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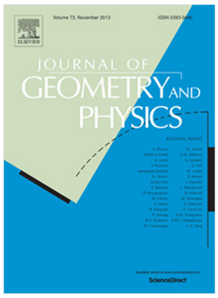

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THE DECAY OF THE $SU(2)$ YANG-MILLS FIELDS ON THE SCHWARZSCHILD BLACK HOLE FOR SPHERICALLY SYMMETRIC SMALL ENERGY INITIAL DATA

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Abstract. We prove uniform decay estimates in the entire exterior of the Schwarzschild black hole for gauge invariant norms on the Yang-Mills fields valued in the Lie algebra associated to the Lie group $SU(2)$. We assume that the initial data are spherically symmetric satisfying a certain Ansatz, and have small energy, which eliminates the stationary solutions which do not decay. In particular, there don't exist any Coulomb type solutions satisfying this Ansatz. We first prove a Morawetz type estimate for the Yang-Mills fields within this setting, using the Yang-Mills equations directly. We then adapt the proof constructed in previous work by the first author to show local energy decay and uniform decay of the L^∞ norm of the middle components in the entire exterior of the Schwarzschild black hole, including the event horizon.

1. Introduction

1.1. **General introduction.** We study the $SU(2)$ Yang-Mills equations on the Schwarzschild metric, with spherically symmetric initial data. The Yang-Mills fields then take values in $su(2)$, the Lie algebra associated to the Lie group. We consider initial data which are spherically symmetric satisfying a certain Ansatz (see e.g. [5], [20], [31]). This Ansatz, which is very frequently used in the literature, is usually called "purely magnetic", and it excludes in particular Coulomb type solutions. Nevertheless, other stationary solutions exist within this Ansatz (see e.g. [5] for a description of these stationary solutions). These stationary solutions are excluded by our small energy assumption. We prove for solutions of the Yang-Mills equations, generated from such an initial data, a Morawetz type estimate. We can then use the method of [19] to prove decay of the fields.

Global existence for Yang-Mills fields on $\mathbb{R}^{3+1}$ was shown by Eardley and Moncrief in a classical result, [12] and [13]. Their result was then generalized by Chruściel and Shatah to general globally hyperbolic curved space-times in [7]. Later, the first author wrote in [18] a new proof that improves the hypotheses of [7].

Our motivation in studying the Yang-Mills equations on the Schwarzschild geometry is twofold. On the one hand, Yang-Mills fields are important from a physical point of view and their study on an important physical space-time like the Schwarzschild metric is therefore an important problem. On the other hand, the Yang-Mills equations are linked to the Einstein equations via the Cartan formalism. One therefore generally hopes to get some insight into the Einstein equations by studying the Yang-Mills equations. This point becomes particularly important in the context of the nonlinear stability problem of these space-times.

The Schwarzschild metric is a solution of the Einstein vacuum equations. The Minkowski metric, that describes flat space-time, can be seen as a special case of the Schwarzschild space-time which itself is part of the Kerr family of solutions of the Einstein vacuum equations. The Schwarzschild family describes spherically symmetric black holes whereas the Kerr family describes rotating black holes, see [22] for a description of these space-times.

The stability of Minkowski space-time was first proved by Christodoulou and Klainerman in [6]. The proof of Christodoulou and Klainerman was later simplified by Lindblad-Rodnianski [25] using wave coordinates. In wave coordinates, the Einstein equations can be written as a covariant wave equation on the metric, with a non-linear term depending on the metric, propagating on the space-time with the metric generated from the solution of this equation. Very recently Hintz and Vasy proved nonlinear stability of the De Sitter Kerr space-time, see [24]. The De Sitter Kerr metric is a solution of the Einstein vacuum equations with positive cosmological constant. The equivalent conjecture for Kerr space-time has not been solved as of today. The conjecture of the nonlinear stability of the (De Sitter) Kerr solution has motivated a lot of work in recent years on proving dispersive properties for solutions of linear hyperbolic equations on (De Sitter) Schwarzschild and (De Sitter) Kerr space-time and a lot of progress has been made on this question. Dispersive properties of linear hyperbolic equations on the De Sitter Kerr metric have been an essential ingredient in the proof of nonlinear stability of the De Sitter Kerr metric by Hintz and Vasy.

Concerning the dispersive properties of the wave equation, we cite the papers of Andersson-Blue [2], Dyatlov [11], Dafermos-Rodnianski [9], Finster-Kamran-Smoller-Yau [14], [15], Tataru-Tohaneanu [29], and Vasy [30] as well as references therein for an overview. The wave equation is of course only a simplification of a linearization of the Einstein equations around the Kerr solution. Recently Dafermos, Holzegel and Rodnianski [10], and Finster, Smoller [16], made important progress in understanding the dispersive properties of the Teukolsky equation on the Schwarzschild and Kerr metric.

Yet, the free scalar wave equation does not admit stationary solutions on the exterior of the Schwarzschild black hole, whereas the Yang-Mills equations admit stationary solutions, which induces new impediment to the problem as one would need to find a way to exclude them in the proof of decay. Stationary solutions already appear for the Maxwell equations which can be understood as a linear version of the Yang-Mills equations (or more precisely as the case where the Lie group is abelian). The Maxwell equation has been studied by Andersson-Blue [3], Blue [4], Ghanem [19], Hintz-Vasy [23] and Sterbenz-Tataru [28]. In the Maxwell case the stationary solutions are Coulomb solutions and because of the linearity of the equation one can get rid of the problem by subtracting a suitable Coulomb solution. Coulomb solutions also exist for the Yang-Mills fields but not within our special purely magnetic Ansatz. Other stationary solutions appear within this Ansatz which are not present in the Maxwell case. There exists an energy gap between the zero curvature solution and these stationary solutions. Thanks to this energy gap, we can show that for small enough energy, the solutions satisfying our special Ansatz decay to zero in an appropriate weighted energy norm. Our results are consistent with what was observed numerically by Bizoń, Rostworowski and Zenginoglu, see [5].

Another important difference between the wave equation and the Maxwell equation is the non-scalar character of the latter. To show decay for the Maxwell fields, one in general uses the scalar wave equation verified by the middle components of the fields, which can be decoupled from the extreme components in the abelian case of the Maxwell equations, see for example the work of Andersson-Blue. However, this separation of the middle components from the extreme components cannot occur in the non-abelian case of the Yang-Mills equations. Therefore, the first author wrote in [19] a new proof of decay for the Maxwell fields which does not pass through the decoupling of the middle-components, associated to any Lie group, without any assumption of spherical symmetry on the initial data, assuming a certain Morawetz type estimate for the middle components. Later on, Andersson, Bäckdahl and Blue obtained a Morawetz type estimate for the derivatives of the extreme components of the Maxwell fields that doesn't rely on the study of the linear wave equation for the middle components, see [1]. Nevertheless, as far as we know, their Morawetz type estimate does not allow one to get decay rates for the Maxwell fields without at least passing through the decoupled scalar wave equation for the middle components. In this paper, we prove a Morawetz type estimate stronger than the one assumed in [19], without passing through the scalar wave equation for the middle components, under some assumptions on the initial data which eliminate the stationary solutions.

One of the advantages of our Ansatz is that it reduces the Yang-Mills equations themselves to a nonlinear scalar wave equation:

$$\partial_t^2 W - \partial_{r^*}^2 W + \frac{(1-\frac{2m}{r})}{r^2} W(W^2-1) = 0.$$

The above equation has two obvious stationary solutions $W_\pm = \pm 1$ which correspond to zero Yang-Mills curvature. However, we note that it doesn't seem to be appropriate to linearize around these stationary solutions, because this would require to control quantities which depend only on $W-1$ or on $W+1$, which are neither natural in this context (in particular, they are not gauge invariant) nor controlled by energy estimates.

The function $P = \frac{(1-\frac{2m}{r})}{r^2}$ that appears in front of the nonlinearity is exactly the same function that appears in front of $-\Delta_{S^2}$ when one studies the linear scalar wave equation without the spherical symmetry assumption. It is therefore not surprising that the difficulties that appear in trying to show a Morawetz type estimate are in some sense similar to the ones linked to trapping for the scalar wave equation, although no trapping appears here because of the spherical symmetry assumption. The solution of the problem is however quite different in this nonlinear setting. The proof of the Morawetz estimate relies on a nonlinear multiplier, see Section 2. Once the Morawetz estimate is established, we adapt the methods of [19] to the current situation. In fact, the paper [19] generalizes the arguments of Dafermos and Rodnianski for the free scalar wave equation, [8], to the Maxwell fields using the Maxwell equations directly, and thereby, it extends to the nonlinear case of the Yang-Mills fields.

While we need our specific Ansatz in order to show the Morawetz estimate, the decay estimates can all be understood as corollaries of a general Morawetz estimate. The arguments in Sections 3 and 4 are in fact more general than what is strictly needed here. This has the advantage, however, that a generalization of our result – for instance, dropping the spherical symmetry assumption – would be reduced to showing the Morawetz estimate in such a more general situation. For details of the calculations of the different tensors and energies, we refer the reader to [19].

1.2. **The exterior of the Schwarzschild black hole.** The exterior Schwarzschild spacetime is given by $\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_{r>2m} \times S^2$ equipped with the metric

$$\begin{aligned} g &= -(1-\frac{2m}{r})dt^2 + \frac{1}{(1-\frac{2m}{r})}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 \\ &= N(-dt^2 + dr^{*2}) + r^2 d\sigma^2 \end{aligned}$$

where

$$N = (1-\frac{2m}{r}) \tag{1.1}$$

$$r^* = r + 2m\log(r-2m) - 3m - 2m\log(m) \tag{1.2}$$

and $d\sigma^2$ is the usual volume element on the sphere. Note that

$$\frac{dr^*}{dr} = N^{-1}, \quad r^*(3m) = 0.$$

The coordinates t, r, θ, ϕ, are called Boyer-Lindquist coordinates. The singularity $r = 2m$ is a coordinate singularity and can be removed by changing coordinates, see [22]. m is the mass of the black hole. We will only be interested in the region outside the black hole, $r > 2m$. If we define,

$$\begin{aligned} v &= t + r^* \\ w &= t - r^* \end{aligned}$$

then, we have,

$$\begin{aligned} g &= -N dv dw + r^2 d\sigma^2 \\ &= -\frac{N}{2} dv \otimes dw - \frac{N}{2} dw \otimes dv + r^2 d\sigma^2 \end{aligned}$$

Let,

$$\frac{\hat{\partial}}{\partial w} = \frac{1}{N}\frac{\partial}{\partial w} \tag{1.3}$$

$$\frac{\hat{\partial}}{\partial v} = \frac{\partial}{\partial v} \tag{1.4}$$

$$\frac{\hat{\partial}}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta} \tag{1.5}$$

$$\frac{\hat{\partial}}{\partial \phi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{1.6}$$

We will also consider the extended Schwarzschild solution. It is obtained by making Kruskal's choice of coordinates:

$$v^{'} = \exp(\frac{v}{4m}) \tag{1.7}$$

$$w^{'} = -\exp(-\frac{w}{4m}) \tag{1.8}$$

and then define,

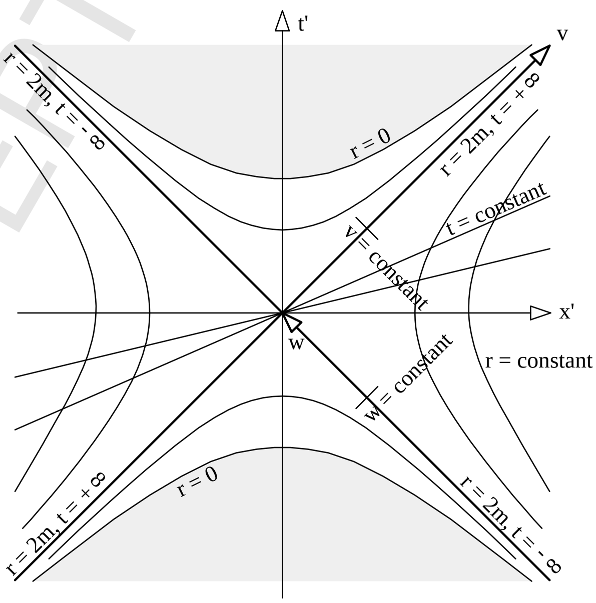
$$t^{'} = \frac{v^{'} + w^{'}}{2} \tag{1.9}$$

$$x^{'} = \frac{v^{'} - w^{'}}{2} \tag{1.10}$$

We get

$$g = \frac{16m^2}{r}\exp(\frac{-r}{2m})(-(dt^{'})^2 + (dx^{'})^2) + r^2(t^{'}, x^{'})d\sigma^2. \tag{1.11}$$

The following figure shows the Kruskal extension of the Schwarzschild metric:



1.3. **The spherically symmetric $SU(2)$ Yang-Mills equations on the Schwarzschild metric.** Let $G = SU(2)$, the real Lie group of 2×2 unitary matrices of determinant 1. The Lie algebra associated to G is $su(2)$, the antihermitian traceless 2×2 matrices. Let τ_j, $j \in \{1, 2, 3\}$, be the following real basis of $su(2)$:

$$\tau_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that

$$[\tau_1, \tau_2] = \tau_3, \quad [\tau_3, \tau_1] = \tau_2, \quad [\tau_2, \tau_3] = \tau_1.$$

We are looking for a connection A, that is a one form with values in the Lie algebra $su(2)$ associated to the Lie group $SU(2)$, which satisfies the Yang-Mills equations which are:

$$\mathbf{D}^{(A)}_{\alpha} F^{\alpha\beta} \equiv \nabla_{\alpha} F^{\alpha\beta} + [A_{\alpha}, F^{\alpha\beta}] = 0, \tag{1.12}$$

where $[.,.]$ is the Lie bracket and $F_{\alpha\beta}$ is the Yang-Mills curvature given by

$$F_{\alpha\beta} = \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha} + [A_{\alpha}, A_{\beta}], \tag{1.13}$$

and where we have used the Einstein raising indices convention with respect to the Schwarzschild metric. We also have the Bianchi identities which are always satisfied in view of the symmetries of the Riemann tensor and the Jacobi identity for the Lie bracket:

$$\mathbf{D}^{(A)}_{\alpha} F_{\mu\nu} + \mathbf{D}^{(A)}_{\mu} F_{\nu\alpha} + \mathbf{D}^{(A)}_{\nu} F_{\alpha\mu} = 0. \tag{1.14}$$

The Cauchy problem for the Yang-Mills equations formulates as the following: given a Cauchy hypersurface Σ in M, and a $\mathcal{G}$-valued one form A_{μ} on Σ, and a $\mathcal{G}$-valued one form E_{μ} on Σ satisfying

$$\left.\begin{array}{rcl} E_t & = & 0, \\ \mathbf{D}^{(A)}_{\mu} E^{\mu} & = & 0 \end{array}\right\} \tag{1.15}$$

we are looking for a $\mathcal{G}$-valued two form $F_{\mu\nu}$ satisfying the Yang-Mills equations such that once $F_{\mu\nu}$ restricted to Σ we have

$$F_{\mu t} = E_{\mu} \tag{1.16}$$

and such that $F_{\mu\nu}$ corresponds to the curvature derived from the Yang-Mills potential A_{μ}, i.e. given by (1.13). Equations (1.15) are the Yang-Mills constraints equations on the initial data.

Any spherically symmetric Yang-Mills potential can be written in the following form after applying a gauge transformation, see [17], [20] and [31],

$$\begin{aligned} A &= [-W_1(t,r)\tau_1 - W_2(t,r)\tau_2]d\theta + [W_2(t,r)\sin(\theta)\tau_1 - W_1(t,r)\sin(\theta)\tau_2]d\phi \\ &+ \cos(\theta)\tau_3 d\phi + A_0(t,r)\tau_3 dt + A_1(t,r)\tau_3 dr. \end{aligned} \tag{1.17}$$

where $A_0(t,r)$, $A_1(t,r)$, $W_1(t,r)$, $W_2(t,r)$ are arbitrary real functions. We then have

$$\left.\begin{array}{rcl}
F_{\theta r} = F_{\mu\nu}(\partial_\theta)^\mu(\partial_r)^\nu &=& (\partial_r W_1 - W_2 A_1)\tau_1 + (\partial_r W_2 + W_1 A_1)\tau_2 \\
F_{\theta t} &=& (\partial_t W_1 - W_2 A_0)\tau_1 + (\partial_t W_2 + W_1 A_0)\tau_2 \\
F_{\phi r} &=& (-\partial_r W_2 \sin(\theta) - W_1 A_1 \sin(\theta))\tau_1 \\
&+& (\partial_r W_1 \sin(\theta) - W_2 A_1 \sin(\theta))\tau_2 \\
F_{\phi t} &=& (-\partial_t W_2 \sin(\theta) + A_0 W_1 \sin(\theta))\tau_1 \\
&+& (\partial_t W_1 \sin(\theta) + A_0 W_2 \sin(\theta))\tau_2 \\
F_{tr} &=& (\partial_t A_1 - \partial_r A_0)\tau_3 \\
F_{\theta\phi} &=& (W_1^2 + W_2^2 - 1)\sin(\theta)\tau_3
\end{array}\right\} \tag{1.18}$$

The Yang-Mills system (1.12) can then be written as

$$\begin{aligned}
0 &= -\frac{1}{r^2}W_1[1-(W_1^2+W_2^2)] + N(-\partial_r^2 W_1 + \partial_r(W_2A_1)) - \frac{1}{N}(-\partial_t^2 W_1 + \partial_t(W_2A_0)) \\
&\quad + \frac{2m}{r^2}(-\partial_r W_1 + W_2 A_1) + NA_1(\partial_r W_2 + W_1 A_1) - \frac{1}{N}A_0(\partial_t W_2 + W_1 A_0) \qquad (1.19) \\
0 &= -\frac{1}{r^2}W_2[1-(W_1^2+W_2^2)] - N(\partial_r^2 W_2 + \partial_r(W_1A_1)) + \frac{1}{N}(\partial_t^2 W_2 + \partial_t(W_1A_0)) \\
&\quad + \frac{2m}{r^2}(-\partial_r W_2 - W_1 A_1) + NA_1(-\partial_r W_1 + W_2 A_1) \\
&\quad - \frac{1}{N}A_0(-\partial_t W_1 + W_2 A_0) \qquad (1.20) \\
0 &= -\frac{2}{r^2}W_1(\partial_r W_2 + W_1 A_1) + \frac{2}{r^2}W_2(\partial_r W_1 - W_2 A_1) \\
&\quad - \frac{1}{N}\partial_t(\partial_t A_1 - \partial_r A_0) \qquad (1.21) \\
0 &= -\frac{2}{r^2}W_1(\partial_t W_2) + \frac{2}{r^2}W_2(\partial_t W_1) - \frac{2N}{r}(\partial_t A_1 - \partial_r A_0) \\
&\quad - N\partial_r(\partial_t A_1 - \partial_r A_0) \qquad (1.22)
\end{aligned}$$

1.4. **The initial data.** We look at initial data prescribed on $t = t_0$ where there exists a gauge transformation such that once applied on the initial data, the potential A can be written for some $c \in \mathbb{R}$ in this gauge as

$$\left.\begin{array}{rcl}
A_t(t=t_0) &=& 0, \\
A_r(t=t_0) &=& 0, \\
A_\theta(t=t_0) &=& -W_1(t_0,r)(\tau_1 + c\tau_2), \\
A_\phi(t=t_0) &=& W_1(t_0,r)(c\sin(\theta)\tau_1 - \sin(\theta)\tau_2) + \cos(\theta)\tau_3,
\end{array}\right\} \tag{1.23}$$

and, we are given in this gauge the following one form E_μ on $t = t_0$:

$$\left.\begin{array}{rcl}
E_\theta(t=t_0) &=& F_{\theta t}(t=t_0) = (\partial_t W_1)(\tau_1 + c\tau_2), \\
E_\phi(t=t_0) &=& F_{\phi t}(t=t_0) = (\partial_t W_1)(-c\sin(\theta)\tau_1 + \sin(\theta)\tau_2) \\
E_r(t=t_0) &=& F_{rt}(t=t_0) = (\partial_t A_1 - \partial_r A_0)\tau_3 = 0, \\
E_t(t=t_0) &=& F_{tt}(t=t_0) = 0.
\end{array}\right\} \tag{1.24}$$

Notice that with this Ansatz the constraint equations (1.15) are automatically fulfilled

$$\begin{aligned}&(\mathbf{D}^{(A)^\theta}E_\theta+\mathbf{D}^{(A)^\phi}E_\phi+\mathbf{D}^{(A)^r}E_r)(t=t_0)\\ =\quad& c\left(-\frac{2}{r^2}W_1(\partial_tW_1)(t_0,r)\tau_3+\frac{2}{r^2}W_1(\partial_tW_1)(t_0,r)\tau_3\right)=0. \qquad (1.25)\end{aligned}$$

Remark 1.1. The principal restriction is $A_t(t=t_0)=A_r(t=t_0)=0$. If we want this to be conserved, then by restricting (1.21) to $t=t_0$ we obtain:

$$W_1\partial_rW_2=W_2\partial_rW_1. \qquad (1.26)$$

Suppose that W_1 and W_2 have no zeros. Then (1.26) gives

$$\partial_r\ln W_2=\partial_r\ln W_1$$

and thus $W_2=cW_1$ for some $c\in\mathbb{R}$.

Now suppose that W is solution of

$$\ddot W-W''+PW(W^2-1)=0, \qquad (1.27)$$

where

$$\dot{}\equiv\partial_t \quad\text{and}\quad {}'\equiv\partial_{r^*}$$

and where

$$P\quad\equiv\quad\frac{N}{r^2}. \qquad (1.28)$$

Then $W_1=\frac{1}{\sqrt{1+c^2}}W$, $W_2=\frac{c}{\sqrt{1+c^2}}W$, $A_0=A_1=0$ are solutions of (1.19) to (1.22). By the uniqueness of the solutions of the Yang-Mills equation F defined by (1.18) is the solution of the Yang-Mills equation. For our special Ansatz the analysis of the Yang-Mills equation therefore reduces to the analysis of (1.27) and this corresponds to the case $c=0$. As all the analysis reduces to this case we will suppose in the following $c=0$ and put $W:=W_1$. With this special Ansatz the data A_μ, E_μ is equivalent to the data $W(t=t_0)$, $\partial_tW(t=t_0)$ and the equation we have to study is (1.27).

1.5. **The fundamental scalar wave equation.** We are looking at solutions such that once the above mentioned gauge transformation is applied the potential A takes the form (1.23) with $c=0$. We recall here the expressions of the components of A and F with our choices:

$$\left.\begin{array}{rcl}A_t&=&0,\\ A_{r^*}&=&0,\\ A_\theta&=&-W\tau_1,\\ A_\phi&=&-W\sin(\theta)\tau_2+\cos(\theta)\tau_3,\end{array}\right\}$$

and

$$\left.\begin{array}{rcl}F_{\theta r^*}&=&W'\tau_1,\\ F_{\theta t}&=&\dot W\tau_1,\\ F_{\phi r^*}&=&W'\sin(\theta)\tau_2,\\ F_{\phi t}&=&\dot W\sin(\theta)\tau_2,\\ F_{tr^*}&=&0,\\ F_{\theta\phi}&=&(W^2-1)\sin(\theta)\tau_3.\end{array}\right\}$$

The principal object of study is now the scalar wave equation

$$\left.\begin{array}{rcl}\ddot{W}-W''+PW(W^2-1) & = & 0,\\ W(0) & = & W_0,\\ \partial_t W(0) & = & W_1.\end{array}\right\}\tag{1.29}$$

It is easy to check that the following energy is conserved, see also [19],

$$\mathcal{E}(W,\dot{W})=\int \dot{W}^2+(W')^2+\frac{P}{2}(W^2-1)^2dr^*.$$

We note by $\dot{H}^k=\dot{H}^k(\mathbb{R},dr^*)$ and $H^k=H^k(\mathbb{R},dr^*)$, the homogeneous and inhomogeneous Sobolev spaces of order k, respectively.

Definition 1.1. (1) We define the spaces L^4_P, resp. L^2_P, as the completion of $C_0^\infty(\mathbb{R})$ for the norm

$$\|v\|^4_{L^4_P}:=\int P|v|^4dr^* \quad \text{resp.} \quad \|v\|^2_{L^2_P}:=\int P|v|^2dr^*.\tag{1.30}$$

(2) We also define for $1\le k\le 2$ the space $\mathcal{H}^k$ as the completion of $C_0^\infty(\mathbb{R})$ for the norm

$$\|u\|^2_{\mathcal{H}^k}=\|u\|^2_{\dot{H}^k}+\|u\|^2_{L^4_P}.\tag{1.31}$$

We note that $\mathcal{H}^k$ is a Banach space which contains all constant functions.

Theorem 1. *Let $(W_0,W_1)\in\mathcal{H}^2\times H^1$. Then there exists a unique strong solution of (1.29) with*

$$\begin{array}{rcl}W & \in & C^1([0,\infty);\mathcal{H}^1)\cap C([0,\infty);\mathcal{H}^2),\\ \partial_t W & \in & C^1([0,\infty);L^2)\cap C([0,\infty);H^1),\\ \sqrt{P}(W^2-1) & \in & C^1([0,\infty);L^2)\cap C([0,\infty);H^1).\end{array}$$

We will prove the theorem in Appendix A. In this appendix, we will also show that $\mathcal{H}^1\times L^2$ is exactly the space of finite energy solutions. Let $\Omega\equiv\{(t',x')\in\mathbb{R}^2|x'>|t'|\}$. We can reformulate the above theorem in the following way

Corollary 1.1. *We suppose that the initial data for the Yang-Mills equations is given after suitable gauge transformation by*

$$\left.\begin{array}{rcl}A_t(0) & = & A_r(0)=0,\\ A_\theta(0) & = & -W_0\tau_1,\\ A_\phi(0) & = & -W_0\sin\theta\tau_2+\cos\theta\tau_3,\\ E_\theta(0) & = & W_1\tau_1,\\ E_\phi(0) & = & W_1\sin\theta\tau_2,\\ E_r(0) & = & E_t(0)=0\end{array}\right\}$$

with $(W_0, W_1) \in \mathcal{H}^2 \times H^1$*. Then, the Yang-Mills equation* (1.12) *admits a unique solution* F *with*

$$\begin{aligned} F_{\theta r^*},\ \frac{1}{\sin\theta}F_{\phi r^*},\ F_{\theta t},\ \frac{1}{\sin\theta}F_{\phi t}, \sqrt{P}\frac{1}{\sin\theta}F_{\theta\phi} &\in C^1([0,\infty);L^2)\cap C([0,\infty);H^1),\\ \forall X,Y \in \{\frac{\hat\partial}{\partial w},\frac{\hat\partial}{\partial v},\frac{\hat\partial}{\partial \theta},\frac{\hat\partial}{\partial \phi}\},\quad F_{\mu\nu}X^\mu Y^\nu(t',x') &\in H^1_{loc}(\Omega),\\ \forall X,Y \in \{\frac{\hat\partial}{\partial w},\frac{\hat\partial}{\partial v},\frac{\hat\partial}{\partial \theta},\frac{\hat\partial}{\partial \phi}\},\quad \lim_{r^*\to\infty} F_{\mu\nu}X^\mu Y^\nu(t,r^*) &= 0. \end{aligned}$$

Proof. We only have to check the last two statements which follow from

$$\begin{aligned} F_{\hat v\hat\theta} &= -\frac{2}{r}\partial_v W\tau_1,\ F_{\hat w\hat\theta} = -\frac{2}{Nr}\partial_w W\tau_1,\ F_{\hat v\hat\phi} = -\frac{2}{r}\partial_v W\tau_2,\\ F_{\hat w\hat\phi} &= -\frac{2}{Nr}\partial_w W\tau_2,\ F_{\hat w\hat v} = 0,\ F_{\hat\theta\hat\phi} = \frac{W^2-1}{r^2}\tau_3 \end{aligned}$$

and the Sobolev embedding $H^1(\mathbb{R}) \subset C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the set of bounded continuous functions equipped with the L^∞ norm. □

Remark 1.2.
(1) Note that our functional setting doesn't impose any specific asymptotic behaviour on the solutions.
(2) Strictly speaking, the initial data are functions on $\mathbb{R}\times S^2$, and W_0 has to be in $\mathcal{H}^2 \otimes L^2(S^2)$. $F_{\theta r^*}$ takes by then values in $H^1 \otimes L^2(S^2)$ etc. We will, in the following, quite often ignore the $L^2(S^2)$ factor which is constant.
(3) Global existence for Yang-Mills fields is of course known in a more general context, see e.g. [18]. Nevertheless, we prefer to give here a theorem for our special Ansatz. The advantage is that the formulation of this theorem and of its proof, is particularly simpler in this special case.
(4) The regularity results imply that we can apply the divergence theorem in the exterior of the black hole. Under some regularity assumptions on the initial data, the result of [18] shows that the traces of suitable normalized components exist up to the horizon. The existence of the traces at the horizon at the regularity level of the above Corollary, follows from our estimates in Section 4. In particular, we can then apply the divergence theorem up to the horizon.

1.6. **Stationary solutions.** Note that $W_\pm = \pm 1$ and $W_\infty = 0$ are obvious stationary solutions of (1.27). The solutions $W_\pm$ correspond to zero Yang-Mills curvature. Other stationary solutions are given by the following theorem, which is implicit in the paper [5] of P. Bizoń, A. Rostworowski and A. Zenginoglu.

Theorem 2. *There exist a decreasing sequence* $\{a_n\}_{n\in\mathbb{N}^{\geq 1}}$, $0 < ... < a_n < a_{n-1} < ... < a_1 = \frac{1+\sqrt{3}}{3\sqrt{3}+5}$*, and* W_n *smooth stationary solutions of* (1.27)*, with*

$$-1 \leq W_n \leq 1,\quad \lim_{x\to-\infty} W_n(x) = a_n,\quad \lim_{x\to\infty} W_n(x) = (-1)^n.$$

For each $n \in \mathbb{N}^{\geq 1}$*, the solution* W_n *has exactly* n *zeros.*

It is implicitly stated in [5], that there is an energy gap between the $W_\pm = \pm 1$ solutions and the next stationary solutions, a statement which is confirmed by our analysis, where we also show that the zero Yang-Mills curvature solution is stable under a small perturbation. In [21], Häfner and Huneau show that all the solutions constructed in Theorem 2 are nonlinearly instable. The paper [21] contains also a detailed proof of Theorem 2.

1.7. **Energy estimates.** Let $< , >$ be an Ad-invariant scalar product on the Lie algebra $su(2)$, i.e. for any $su(2)$-valued tensors A, B, and C, we have

$$< [A, B], C > = < A, [B, C] >$$

where the Lie bracket $[., .]$ can be defined as corresponding to commutation of matrices.

Let F be the Yang-Mills curvature solution to the Yang-Mills equations. We consider the Yang-Mills energy-momentum tensor, see [18],

$$T_{\mu\nu}(F) = < F_{\mu\beta}, F_{\nu}{}^{\beta} > - \frac{1}{4}\mathbf{g}_{\mu\nu} < F_{\alpha\beta}, F^{\alpha\beta} >$$

We have (see [19, page 17]):

$$\nabla^{\nu} T_{\mu\nu} = 0. \tag{1.32}$$

Considering now a vector field X^{ν} we let

$$J_{\mu}(X) = X^{\nu} T_{\mu\nu} = T_{\mu X}.$$

Using (1.32) we obtain

$$\nabla^{\mu} J_{\mu}(X) = \pi^{\mu\nu}(X) T_{\mu\nu},$$

where $\pi^{\mu\nu}$ is the deformation tensor defined by

$$\pi^{\mu\nu}(X) = \frac{1}{2}(\nabla^{\mu} X^{\nu} + \nabla^{\nu} X^{\mu}). \tag{1.33}$$

Applying the divergence theorem on $J_{\mu}(X)$ in the region B bounded to the past by Σ_{t_1} and to the future by Σ_{t_2}, we obtain:

$$\begin{aligned} \int_B \pi^{\mu\nu}(X) T_{\mu\nu} dV_B &= \int_{\Sigma_{t_1}} J_{\mu}(X) n^{\mu} dV_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} J_{\mu}(X) n^{\mu} dV_{\Sigma_{t_2}} \\ &= E_F^{(X)}(t_1) - E_F^{(X)}(t_2) \end{aligned} \tag{1.34}$$

where n^{μ} are the unit normal to the hypersurfaces Σ_t, and

$$E_F^{(X)}(t) = \int_{\Sigma_t} J_{\mu}(X) n^{\mu} dV_{\Sigma_t}. \tag{1.35}$$

We refer to (1.34) as the energy identity associated to the vector field X as multiplier. If X is Killing, then the deformation tensor $\pi^{\mu\nu}(X)$ is zero and (1.34) gives a conserved energy. In the Schwarzschild case, the vector field ∂_t is Killing and we obtain the conserved energy:

$$\begin{aligned} E_F^{(\frac{\partial}{\partial t})} &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 2[N^2(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2 \\ &+ N(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)].r^2 \sin(\theta) d\theta d\phi dr^*. \end{aligned}$$

where $|\,.\,|$ is the norm associated to the Ad-invariant scalar product $<\,,\,>$. Note that in our special case the energy simply reads

$$E_F^{(\frac{\partial}{\partial t})} = \int_{\mathbb{R}}\int_{S^2} \dot{W}^2 + (W')^2 + \frac{P}{2}(W^2-1)^2 dr^* d\sigma^2.$$

We will often apply the divergence theorem to vector fields of the form

$$X = X^w(v,w)\frac{\partial}{\partial w} + X^v(v,w)\frac{\partial}{\partial v}$$

Then we have, see [19, page 113],

$$\begin{aligned} & \pi^{\alpha\beta}(X)T_{\alpha\beta}(F) \\ = & (|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2)(-2N\partial_v X^w) + (|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2)(\frac{-2}{N}\partial_w X^v) \\ & +(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)(-2[\partial_v X^v + \partial_w X^w + \frac{(3\mu-2)}{2r}(X^v - X^w)]) \end{aligned}$$

where

$$\mu \equiv \frac{2m}{r}. \tag{1.36}$$

1.8. **Main result.** Because of the existence of stationary solutions with finite conserved energy, other than the zero curvature solution, there can't be any Morawetz estimate that holds for all finite energy solutions. Nevertheless, we obtain a Morawetz estimate that holds for initial data with small enough energy. More precisely, we will prove the following theorem :

Theorem 3. *There exists $\epsilon > 0$ with the following property. For all solutions F of (1.12) with initial data as in Corollary 1.1 and such that $E_F^{(\frac{\partial}{\partial t})}(t=t_0) < \epsilon$, we have for all t,*

$$\begin{aligned} & \int_{t=t_0}^{t}\int_{r^*=-\infty}^{\infty}\int_{S^2} [PN^2(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + \frac{N}{r}(|F_{\hat{v}\hat{w}}|^2 + |F_{\hat{\phi}\hat{\theta}}|^2) + P(|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2)]r^2 d\sigma^2 dr^* dt \\ = & \int_{t=t_0}^{t}\int_{r^*=-\infty}^{\infty}\int_{S^2} P\dot{W}^2 + P(W')^2 + \frac{P}{2r}(W^2-1)^2 dr^* d\sigma^2 dt \\ \lesssim & E_F^{(\frac{\partial}{\partial t})}(t=t_0) \end{aligned} \tag{1.37}$$

and, the local energy decays as,

$$E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq r_2^*) \leq C(r_1^*, r_2^*)\frac{(E_F^{(\frac{\partial}{\partial t})}(t_0) + E_F^{(K)}(t_0))}{t}. \tag{1.38}$$

Furthermore, under some additional regularity assumptions on the initial data, for all $r \geq 2m$ (including the event horizon), we have,

$$|F_{\hat{\theta}\hat{\phi}}| + |F_{\hat{v}\hat{w}}| = |\frac{W^2-1}{r^2}|(v,w,\theta,\phi) \leq \frac{C}{(\max\{1,v\})^{\frac{1}{2}}}. \tag{1.39}$$

More precisely, let $R > 2m$. Then we have in the region $r \geq R > 2m$ (away from the event horizon),

$$|F_{\hat{\theta}\hat{\phi}}| + |F_{\hat{v}\hat{w}}| = |\frac{W^2-1}{r^2}|(v,w,\theta,\phi) \lesssim \frac{E_1}{(1+|v|)^{\frac{1}{2}}}, \tag{1.40}$$

$$|F_{\hat{\theta}\hat{\phi}}| + |F_{\hat{v}\hat{w}}| = |\frac{W^2-1}{r^2}|(v,w,\theta,\phi) \quad \lesssim \quad \frac{E_1}{(1+|w|)^{\frac{1}{2}}}, \tag{1.41}$$

and in the region $2m \leq r \leq R$ (near the event horizon),

$$|F_{\hat{\theta}\hat{\phi}}| + |F_{\hat{v}\hat{w}}| \lesssim |W^2-1|(v,w,\theta,\phi) \lesssim \frac{E_2}{(\max\{1,v\})^{\frac{1}{2}}}, \tag{1.42}$$

with

$$\begin{aligned} E_1 &= [E_F^{(\frac{\partial}{\partial t})}(t=t_0) + E_F^{(K)}(t=t_0)]^{\frac{1}{2}}, \\ E_2 &= [(E_F^{(\frac{\partial}{\partial t})}(t=t_0) + E_F^{(K)}(t=t_0)+1)^2 + E_F^{\#(\frac{\partial}{\partial t})}(t=t_0)]^{\frac{1}{2}}, \end{aligned}$$

where,

$$\begin{aligned} E_F^{(\frac{\partial}{\partial t})}(t=t_0) &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 2[N^2(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2 \\ &\quad + N(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)].r^2\sin(\theta)d\theta d\phi dr^*, \\ E_F^{(K)}(t_i) &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (w^2N^2[|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2] + v^2[|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2] \\ &\quad + (\omega^2+v^2)N[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2])r^2\sin(\theta)d\theta d\phi dr^*, \\ E_F^{\#(\frac{\partial}{\partial t})}(t=t_0) &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} [N(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + (|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2) \\ &\quad + (|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)].r^2\sin(\theta)d\theta d\phi dr^*(t=t_0). \end{aligned}$$

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2. Proof of the Morawetz estimate (1.37)

Recall that $\mu = \frac{2m}{r}$, $\quad N = 1-\mu$, $\quad P = \frac{N}{r^2}$. We have

$$P' = PV, \quad \text{where} \quad V = \frac{3\mu-2}{r}.$$

We compute

$$V' = N\partial_r\left(\frac{3\mu-2}{r}\right) = \frac{2N}{r^2}(1-3\mu) = 2P(1-3\mu).$$

The following proposition will be useful

Proposition 2.1.

$$\|\sqrt{P}(W^2-1)\|_{L^\infty(\mathbb{R})} \lesssim \sqrt{E_F^{(\frac{\partial}{\partial t})}} + E_F^{(\frac{\partial}{\partial t})}$$

Proof. We use the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$:

$$\begin{aligned}\|\sqrt{P}(W^2-1)\|^2_{L^\infty(\mathbb{R})} &\lesssim \int P(W^2-1)^2 + \int ((\sqrt{P})')^2(W^2-1)^2 + \int P((W^2-1)')^2 \\ &= E_F^{(\frac{\partial}{\partial t})} + \frac{1}{4}\int PV^2(W^2-1)^2 + \int 4PW^2(W')^2 \\ &\lesssim (1+\|\sqrt{P}(W^2-1)\|_{L^\infty(\mathbb{R})})E_F^{(\frac{\partial}{\partial t})}\end{aligned}$$

The proposition follows. □

2.1. **A first multiplier.** Let

$$G = f(r^*)\frac{\partial}{\partial r^*} \quad = \quad -f(r^*)\frac{\partial}{\partial w} + f(r^*)\frac{\partial}{\partial v}$$

where

$$f = \frac{1}{3m} - \frac{1}{r} \tag{2.1}$$

We have

$$f' = \frac{N}{r^2} = P. \tag{2.2}$$

We obtain (see [19, (49)]):

$$\begin{aligned}T^{\alpha\beta}(F)\pi_{\alpha\beta}(G)dVol &= T^{\alpha\beta}(F)\pi_{\alpha\beta}(G)Nr^2dr^*d\sigma^2dt \\ &= [(N^2|F_{\hat{w}\hat{\theta}}|^2 + N^2|F_{\hat{w}\hat{\phi}}|^2 + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2)f' \\ &\quad -2N(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)(f' + \frac{f}{r}(3\mu-2))]r^2dr^*d\sigma^2dt \\ &= [\frac{1}{2}P\dot{W}^2 + \frac{1}{2}P(W')^2 - \frac{1}{4}P(Vf+f')(W^2-1)^2]dr^*d\sigma^2dt\end{aligned}$$

and (see [19, (52)]):

$$\begin{aligned}E_F^{(G)}(t) &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\mathcal{S}^2} -f[\frac{1}{r^2} < F_{t\theta}, F_{r^*\theta} > + \frac{1}{r^2\sin^2\theta} < F_{t\phi}, F_{r^*\phi} >]r^2d\sigma^2dr^* \\ &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\mathcal{S}^2} -fN[\frac{1}{r^2} < F_{t\theta}, F_{r\theta} > + \frac{1}{r^2\sin^2\theta} < F_{t\phi}, F_{r\phi} >]r^2d\sigma^2dr^* \\ &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\mathcal{S}^2} -fP(\partial_t W)(\partial_r W)r^2d\sigma^2dr^* \\ &= \int_{r^*=-\infty}^{r^*=\infty}\int_{\mathcal{S}^2} -f\dot{W}W'd\sigma^2dr^* \end{aligned} \tag{2.3}$$

Applying the divergence theorem in the region $t_0 \leq t' \leq t$ and differentiating with respect to t we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}}\dot{W}fW'dr^* - \frac{1}{4}\int_{\mathbb{R}}P(Vf+P)(W^2-1)^2dr^* + \frac{1}{2}\int_{\mathbb{R}}P(W')^2dr^* + \frac{1}{2}\int_{\mathbb{R}}P\dot{W}^2dr^* = 0.$$

Note that

$$-Vf = 2\frac{(r-3m)^2}{3mr^3} \geq 0.$$

2.2. **A second multiplier.** Let h be a smooth function with compact support. We multiply the Yang-Mills equation (1.27) by $hW(W^2-1)$ and integrate by parts. We obtain

$$0 = \int_{\mathbb{R}} \ddot{W}hW(W^2-1)dr^* - \int_{\mathbb{R}} W''hW(W^2-1)dr^* + \int_{\mathbb{R}} PhW^2(W^2-1)^2dr^*.$$

(1) First term. We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \dot{W}hW(W^2-1)dr^* &= \int_{\mathbb{R}} \ddot{W}hW(W^2-1)dr^* + \int_{\mathbb{R}} \dot{W}^2h(W^2-1)dr^* \\ &\quad +2\int_{\mathbb{R}} \dot{W}^2hW^2dr^*.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} \ddot{W}hW(W^2-1)dr^* &= \frac{d}{dt}\int_{\mathbb{R}} \dot{W}hW(W^2-1)dr^* - \int_{\mathbb{R}} h\dot{W}^2(W^2-1)dr^* \\ &\quad -2\int_{\mathbb{R}} h\dot{W}^2W^2dr^*.\end{aligned}$$

(2) Second term. We have

$$\begin{aligned}-\int_{\mathbb{R}} W''hW(W^2-1)dr^* &= \int_{\mathbb{R}} W'h'W(W^2-1)dr^* + \int_{\mathbb{R}} (W')^2h(W^2-1)dr^* \\ &\quad +2\int_{\mathbb{R}} h(W')^2W^2dr^*.\end{aligned}$$

We have

$$\int_{\mathbb{R}} W'h'W(W^2-1)dr^* = \frac{1}{4}\int_{\mathbb{R}} h'\frac{d}{dr^*}(W^2-1)^2dr^* = -\frac{1}{4}\int_{\mathbb{R}} h''(W^2-1)^2dr^*.$$

Thus

$$\begin{aligned}-\int_{\mathbb{R}} W''hW(W^2-1)dr^* &= -\frac{1}{4}\int_{\mathbb{R}} h''(W^2-1)^2dr^* \\ &\quad +\int_{\mathbb{R}} (W')^2h(W^2-1)dr^* + 2\int_{\mathbb{R}} h(W')^2W^2dr^*.\end{aligned}$$

(3) Third term. We have

$$\int_{\mathbb{R}} PhW^2(W^2-1)^2dr^* = \int_{\mathbb{R}} Ph(W^2-1)^2dr^* + \int_{\mathbb{R}} Ph(W^2-1)^3dr^*.$$

(4) Summarizing we have

$$\frac{d}{dt}\int_{\mathbb{R}} \dot{W}hW(W^2-1)dr^* - \int_{\mathbb{R}} h\dot{W}^2(W^2-1)dr^* - 2\int_{\mathbb{R}} h\dot{W}^2W^2dr^*$$
$$-\frac{1}{4}\int_{\mathbb{R}} h''(W^2-1)^2dr^* + \int_{\mathbb{R}}(W')^2h(W^2-1)dr^* + 2\int_{\mathbb{R}} h(W')^2W^2dr^*$$
$$+\int_{\mathbb{R}} Ph(W^2-1)^2dr^* + \int_{\mathbb{R}} Ph(W^2-1)^3dr^*$$
$$+\frac{1}{2}\int_{\mathbb{R}} P\dot{W}^2dr^* + \frac{1}{2}\int_{\mathbb{R}} P(W')^2dr^* - \frac{1}{4}\int_{\mathbb{R}} P(Vf+f')(W^2-1)^2dr^*$$
$$+\frac{d}{dt}\int_{\mathbb{R}} \dot{W}fW'dr^* = 0 \tag{2.4}$$

2.3. **Choice of h.** Let $\chi(r^*) \in C_0^\infty((-2,2))$, $\chi \geq 0$, $\chi(r^*) \equiv 1$ on $[-1,1]$. For a smooth function $\psi(r^*)$ we put $\psi_a(r^*) = \psi\left(\frac{r^*}{a}\right)$. We now choose

$$h(r^*) = \frac{1}{4}(1-\delta)P\chi_a(r^*). \tag{2.5}$$

Here $\delta > 0$ will be chosen later on. We compute

$$h' = \frac{1}{4}(1-\delta)P'\chi_a + \frac{1}{4a}(1-\delta)P\chi_a', \tag{2.6}$$
$$h'' = \frac{1}{4}(1-\delta)P''\chi_a + \frac{1}{2a}(1-\delta)P'\chi_a' + \frac{(1-\delta)}{4a^2}P\chi_a''. \tag{2.7}$$

Here $\chi_a' = (\chi')_a$, $\chi_a'' = (\chi'')_a$ and thus there exists a constant $C > 0$ such that

$$\forall r^* \in \mathbb{R},\, a > 0, \quad |\chi_a'(r^*)| \leq C, \quad |\chi_a''(r^*)| \leq C.$$

We have

$$P' = PV,\, P'' = PV^2 + PV' = PV^2 + 2P\frac{N}{r^2}(1-3\mu) = PV^2 + 2P^2(1-3\mu).$$

2.4. **Estimate of the nonlinear contribution.** We have

$$-\frac{1}{4}\int h''(W^2-1)^2 = \frac{1-\delta}{16}\int P(\chi_a(-V^2-V') - \frac{2}{a}V\chi_a' - \frac{1}{a^2}\chi_a'')(W^2-1)^2.$$

Apart from an error term which is small when the energy is small and which will be treated in Section 2.6, the nonlinear contribution is

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}} P(-Vf + (\chi_a(1-\delta)-1)P)(W^2-1)^2dr^* \\
&+ \frac{1-\delta}{16}\int_{\mathbb{R}} P(\chi_a(-V^2-V') - \frac{2}{a}V\chi_a' - \frac{1}{a^2}\chi_a'')(W^2-1)^2dr^* \\
&= \int_{\mathbb{R}} \frac{1}{4}P\chi_a(-Vf - \delta P - \frac{1-\delta}{4}(V'+V^2))(W^2-1)^2dr^* \\
&+ \int_{\mathbb{R}} \frac{1}{4}P\{(1-\chi_a)(-Vf-P) - \frac{1-\delta}{2a}V\chi_a' - \frac{1-\delta}{4a^2}\chi_a''\}(W^2-1)^2dr^*.
\end{aligned}$$

Lemma 2.1. *Let $1 > \epsilon > 0$. Then for a large enough we have uniformly in $0 < \delta < 1/2$:*

$$-Vf - \frac{(1-\delta)}{2a}V\chi_a' - \frac{1-\delta}{4a^2}\chi_a'' \geq -(1-\epsilon)Vf.$$

Proof.

- We first consider
$$-\frac{\epsilon}{2}Vf - \frac{(1-\delta)}{2a}V\chi_a' = V(-\frac{\epsilon}{2}f - \frac{(1-\delta)}{2a}\chi_a').$$
  - For $|r^*| \leq a$ $V(-\epsilon/2f - \frac{1-\delta}{2a}\chi_a') = -\epsilon/2Vf$ is positive.
  - For $r^* \leq -a$ and a large enough $V(r^*) > 0$. As $-\epsilon/2f(2m) = \frac{\epsilon}{12m}$ the whole expression is positive for a large enough.
  - For $r^* \geq a$ and a large enough $V(r^*) < 0$. Noting that $\lim_{r\to\infty} f(r) = \frac{1}{3m}$ we see that the whole expression is positive for a large enough.
- We now consider
$$-\frac{\epsilon}{2}Vf - \frac{1-\delta}{4a^2}\chi_a''$$
For $r^* \geq a$ and a large enough we have
$$-Vf \gtrsim \frac{1}{a}$$
which gives the estimate for $r^* \geq a$, a large. For $r^* \leq -a$ we use that $-Vf(2m) = \frac{1}{12m^2}$.

□

Lemma 2.2. *Let $1 > \epsilon > 0$. For a large enough we have*

$$(-(1-\epsilon)Vf - P)(1-\chi_a) \geq -(1-2\epsilon)Vf(1-\chi_a).$$

Proof. We first consider the case $r^* \geq a$. We have

$$-Vf = \frac{2}{3mr} + \mathcal{O}(r^{-2}),\ P = \mathcal{O}(r^{-2}), \quad r \to \infty$$

which shows the inequality for $r^* \geq a$ and a sufficiently large. For $r^* \leq -a$ and a large the inequality follows from

$$-Vf(2m) = \frac{1}{12m^2},\ P(2m) = 0.$$

□

We have the following

Proposition 2.2. *There exists $1/2 > \epsilon > 0$ with the following property. For all $a > 0$ there exists $\delta = \delta(a) > 0$ such that we have*

$$\chi_a(-(1-\epsilon)Vf - \delta P - \frac{1}{4}(1-\delta)V^2 - \frac{1}{4}(1-\delta)V') \gtrsim \chi_a.$$

Proof. First recall that

$$V' = \frac{2N}{r^2}(1-3\mu).$$

We claim that it is sufficient to show

$$-4Vf - V^2 - \frac{2N}{r^2}(1-3\mu) > 0 \quad \text{on} \quad \mathbb{R}_r^{\geq 2m}. \tag{2.8}$$

Let us first show that Proposition 2.2 follows from (2.8). To see this, we first notice that if (2.8) is satisfied, we also have

$$-4(1-\epsilon)Vf - V^2 - \frac{2N}{r^2}(1-3\mu) > 0 \quad \text{on} \quad \mathbb{R}_r^{\geq 2m} \tag{2.9}$$

for $\epsilon > 0$ small enough. Indeed, there exists $C_0 > 3m$ such that for $r \geq C_0$, we have

$$-4(1-\epsilon)Vf - V^2 - \frac{2N}{r^2}(1-3\mu) \geq (-3+4\epsilon)Vf > 0$$

if $\epsilon < 3/4$. This follows from

$$\begin{aligned} -Vf &= \frac{2}{3mr} + \mathcal{O}(r^{-2}),\ r \to \infty, \\ -V^2 &= \mathcal{O}(r^{-2}),\ r \to \infty, \\ -\frac{2N}{r^2}(1-3\mu) &= \mathcal{O}(r^{-2}),\ r \to \infty. \end{aligned}$$

On the compact interval $[2m, C_0]$, we can add a small (negative) multiple of $-fV$ without changing the positivity. Thus, we have proved that (2.8) implies (2.9) for $\epsilon > 0$ small enough, and now, we would like to prove that (2.9) implies Proposition 2.2. Let us fix $\epsilon > 0$ such that (2.9) is satisfied. Then, the left hand side of (2.9), is $\geq \epsilon_a > 0$ on the support of χ_a for some constant ϵ_a. However, on the support of χ_a the functions $V', -P$ are bounded, so we can add small multiplies of them so as the whole expression would still be $\geq \epsilon_a/2$. Thus, this shows that (2.8) implies Proposition 2.2. Hence, it remains to prove (2.8). In fact, we have

$$\begin{aligned} -4Vf - V^2 - \frac{2}{r^2}(1-3\mu)N &= -4\frac{3\mu-2}{r}\left(\frac{1}{3m} - \frac{1}{r}\right) - \frac{(6m-2r)^2}{r^4} \\ &- \frac{2}{r^2}(1-\mu)(1-3\mu). \end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned} (8r^2 - 24mr)\left(\frac{r}{3m} - 1\right) - (6m-2r)^2 - 2(r-2m)(r-6m) &> 0 \quad \forall r \in \mathbb{R}^{\geq 2m} \\ \Leftrightarrow 8r^3 - 66mr^2 + 192m^2 r - 180m^3 &> 0 \quad \forall r \in \mathbb{R}^{\geq 2m}. \end{aligned}$$

Let $g(r) = 8r^3 - 66mr^2 + 192m^2 r - 180m^3$. We have

$$g(2m) = 4m^3, \quad \lim_{r\to\infty} g(r) = \infty.$$

We compute

$$g'(r) = 24\left(r^2 - \frac{11}{2}mr + 8m^2\right) > 0.$$

It follows that $g(r) > 0$ for all $r \geq 2m$. $\square$

Proposition 2.3. *For $a > 0$ large enough and δ small enough we have*

$$\begin{aligned} &\chi_a\left(-Vf - \delta P - \frac{1}{4}(1-\delta)V^2 - \frac{1}{4}(1-\delta)V'\right) \\ &+ \quad (1-\chi_a)(-Vf - P) - \frac{1-\delta}{2a}V\chi_a' - \frac{1-\delta}{4a^2}\chi_a'' \gtrsim \frac{1}{r}. \end{aligned}$$

Proof. We first choose $\epsilon > 0$ as in Proposition 2.2. Then for a large enough we have by Lemma 2.1 uniformly in $0 < \delta < 1/2$:

$$-Vf - \frac{1-\delta}{2a}V\chi_a' - \frac{1-\delta}{4a^2}\chi_a'' \geq -(1-\epsilon)Vf. \tag{2.10}$$

Using Lemma 2.2 we obtain by choosing a possibly larger :

$$(1-\chi_a)(-(1-\epsilon)Vf - P) \geq -(1-2\epsilon)Vf(1-\chi_a). \tag{2.11}$$

For a large enough we have for $r^* \geq a$

$$-Vf \gtrsim \frac{1}{r} \tag{2.12}$$

and for $r^* \leq -a$

$$-Vf \gtrsim 1 \gtrsim \frac{1}{r}. \tag{2.13}$$

We fix a such that (2.10)-(2.13) are fulfilled. We now apply Proposition 2.2 and obtain by choosing $\delta > 0$ small enough :

$$\begin{aligned}
&\chi_a(-Vf - \delta P - \frac{1}{4}(1-\delta)V^2 - \frac{1}{4}(1-\delta)V') \\
+ \quad &(1-\chi_a)(-Vf - P) - \frac{1-\delta}{2a}V\chi_a' - \frac{1-\delta}{4a^2}\chi_a'' \gtrsim -Vf(1-\chi_a) + \chi_a.
\end{aligned}$$

Using (2.12) and (2.13) we see that

$$-Vf(1-\chi_a) + \chi_a \gtrsim \frac{1}{r}.$$

□

2.5. **Estimates on the linear contribution.** We now fix a, δ as in Proposition 2.3.

2.5.1. *Space derivatives.* We have from the definition of h in (2.5),

$$\begin{aligned}
&\int_{\mathbb{R}} (W')^2 h(W^2-1)dr^* + 2\int_{\mathbb{R}} h(W')^2W^2dr^* + \frac{1}{2}\int_{\mathbb{R}} P(W')^2dr^* \\
= \quad &\frac{1}{4}(1-\delta)\int_{\mathbb{R}} P(W')^2(W^2-1)\chi_a dr^* + \frac{1}{2}(1-\delta)\int_{\mathbb{R}} P(W')^2(W^2-1)\chi_a dr^* \\
&+\frac{1}{2}(1-\delta)\int_{\mathbb{R}} P(W')^2\chi_a dr^* + \frac{1}{2}\int_{\mathbb{R}} P(W')^2dr^* \\
= \quad &\frac{3}{4}(1-\delta)\int_{\mathbb{R}} P(W')^2(W^2-1)\chi_a dr^* + (1-\frac{1}{2}\delta)\int_{\mathbb{R}} P(W')^2\chi_a dr^* \\
&+\frac{1}{2}\int_{\mathbb{R}} P(W')^2(1-\chi_a)dr^* \\
\gtrsim \quad &\int_{\mathbb{R}} P(W')^2dr^* - \left|\int_{\mathbb{R}} P(W')^2(W^2-1)\chi_a dr^*\right|.
\end{aligned}$$

2.5.2. *Time derivatives.* We have from the definition of h in (2.5),

$$\begin{aligned}
& -\int_{\mathbb{R}} h\dot{W}^2(W^2-1)dr^* - 2\int_{\mathbb{R}} h\dot{W}^2W^2dr^* + \frac{1}{2}\int_{\mathbb{R}} P\dot{W}^2dr^* \\
= & -\frac{3}{4}(1-\delta)\int_{\mathbb{R}} P\chi_a\dot{W}^2(W^2-1)dr^* - \frac{1}{2}(1-\delta)\int_{\mathbb{R}} P\chi_a\dot{W}^2dr^* + \frac{1}{2}\int_{\mathbb{R}} P\dot{W}^2dr^* \\
= & -\frac{3}{4}(1-\delta)\int_{\mathbb{R}} P\chi_a\dot{W}^2(W^2-1)dr^* + \frac{1}{2}\delta\int_{\mathbb{R}} P\chi_a\dot{W}^2dr^* + \frac{1}{2}\int_{\mathbb{R}} P\dot{W}^2(1-\chi_a)dr^* \\
\gtrsim & \int_{\mathbb{R}} P\dot{W}^2dr^* - \left|\int_{\mathbb{R}} P\chi_a\dot{W}^2(W^2-1)dr^*\right|
\end{aligned}$$

2.6. **Small energy error terms.** We again fix a, δ as in Proposition 2.3.

Proposition 2.4. *For all $\epsilon_0 > 0$ there exists $\epsilon_1 > 0$ such that for $E_F^{(\frac{\partial}{\partial t})} \leq \epsilon_1$ we have*

$$\left|\int_{\mathbb{R}} P\chi_a(W')^2(W^2-1)dr^*\right| \leq \epsilon_0\int_{\mathbb{R}} P(W')^2dr^*, \tag{2.14}$$

$$\left|\int_{\mathbb{R}} \chi_a P\dot{W}^2(W^2-1)dr^*\right| \leq \epsilon_0\int_{\mathbb{R}} P\dot{W}^2dr^*, \tag{2.15}$$

$$\left|\int_{\mathbb{R}} P^2\chi_a(W^2-1)^3dr^*\right| \leq \epsilon_0\int_{\mathbb{R}} \frac{P}{r}(W^2-1)^2dr^*. \tag{2.16}$$

Proof. We have

$$\begin{aligned}
\left|\int_{\mathbb{R}} P\chi_a(W')^2(W^2-1)dr^*\right| &\leq \|\chi_a(W^2-1)\|_{L^\infty}\int_{\mathbb{R}} P(W')^2dr^* \\
&\lesssim \|\sqrt{P}(W^2-1)\|_{L^\infty}\int_{\mathbb{R}} P(W')^2dr^* \\
&\lesssim \left(\sqrt{E_F^{(\frac{\partial}{\partial t})}} + E_F^{(\frac{\partial}{\partial t})}\right)\int_{\mathbb{R}} P(W')^2dr^*.
\end{aligned}$$

This shows (2.14). The proof for (2.15) is strictly analogous. To prove (2.16) we estimate

$$\left|\int_{\mathbb{R}} P^2\chi_a(W^2-1)^3dr^*\right| \lesssim \left(\sqrt{E_F^{(\frac{\partial}{\partial t})}} + E_F^{(\frac{\partial}{\partial t})}\right)\int_{\mathbb{R}} \frac{P}{r}(W^2-1)^2dr^*.$$

□

2.7. **End of the proof of the estimate (1.37).** We choose a large enough and $\delta > 0$ small enough such that the estimate in Proposition 2.3 is fulfilled. Once a, δ fixed in this way we choose the energy small enough such that

(1) The nonlinear contribution in Section 2.4 plus the error term $\int Ph(W^2-1)^3$ dominate
$$\int_{\mathbb{R}} \frac{P}{r}(W^2-1)^2dr^*.$$

(2) The space derivatives in Section 2.5.1 dominate

$$\int P(W')^2 dr^*.$$

(3) The time derivatives in Section 2.5.2 dominate

$$\int P\dot{W}^2 dr^*.$$

Then, integrating in t, we obtain that for initial data with small enough energy,

$$\begin{aligned}
&\int_{t=t_0}^{t}\int_{\mathbb{R}} P\dot{W}^2 + P(W')^2 + \frac{P}{r}(W^2-1)^2 dr^* \\
\lesssim\ & \left|\int_{\mathbb{R}} \dot{W}P\chi_a W(W^2-1)dr^*\right|_{t=t_0}^{t} + \left|\int_{\mathbb{R}} \dot{W}fW'dr^*\right|_{t=t_0}^{t}.
\end{aligned}$$

We have by the Cauchy-Schwarz inequality

$$\begin{aligned}
\left|\int_{\mathbb{R}} \dot{W}fW'dr^*\right| &\lesssim \left(\int_{\mathbb{R}} \dot{W}^2 dr^*\right)^{1/2}\left(\int_{\mathbb{R}} (W')^2 dr^*\right)^{1/2} \lesssim E_F^{(\frac{\partial}{\partial t})}(t=t_0), \\
\left|\int_{\mathbb{R}} \dot{W}P\chi_a W(W^2-1)dr^*\right| &\lesssim \left(\int_{\mathbb{R}} \dot{W}^2 dr^*\right)^{1/2}\left(\int_{\mathbb{R}} P^2(W^2-1)^2\chi_a^2 W^2 dr^*\right)^{1/2} \\
&\lesssim \left(\int_{\mathbb{R}} \dot{W}^2 dr^*\right)^{1/2} \|P\chi_a^2 W^2\|_{L^\infty}^{1/2}\left(\int_{\mathbb{R}} P(W^2-1)^2 dr^*\right)^{1/2} \\
&\lesssim E_F^{(\frac{\partial}{\partial t})}(t=t_0)\left(E_F^{(\frac{\partial}{\partial t})}(t=t_0) + \sqrt{E_F^{(\frac{\partial}{\partial t})}(t=t_0)} + 1\right).
\end{aligned}$$

where in the last inequality we have used Proposition 2.1 for energy small enough. This finishes the proof of the estimate. □

3. The Proof of Local Energy Decay on $t = constant$ Hypersurfaces

The proof of the local energy decay follows the arguments of [19] which we adapt to the situation where the Morawetz estimate is only available et low energies. We refer to [19] for the details of the calculations.

3.1. The vector field K.

Let

$$K = -w^2\frac{\partial}{\partial w} - v^2\frac{\partial}{\partial v} \tag{3.1}$$

We then compute (see [19, equation (44)])

$$\pi^{\alpha\beta}(K)T_{\alpha\beta}(F) = 4t[2 + \frac{(3\mu-2)r^*}{r}].[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2] \tag{3.2}$$

and define

$$J_F^{(K)}(t_i \leq t \leq t_{i+1}) = \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathcal{S}^2} \pi^{\alpha\beta}(K) T_{\alpha\beta}(F) dVol \tag{3.3}$$

We also have [19, equation (47)] :

$$\begin{aligned} E_F^{(K)}(t_i) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathcal{S}^2} [w^2 N(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + v^2(\frac{1}{N}|F_{\hat{v}\hat{\theta}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\phi}}|^2) \\ &\quad +(w^2+v^2)(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)] r^2 N d\sigma^2 dr^* \end{aligned} \tag{3.4}$$

3.2. **Local energy decay.** The following estimate controls the bulk term :

Proposition 3.1.

$$J_F^{(K)}(t_i \leq t \leq t_{i+1}) \lesssim t_{i+1} \int_{t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{R_0^*} \int_{\mathcal{S}^2} \frac{N}{r} (|F_{\hat{w}\hat{v}}|^2 + |F_{\hat{\theta}\hat{\phi}}|^2) r^2 d\sigma^2 dr^* dt \tag{3.5}$$

where $2m < r_0 \leq 3m \leq R_0$

Proof. First note that we have for $|r_*| >> 1$:

$$2 + \frac{(3\mu - 2)r_*}{r} < 0.$$

On the remaining compact interval $\frac{N}{r}$ is strictly positive. □

We now estimate the local energy in terms of $E_F^{(K)}$

Proposition 3.2.

$$E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq r_2^*) \leq C(r_1^*, r_2^*) \frac{E_F^{(K)}(t)}{t^2} \tag{3.6}$$

Proof. We have for t sufficiently large :

$$\begin{aligned} & E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq r_2^*)(t) \\ = & \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathcal{S}^2} (N|F_{\hat{w}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\phi}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\theta}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\phi}}|^2 + |F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2).Nr^2 d\sigma^2 dr^*(t) \\ \lesssim & \frac{E_F^{(K)}(t)}{\min_{\{r_1^* \leq r^* \leq r_2^*\}} w^2(t, r_*)} + \frac{E_F^{(K)}(t)}{\min_{\{r_1^* \leq r^* \leq r_2^*\}} v^2(t, r_*)} \end{aligned} \tag{3.7}$$

□

We eventually obtain the following local energy decay

Proposition 3.3. *We have*

$$E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq r_2^*) \quad \leq \quad C(r_1^*, r_2^*)\frac{(E_F^{(K)}(t_0) + |E_F^{(\frac{\partial}{\partial t})}(t_0)|)}{t} \tag{3.8}$$

Proof. Using (3.6), (3.5), the divergence theorem, and the Morawetz estimate (3), we get (3.8). □

3.3. **Decay for the middle components away from the horizon.**

Proposition 3.4. *Let $R > 2m$. We have for all $r \geq R$,*

$$|F_{\hat{\theta}\hat{\phi}}(v, w, \theta, \phi)| = \left|\frac{W^2(v, w) - 1}{r^2}\right| \quad \lesssim \quad \frac{E_1}{\sqrt{1 + |v|}}$$

and

$$|F_{\hat{\theta}\hat{\phi}}(v, w, \theta, \phi)| = \left|\frac{W^2(v, w) - 1}{r^2}\right| \quad \lesssim \quad \frac{E_1}{\sqrt{1 + |w|}}$$

where,

$$E_1 \quad = \quad [|E_F^{(\frac{\partial}{\partial t})}(t = t_0)| + E_F^{(K)}(t_0)]^{\frac{1}{2}}$$

Proof.

We consider the region $w \geq 1$, $r \geq R$, where R is fixed.

Let, r_F be a value of r such that $R^* \leq r_F^* \leq R^* + 1$, and to be determined later. Let $r^* \geq r_F^*$. We have,

$$\begin{aligned}
\int_{S^2} r^2 |F_{\hat{\theta}\hat{\phi}}|^2(t, r, \theta, \phi) d\sigma^2 \quad &= \quad \int_{S^2} r^2 |F_{\hat{\theta}\hat{\phi}}|^2(t, r_F, \theta, \phi) d\sigma^2 + \int_{S^2}\int_{\overline{r}^* = r_F^*}^{\overline{r}^* = r^*} \partial_{r^*}[r^2 |F_{\hat{\theta}\hat{\phi}}|^2](t, r, \theta, \phi) d\overline{r}^* d\sigma^2 \\
&= \quad \int_{S^2} r_F^2 |F_{\hat{\theta}\hat{\phi}}|^2(t, r_F, \theta, \phi) d\sigma^2 + \int_{S^2}\int_{\overline{r}^* = r_F^*}^{\overline{r}^* = r^*} 2r |F_{\hat{\theta}\hat{\phi}}|^2(t, r, \theta, \phi) N d\overline{r}^* d\sigma^2 \\
&\quad + 2\int_{S^2}\int_{\overline{r}^* = r_F^*}^{\overline{r}^* = r^*} r^2 \partial_{r^*} |F_{\hat{\theta}\hat{\phi}}|^2(t, r, \theta, \phi) d\overline{r}^* d\sigma^2
\end{aligned}$$

From (3.7) we obtain

$$\int_{r^* = r_1^*}^{r^* = r_2^*} \int_{S^2} |F_{\hat{\theta}\hat{\phi}}|^2(t, \overline{r}, \theta, \phi) N r^2 d\sigma^2 d\overline{r}^* \lesssim \frac{E_F^{(K)}(t)}{\min_{\{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_F^{(K)}(t)}{\min_{\{r_1^* \leq r^* \leq r_2^*\}} v^2} \tag{3.9}$$

Therefore,

$$\int_{\overline{r}^*=R^*}^{\overline{r}^*=R^*+1} \int_{\mathcal{S}^2} |F_{\hat{\theta}\hat{\phi}}|^2(t,\overline{r},\theta,\phi) N r^2 d\sigma^2 d\overline{r}^* \lesssim \frac{E_F^{(K)}(t)}{t^2}$$

There exists r_F, such that $R^* \leq r_F^* \leq R^*+1$ and,

$$\int_{\mathcal{S}^2} r_F^2 |F_{\hat{\theta}\hat{\phi}}|^2(t,r_F,\theta,\phi) d\sigma^2 \quad \lesssim \quad \frac{E_F^{(K)}(t)}{t^2(R^*+1-R^*)N(R)}$$

which gives,

$$\int_{\mathcal{S}^2} r_F^2 |F_{\hat{\theta}\hat{\phi}}|^2(t,r_F,\theta,\phi) d\sigma^2 \quad \lesssim \quad \frac{E_F^{(K)}(t)}{t^2} \tag{3.10}$$

On the other hand, from (3.9) and from looking at the region of integration in Penrose diagram, it is easy to see that

$$\begin{aligned} \int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} \int_{\mathcal{S}^2} \overline{r} |F_{\hat{\theta}\hat{\phi}}|^2 N(t,\overline{r},\theta,\phi) d\sigma^2 d\overline{r}^* \quad &\lesssim \quad \int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} \frac{\overline{r}}{R} \int_{\mathcal{S}^2} \overline{r} |F_{\hat{\theta}\hat{\phi}}|^2 N(t,\overline{r},\theta,\phi) d\sigma^2 d\overline{r}^* \\ &\lesssim \quad \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2} \end{aligned} \tag{3.11}$$

Thus,

$$\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} \int_{\mathcal{S}^2} \overline{r} |F_{\hat{\theta}\hat{\phi}}|^2 N(t,\overline{r},\theta,\phi) d\sigma^2 d\overline{r}^* \quad \lesssim \quad \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2} \tag{3.12}$$

Now, we want to estimate the term:

$$\int_{\mathcal{S}^2} \int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} r^2 \nabla_{r^*} |F_{\hat{\theta}\hat{\phi}}|^2(t,r,\theta,\phi) d\overline{r}^* d\sigma^2$$

Using Cauchy-Schwarz, we obtain

$$\begin{aligned} &\int_{\mathcal{S}^2} \int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} r^2 \partial_{r^*} |F_{\hat{\theta}\hat{\phi}}|^2(t,r,\theta,\phi) d\overline{r}^* d\sigma^2 \\ \lesssim \quad &\Big(\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} \int_{\mathcal{S}^2} \overline{r}^2 |\partial_r F_{\hat{\theta}\hat{\phi}}|^2 N^2 d\sigma^2 d\overline{r}^* \Big)^{\frac{1}{2}} \Big(\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} \int_{\mathcal{S}^2} \overline{r}^2 |F_{\hat{\theta}\hat{\phi}}|^2 d\sigma^2 d\overline{r}^* \Big)^{\frac{1}{2}} \end{aligned}$$

We have,

$$\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} \int_{\mathcal{S}^2} \overline{r}^2 |F_{\hat{\theta}\hat{\phi}}|^2(t,\overline{r},\theta,\phi) d\sigma^2 d\overline{r}^* \quad \lesssim \quad \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}$$

Thus,

$$[\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*}\int_{\mathcal{S}^2}\overline{r}^2|F_{\hat{\theta}\hat{\phi}}|^2 d\sigma^2 d\overline{r}^*]^{\frac{1}{2}} \lesssim \frac{\sqrt{E_F^{(K)}(t)}}{t} + \frac{\sqrt{E_F^{(K)}(t)}}{w} \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
& \int_{\mathcal{S}^2}\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} r^2|\partial_r F_{\hat{\theta}\hat{\phi}}|^2(t,r,\theta,\phi)N^2 d\overline{r}^* d\sigma^2 \\
= & \int_{\mathcal{S}^2}\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} r^2[\frac{-2}{r^3}(W^2-1)+2\frac{W\partial_r W}{r^2}]^2(t,r,\theta,\phi)N^2 d\overline{r}^* d\sigma^2 \\
\lesssim & \int_{\mathcal{S}^2}\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} [\frac{N^2}{r^4}(W^2-1)^2+\frac{N^2(W^2-1+1)}{r^2}|\partial_r W|^2]d\overline{r}^* d\sigma^2 \\
\lesssim & \int_{\mathcal{S}^2}\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} [\frac{N}{r^2}(W^2-1)^2+\frac{(W^2-1+1)}{r^2}|\partial_r W|^2 N^2]d\overline{r}^* d\sigma^2 \\
\lesssim & (\|\sqrt{P}(W^2-1)\|_{L^\infty(\mathbb{R})}+1)\int_{\mathcal{S}^2}\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} [\frac{N}{r^2}(W^2-1)^2+|\partial_{r^*}W|^2]d\overline{r}^* d\sigma^2 \\
\lesssim & (E_F^{(\frac{\partial}{\partial t})}+\sqrt{E_F^{(\frac{\partial}{\partial t})}}+1)\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} [\frac{N}{r^2}(W^2-1)^2+|\partial_{r^*}W|^2]d\overline{r}^* d\sigma^2.
\end{aligned}$$

Here we have used Proposition 2.1. Thus, by using (3.9) and from looking again at the region of integration in Penrose diagram, we obtain

$$\int_{\mathcal{S}^2}\int_{\overline{r}^*=r_F^*}^{\overline{r}^*=r^*} r^2|\partial_r F_{\hat{\theta}\hat{\phi}}|^2(t,r,\theta,\phi)N^2 d\overline{r}^* d\sigma^2 \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2} \tag{3.14}$$

Finally, we obtain,

$$r^2|F_{\hat{\theta}\hat{\phi}}|^2(t,r,\theta,\phi) \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}$$

Thus,

$$|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi) \lesssim \frac{[E_F^{(K)}(t)]^{\frac{1}{2}}}{rt} + \frac{[E_F^{(K)}(t)]^{\frac{1}{2}}}{rw}$$

Thus,

$$|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi) \lesssim \frac{[E_F^{(K)}(t)]^{\frac{1}{2}}}{rt} + \frac{[E_F^{(K)}(t)]^{\frac{1}{2}}}{rw}$$

Using (3.5) and the Morawetz estimate we see that

$$[E_F^{(K)}(t)]^{\frac{1}{2}} \lesssim E_1\sqrt{t}.$$

If $r^* \leq (1-\epsilon)t$, $1 > \epsilon > 0$ we have $w = t - r^* \geq \epsilon t$. If $r^* \geq (1-\epsilon)t$ we have

$$\frac{\sqrt{t}}{rw} \leq \frac{1}{\sqrt{r}w}.$$

Summarizing we obtain in the region $w \geq 1$, $r \geq R$:

$$|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi) \lesssim \frac{E_1}{\sqrt{rt}} + \frac{E_1}{\sqrt{rw}}.$$

Consider first the part of the region where $t \geq 1$:

- In the region $r \geq R$, $t \geq 1$, $w \geq 1$ we have
$$w = t - r^* \lesssim r + t \lesssim rt$$
and thus we obtain the estimate
$$|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi)| \lesssim \frac{E_1}{\sqrt{w}}.$$
- To obtain the estimate in v we distinguish two cases
  (1) $v \leq 1$. The region $w \geq 1$, $v \leq 1$ is a compact region and $|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi)|$ is uniformly bounded there.
  (2) Now consider $v \geq 1$, $w \geq 1$, $r \geq R$, $t \geq 1$. We have
$$v = r^* + t \lesssim r + t \lesssim rt,$$
$$v \lesssim r + t \lesssim Cr + t - r^* \lesssim r + w \lesssim rw.$$
  Thus for $v \geq 1$ we have
$$|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi)| \lesssim \frac{E_1}{\sqrt{v}}$$

For $t \leq 1$, the intersection with the region $w \geq 1$, $r \geq R$, is compact, hence, $|F_{\hat{\theta}\hat{\phi}}|(v,w,\theta,\phi)|$ is uniformly bounded in this region.

Somewhat similar arguments for the other regions, see [19] for details, gives the stated result.

□

4. Decay of the Energy to Observers Traveling to the Black Hole on $v = \textit{constant}$ Hypersurfaces Near the Horizon

4.1. **The vector field H.**

Let

$$H = -\frac{h(r^*)}{N}\frac{\partial}{\partial w} - h(r^*)\frac{\partial}{\partial v} \tag{4.1}$$

We have, see [19, equation (88)],

$$\begin{aligned}
&\pi^{\alpha\beta}(H)T_{\alpha\beta}(F)\\
=\ &(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2)(h^{'} - \frac{\mu}{r}h) + (|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2)(\frac{-1}{N}h^{'})\\
&+[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].[-2(\frac{1}{2N}[(h^{'} - Nh^{'}) - \frac{\mu}{r}h]) + \frac{(2-3\mu)}{Nr}(h - Nh)]
\end{aligned}$$

This gives the following bulk term

$$\begin{aligned}
&I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \equiv \int_{v=v_i}^{v=v_{i+1}}\int_{w=w_i}^{w=w_{i+1}}\int_{\mathcal{S}^2}\pi^{\alpha\beta}T_{\alpha\beta}(F)dVol\\
=\ &\int_{v=v_i}^{v=v_{i+1}}\int_{w=w_i}^{w=\infty}\int_{\mathcal{S}^2}([|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2](h^{'} - \frac{\mu}{r}h) + ([|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2](\frac{-h^{'}}{N})\\
+\ &[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].\mu[\frac{-1}{N}h^{'} + \frac{3}{r}h]).r^2d\sigma^2Ndwdv
\end{aligned} \tag{4.2}$$

We also compute the flux of H along the null hypersurfaces [19, equation (92)]

$$\begin{aligned}
&F_F^{(H)}(v = v_i)(w_i \leq w \leq w_{i+1})\\
&= \int_{w=w_i}^{w=w_{i+1}}\int_{\mathcal{S}^2} -2Nh(r^*)[|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2 + |F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2]r^2d\sigma^2dw
\end{aligned} \tag{4.3}$$

and [19, equation (90)]

$$\begin{aligned}
&F_F^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1})\\
&= \int_{v=v_i}^{v=v_{i+1}}\int_{\mathcal{S}^2} -2h(r^*)[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2]r^2d\sigma^2dv
\end{aligned} \tag{4.4}$$

We will also need the energy [19, page 76]

$$\begin{aligned}
E_F^{(H)}(t) = &\int_{r^*=-\infty}^{r^*=\infty}\int_{\mathcal{S}^2}[hN(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + h(N+1)(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2)\\
&+h(|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2)]r^2d\sigma^2dr^*.
\end{aligned}$$

We will also need the energy

$$\begin{aligned}
&E_F^{\#(\frac{\partial}{\partial t})}(t)\\
&= \int_{r^*=-\infty}^{r^*=\infty}\int_{\mathcal{S}^2}[N|F_{\hat{w}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\phi}}|^2 + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2 + |F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].r^2d\sigma^2dr^*
\end{aligned}$$

Note that $E_F^{\#(\frac{\partial}{\partial t})}(t)$ is equivalent to $E_F^{(H)}(t)$ in regions where $h = 1$. We are going to choose h such that

$$\lim_{r_* \to -\infty} h(r^*) = 1$$

and for all $r > 2m$:

$$h \geq 0$$

For $2m < r_1$ and r_1 sufficiently close to $2m$ we can choose h such that h is supported in $2m < r \leq 1.2 r_1$ and such that it satisfies on $(2m, r_1]$

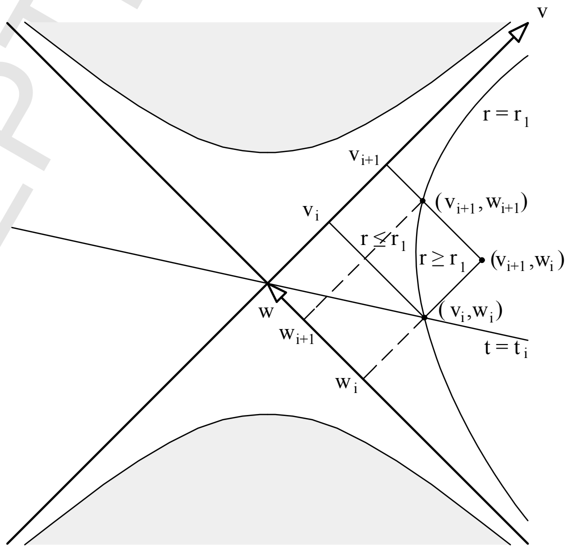
$$\begin{aligned} h &> 0 & (4.5) \\ h' &\geq 0 & (4.6) \\ \frac{\mu}{r} h - h' &\gtrsim h & (4.7) \\ \mu[\frac{h'}{N} - \frac{3}{r} h] &\gtrsim h & (4.8) \end{aligned}$$

A suitable choice is for example $h(r^*) = exp\left(\int_{-\infty}^{r_*} \delta \frac{\mu(s)}{r(s)} ds\right)$ $(0 < \delta < 1)$ on $2m < r \leq r_1$ and then extended smoothly.

Applying the divergence theorem for $F_{\mu\nu}$ in a rectangle in the Penrose diagram representing the exterior of the Schwarzschild space-time of which one side contains the horizon, say in the region $[w_i, \infty] \times [v_i, v_{i+1}]$, then, we have

$$\begin{aligned} & -F_F^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_F^{(H)}(v = v_i)(w_i \leq w \leq \infty) \\ &= \; -I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\ &- \; F_F^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) - F_F^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty). \quad (4.9) \end{aligned}$$

The following figure shows the domain of integration:



Remark 4.1. At this stage, it is not clear that $F_F^{(H)}(w=\infty)(v_i \leq v \leq v_{i+1})$ is finite at the regularity level of Corollary 1.1. To show that it is well defined, we can either first work on the regularity level of [18], or first work in a region $[w_i, w] \times [v_i, v_{i+1}]$ and then consider the limit $w \to \infty$. Our estimates show then in particular that $F_F^{(H)}(w=\infty)(v_i \leq v \leq v_{i+1})$ defined as the above limit is finite at the regularity level of Corollary 1.1. We omit these details.

4.2. **The Main Estimates.** Let

$$\begin{aligned} w_i &= t_i - r_1^* \\ v_i &= t_i + r_1^* \end{aligned}$$

where t_i is a sequence of positive numbers with

$$t_i < t_{i+1} \leq 1.1 t_i, \quad \sum_{i=0}^{\infty} \frac{1}{t_i} < \infty$$

and r_1 is as determined in the construction of the vector field H. We have $r(w_i, v_i) = r_1$.

4.2.1. *Controlling the flux of H away from the horizon.* Away from the horizon we can control the flux of the vector field H by the flux of the vector field ∂_t:

Proposition 4.1. *For $v_{i+1} \geq v_i$, we have*

$$-F_F^{(H)}(w=w_i)(v_i \leq v \leq v_{i+1}) \lesssim F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i \leq v \leq v_{i+1}) \tag{4.10}$$

Proof. We have (see [19, page 67]),

$$F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i \leq v \leq v_{i+1}) = \int_{v=v_i}^{v=v_{i+1}} \int_{S^2} 2[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\theta}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\phi}}|^2] r^2 N d\sigma^2 dv$$

The region $w = w_i$ and $v_i \leq v \leq v_{i+1}$ is in the region $r \geq r_1$ as $r(w_i, v_i) = r_1$, and $v_{i+1} \geq v_i$. Thus, in this region

$$\frac{h(r^*)}{N} \lesssim 1$$

which gives immediately (4.10).

□

4.2.2. *Controlling the bulk term generated from H away from the horizon.* By construction of the vector field H the bulk has good sign in the region $r \leq r_1$. In the region $r \geq r_1$ it can be controlled by the Morawetz estimate. We obtain :

Proposition 4.2. *We have*

$$
\begin{aligned}
&|I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \geq r_1)| \\
&\lesssim \quad |E_F^{(\frac{\partial}{\partial t})}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)| + \frac{(1 + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^{(\frac{\partial}{\partial t})}(t = t_0))^2}{t_i},
\end{aligned}
$$

where

$$
\begin{aligned}
&E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) \\
&= \quad \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathcal{S}^2} [N|F_{\hat{w}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\phi}}|^2 + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2 + |F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].r^2 d\sigma^2 dr^*(t = t_0)
\end{aligned}
$$

It will be important in the following that the non explicitly decaying term on the RHS is a local energy rather than a global energy as given by a direct application of the Morawetz estimate. To obtain this we construct a solution of the Yang-Mills equation with compactly supported data on $t = t_i$ and which coincides with our solution in the region we are interested in. More precisely let $\hat{F}$ be the curvature associated to gauge transformations applied to the following potential $\hat{A}$,

$$
\hat{A} \quad = \quad \hat{W}(t,r)\tau_1 d\theta + \hat{W}(t,r)\sin(\theta)\tau_2 d\phi + \cos(\theta)\tau_3 d\phi \tag{4.11}
$$

where $\hat{W}$ is defined as the solution to the following Cauchy problem

$$
\partial_t^2 \hat{W} - \partial_{r^*}^2 \hat{W} - P\hat{W}[1 - \hat{W}^2] = 0
$$

where

$$
\hat{W}(t = t_i, r^*) = \hat{\chi}(\frac{2r^*}{t_i})W(t = t_i, r^*)
$$

$$
\partial_t \hat{W}(t = t_i, r^*) = \hat{\chi}(\frac{2r^*}{t_i})\partial_t W(t = t_i, r^*)
$$

and $\hat{\chi}$ is a smooth cut-off function equal to one on $[-1, 1]$ and zero outside $[-\frac{3}{2}, \frac{3}{2}]$. We have that the expression of F in the Ansatz (1.23), and the expression of $\hat{F}$ in the Ansatz (4.11), verify the following:

For $-\frac{t_i}{2}, \leq r^* \leq \frac{t_i}{2}$ $\quad \hat{F}_{\hat{r^*}\hat{t}}(t = t_i, r^*) = F_{\hat{r^*}\hat{t}}(t = t_i, r^*)$, $\hat{F}_{\hat{\theta}\hat{\phi}}(t = t_i, r^*) = F_{\hat{\theta}\hat{\phi}}(t = t_i, r^*)$,

and for $t_i \leq t \leq t_{i+1}$ $\quad \nabla^\mu \hat{F}_{\mu\nu} + [\hat{A}^\mu, \hat{F}_{\mu\nu}] = 0.$

We note that the Bianchi identities will be satisfied, since in this case, the curvature derives from a potential.

The following lemma controls the energy of $\hat{F}$ in terms of a local energy of F + decaying terms. It shows in particular that we can apply the Morawetz estimate to $\hat{F}$ if t_i is sufficiently large.

Lemma 4.1. *We have*

$$
\begin{aligned}
E_{\hat{F}}^{(\frac{\partial}{\partial t})}(t = t_i) \quad &= \quad \int_{r*=-\infty}^{\infty} \int_{\mathcal{S}^2} (|\partial_t \hat{W}(t,r)|^2 + |\partial_{r^*} \hat{W}(t,r)|^2 + \frac{N[\hat{W}^2(t,r) - 1]^2}{2r^2})dr^* d\sigma^2 \\
&\lesssim \quad E_F^{(\frac{\partial}{\partial t})}(-\frac{3t_i}{4} \leq r^* \leq \frac{3t_i}{4}) + \frac{(E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^{(\frac{\partial}{\partial t})}(t = t_0) + 1)^2}{t_i}
\end{aligned}
$$

Proof.

$$\begin{aligned}
& |\partial_{r^*}\hat{W}|^2(t=t_i,r^*) \\
= \quad & |\partial_{r^*}[\hat{\chi}(\frac{2r^*}{t_i})W(t=t_i,r^*)]|^2 \\
= \quad & |\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})W+\hat{\chi}(\frac{2r^*}{t_i})\partial_{r^*}W|^2 \\
\lesssim \quad & |\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})W|^2+|\hat{\chi}(\frac{2r^*}{t_i})\partial_{r^*}W|^2 \\
\lesssim \quad & |\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)|+|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2+|\hat{\chi}(\frac{2r^*}{t_i})\partial_{r^*}W|^2 \\
\lesssim \quad & |\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2+|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2+|\hat{\chi}(\frac{2r^*}{t_i})\partial_{r^*}W|^2
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{r*=-\infty}^{\infty}\int_{\mathcal{S}^2}|\partial_{r^*}\hat{W}(t,r)|^2dr^*d\sigma^2 \lesssim E_F^{(\frac{\partial}{\partial t})}(-\frac{3t_i}{4}\le r^*\le\frac{3t_i}{4}) \\
& +\int_{r*=\frac{-3t_i}{4}}^{\frac{-t_i}{2}}(\frac{1}{t_i^2}+|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2)dr^*+\int_{r*=\frac{t_i}{2}}^{\frac{3t_i}{4}}(\frac{1}{t_i^2}+|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2)dr^* \\
\lesssim \quad & E_F^{(\frac{\partial}{\partial t})}(-\frac{3t_i}{4}\le r^*\le\frac{3t_i}{4})+\frac{1}{t_i} \\
& +\int_{r*=\frac{-3t_i}{4}}^{\frac{-t_i}{2}}|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2dr^*+\int_{r*=\frac{t_i}{2}}^{\frac{3t_i}{4}}|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2dr^*
\end{aligned}$$

We need to control the integrals

$$\int_{r*=\frac{-3t_i}{4}}^{\frac{-t_i}{2}}|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2dr^*+\int_{r*=\frac{t_i}{2}}^{\frac{3t_i}{4}}|\frac{2}{t_i}\hat{\chi}'(\frac{2r^*}{t_i})|^2(W^2-1)^2dr^*$$

We have by a Sobolev inequality, for $r\ge 2m$, $r\le r_1$,

$$\begin{aligned}
\|(W^2-1)\|_{L^\infty}^2 \quad \lesssim \quad & \int_{r^*=-\infty}^{\infty}(W^2-1)^2dr^*+\int_{r^*=-\infty}^{\infty}(\partial_{r^*}(W^2-1))^2dr^* \\
\lesssim \quad & E_F^{\#(\frac{\partial}{\partial t})}(t=t_0)+\int_{r^*=-\infty}^{\infty}W^2(\partial_{r^*}W)^2dr^* \\
\lesssim \quad & E_F^{\#(\frac{\partial}{\partial t})}(t=t_0)+\int_{r^*=-\infty}^{\infty}(W^2-1)(\partial_{r^*}W)^2dr^* \\
\lesssim \quad & (1+\|(W^2-1)\|_{L^\infty})(E_F^{\#(\frac{\partial}{\partial t})}(t=t_i)+E_F^{(\frac{\partial}{\partial t})}(t=t_0)).
\end{aligned}$$

It follows

$$\|(W^2-1)^2\|_{L^\infty}^2\lesssim(1+E_F^{\#(\frac{\partial}{\partial t})}(t=t_i)+E_F^{(\frac{\partial}{\partial t})}(t=t_0))^2.$$

By using the divergence theorem in the region $\{t_0\le t\le t_i\}$, we obtain

$$\begin{aligned} & -I_F^{(H)}(t_0 \leq t \leq t_i)(r \leq r_1) + E_F^{(H)}(t_i) \\ = \quad & E_F^{(H)}(t_0) + I_F^{(H)}(t_0 \leq t \leq t_i)(r \geq r_1) \end{aligned} \tag{4.12}$$

Now we use that $E_F^{(H)}(t_i)$ is equivalent to $E_F^{\#(\frac{\partial}{\partial t})}(t = t_i)$ on $r \leq r_1$ and controlled by $E_F^{(\frac{\partial}{\partial t})}(t = t_i)$ on $r \geq r_1$. Due to the positivity of the terms on the left hand side and using the Morawetz estimate (1.37), we get

$$E_F^{\#(\frac{\partial}{\partial t})}(t = t_i) \lesssim E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^{(\frac{\partial}{\partial t})}(t = t_0)$$

On the other hand,

$$\begin{aligned} 0 \quad & \leq \quad \hat{W}^2 \leq W^2 \\ -1 \quad & \leq \quad \hat{W}^2 - 1 \leq W^2 - 1 \end{aligned}$$

Hence,

$$|\hat{W}^2 - 1|^2 \leq 1 + |W^2 - 1|^2 \lesssim (1 + E_F^{\#(\frac{\partial}{\partial t})}(t = t_i) + E_F^{(\frac{\partial}{\partial t})}(t = t_0))^2$$

If $\hat{W}^2(t,r) - 1 \leq 0$, then $(\hat{W}^2(t,r) - 1)^2 \leq 1$, and

$$\begin{aligned} & \int_{r^*=-\infty}^{\infty} \int_{\mathcal{S}^2} \frac{N(\hat{W}^2 - 1)^2(t,r)}{2r^2} dr^* d\sigma^2 \\ \lesssim \quad & \int_{r^*=-\frac{t_i}{2}}^{\frac{t_i}{2}} \int_{\mathcal{S}^2} \frac{N(W^2 - 1)^2}{2r^2} dr^* d\sigma^2 + \|(\hat{W}^2 - 1)\|_{L^\infty}^2 (\int_{r^*=\frac{t_i}{2}}^{\infty} \int_{\mathcal{S}^2} \frac{N}{2r^2} dr^* d\sigma^2 + \int_{r^*=-\infty}^{\frac{-t_i}{2}} \int_{\mathcal{S}^2} \frac{N}{2r^2} dr^* d\sigma^2) \\ \lesssim \quad & E_F^{(\frac{\partial}{\partial t})}(-\frac{3t_i}{4} \leq r^* \leq \frac{3t_i}{4}) + \frac{(1 + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^{(\frac{\partial}{\partial t})}(t = t_0))^2}{t_i} \end{aligned}$$

□

Proof of Proposition 4.2. Recall that

$$\begin{aligned} & \Big| I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \geq r_1) \Big| \\ = \quad & \Big| \int\int_{v_i \leq v \leq v_{i+1}, w_i \leq w \leq w_{i+1}, r \geq r_1} \int_{\mathcal{S}^2} ([|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2](h' - \frac{\mu}{r} h) + ([|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2](\frac{-h'}{N}) \\ + \quad & [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].\mu[\frac{-1}{N}h' + \frac{3}{r}h]).r^2 d\sigma^2 N dw dv \Big| \\ \lesssim \quad & \int_{t_1}^{t_{i+1}} \int_{\mathbb{R}_{r_*}} \int_{S^2} \chi_{r_1^* \leq r_* \leq 1.2 r_1^*}(r^*)[PN^2(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2)] d\sigma^2 dr^* dt \\ + \quad & \int_{t_1}^{t_{i+1}} \int_{\mathbb{R}_{r_*}} \int_{S^2} \chi_{r_1^* \leq r_* \leq 1.2 r_1^*}(r^*)[\frac{N}{r}(|F_{\hat{v}\hat{w}}|^2 + |F_{\hat{\phi}\hat{\theta}}|^2) + P|F_{\hat{v}\hat{\theta}}|^2 + P|F_{\hat{v}\hat{\phi}}|^2] r^2 d\sigma^2 dr^* dt. \end{aligned}$$

By finite speed of propagation this equals for t_i sufficiently large

$$\int_{t_1}^{t_{i+1}}\int_{\mathbb{R}_{r_*}}\int_{S^2}\chi_{r_1^*\le r_*\le 1.2r_1^*}(r^*)[PN^2(|\hat{F}_{\hat{w}\hat{\theta}}|^2+|\hat{F}_{\hat{w}\hat{\phi}}|^2)]d\sigma^2dr^*dt$$
$$+\int_{t_1}^{t_{i+1}}\int_{\mathbb{R}_{r_*}}\int_{S^2}\chi_{r_1^*\le r_*\le 1.2r_1^*}(r^*)[\frac{N}{r}(|\hat{F}_{\hat{v}\hat{w}}|^2+|\hat{F}_{\hat{\phi}\hat{\theta}}|^2)+P|\hat{F}_{\hat{v}\hat{\theta}}|^2+P|\hat{F}_{\hat{v}\hat{\phi}}|^2]r^2d\sigma^2dr^*dt. \tag{4.13}$$

By Lemma 4.1 we can apply the Morawetz estimate to $\hat{F}$. Applying the Morawetz estimate to the term (4.13) and applying again Lemma 4.1 gives the Proposition. □

4.2.3. *Estimate on the bulk and the flux generated from H near the horizon.* Let us summarize the estimates we have obtained so far :

Proposition 4.3. *We have*

$$\begin{aligned}
&-I_F^{(H)}(v_i\le v\le v_{i+1})(w_i\le w\le\infty)(r\le r_1)\\
&-F_F^{(H)}(v=v_{i+1})(w_i\le w\le\infty)-F_F^{(H)}(w=\infty)(v_i\le v\le v_{i+1})\\
\lesssim\ & F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i\le v\le v_{i+1})-F_F^{(H)}(v=v_i)(w_i\le w\le\infty)\\
&+E_F^{(\frac{\partial}{\partial t})}(-(0.85)t_i\le r^*\le(0.85)t_i)(t=t_i)+\frac{(E_F^{\#(\frac{\partial}{\partial t})}(t=t_0)+E_F^{(\frac{\partial}{\partial t})}(t=t_0)+1)^2}{t_i}
\end{aligned}\tag{4.14}$$

Proof.

Let us first recall (4.9) :

$$\begin{aligned}
&-I_F^{(H)}(v_i\le v\le v_{i+1})(w_i\le w\le\infty)\\
&-F_F^{(H)}(w=\infty)(v_i\le v\le v_{i+1})-F_F^{(H)}(v=v_{i+1})(w_i\le w\le\infty)\\
=\ &-F_F^{(H)}(w=w_i)(v_i\le v\le v_{i+1})-F_F^{(H)}(v=v_i)(w_i\le w\le\infty)
\end{aligned}$$

From (4.10), we have,

$$-F_F^{(H)}(w=w_i)(v_i\le v\le v_{i+1})\lesssim F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i\le v\le v_{i+1})$$

Thus, we obtain

$$\begin{aligned}
&-I_F^{(H)}(v_i\le v\le v_{i+1})(w_i\le w\le\infty)\\
&-F_F^{(H)}(w=\infty)(v_i\le v\le v_{i+1})-F_F^{(H)}(v=v_{i+1})(w_i\le w\le\infty)\\
\le\ & CF_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i\le v\le v_{i+1})-F_F^{(H)}(v=v_i)(w_i\le w\le\infty)
\end{aligned}\tag{4.15}$$

(where C is a constant).

From Proposition 4.2, we get (4.14).

□

4.2.4. *Estimate on the energy for observers travelling to the black hole.*

Proposition 4.4. *We have*

$$\begin{aligned}
&\inf_{v_i \leq v \leq v_{i+1}} -F_F^{(H)}(v)(w_i \leq w \leq \infty) \\
\lesssim \;& \frac{-I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1)}{(v_{i+1} - v_i)} + \sup_{v_i \leq v \leq v_{i+1}} F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned} \tag{4.16}$$

Proof.

Using (4.5)-(4.8) in the region $r \leq r_1$, we get for $v \geq v_i$,

$$\begin{aligned}
& -F_F^{(H)}(v)(w_i \leq w \leq \infty)(r \leq r_1) \\
\lesssim \;& \int_{w=w_i, r\leq r_1}^{w=\infty} \int_{\mathcal{S}^2} ([|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2]([\frac{\mu}{r}h - h^{'}]) + [|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2](\frac{h^{'}}{N}) \\
& + [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].\mu[\frac{1}{N}h^{'} - \frac{3}{r}h]).r^2 d\sigma^2 N dw
\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}
& F_F^{(\frac{\partial}{\partial t})}(v = v_i)(w_i \leq w \leq w_{i+1}) \\
= \;& \int_{w=w_i}^{w=w_{i+1}} \int_{\mathcal{S}^2} 2[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\phi}}|^2]r^2 N d\sigma^2 dw
\end{aligned}$$

Thus, from the boundedness of h, h', we have in $r \geq r_1$,

$$-F_F^{(H)}(v)(w_i \leq w \leq \infty)(r \geq r_1) \quad \lesssim \quad F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)$$

Thus,

$$\begin{aligned}
& -F_F^{(H)}(v)(w_i \leq w \leq \infty) \\
\lesssim \;& \int_{w=w_i, r\leq r_1}^{w=\infty} \int_{\mathcal{S}^2} ([|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2]([\frac{\mu}{r}h - h^{'}]) + ([|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2](\frac{h^{'}}{N}) \\
& + [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].\mu[\frac{1}{N}h^{'} - \frac{3}{r}h]).r^2 d\sigma^2 N dw + F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned}$$

We have,

$$\begin{aligned}
& (v_{i+1}-v_i) \inf_{v_i \leq v \leq v_{i+1}} -F_F^{(H)}(v)(w_i \leq w \leq \infty) \\
\leq & \int_{v=v_i}^{v=v_{i+1}} -F_F^{(H)}(v)(w_i \leq w \leq \infty) dv \\
\lesssim & \int_{v=v_i, r\leq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathcal{S}^2} ([|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2]([\frac{\mu}{r}h - h^{'}]) + ([|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2](\frac{h^{'}}{N}) \\
& +[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].\mu[\frac{1}{N}h^{'} - \frac{3}{r}h]).r^2 d\sigma^2 N dw dv + \int_{v=v_i}^{v=v_{i+1}} F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) dv \\
\lesssim & -I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\
& +(v_{i+1}-v_i) \sup_{v_i \leq v \leq v_{i+1}} F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned}$$

□

4.2.5. *Bounding the bulk term generated from H near the horizon.*

Proposition 4.5. *We have*

$$0 \leq -I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \quad \lesssim \quad (E_F^{(\frac{\partial}{\partial t})} + E_F^{\#(\frac{\partial}{\partial t})}(t=t_0) + 1)^2 \tag{4.17}$$

where,

$$\begin{aligned}
& E_F^{\#(\frac{\partial}{\partial t})}(t=t_0) \\
= & \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathcal{S}^2} [N|F_{\hat{w}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\phi}}|^2 + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2 + |F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].r^2 d\sigma^2 dr^*(t=t_0)
\end{aligned} \tag{4.18}$$

Proof. We have

$$\begin{aligned}
E_F^{(H)}(t) \quad = & \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathcal{S}^2} [hN(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + h(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2) + hN(|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2) \\
& +h(|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2)]r^2 d\sigma^2 dr^*
\end{aligned}$$

By using the divergence theorem in the region $\{v \leq v_0, t_0 \leq t \leq \infty, r \leq r_1\}$, we obtain

$$\begin{aligned}
& -I_F^{(H)}(v \leq v_0)(t_0 \leq t \leq \infty)(r \leq r_1) \\
& -F_F^{(H)}(v=v_0)(w_0 \leq w \leq \infty) - F_F^{(H)}(w=\infty)(-\infty \leq v \leq v_0) \\
= \quad & E_F^{(H)}(t_0)(r \leq r_1)
\end{aligned}$$

Due to the positivity of the terms on the left hand side, we get

$$-F_F^{(H)}(v=v_0)(w_0 \leq w \leq \infty) \lesssim E_F^{(H)}(t_0) \quad \lesssim \quad E_F^{\#(\frac{\partial}{\partial t})}(t=t_0) \tag{4.19}$$

From the divergence theorem and the fact that $\frac{\partial}{\partial t}$ is Killing, it is easy to see that by integrating in a suitable region and using the positivity of the energy we get,

$$\begin{aligned} F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i \leq v \leq v_{i+1}) &= F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i \leq v \leq v_{i+1})(r \geq r_1) \\ &\lesssim E_F^{(\frac{\partial}{\partial t})}(t_i) \end{aligned} \tag{4.20}$$

From (4.14) and (4.20) we get,

$$\begin{aligned} & -I_F^{(H)}(v_0 \leq v \leq v_1)(w_0 \leq w \leq \infty)(r \leq r_1) \\ & -F_F^{(H)}(v=v_1)(w_0 \leq w \leq \infty) - F_F^{(H)}(w=\infty)(v_0 \leq v \leq v_1) \\ \lesssim\ & F_F^{(\frac{\partial}{\partial t})}(w=w_0)(v_0 \leq v \leq v_1) - F_F^{(H)}(v=v_0)(w_0 \leq w \leq \infty) \\ & +E_F^{(\frac{\partial}{\partial t})}(-(0.85)t_0 \leq r^* \leq (0.85)t_0)(t=t_0) + \frac{(1+E_F^{\#(\frac{\partial}{\partial t})}(t=t_0)+E_F^{(\frac{\partial}{\partial t})}(t=t_0))^2}{t_0} \\ \lesssim\ & (1+E_F^{(\frac{\partial}{\partial t})}+E_F^{\#(\frac{\partial}{\partial t})}(t=t_0))^2. \end{aligned}$$

On the other hand, for all $r \leq r_1$, we have by (4.5)-(4.8),

$$h' - \frac{\mu}{r}h \leq 0 \tag{4.21}$$

$$\frac{-h'}{N} \leq 0 \tag{4.22}$$

$$\mu[\frac{-1}{N}h' + \frac{3}{r}h] \leq 0 \tag{4.23}$$

Thus,

$$\begin{aligned} & I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\ =\ & \int_{v=v_i, r \leq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathcal{S}^2} ([|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2](h' - \frac{\mu}{r}h) + ([|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2](\frac{-h'}{N}) \\ & +[|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].\mu[\frac{-1}{N}h' + \frac{3}{r}h]).r^2 d\sigma^2 N dw dv \\ \leq\ & 0 \end{aligned} \tag{4.24}$$

Hence, by recurrence from inequality (4.14), and using (4.20), we obtain for all integers i

$$\begin{aligned} & -I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\ & -F_F^{(H)}(v=v_{i+1})(w_i \leq w \leq \infty) - F_F^{(H)}(w=\infty)(v_i \leq v \leq v_{i+1}) \\ \lesssim\ & F_F^{(\frac{\partial}{\partial t})}(w=w_i)(v_i \leq v \leq v_{i+1}) - F_F^{(H)}(v=v_i)(w_i \leq w \leq \infty) \\ & +E_F^{(\frac{\partial}{\partial t})}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t=t_i) + \sum_{i=0}^{\infty} \frac{(1+E_F^{\#(\frac{\partial}{\partial t})}(t=t_0)+E_F^{(\frac{\partial}{\partial t})}(t=t_0))^2}{t_i} \\ \lesssim\ & (1+E_F^{(\frac{\partial}{\partial t})}(t=t_i)+E_F^{\#(\frac{\partial}{\partial t})}(t=t_0))^2 \end{aligned}$$

Due to sign of h, and the definition of h, we have that the terms in each of the integrands on the left hand side are positive, hence, we obtain (4.17).

$\square$

4.2.6. *Decay of the flux of H.*

Proposition 4.6. *For all v, let*

$$w_0(v) = v - 2r_1^*$$

Let

$$v_+ = \max\{1, v\} \tag{4.25}$$

We have,

$$-F_F^{(H)}(v)(w_0(v) \leq w \leq \infty) \lesssim \frac{[(E_F^{(\frac{\partial}{\partial t})} + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + 1)^2 + E_F^{(K)}(t_0)]}{v_+} \tag{4.26}$$

and,

$$-F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v) \lesssim \frac{[(E_F^{(\frac{\partial}{\partial t})} + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + 1)^2 + E_F^{(K)}(t_0)]}{v_+} \tag{4.27}$$

For the proof we need several lemmas. In this section we choose

$$t_i = (1.1)^i t_0,\ t_0 > 0.$$

Lemma 4.2. *We have,*

$$\sup_{v_i \leq v \leq v_{i+1}} F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \lesssim \frac{E_F^{(K)}(t_i)}{t_i^2} \tag{4.28}$$

Proof. By integrating in a well chosen region and using the divergence theorem we get that,

$$\begin{aligned} &\sup_{v_i \leq v \leq v_{i+1}} F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \\ \lesssim &\int_{r^*=r_1^*-0.1t_i}^{r^*=r_1^*+0.1t_i} \int_{\mathcal{S}^2} (N|F_{\hat{w}\hat{\theta}}|^2 + N|F_{\hat{w}\hat{\phi}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\theta}}|^2 + \frac{1}{N}|F_{\hat{v}\hat{\phi}}|^2 + |F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2).Nr^2 d\sigma^2 dr^*(t_i) \end{aligned} \tag{4.29}$$

Thus using (3.7) to estimate (4.29) gives (4.28).

□

By Proposition 4.4 and Proposition 4.5 we obtain :

$$\inf_{v_i \leq v \leq v_{i+1}} F_F^{(H)}(v)(w_i \leq w \leq \infty) \lesssim \frac{1}{(v_{i+1} - v_i)} (E_F^{(\frac{\partial}{\partial t})} + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + 1)^2 + \frac{E_F^{(K)}(t_i)}{t_i^2} \tag{4.30}$$

and thus, there exists a $v_i^{\#} \in [v_i, v_{i+1}]$ where the above inequality holds.

We have,

$$v_{i+1} - v_i = 0.1 t_i$$

Let,

$$w_i^{\#} = v_i^{\#} - 2r_1^* \tag{4.31}$$

Note that $w_i^{\#} \geq w_i$.

Therefore we have using (4.30) and the positivity of $-F_F^{(H)}(v_i^{\#})(w_i \leq w \leq w_i^{\#})$

$$\begin{aligned} -F_F^{(H)}(v_i^{\#})(w_i^{\#} \leq w \leq \infty) &\lesssim -F_F^{(H)}(v_i^{\#})(w_i \leq w \leq \infty) \\ &\lesssim \frac{1}{t_i}(E_F^{(\frac{\partial}{\partial t})} + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + 1)^2 + \frac{E_F^{(K)}(t_i)}{t_i^2} \end{aligned}$$

From (4.14), applied in the region $[w_i^{\#}, \infty] \times [v_i^{\#}, v_{i+1}]$, we get due to the positivity of $-I_F^{(H)}(v_i^{\#} \leq v \leq v_{i+1})(w_i^{\#} \leq w \leq \infty)(r \leq r_1)$, and $-F_F^{(H)}(w = \infty)(v_i^{\#} \leq v \leq v_{i+1})$, that,

$$\begin{aligned} & -F_F^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty) \\ \lesssim\ & F_F^{(\frac{\partial}{\partial t})}(w = w_i^{\#})(v_i^{\#} \leq v \leq v_{i+1}) - F_F^{(H)}(v = v_i^{\#})(w_i^{\#} \leq w \leq \infty) \\ & + E_F^{(\frac{\partial}{\partial t})}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i) + \frac{(1 + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^{(\frac{\partial}{\partial t})}(t = t_0))^2}{t_i} \end{aligned}$$

Lemma 4.3.

$$F_F^{(\frac{\partial}{\partial t})}(w = w_i^{\#})(v_i^{\#} \leq v \leq v_{i+1}) \quad \lesssim \quad \frac{E_F^{(K)}(t_i)}{t_i^2}$$

Proof. By applying the divergence theorem in a well chosen region, we get,

$$F_F^{(\frac{\partial}{\partial t})}(w = w_i^{\#})(v_i^{\#} \leq v \leq v_{i+1}) \quad \lesssim \quad E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i)$$

By Proposition 3.3 we have

$$\begin{aligned} & E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i) \\ \lesssim \quad & \frac{E_F^{(K)}(t_i)}{\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2} + \frac{E_F^{(K)}(t_i)}{\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i + r^*|^2} \end{aligned} \tag{4.32}$$

For $r^* \in [r_1^*, (0.1)t_i + r_1^*]$, and t_i large enough,

$$\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2 \geq |(0.9)t_i - r_1^*|^2$$

Therefore,

$$E_F^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i) \quad \lesssim \quad \frac{E_F^{(K)}(t_i)}{t_i^2}$$

□

We now apply again the divergence theorem and obtain using Lemma 4.3, Proposition 4.5 and (4.30)

$$-F_F^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty) \quad \lesssim \quad \frac{1}{t_i}(E_F^{(\frac{\partial}{\partial t})} + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + 1)^2 + \frac{E_F^{(K)}(t_0)}{t_i^2}$$

and thus,

$$\begin{aligned} -F_F^{(H)}(v = v_{i+1})(w_{i+1} \leq w \leq \infty) \quad & \lesssim \quad -F_F^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty) \\ & \lesssim \quad \frac{(E_F^{(\frac{\partial}{\partial t})}(t_0) + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + 1)^2 + E_F^{(K)}(t_0)}{t_i} \end{aligned} \tag{4.33}$$

(due to the positivity of $-F_F^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty)$).

4.3. Decay for the middle components.

Proposition 4.7. *Let v_+ be as defined in (4.25), we have for all r, such that $2m \leq r \leq r_1$,*

$$|F_{\hat{\theta}\hat{\phi}}(v, w, \omega)| = |\frac{W^2(v, w) - 1}{r^2}| \lesssim \frac{E_1}{\sqrt{v_+}}$$

where,

$$E_1 = [(1 + E_F^{(\frac{\partial}{\partial t})}(t = t_0) + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0))^2 + E_F^{(K)}(t_0)]^{\frac{1}{2}}$$

Proof.

We have by a Sobolev inequality, for $r \geq 2m$, $r \leq r_1$,

$$\begin{aligned}
\|(W^2 - 1)\|^2_{L^\infty} &\lesssim \int_{\overline{v}=v-1}^{\overline{v}=v} \int_{\mathcal{S}^2} (W^2 - 1)^2 d\sigma^2 d\overline{v} + \int_{\overline{v}=v-1}^{\overline{v}=v} \int_{\mathcal{S}^2} (\partial_v (W^2 - 1))^2 d\sigma^2 d\overline{v} \\
&\lesssim -F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v) + \int_{\overline{v}=v-1}^{\overline{v}=v} \int_{\mathcal{S}^2} W^2 (\partial_v W)^2 d\sigma^2 d\overline{v} \\
&\lesssim -F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v) + \int_{\overline{v}=v-1}^{\overline{v}=v} \int_{\mathcal{S}^2} (W^2 - 1)(\partial_v W)^2 d\sigma^2 d\overline{v} \\
&\lesssim (1 + \|(W^2 - 1)\|_{L^\infty}).(-F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v))
\end{aligned}$$

(where we used what we already proved, and the fact that r is bounded in the region $0 < 2m \leq r \leq r_1$). Hence,

$$\|(W^2 - 1)\|_{L^\infty} \lesssim \sqrt{|F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v))|} + |F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v))|$$

Consequently, we have,

$$|F_{\hat{\theta}\hat{\phi}}(v, w, \omega)|^2 \lesssim -F_F^{(H)}(w)(v - 1 \leq \overline{v} \leq v)$$

Thus,

$$|F_{\hat{\theta}\hat{\phi}}(v, w_0, \omega)| \lesssim \frac{E_1}{(v_+)^{\frac{1}{2}}}$$

□

Appendix A. Proof of Theorem 1

In this appendix, we prove Theorem 1. In the following $\|.\|$ always stands for $\|.\|_{L^2}$, and $\int$ for $\int_{\mathbb{R}}$. The following lemma collects some useful estimates; it proves in particular that the space $\mathcal{H}^1 \times L^2$ is exactly the space of finite energy solutions.

Lemma A.1. *We have*

$$\|W\|^4_{L^4_P} \lesssim \int \frac{P}{2}(W^2-1)^2 dr^* + 1, \tag{A.1}$$

$$\int \frac{P}{2}(W^2-1)^2 dr^* \lesssim \|W\|^4_{L^4_P} + 1, \tag{A.2}$$

$$\|W\|_{L^2_P} \lesssim \|W\|_{L^4_P}, \tag{A.3}$$

$$\|\sqrt{P}W^2\|_{L^\infty} \lesssim \|W'\|^2_{L^2} + \|W\|^2_{L^4_P}. \tag{A.4}$$

Proof. We first show (A.1). We estimate

$$\begin{aligned}
\|W\|^4_{L^4_P} &= \int PW^4 = \int P(W^2-1)(W^2+1) + \int P \\
&= \int P(W^2-1)^2 + 2\int P(W^2-1) + \int P \\
&\leq \int P(W^2-1)^2 + 2\left(\int P\right)^{1/2}\left(\int P(W^2-1)^2\right)^{1/2} + \int P \\
&\lesssim \int \frac{P}{2}(W^2-1)^2 + 1.
\end{aligned}$$

Here, we have used the Cauchy-Schwarz inequality in the third line.

(A.2) follows from

$$\int \frac{P}{2}(W^2-1)^2 \lesssim \int PW^4 + \int P.$$

(A.3) follows from the Cauchy-Schwarz inequality

$$\int PW^2 \leq \left(\int P\right)^{1/2}\left(\int PW^4\right)^{1/2}.$$

To show (A.4) we use the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$. This gives

$$\begin{aligned}
\|\sqrt{P}W^2\|^2_{L^\infty} &\lesssim \int PW^4 + \int P^{-1}(P')^2W^4 + \int PW^2(W')^2 \\
&\lesssim \|W\|^4_{L^4_P} + \|W'\|^2\|\sqrt{P}W^2\|_{L^\infty}.
\end{aligned}$$

Here we have used $P^{-1}P' \lesssim P$. This quadratic inequality implies the result. □

We now write the Yang-Mills equation as a first order system. We put

$$\Psi = (W, \frac{1}{i}\partial_t W).$$

If W is solution of the Yang-Mills equation, then $\Psi = (\psi_1, \psi_2)$ solves

$$\left\{ \begin{array}{rcl} \partial_t \Psi & = & A\Psi + F(\Psi), \\ \Psi(0) & = & (W_0, \frac{1}{i} W_1) =: \Psi_0, \end{array} \right. \tag{A.5}$$

Here,

$$A = i \begin{pmatrix} 0 & \mathbb{1} \\ -\partial^2_{r^*} & 0 \end{pmatrix}, \quad F(\Psi) = \begin{pmatrix} 0 \\ iP\psi_1(\psi_1^2 - 1) \end{pmatrix}.$$

Let $X = \mathcal{H}^1 \times L^2$. Because of (A.1) and (A.2) X is exactly the space of finite energy solutions. Note that the natural domain of A is

$$D(A) = \mathcal{H}^2 \times H^1 =: Z.$$

Remark A.1. (1) X is defined as a complex Hilbert space. Nevertheless we are looking for real solutions of (1.29) and therefore for solutions of (A.5) with real first component and purely imaginary second component. This subspace is of course preserved by the evolution and we can in the following suppose in our estimates that ψ_1 and $i\psi_2$ are real.

(2) Note that in this setting the conserved energy writes

$$\mathcal{E}(\Psi) = \int |\psi_2|^2 + |\psi_1'|^2 + \frac{P}{2}(\psi_1^2 - 1)^2 dr^*.$$

Theorem 1 will follow from

Theorem 4. *Let $\Psi_0 \in Z$. Then* (A.5) *has a unique strong solution*

$$\Psi \in C^1([0,\infty); X) \cap C([0,\infty); Z).$$

Proof of Theorem 1 supposing Theorem 4. The only point that doesn't follow immediately from Theorem 4 is

$$\sqrt{P}(W^2 - 1) \in C([0,\infty); H^1).$$

We compute

$$(\sqrt{P}(W^2 - 1))' = \frac{1}{2} P^{-1/2} P'(W^2 - 1) + 2\sqrt{P} W W'.$$

As $P^{-1/2}P' \lesssim \sqrt{P}$ the first term is continuous by the property

$$\sqrt{P}(W^2 - 1) \in C^1([0,\infty); L^2).$$

For the second term we compute

$$\sqrt{P}((WW')(t) - (WW')(t_0)) = \sqrt{P}(W(t) - W(t_0))W'(t_0) + \sqrt{P}W(t)(W'(t) - W'(t_0))$$

and thus

$$\begin{array}{rcl} \|\sqrt{P}((WW')(t) - (WW')(t_0)\|^2 & = & \int P(W(t) - W(t_0))^2 W'^2(t_0) + \int PW^2(t)(W'(t) - W'(t_0))^2 \\ & \lesssim & \|\sqrt{P}(W(t) - W(t_0))^2\|_{L^\infty} \|W'(t_0)\|^2 \\ & + & \|\sqrt{P}W^2(t)\|_{L^\infty} \|W'(t) - W'(t_0)\|^2 \\ & \lesssim & (\|W'(t) - W'(t_0)\|^2 + \|W(t) - W(t_0)\|^2_{L^4_P}) \|W'(t_0)\|^2 \\ & + & (\|W'(t)\|^2 + \|W(t)\|^2_{L^4_P}) \|W'(t) - W'(t_0)\|^2 \to 0,\ t \to 0. \end{array}$$

Here we have also used (A.4). $\square$

It therefore remains to show Theorem 4. We start by studying the linear part:

Lemma A.2. *A is the generator of a C^0- semigroup on X.*

Proof. First note that $e^{tA}(\psi_1, \psi_2)$ defined as the solution at time t for initial data (ψ_1, ψ_2) sends $C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ into itself because of finite propagation speed. We want to extend this propagator to X. First note that

$$\|e^{tA}(\psi_1, \psi_2)\|_{\dot{H}^1 \times L^2} = \|e^{tA}(\psi_1, \psi_2)\|_{\dot{H}^1 \times L^2} \tag{A.6}$$

because $-iA$ is selfadjoint on $\dot{H}^1 \times L^2$. Now, we have for $(\psi_1, \psi_2) \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$

$$(e^{tA}(\psi_1, \psi_2))_1 = \frac{1}{2}(\psi_1(r^* + t) + \psi_1(r^* - t) + i \int_{r^*-t}^{r^*+t} \psi_2(s)ds). \tag{A.7}$$

We estimate

$$\|\psi_1(r^* + t) + \psi_1(r^* - t)\|_{L^4_P} \lesssim \|\psi_1\|_{L^4_P}, \tag{A.8}$$

$$\begin{aligned} \int P \left| \int_{r^*-t}^{r^*+t} \psi_2(s)ds \right|^4 dr^* &\leq \int P \left(\int_{r^*-t}^{r^*+t} |\psi_2(s)|^2 ds \right)^2 4t^2 dr^* \\ &\leq 4t^2 \int P dr^* \|\psi_2\|^4 \lesssim t^2 \|\psi_2\|^4. \end{aligned} \tag{A.9}$$

Therefore, e^{tA} extends to a semigroup on X. It remains to check that it is strongly continuous. Again, because of the selfadjointness of $-iA$, we have for $(\psi_1, \psi_2) \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$

$$\|(e^{tA} - \mathbb{1})(\psi_1, \psi_2)\|_{\dot{H}^1 \times L^2} \to 0,\ t \to 0.$$

Using (A.7) and the Lebesgue lemma, we see that for $(\psi_1, \psi_2) \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$

$$\|((e^{tA} - \mathbb{1})(\psi_1, \psi_2))_1\|_{L^4_P} \to 0,\ t \to 0.$$

Strong continuity follows now from a density argument, using (A.6), (A.8) and (A.9).

□

For the nonlinear part, we need:

Lemma A.3. *$F : X \to X$ is continuously differentiable.*

Proof. We first establish that F sends X into X. This follows from the computation

$$\|F(\Psi)\|_X \leq \|\sqrt{P}\psi_1\|(\|\sqrt{P}\psi_1^2\|_{L^\infty} + 1) \leq \|\psi_1\|_{L^4_P}(\|\Psi\|_X^2 + 1).$$

Here, we have used (A.3) and (A.4). Let $h = (h_1, h_2) \in X$. We compute

$$\frac{1}{i}(F(\Psi + h) - F(\psi)) = \begin{pmatrix} 0 & 0 \\ P(3\psi_1^2 - 1) & 0 \end{pmatrix} h + \begin{pmatrix} 0 \\ 3P\psi_1 h_1^2 + P h_1^3 \end{pmatrix}$$

We first have to check that the matrix on the R.H.S. defines a linear bounded operator on X. We estimate

$$\begin{aligned} \|P\psi_1^2 h_1\| &\leq \|\sqrt{P}\psi_1^2\|_{L^\infty} \|\sqrt{P}h_1\| \leq \|\Psi\|_X^2 \|h_1\|_{L^4_P}, \\ \|Ph_1\| &\leq \|\sqrt{P}h_1\| \leq \|h_1\|_{L^4_P}. \end{aligned}$$

Here, we have used (A.3) and (A.4). It remains to show that the second term is of order $\mathcal{O}(\|h\|_X^2)$. We estimate using again (A.3) and (A.4)

$$\begin{aligned}\|P\psi_1 h_1^2\| &\le \|\sqrt{P}h_1^2\|_{L^\infty}\|\sqrt{P}\psi_1\| \le \|h\|_X^2\|\psi_1\|_{L^4_P},\\ \|Ph_1^3\| &\le \|\sqrt{P}h_1\|\|\sqrt{P}h_1^2\|_{L^\infty} \lesssim \|h\|_X^3.\end{aligned}$$

Let $\mathcal{B}(X)$ be the space of bounded linear operators on X. It remains to show that

$$\Psi \mapsto \tilde{\mathcal{L}}(\Psi) = \begin{pmatrix} 0 & 0 \\ 3P\psi_1^2 - 1 & 0 \end{pmatrix}$$

is continuous as an application from X to $\mathcal{B}(X)$. This obviously follows from the continuity of

$$\Psi \mapsto \mathcal{L}(\Psi) = \begin{pmatrix} 0 & 0 \\ P\psi_1^2 & 0 \end{pmatrix}.$$

We estimate

$$\begin{aligned}\|(\mathcal{L}(\Psi) - \mathcal{L}(\Phi))h\|_X^2 &= \|P(\psi_1^2 - \phi_1^2)h_1\|_{L^2}^2\\ &\le \|\sqrt{P}(\psi_1 - \phi_1)\|_{L^2}^2\|\sqrt{P}(\psi_1 + \phi_1)h_1\|_{L^\infty}^2\\ &\le \|\Psi - \Phi\|_X^2\|\sqrt{P}(\psi_1 + \phi_1)h_1\|_{L^\infty}^2.\end{aligned}$$

Here we have used (A.3). By the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and the fact that $P^{-1}(P')^2 \lesssim P$ we obtain

$$\begin{aligned}\|\sqrt{P}(\psi_1 + \phi_1)h_1\|_{L^\infty}^2 &\lesssim \int P(\psi_1 + \phi_1)^2 h_1^2 + \int P(\psi_1' + \phi_1')^2 h_1^2 + \int P(\psi_1 + \phi_1)^2 (h_1')^2\\ &\lesssim \left(\int P(\psi_1 + \phi_1)^4\right)^{1/2}\left(\int Ph_1^4\right)^{1/2}\\ &+ \|\sqrt{P}h_1^2\|_{L^\infty}^2 \int (\psi_1')^2 + (\phi_1')^2 dr^* + \|\sqrt{P}(\psi_1 + \phi_1)^2\|_{L^\infty}\int (h_1')^2 dr^*\\ &\lesssim (\|\Psi\|_X^2 + \|\Phi\|_X^2)\|h\|_X^2.\end{aligned}$$

Here we have used the Cauchy-Schwarz inequality and (A.4). Summarizing we obtain

$$\|(\mathcal{L}(\Psi) - \mathcal{L}(\Phi))h\|_X^2 \lesssim \|\Psi - \Phi\|_X^2(\|\Psi\|_X^2 + \|\Phi\|_X^2)\|h\|_X^2$$

and thus

$$\|\mathcal{L}(\Psi) - \mathcal{L}(\Phi)\|_{\mathcal{B}(X)} \lesssim \|\Psi - \Phi\|_X(\|\Psi\|_X + \|\Phi\|_X),$$

which is the required estimate. □

We will also need the following a priori estimate

Proposition A.1. *There exists $C > 0$ such that for all solutions $\Psi \in C([0,\infty); Z) \cap C^1([0,\infty); X)$ of* (A.5) *we have uniformly in T:*

$$\|\Psi(t)\|_X \le C(\|\Psi_0\|_X + 1).$$

Proof. This follows from the conservation of energy. We have

$$\|\Psi(t)\|_X^4 \lesssim \mathcal{E}(\Psi) + 1 = \mathcal{E}(\Psi_0) + 1 \lesssim (\|\Psi_0\|_X^4 + 1).$$

□

Proof of Theorem 4 By [27, Theorem 6.1.5] (A.5) has a unique strong solution on some interval $[0, T]$. By [27, Theorem 6.1.4] and the a priori estimate of Proposition A.1, this solution is global. $\square$

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