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On the normal scalar curvature conjecture in Kenmotsu statistical manifolds

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ABSTRACT

In this paper, we prove DDVV conjecture (the generalized Wintgen inequality) for statistical submanifolds of Kenmotsu statistical manifolds of constant φ-sectional curvature. Further, we give some applications of derived inequality.

1. Introduction

In differential geometry, one of most fundamental research problem is to find the relationships between intrinsic and extrinsic invariants. For example; on the surfaces M^2 of the Euclidean space $\mathbb{E}^3$, the Euler inequality

$$G \leq \|H\|^2$$

is fulfilled, where G is the Gauss curvature (an intrinsic invariant) of M^2 and $\|H\|^2$ is the squared mean curvature (an extrinsic invariant) of M^2. Furthermore, the equality holds of above inequality if and only if M^2 is totally umbilical, or still, by a theorem of Meusnier, if and only if M^2 is a plane $\mathbb{E}^2$ or, it is a sphere $\mathbb{S}^2$ in $\mathbb{E}^3$.

In [24], P. Wintgen proved that the Gauss curvature G, the squared mean curvature $\|H\|^2$ and the normal curvature $G^{\perp}$ of any surface M^2 in $\mathbb{E}^4$ always satisfy the inequality

$$G + G^{\perp} \leq \|H\|^2 \tag{1.1}$$

and the equality holds if and only if the ellipse of curvature of M^2 in $\mathbb{E}^4$ is a circle. The inequality (1.1) is called Wintgen inequality and the Whitney 2-sphere satisfies the equality case of Wintgen inequality.

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Later, the Wintgen inequality was extended for the surfaces M^2 of codimension m in real space form $\tilde{M}^{m+2}(c)$ in [22] and [12] independently as:

$$G + G^{\perp} \leq \|H\|^2 + c.$$

The equality case was also investigated.

In 1999, De Smet, Dillen, Verstraelen, Vrancken [7] developed the generalized Wintgen inequality named as *DDVV conjecture* for the submanifolds in real space forms as follows:

Conjecture 1. *Let $f : M^n \to \tilde{M}^{n+m}(c)$ be an isometric immersion, where $\tilde{M}^{n+m}(c)$ is a real space form of constant sectional curvature c. Then*

$$\rho + \rho^{\perp} \leq \|H\|^2 + c.$$

where ρ is the normalized scalar curvature (intrinsic invariant) and $\rho^{\perp}$ is the normalized scalar normal curvature (extrinsic invariant).

If K and $R^{\perp}$ the sectional curvature function and the normal curvature tensor on M^n, respectively in $\tilde{M}^{n+m}(c)$, then the normalized scalar curvature ρ is given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} K(e_i, e_j) \tag{1.2}$$

where τ is the scalar curvature, and the normalized scalar normal curvature $\rho^{\perp}$ by

$$\rho = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n} \sum_{n+1\leq r<s\leq m+n} \left(R^{\perp}(e_i, e_j, e_r; e_s)\right)^2} \tag{1.3}$$

The Conjecture 1 was proven in [7] for a submanifold M^n of arbitrary dimension $n \geq 2$ and codimension 2 in the real space form $\tilde{M}^{n+2}(c)$ of constant sectional curvature c. Later, the DDVV conjecture was proved for general case in [15] and in [11] independently.

For a normally flat submanifold, i.e., $R^{\perp} = 0$, this conjecture is proved by B.-Y. Chen in [5,6]. Hence, the conjecture is true for the hypersurfaces of real space forms. In [16,17], some related inequalities are derived for normal scalar curvature in complex space forms.

Recently, I. Mihai proved DDVV conjecture for Lagrangian submanifolds in complex space forms [18] and Legendrian submanifolds in Sasakian space forms [19].

Next, in 1985, the concept of statistical manifolds was introduced by Amari [1] which provide a setting for the field of information geometry. The applications of statistical manifold in information geometry attracts the attention of distinguished geometers. Statistical manifolds are Riemannian manifolds with an affine connection besides the Levi-Civita connection. Moreover, submanifolds of statistical manifolds have been demonstrated by various different authors (for instance, see [4,8,9]). Recently, in [10] Furuhata et al. demonstrated Kenmotsu statistical manifolds and warped products.

The Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature has been studied in (for instance, see [2–4,20,21]).

In this paper, we establish the generalized Wintgen inequality (DDVV conjecture) for statistical submanifolds in Kenmotsu statistical manifolds of constant φ-sectional curvature. As a consequence, we give some applications of derived inequality.

2. Statistical manifolds and their submanifolds

In this section, we provide some basic formulas and preliminaries on statistical manifold and their submanifolds.

A *statistical manifold* is a Riemannian manifold $(\tilde{N}^{n+k}, \tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*)$ with a couple of torsion free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ fascinating

$$(\tilde{\nabla}_X \tilde{g})(Y, Z) = (\tilde{\nabla}_Y \tilde{g})(X, Z), \tag{2.1}$$

$$X\tilde{g}(Y, Z) = \tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(Y, \tilde{\nabla}^*_X Z), \tag{2.2}$$

for $X, Y, Z \in \Gamma(T\tilde{N})$. An affine connection $\tilde{\nabla}^*$ is the dual (or conjugate) connection, i.e., $(\tilde{\nabla}^*)^* = \tilde{\nabla}$.

The definition of the dual (or conjugate) connection $\tilde{\nabla}^*$ can be seen in the following remark:

Remark 1 (*[23]*). $(\tilde{\nabla}^*, \tilde{g})$ is also a *statistical structure* where $\tilde{\nabla}^*$ is defined as

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^{\circ}, \tag{2.3}$$

Here, $\tilde{\nabla}^{\circ}$ is the Levi-Civita connection for $\tilde{N}$.

Now, first we give the definition of statistical submanifolds of a statistical manifold and then we give some notations and general formulas.

Let M^n be a submanifold of a statistical manifold $(\tilde{N}^{2m+1}, \tilde{g})$. Then, (M^n, g) is the statistical submanifold with:

$$\begin{cases} \textit{induced connections}, & \nabla, \nabla^*; \\ \textit{second fundamentalforms}, & h, h^*; \\ \textit{shape operators}, & A, A^*; \\ \textit{normal connections}, & \nabla^\perp, \nabla^{*\perp}. \end{cases}$$

Moreover, the induced metric g is unique, ∇ and ∇^* are induced conjugate statistical connections on the submanifold M.

Let us denote $\Gamma(TM)$ and $\Gamma(T^\perp M)$ to be the set of all sections of tangent and normal bundle to M respectively. Then, with $X, Y \in \Gamma(TM)$, the Gauss formulas for the connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are outlined by [23]

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y). \tag{2.5}$$

respectively, where h and h^* are bilinear maps from which the bilinear transformations A_υ and A^*_υ are given by [23]

$$g(A_\upsilon X, Y) = g(h(X, Y), \upsilon), \tag{2.6}$$

$$g(A^*_\upsilon X, Y) = g(h^*(X, Y), \upsilon), \tag{2.7}$$

for any $\upsilon \in \Gamma(TM^\perp)$. Furthermore, the Weingarten formulas for the connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ follow [23]

$$\tilde{\nabla}_X \upsilon = -A^*_\upsilon X + \nabla^\perp_X \upsilon, \tag{2.8}$$

$$\tilde{\nabla}^*_X \upsilon = -A_\upsilon X + \nabla^{*\perp}_X \upsilon, \tag{2.9}$$

respectively, where the normal dual connections $\nabla^\perp$ and $\nabla^{*^\perp}$ are the Riemannian dual connections on $M^\perp$.

Let us denote $\tilde{R}$ and $\tilde{R}^*$ (R and R^*) to be the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$ (∇ and ∇^*), respectively. Then, the basic fundamental equations namely Gauss equations and Ricci equations respectively are [23]

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \tilde{g}(h(X, Z), h^*(Y, W)) - g(h^*(X, W), h(Y, Z)), \tag{2.10}$$

$$\tilde{g}(\tilde{R}^*(X, Y)Z, W) = g(R^*(X, Y)Z, W) + \tilde{g}(h^*(X, Z), h(Y, W)) - g(h(X, W), h^*(Y, Z)), \tag{2.11}$$

and

$$g(R^\perp(X, Y)\upsilon_1, \upsilon_2) = g(\tilde{R}(X, Y)\upsilon_1, \upsilon_2) + g([A^*_{\upsilon_1}, A_{\upsilon_2}]X, Y), \tag{2.12}$$

$$g(R^{*\perp}(X, Y)\upsilon_1, \upsilon_2) = g(\tilde{R}^*(X, Y)\upsilon_1, \upsilon_2) + g([A_{\upsilon_1}, A^*_{\upsilon_2}]X, Y), \tag{2.13}$$

where $R^\perp$ is the curvature tensor of normal connection $\nabla^\perp$ on $\Gamma(T^\perp M)$ and $\upsilon_1, \upsilon_2 \in \Gamma(T^\perp M)$.

The curvature tensor fields $\tilde{R}$ and $\tilde{R}^*$ of dual connections satisfy

$$\tilde{g}(\tilde{R}^*(X, Y)Z, W) = -g(\tilde{R}(X, Y)W, Z). \tag{2.14}$$

Now, let $\{e_i\}_1^n$ and $\{e_r\}_{n+1}^{2m+1}$ be an orthonormal tangent and an orthonormal normal frames respectively on M^n. The mean curvature vector fields H and H^* have the following forms [14].

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}\sum_{r=n+1}^{2m+1}\left(\sum_{i=1}^{n} h^r_{ii}\right) e_r, \tag{2.15}$$

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n}\sum_{r=n+1}^{2m+1}\left(\sum_{i=1}^{n} h^{*r}_{ii}\right) e_r, \tag{2.16}$$

where $h^r_{ij} = \tilde{g}\big(h(e_i, e_j), e_r\big)$ and $h^{*r}_{ij} = \tilde{g}\big(h^*(e_i, e_j), e_r\big)$.

Moreover, the squared mean curvatures are stated by [14]

$$\|H\|^2 = \frac{1}{n^2}\sum_{r=n+1}^{2m+1}\left(\sum_{i=1}^{n} h^r_{ii}\right)^2, \; \|H^*\|^2 = \frac{1}{n^2}\sum_{r=n+1}^{2m+1}\left(\sum_{i=1}^{n} h^{*r}_{ii}\right)^2.$$

We set

$$\|h\|^2 = \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2, \ \|h^*\|^2 = \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{*r})^2.$$

The Casorati curvatures $\mathcal{C}$ and $\mathcal{C}^*$ of the submanifold M can be expressed as

$$\mathcal{C} = \frac{1}{n}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 = \frac{\|h\|^2}{n}, \ \mathcal{C}^* = \frac{1}{n}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{*r})^2 = \frac{\|h^*\|^2}{n}.$$

For the orthonormal vector fields $X, Y \in \Gamma(TM)$, the sectional curvature K on statistical submanifold M^n of statistical manifold $\tilde{N}^{2m+1}$ is given by [2]

$$K(X,Y) = \frac{1}{2}\Big[g\big(R(X,Y)Y,X\big) + g\big(R^*(X,Y)Y,X\big)\Big]. \tag{2.17}$$

3. Kenmotsu statistical manifolds

Recently, one of the well-known geometer Furuhata [10] in 2017 deliberate a statistical structure on an almost contact metric manifold named as Kenmotsu statistical structure.

Let $\tilde{N}$ be a $(2m+1)$-dimensional smooth manifold with a Riemannian metric $g \in \Gamma(T\tilde{N}^{(0,2)})$, a structure tensor field $\varphi \in \Gamma(T\tilde{N}^{(1,1)})$ and a structure vector field $\xi \in \Gamma(T\tilde{N})$ such that

(1) (g, φ, ξ) defines an *almost contact metric structure* on $\tilde{N}$ satisfying [13]

$$\begin{cases} \eta(\xi) = 1, \ \eta\circ\varphi = 0, \ \varphi\xi = 0, \\ \varphi^2 X = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{cases} \tag{3.1}$$

for $X, Y \in \Gamma(T\tilde{N})$

(2) (g, φ, ξ) defines *Kenmotsu structure* on $\tilde{N}$ if (g, φ, ξ) is an almost contact metric structure on $\tilde{N}$ and satisfies

$$(\tilde{\nabla}_X^g\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{3.2}$$

where $X, Y \in \Gamma(T\tilde{N})$.

Then, $(\tilde{N}, g, \varphi, \xi)$ is called a *Kenmotsu manifold*. Moreover, we have

$$\tilde{\nabla}_X^g\xi = X - \eta(X)\xi, \tag{3.3}$$

for any $X \in \Gamma(T\tilde{N})$.

Definition 1 (*[10]*). Let $(\tilde{N}, g, \varphi, \xi)$ be a Kenmotsu manifold with statistical structure $(\tilde{\nabla} = \tilde{\nabla}^\circ + \mathcal{K}, g)$ on $\tilde{N}$. Then, $(\tilde{\nabla}, g, \varphi, \xi)$ is called *Kenmotsu statistical structure* on $\tilde{N}$ if

$$\mathcal{K}(X, \varphi Y) + \varphi\mathcal{K}(X,Y) = 0, \tag{3.4}$$

for $X, Y \in \Gamma(T\tilde{N})$. A manifold equipped with a Kenmotsu statistical structure is known as *Kenmotsu statistical manifold.*

Remark 2 (*[10]*). If $(\tilde{N}, \tilde{\nabla}, g, \varphi, \xi)$ is a Kenmotsu statistical manifold, then so is $(\tilde{N}, \tilde{\nabla}^*, g, \varphi, \xi)$.

Definition 2 (*[10]*). A Kenmotsu statistical manifold $(\tilde{N}, \tilde{\nabla}, g, \varphi, \xi)$ is said to be of constant $\varphi-$sectional curvature $c \in \mathbb{R}$ if

$$\begin{aligned}\tilde{R}(X,Y)Z = \left(\frac{c-3}{4}\right)\Big\{g(Y,Z)X - g(X,Z)Y\Big\} + \left(\frac{c+1}{4}\right)\Big\{&g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \\ &- \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y + \eta(Y)g(Z,X)\xi - \eta(X)g(Z,Y)\xi\Big\},\end{aligned} \tag{3.5}$$

for $X, Y, Z \in \Gamma(T\tilde{N})$ and symbolized by $\tilde{N}^{2m+1}(c)$.

Then, from (3.5), we obtain

Theorem 1. *On a Kenmotsu statistical manifold $\tilde{N}$, we have the following relations:*

(1) $\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y$
(2) $\tilde{R}(\xi,X)Y = g(X,Y)\xi - \eta(Y)X$

(3) $\tilde{R}(\varphi X, \xi)Y = \eta(Y)\varphi X - g(\varphi X, Y)\xi$
(4) $\tilde{R}(X, \varphi Y)\xi + \tilde{R}(\xi, X)\varphi Y = -\tilde{R}(\varphi Y, \xi)X$

for any $X, Y \in \Gamma(T\tilde{N})$.

On putting $X = W = e_i$ in (2.10) and considering (3.5), the expression for the Ricci curvature of M can be derived as

$$\begin{aligned}\mathrm{Ric}(Y,Z) &= \left(\frac{c-3}{4}\right)(n-1)g(Y,Z) + \left(\frac{c+1}{4}\right)\left[3g(\varphi Y, \varphi Z) - (n-2)\eta(Y)\eta(Z) - g(Y,Z)\right] \\ &\quad + \sum_{r=1}^{2m+1-n}\left[g(A_{e_r}Y, Z)\mathrm{tr}A^*_{e_r} - g(A^*_{e_r}Y, A_{e_r}Z)\right] \\ &= \left(\frac{c-3}{4}\right)(n-1)g(Y,Z) + \left(\frac{c+1}{4}\right)\left[2g(Y,Z) - (n+1)\eta(Y)\eta(Z)\right] \\ &\quad + \sum_{r=1}^{2m+1-n}\left[g(A_{e_r}Y, Z)\mathrm{tr}A^*_{e_r} - g(A^*_{e_r}Y, A_{e_r}Z)\right].\end{aligned}$$

Similarly, we have the dual Ricci curvature Ric^* of M,

$$\begin{aligned}\mathrm{Ric}^*(Y,Z) &= \left(\frac{c-3}{4}\right)(n-1)g(Y,Z) + \left(\frac{c+1}{4}\right)\left[2g(Y,Z) - (n+1)\eta(Y)\eta(Z)\right] \\ &\quad + \sum_{r=1}^{2m+1-n}\left[g(A^*_{e_r}Y, Z)\mathrm{tr}A_{e_r} - g(A_{e_r}Y, A^*_{e_r}Z)\right].\end{aligned}$$

Thus, we have the following useful result.

Theorem 2. *Let M^n be a statistical submanifold of a Kenmotsu statistical manifold $\tilde{N}^{2m+1}(c)$. Then, the Ricci tensor $\mathcal{Q}$ and the dual Ricci tensor $\mathcal{Q}^*$ of M satisfy the following:*

$$\mathcal{Q}(X) = \left(\frac{c-3}{4}\right)(n-1)X + \left(\frac{c+1}{4}\right)\left[2X - (n+1)\eta(X)\xi\right] + \sum_{r=1}^{2m+1-n}\left[(trA^*_{e_r})A_{e_r}X - A_{e_r}A^*_{e_r}X\right],$$

$$\mathcal{Q}^*(X) = \left(\frac{c-3}{4}\right)(n-1)X + \left(\frac{c+1}{4}\right)\left[2X - (n+1)\eta(X)\xi\right] + \sum_{r=1}^{2m+1-n}\left[(trA_{e_r})A^*_{e_r}X - A^*_{e_r}A_{e_r}X\right],$$

for all $X \in \Gamma(TM)$.

Now, from (1.2) and (2.17) together with (2.10) and (2.11) and (2.14), we obtain

$$\begin{aligned}2\tau &= \left(\frac{c-3}{2}\right)n(n-1) + \left(\frac{c+1}{2}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + \sum_{1\le i<j\le n}\Big[g\big(h(e_j,e_j), h^*(e_i,e_i)\big) - g\big(h(e_i,e_j), h^*(e_j,e_i)\big) \\ &\quad + g\big(h^*(e_j,e_j), h(e_i,e_i)\big) - g\big(h^*(e_i,e_j), h(e_j,e_i)\big)\Big] \\ &= \left(\frac{c-3}{2}\right)n(n-1) + \left(\frac{c+1}{2}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + 2n^2 g(H,H^*) - 2\sum_{1\le i<j\le n} g(h(e_i,e_j), h^*(e_j,e_i)),\end{aligned} \tag{3.6}$$

which gives

$$\begin{aligned}2\tau &\ge \left(\frac{c-3}{2}\right)n(n-1) + \left(\frac{c+1}{2}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + 2n^2 g(H,H^*) - 2\|h\|\|h^*\| \\ &= \left(\frac{c-3}{2}\right)n(n-1) + \left(\frac{c+1}{2}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + 2n^2 g(H,H^*) - 2n\sqrt{\mathcal{C}\mathcal{C}^*}.\end{aligned} \tag{3.7}$$

Moreover, we know $h^s_{ij} = g(h(e_i,e_j), e_s)$. Then, Eq. (3.6) can be expressed as

$$\rho = \left(\frac{c-3}{2}\right) + \left(\frac{c+1}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + \frac{1}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\le i<j\le n}\left[h^s_{jj}h^{*s}_{ii} - 2h^s_{ij}h^{*s}_{ij} + h^{*s}_{jj}h^s_{ii}\right]. \tag{3.8}$$

Thus, from (1.2) and (3.7), we have

Theorem 3. *Let M^n be a statistical submanifold of Kenmotsu statistical manifold $\tilde{N}^{2m+1}(c)$ of constant φ-sectional curvature c. Then, the normalized scalar curvature satisfies*

$$\rho \geq \left(\frac{c-3}{2}\right) + \left(\frac{c+1}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + \left(\frac{2n}{n-1}\right) g(H, H^*) - \left(\frac{2}{n-1}\right)\sqrt{CC^*}.$$

As a consequence, we have the following applications of Theorem 3.

1. If θ is the angle between the mean curvature vectors H and H^* of M, then from Theorem 3, we derive

$$\rho \geq \left(\frac{c-3}{2}\right) + \left(\frac{c+1}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + \left(\frac{2n}{n-1}\right) \|H\|\|H^*\|\cos\theta - \left(\frac{2}{n-1}\right)\sqrt{CC^*}. \tag{3.9}$$

2. If the mean curvature vectors H and H^* of M are parallel, then the normalized scalar curvature satisfies

$$\rho \geq \left(\frac{c-3}{2}\right) + \left(\frac{c+1}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) + \left(\frac{2n}{n-1}\right) \|H\|\|H^*\| - \left(\frac{2}{n-1}\right)\sqrt{CC^*}.$$

3. If the mean curvature vectors H and H^* of M are orthogonal, then the normalized scalar curvature ρ satisfies

$$\rho \geq \left(\frac{c-3}{2}\right) + \left(\frac{c+1}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right) - \left(\frac{2}{n-1}\right)\sqrt{CC^*}.$$

4. Generalized Wintgen inequality

In this section, we derive the main result namely the generalized Wintgen inequality for statistical submanifolds of Kenmotsu statistical manifolds. First, we have the following relation for scalar normal curvature $\tau^\perp$ [2]

$$\tau^\perp = \frac{1}{2}\left\{\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[g\left(R^\perp(e_i,e_j)\xi_s,\xi_t\right) + g\left(R^{*\perp}(e_i,e_j)\xi_s,\xi_t\right)\right]^2\right\}^{\frac{1}{2}}. \tag{4.1}$$

By the virtue of (2.12) and (2.13) for the dual connections ∇ and ∇^*,

$$\begin{aligned}
\tau^\perp &= \frac{1}{2}\left\{\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[g\left([A^*_{e_s}, A_{e_t}]e_i, e_j\right) + g\left([A_{e_s}, A^*_{e_t}]e_i, e_j\right)\right]^2\right\}^{\frac{1}{2}}\\
&= \frac{1}{2}\left\{\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[\sum_{k=1}^n\left(h^t_{ik}h^{*s}_{jk} - h^t_{jk}h^{*s}_{ik} + h^{*t}_{ik}h^s_{jk} - h^s_{ik}h^{*t}_{jk}\right)\right]^2\right\}^{\frac{1}{2}}\\
&= \frac{1}{2}\left\{\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[\sum_{k=1}^n\left(4\left(h^{\circ t}_{ik}h^{\circ s}_{jk} - h^{\circ s}_{ik}h^{\circ t}_{jk}\right) + \left(h^s_{ik}h^t_{jk} - h^t_{ik}h^s_{jk}\right) + \left(h^{*s}_{ik}h^{*t}_{jk} - h^{*t}_{ik}h^{*s}_{jk}\right)\right)\right]^2\right\}^{\frac{1}{2}}\\
&\leq \frac{\sqrt{3}}{2}\left\{\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left(16\left[\sum_{k=1}^n\left(h^{\circ t}_{ik}h^{\circ s}_{jk} - h^{\circ s}_{ik}h^{\circ t}_{jk}\right)\right]^2 + \left[\sum_{k=1}^n\left(h^s_{ik}h^t_{jk} - h^t_{ik}h^s_{jk}\right)\right]^2\right.\right.\\
&\quad \left.\left. + \left[\sum_{k=1}^n\left(h^{*s}_{ik}h^{*t}_{jk} - h^{*t}_{ik}h^{*s}_{jk}\right)\right]^2\right)\right\}^{\frac{1}{2}},
\end{aligned} \tag{4.2}$$

where we have used the Cauchy–Schwarz inequality $3(a^2+b^2+c^2)-(a+b+c)^2 \geq 0,\ \forall a,b,c \in \mathbb{R}$. Moreover, we have a well-known algebraic inequality [15]

$$\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[\frac{1}{2n}\left(\left(h^s_{ii} - h^s_{jj}\right)^2\right) + \left(h_{ij}\right)^2\right] \geq \left[\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left(\sum_{k=1}^n\left(h^s_{jk}h^t_{ik} - h^s_{ik}h^t_{jk}\right)\right)^2\right]^{\frac{1}{2}}. \tag{4.3}$$

Similarly above relation also holds for the connection ∇^* and ∇° as follows.

$$\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[\frac{1}{2n}\left(\left(h^{*s}_{ii} - h^{*s}_{jj}\right)^2\right) + \left(h^*_{ij}\right)^2\right] \geq \left[\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left(\sum_{k=1}^n\left(h^{*s}_{jk}h^{*t}_{ik} - h^{*s}_{ik}h^{*t}_{jk}\right)\right)^2\right]^{\frac{1}{2}} \tag{4.4}$$

and

$$\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left[\frac{1}{2n}\left(\left(h^{\circ s}_{ii} - h^{\circ s}_{jj}\right)^2\right) + \left(h^\circ_{ij}\right)^2\right] \geq \left[\sum_{1\leq s<t\leq 2m+1-n}\ \sum_{1\leq i<j\leq n}\left(\sum_{k=1}^n\left(h^{\circ s}_{jk}h^{\circ t}_{ik} - h^{\circ s}_{ik}h^{\circ t}_{jk}\right)\right)^2\right]^{\frac{1}{2}}. \tag{4.5}$$

Thus, using (4.2)–(4.5) yields

$$\tau^{\perp} \leq \frac{\sqrt{3}}{2}\Bigg\{\frac{1}{2n}\sum_{1\leq s<t\leq 2m+1-n}\sum_{1\leq i<j\leq n}\Big[16\big(h_{ii}^{\circ s}-h_{jj}^{\circ s}\big)^2+\big(h_{ii}^{*s}-h_{jj}^{*s}\big)^2+\big(h_{ii}^{s}-h_{jj}^{s}\big)^2\Big]$$
$$+\sum_{1\leq s<t\leq 2m+1-n}\sum_{1\leq i<j\leq n}\Big[16\big(h_{ij}^{\circ s}\big)^2+\big(h_{ij}^{*s}\big)^2+\big(h_{ij}^{s}\big)^2\Big]\Bigg\}. \tag{4.6}$$

Furthermore, the norm of H can be rewritten as

$$n^2\|H\|^2=\sum_{s=1}^{2m+1-n}\left(\sum_{1\leq i<j\leq n}h_{ii}^{s}\right)^2=\frac{1}{n-1}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\big(h_{ii}^{s}-h_{jj}^{s}\big)^2+\frac{2n}{n-1}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}h_{ii}^{s}h_{jj}^{s}. \tag{4.7}$$

Similarly, we have the same relation for the connections ∇^* and ∇° as

$$n^2\|H\|^2=\sum_{s=1}^{2m+1-n}\left(\sum_{1\leq i<j\leq n}h_{ii}^{*s}\right)^2=\frac{1}{n-1}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\big(h_{ii}^{*s}-h_{jj}^{*s}\big)^2+\frac{2n}{n-1}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}h_{ii}^{*s}h_{jj}^{*s} \tag{4.8}$$

and

$$n^2\|H\|^2=\sum_{s=1}^{2m+1-n}\left(\sum_{1\leq i<j\leq n}h_{ii}^{\circ s}\right)^2=\frac{1}{n-1}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\big(h_{ii}^{\circ s}-h_{jj}^{\circ s}\big)^2+\frac{2n}{n-1}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}h_{ii}^{\circ s}h_{jj}^{\circ s}. \tag{4.9}$$

Substituting above relation for connections ∇, ∇^* and ∇°, the relation (1.3) together with (4.6) yields

$$\rho^{\perp}\leq\sqrt{3}\Bigg\{\frac{1}{2}\big(\|H\|^2+16\|H^\circ\|^2+\|H^*\|^2\big)-\frac{1}{n(n-1)}\Bigg[\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\big(h_{ii}^{s}h_{jj}^{s}+h_{ii}^{*s}h_{jj}^{*s}+16h_{ii}^{\circ s}h_{jj}^{\circ s}\big)\Bigg]$$
$$+\frac{1}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\Big[16\big(h_{ij}^{\circ s}\big)^2+\big(h_{ij}^{*s}\big)^2+\big(h_{ij}^{s}\big)^2\Big]\Bigg\}$$
$$=\frac{\sqrt{3}}{2}\big[\|H\|^2+16\|H^\circ\|^2+\|H^*\|^2\big]-\frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\Big[20h_{ii}^{\circ s}h_{jj}^{\circ s}-20n\mathcal{C}^\circ-\big(h_{ii}^{s}h_{jj}^{*s}+h_{ii}^{*s}h_{jj}^{s}-2h_{ij}^{s}h_{ij}^{*s}\big)\Big].$$

Inserting (3.8) in above relation, we have

$$\rho^{\perp}\leq\frac{\sqrt{3}}{2}\big[\|H\|^2+16\|H^\circ\|^2+\|H^*\|^2\big]+\sqrt{3}\left(\rho-\frac{c-3}{2}\right)-\frac{\sqrt{3}(c+1)}{2n(n-1)}\big(3\|\varphi\|^2-2(n-1)\big)$$
$$-\frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\Big[20h_{ii}^{\circ s}h_{jj}^{\circ s}-20n\mathcal{C}^\circ\Big],$$

which implies that

$$\rho^{\perp}-\sqrt{3}\rho\leq\frac{\sqrt{3}}{2}\big[4\|H\|^2+4\|H^*\|^2+4\big(2g(H,H^*)+\|H\|^2+\|H^*\|^2\big)\big]-\frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\Big[20h_{ii}^{\circ s}h_{jj}^{\circ s}-20n\mathcal{C}^\circ\Big]$$
$$-\sqrt{3}\left(\frac{c-3}{2}\right)-\left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\big(3\|\varphi\|^2-2(n-1)\big).$$

Finally, we derive

$$\rho^{\perp}-\sqrt{3}\rho\leq\frac{5\sqrt{3}}{2}\big(\|H\|^2+\|H^*\|^2\big)+4\sqrt{3}g(H,H^*)-\sqrt{3}\left(\frac{c-3}{2}\right)-\frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\Big[20h_{ii}^{\circ s}h_{jj}^{\circ s}-20n\mathcal{C}^\circ\Big]$$
$$-\left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\big(3\|\varphi\|^2-2(n-1)\big).$$

Hence, we are able to state the generalized Wintgen inequality (DDVV conjecture) involving normalized scalar curvature and normalized scalar normal curvature with Casorati curvature for a statistical submanifold of a Kenmotsu statistical manifold.

Theorem 4. *Let M^n be a statistical submanifold of a Kenmotsu statistical manifold $\tilde{N}^{2m+1}(c)$. Then,*

$$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 4\sqrt{3}g(H, H^*) - \sqrt{3}\left(\frac{c-3}{2}\right) - \frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\left[20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right] - \left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right). \tag{4.10}$$

5. Some applications of generalized Wintgen inequality

We have the following applications of Theorem 4.

Corollary 1. *Let M^n be a statistical submanifold of Kenmotsu statistical manifold $\tilde{N}^{2m+1}(c)$ such that the angle between H and H^* is θ. Then,*

$$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 4\sqrt{3}\|H\|\|H^*\|\cos\theta - \sqrt{3}\left(\frac{c-3}{2}\right) - \frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\left[20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right] - \left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right). \tag{5.1}$$

Corollary 2. *Let M^n be a statistical submanifold of Kenmotsu statistical manifold $\tilde{N}^{2m+1}(c)$. Then,*

H and H^* on M^n	Inequalities
Parallel	$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}(\lvert\lvert H\rvert\rvert+\lvert\lvert H^*\rvert\rvert)^2 - \sqrt{3}\left(\frac{c-3}{2}\right) - \frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\left(20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right) - \left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\left(3\lvert\lvert\varphi\rvert\rvert^2 - 2(n-1)\right)$
Orthogonal	$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\lvert\lvert H\rvert\rvert^2+\lvert\lvert H^*\rvert\rvert^2\right) - \sqrt{3}\left(\frac{c-3}{2}\right) - \frac{\sqrt{3}}{n(n-1)}\sum_{s=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\left(20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right) - \left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\left(3\lvert\lvert\varphi\rvert\rvert^2 - 2(n-1)\right)$

Corollary 3. *Let M^n be a statistical submanifold of Kenmotsu statistical manifold $\tilde{N}^{2m+1}(c)$ such that M^n is totally geodesic with respect to ∇°. Then,*

$$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 4\sqrt{3}g(H, H^*) - \sqrt{3}\left(\frac{c-3}{2}\right) - \left(\frac{\sqrt{3}(c+1)}{2n(n-1)}\right)\left(3\|\varphi\|^2 - 2(n-1)\right). \tag{5.2}$$

Remark 3 ([10]). Here, one can easily see that local version of any Kenmotsu manifold. Let (M_0, g_0, J) be an almost Hermitian manifold. Denote $\tilde{N} = M_0 \times \mathbb{R}, \tilde{g} = e^{2t}g_0 + (dt)^2$, the structure vector field $\xi = \frac{\partial}{\partial t} \in \Gamma(T\tilde{N})$, the structure tensor field $\varphi \in \Gamma(T\tilde{N}^{(1,1)})$. Then

(1) The triple $(\tilde{g}, \varphi, \xi)$ is an almost contact metric structure on $\tilde{N}$.
(2) The pair (g, J) is a Kaehler structure on M_0 if and only if the triple $(\tilde{g}, \varphi, \xi)$ is a Kenmotsu structure on $\tilde{N}$.

Remark 4 ([10]). Let (M_0, g_0, J) be a Kaehler manifold and $(\tilde{N} = M_0 \times \mathbb{R}, \tilde{g}, \varphi, \xi)$ be the Kenmotsu manifold as given in Remark 3. Let $(\tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g})$ be a statistical structure on $\tilde{N}$. We denote $\Lambda \in \Gamma(T\tilde{N}^{(0,2)} \otimes TM_0), \lambda \in \Gamma(TM_0^{(0,2)})$ and $\mathcal{K} \in \Gamma(TM_0^{(1,2)})$ defined by $\mathcal{K}(X, Y) = \Lambda(X, Y) + \lambda(X, Y)\xi$ and $\mathcal{K}(U, V) = \Lambda(U, V)$ for $X, Y \in \Gamma(T\tilde{N})$ and $U, V \in \Gamma(TM_0)$. Then, the following are equivalent:

(1) $(\tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g})$ is a Kenmotsu statistical structure on $\tilde{N}$.
(2) $(\nabla = \nabla^{g_0} + \mathcal{K}, g_0, J)$ is a holomorphic statistical structure on M_0 and $\Lambda(X, \xi) = 0, \lambda(X, V) = 0$ hold for $X \in \Gamma(TM)$ and $V \in \Gamma(TM_0)$.

Further, if the Kenmotsu statistical manifold $(\tilde{N} = M_0 \times \mathbb{R}, \tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g}, \varphi, \xi)$ is of constant φ-sectional curvature c then it implies that, $c = -1$ and $(M_0, \nabla = \nabla^{g_0} + \mathcal{K}, g_0, J)$ is of constant holomorphic sectional curvature 0.

Thus, we have the following results

Theorem 5. *Let $(\tilde{N} = M_0 \times \mathbb{R}, \tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g}, \varphi, \xi)$ be the Kenmotsu statistical manifold of constant φ-sectional curvature c as given in Remark 4. If M^n is a statistical submanifold of $\tilde{N}$. Then,*

$$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 4\sqrt{3}g(H, H^*) + 2\sqrt{3} - \frac{\sqrt{3}}{n(n-1)} \sum_{s=1}^{2m+1-n} \sum_{1\leq i<j\leq n} \left[20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right]. \tag{5.3}$$

Proposition 1. *Let $(\tilde{N} = M_0 \times \mathbb{R}, \tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g}, \varphi, \xi)$ be the Kenmotsu statistical manifold of constant φ-sectional curvature c as given in Remark 4. If M^n is a statistical submanifold of $\tilde{N}$ such that the angle between H and H^* is θ. Then,*

$$\begin{aligned}\rho^{\perp} - \sqrt{3}\rho \leq{}& \frac{5\sqrt{3}}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 4\sqrt{3}\|H\|\|H^*\|\cos\theta + 2\sqrt{3} \\ & - \frac{\sqrt{3}}{n(n-1)} \sum_{s=1}^{2m+1-n} \sum_{1\leq i<j\leq n} \left[20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right].\end{aligned} \tag{5.4}$$

Corollary 4. *Let $(N = M_0 \times \mathbb{R}, \tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g}, \varphi, \xi)$ be the Kenmotsu statistical manifold of constant φ-sectional curvature c as given in Remark 4. If M^n is a statistical submanifold of $\tilde{N}$. Then,*

H and H^* on M^n	Inequalities
Parallel	$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|\|H\|\|+\|\|H^*\|\|\right)^2 + 2\sqrt{3} - \frac{\sqrt{3}}{n(n-1)} \sum_{s=1}^{2m+1-n} \sum_{1\leq i<j\leq n} \left(20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right)$
Orthogonal	$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|\|H\|\|^2+\|\|H^*\|\|^2\right) + 2\sqrt{3} - \frac{\sqrt{3}}{n(n-1)} \sum_{s=1}^{2m+1-n} \sum_{1\leq i<j\leq n} \left(20h_{ii}^{os}h_{jj}^{os} - 20nC^{\circ}\right)$

Corollary 5. *Let $(\tilde{N} = M_0 \times \mathbb{R}, \tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g}, \varphi, \xi)$ be the Kenmotsu statistical manifold of constant φ-sectional curvature c as given in Remark 4. If M^n is a statistical submanifold of $\tilde{N}$ such that M^n is totally geodesic with respect to ∇°. Then,*

$$\rho^{\perp} - \sqrt{3}\rho \leq \frac{5\sqrt{3}}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 4\sqrt{3}g(H, H^*) + 2\sqrt{3}.$$

Example 1. Consider the Kenmotsu manifold $(\mathbb{H}^{2m+1}, \tilde{g}, \varphi, \xi)$ (see [10], Example 3.3) where $\mathbb{H}^{2m+1} = (x^1, \ldots, x^n, y^1, \ldots, y^n, z) \in \mathbb{R}^{2m+1} : z > 0$.

Here, we denote

$$\tilde{\mathcal{K}}(X, Y) = f\eta(X)\eta(Y)\xi,$$

for any $X, Y \in \Gamma(T\mathbb{H}^{2m+1})$ and $f \in C^{\infty}(\mathbb{H}^{2m+1})$, where η is the 1-form on $\mathbb{H}^{2m+1}$. Then, $(\mathbb{H}^{2m+1}, \tilde{\nabla} = \tilde{\nabla}^{\tilde{g}} + \tilde{\mathcal{K}}, \tilde{g}, \varphi, \xi)$ is a Kenmotsu statistical manifold with constant φ-sectional curvature $c = -1$ (see ([10], Example 3.10)).

Next, let M^n be a submanifold of $\mathbb{H}^{2m+1}$. Then, the inequalities (5.3) and (5.4) are satisfied.

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