore,

$$[\hat{A}, \hat{B}] = [A, B]_c - \pi_1(A)\theta^i(B)\tilde{e}_i + \pi_1(B)\theta^i(A)\tilde{e}_i - \theta^i(A)\theta^j(B)\widetilde{[e_i, e_j]}.$$

³ Since $\pi_1(\hat{A})(qg)$ is complex, there are actually two curves here, and the following equation should be interpreted as a compact form to combine two equations, with one for the real part and the other for the imaginary part of $\pi_1(\hat{A})(qg)$.

Noting that by Eq. (2.1),

$$d_L \theta^i(A, B) = \pi_1(A) \theta^i(B) - \pi_1(B) \theta^i(A) - \theta^i([A, B]_c),$$

we get

$$\begin{aligned} [\hat{A}, \hat{B}] &= [A, B]_c - \theta^i([A, B]_c) \tilde{e}_i - d_L \theta^i(A, B) \tilde{e}_i - \theta^i(A) \theta^j(B) [\widetilde{e_i, e_j}] \\ &= [\widetilde{A, B}]_c - d_L \theta^i(A, B) \tilde{e}_i - \theta^i(A) \theta^j(B) [\widetilde{e_i, e_j}]. \end{aligned}$$

Since \mathfrak{A} is integrable, we must have

$$d_L \theta^i(A, B) \tilde{e}_i + \theta^i(A) \theta^j(B) [\widetilde{e_i, e_j}] = 0.$$

Therefore,

$$\pi_1([\hat{A}, \hat{B}] - [\widetilde{A, B}]_c) = 0. \quad \square$$

So, given ρ , all our previous examples of GH principal G -bundles produce GH vector bundles.

Remark. It is easy to see that D is adapted around a regular point if the Higgs field Φ associated to \mathfrak{A} vanishes there.

Finally, we point out the particularity of the case $G = \mathbb{C}^*$, which can be expected from the observation in [10] that the total space of a GH line bundle has a canonical GC structure. Let $(\mathbf{P}, \mathbf{E}, \varphi)$ be a generalized principal \mathbb{C}^* -bundle.

Theorem 4.14. *If \mathbf{P} is a GH principal \mathbb{C}^* -bundle, then the total space of \mathbf{P} acquires a canonical \mathbb{C}^* -invariant GC structure \mathfrak{L} . Conversely, given a \mathbb{C}^* -invariant GC structure \mathfrak{L} on \mathbf{P} such that $\mathfrak{L} \cap K_{\mathbb{C}} = K_a$, \mathbf{P} is a GH principal \mathbb{C}^* -bundle in a canonical way.*

Proof. If \mathbf{P} is a GH principal \mathbb{C}^* -bundle, we show how to determine the GC structure \mathfrak{L} by the corresponding \mathfrak{A} . Note that $\mathfrak{A} \oplus K_h \subset \mathfrak{A}^\perp$ and that $\mathfrak{A}^\perp/\mathfrak{A}$ is of rank 2. The pairing on \mathbf{E} then induces a nondegenerate pairing on $\mathfrak{A}^\perp/\mathfrak{A}$, w.r.t. which $(K_h + \mathfrak{A})/\mathfrak{A}$ is isotropic. There is a unique line $l \subset \mathfrak{A}^\perp/\mathfrak{A}$, which is isotropic and has nondegenerate pairing with $(K_h + \mathfrak{A})/\mathfrak{A}$. The preimage of l in \mathfrak{A}^\perp is precisely the required \mathfrak{L} .

As for integrability of \mathfrak{L} , suppose that \mathbf{P} is of a product form $W \times \mathbb{C}^*$ (W is small enough) and choose splittings such that both the curvatures of \mathbf{E} and E are zero. In the following, Courant brackets are just written as $[\cdot, \cdot]$.

Let w be the standard coordinate of \mathbb{C}^* and e the canonical generator of \mathbb{C} . Then \mathfrak{L} is generated by

$$\{a_i - \theta(a_i)\varphi(e), \varphi(\bar{e}), dw + w\theta\},$$

where $\{a_i\}$ is a frame of L and θ is a section of $p^*(\bar{L})$. Note that $\pi_1(\varphi(e)) = w\partial_w$, and $\theta(a_i)$ does not depend on w since \mathfrak{A} is \mathbb{C}^* -invariant and \mathbb{C}^* is abelian. We only need to compute $[a_i - \theta(a_i)\varphi(e), dw + w\theta]$:

$$[a_i - \theta(a_i)\varphi(e), dw + w\theta] = w[a_i, \theta] - w d\theta(a_i) + w[\pi_2(\theta)\theta(a_i)]\varphi(e) - \theta(a_i)[\varphi(e), dw + w\theta].$$

Since \mathfrak{L} is obviously \mathbb{C}^* -invariant, the last term lies in \mathfrak{L} .

Claim 1: $[a_i, \theta] - d\theta(a_i) \in \Gamma(L)$. In fact, by Eq. (2.1),

$$\begin{aligned} \langle [a_i, \theta] - d\theta(a_i), a_j \rangle &= \langle [a_i, \theta], a_j \rangle - \frac{1}{2} \pi_2(a_j) \theta(a_i) \\ &= \frac{1}{2} \pi_2(a_i) \theta(a_j) - \frac{1}{2} \theta([a_i, a_j]) - \frac{1}{2} \pi_2(a_j) \theta(a_i) \\ &= \frac{1}{2} d_L \theta(a_i, a_j). \end{aligned}$$

Since integrability of \mathfrak{A} implies that $d_L \theta = 0$ (see the proof of Theorem 4.13), our claim follows.

Claim 2: $\theta([a_i, \theta] - d\theta(a_i)) = -\pi_2(\theta)\theta(a_i)$. This follows from a simple calculation by using the axioms defining a Courant algebroid.

Combining the above results, we deduce that \mathfrak{L} is integrable.

Conversely, if \mathfrak{L} is a \mathbb{C}^* -invariant GC structure on \mathbf{P} such that $\mathfrak{L} \cap K_{\mathbb{C}} = K_a$, then the canonical $\mathfrak{A} := \mathfrak{L} \cap K_h^\perp$. It is easy to check that \mathfrak{A} is integrable. \square

5. Reduction of GH structures

We continue to use the main notation of the previous section. On a GC manifold (M, L) , if there is an extended action of a group G_1 (with its Lie algebra \mathfrak{g}_1), preserving L , then under some mild conditions, L descends to a GC structure L_r in some reduced manifold M_r . Let \mathfrak{A} be a given GH principal G -bundle over M . This section is then devoted to partially answering the question: if \mathfrak{A} is compatible with the G_1 -action (see the following), then does \mathfrak{A} descend to a GH principal G -bundle \mathfrak{A}_r over M_r ?

Definition 5.1. A G_1 -generalized principal G -bundle over a G_1 -manifold M is a generalized principal G -bundle $(\mathbf{P}, \mathbf{E}, \varphi)$, equipped with a compatible extended G_1 -action (α, ψ) on \mathbf{E} in the following sense:

- (i) \mathbf{P} as an ordinary principal G -bundle is G_1 -equivariant, i.e. the projection p is G_1 -equivariant and the actions of G and G_1 on \mathbf{P} commute,
- (ii) the extended G_1 -action (α, ψ) on \mathbf{E} commutes with the extended G -action φ .

The two extended actions φ, ψ are called orthogonal if $\langle \psi(\alpha), \varphi(\mathfrak{g}) \rangle = 0$. If this is the case and ψ is also isotropic, then \mathbf{E} carries an isotropic extended $G_1 \times G$ -action $(\alpha \oplus \mathfrak{g}, \psi \oplus \varphi)$. In the following, assume that G_1 is compact.

Lemma 5.2. Let $(\mathbf{P}, \mathbf{E}, \varphi)$ be a G_1 -generalized principal G -bundle over M . Assume that G_1 acts on M freely and the extended G_1 -action ψ is trivial, isotropic and orthogonal to φ . Then $(\mathbf{P}, \mathbf{E}, \varphi)$ descends to a generalized principal G -bundle $(\mathbf{P}_r, \mathbf{E}_r, \varphi_r)$ over $M_r := M/G_1$.

Proof. Let K_1 and K be the subbundles of \mathbf{E} generated by $\psi(\mathfrak{g}_1)$ and $\varphi(\mathfrak{g})$ respectively. It is easy to check that $\mathbf{P}_r := \mathbf{P}/G_1$ is an ordinary principal G -bundle over M_r , and \mathbf{E} descends to $\mathbf{E}_r := \frac{K_1^\perp}{K_1}/G_1$. We need to show φ descends to an isotropic and trivially extended G -action φ_r on \mathbf{E}_r . Since the two extended actions are orthogonal, $K \subset K_1^\perp$. Note that K is G_1 -invariant. Therefore $K^r := \frac{K+K_1}{K_1}/G_1$ is isotropic in \mathbf{E}_r . If $\mathfrak{g} \in \mathfrak{g}$, then $\varphi(\mathfrak{g})$ is G_1 -invariant and descends to a section of K^r , defining the trivially extended G -action φ_r . Then K^r is generated by $\varphi_r(\mathfrak{g})$ and φ_r integrates to the natural G -action on $\frac{K_1^\perp}{K_1}/G_1$, i.e. \mathbf{E}_r . \square

Remark. \mathbf{E} and \mathbf{E}_r descend to Courant algebroids E over M and E_r over M_r respectively. E also carries an isotropic trivially extended G_1 -action ψ_r and can be further reduced. However, the reduced Courant algebroid of E is precisely E_r . In fact, E_r is the reduced Courant algebroid of \mathbf{E} carrying the isotropic trivially extended $G_1 \times G$ -action $(\mathfrak{g}_1 \oplus \mathfrak{g}, \psi \oplus \varphi)$. Besides, if G_1 is not compact, but the $G_1 \times G$ action is proper, the lemma still holds.

In the following, let G be a complex Lie group, let the assumptions of Lemma 5.2 hold and we continue to use the notation in the proof.

Theorem 5.3. If \mathfrak{A} is a G_1 -invariant GH structure in \mathbf{P} such that both $\mathfrak{A} \cap K_{1\mathbb{C}}$ and $\mathfrak{A} \cap K_{1\mathbb{C}}^\perp$ have constant rank, then \mathfrak{A} descends to a G -invariant integrable generalized distribution $\mathfrak{A}_r \subset \mathbf{E}_{r\mathbb{C}}$. \mathfrak{A}_r is a GH structure in $(\mathbf{P}_r, \mathbf{E}_r, \varphi_r)$ iff

$$\operatorname{rk} \frac{\mathfrak{A} \cap K_{1\mathbb{C}}^\perp}{\mathfrak{A} \cap K_{1\mathbb{C}}} = \dim M_r + \dim \mathfrak{g}_h, \quad \text{and} \quad \mathcal{J}K_1 \cap K_1^\perp \subset K_1, \quad (5.1)$$

where \mathcal{J} is the canonical complex structure in K^\perp associated with \mathfrak{A} .⁴

Proof. The existence and integrability of \mathfrak{A}_r follows by Lemma 2.9. Since $K_a \subset \mathfrak{A}$ is G_1 -invariant, K_a^r generated by $\varphi_r(\mathfrak{g}_a)$ is contained in \mathfrak{A}_r . (5.1) is the sufficient and necessary condition for $\mathfrak{A}_r \oplus \tilde{\mathfrak{A}}_r = K_{\mathbb{C}}^r$.⁵ This can be proved along the same line of the proof of [11, Lemma 5.1] by replacing the GC structure there by the canonical complex structure \mathcal{J} in K^\perp associated with \mathfrak{A} and \tilde{K} there by K_1 . That \mathfrak{A}_r descends to a GC structure on M_r follows from Remark (1) below Definition 4.4. \square

Remark. Let L and L_r be the underlying GC structures associated to \mathfrak{A} and \mathfrak{A}_r respectively. It seems that L does not necessarily descend to L_r , though E does descend to E_r .

The following example is motivated by an observation in [11] that the reduced manifold M_r of a complex manifold M is not necessarily of complex type. So starting with co-Higgs principal bundles over M , we may construct GH principal bundles over M_r , which are no longer co-Higgs principal bundles.

Example 5.4. Consider \mathbb{C}^3 equipped with its standard complex coordinates $(z_k = x_k + iy_k, k = 1, 2, 3)$. Let Λ be the lattice generated by the standard basis $\{\delta_j\}$ of $\mathbb{R}^6 (\cong \mathbb{C}^3)$, and let

$$A_j = \begin{pmatrix} \lambda_j & c_j \\ 0 & \lambda_j \end{pmatrix}, \quad \lambda_j \neq 0, \quad c_j \quad (j = 2, 3, 5, 6) \quad \text{are constant.}$$

These A_j commute with each other and define a Λ -action on $\mathbb{C}^3 \times GL(2, \mathbb{C})$ by

$$\sum_{j=1}^6 n_j \delta_j \cdot (z, g) = \left(z + \sum_{j=1}^6 n_j \delta_j, e^{-2\pi i n_6 z_3} e^{\pi i n_6 (n_6 - 1)} \prod_{k=2,3,5,6} A_j^{n_j} g \right).$$

⁴ Note that $K_1 \subset K^\perp$.

⁵ However, to reduce a GC structure L on M , the corresponding condition concerning $\operatorname{rk} \frac{L \cap K_{1\mathbb{C}}^\perp}{L \cap K_{1\mathbb{C}}}$ holds trivially.

The quotient of this action is the total space of a holomorphic principal $GL(2, \mathbb{C})$ -bundle $p : \mathbf{P} \rightarrow \mathbb{C}^3/\Lambda$.⁶ By $[(z, g)]$ denote the point in \mathbf{P} corresponding to (z, g) . Let the 2-torus T^2 act on \mathbf{P} by

$$(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \cdot [(z, g)] = [(z + \theta_1\delta_1 + \theta_2\delta_4, g)],$$

and on \mathbb{C}^3/Λ by

$$(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \cdot z = z + \theta_1\delta_1 + \theta_2\delta_4 \mod \Lambda.$$

Then p is T^2 -equivariant w.r.t. these two actions. As in Example 4.2, with $H = 0$ one can define a generalized principal $GL(2, \mathbb{C})$ -bundle $(\mathbf{P}, \mathbf{E}, \varphi)$. Let ψ be the extended T^2 -action on \mathbf{E} defined by

$$\psi(\alpha_1) = \partial_{x_1} + dx_2, \quad \psi(\alpha_2) = \partial_{y_2} + dy_1,$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis of \mathbb{R}^2 (the Lie algebra of T^2).

Obviously, the holomorphic structure in \mathbf{P} is T^2 -invariant. Equip \mathbf{P} with a suitable T^2 -invariant Higgs field, e.g. in terms of (z, g) ($|x_k|, |y_k| < 1$), $\Phi(z, g) = \text{Ad}_{g^{-1}}(\sum_{k=1}^3 \phi_k \partial_{z_k})$, where ϕ_k are three 2×2 constant matrices commuting with each other (It is easy to check that Φ does extend to a global Higgs field). Let \mathfrak{A} be the corresponding GH structure in \mathbf{P} . Then it can be checked that all the assumptions of Theorem 5.3 hold and \mathfrak{A} descends to a GH structure $\mathfrak{A}_r \subset \mathbf{E}_{r\mathbb{C}}$. Note that along the same line in [11], the reduced manifold $(\mathbb{C}^3/\Lambda)/T^2$ is a GC 4-manifold of type 1. Therefore \mathbf{P}_r with \mathfrak{A}_r is however not a co-Higgs principal bundle.

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⁶ \mathbf{P} is actually the tensor product of a flat principal $GL(2, \mathbb{C})$ -bundle and a holomorphic principal \mathbb{C}^* -bundle.