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Noether symmetries and conserved quantities for spaces with a section of zero curvature

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ABSTRACT

In an earlier paper (Feroze, 2010 [21]), the existence of new conserved quantities (Noether invariants) for spaces of different curvatures was discussed. There, it was conjectured that the number of new conserved quantities for spaces with an m-dimensional section of zero curvature is m. Here, along with the proof of this conjecture, the form of the new conserved quantities is also presented. For the illustration of the theorem, an example of conformally flat spacetime is constructed which also demonstrates that the conformal Killing vectors (CKVs), in general, are not symmetries of the Lagrangian for the geodesic equation.

1. Introduction

Noether symmetries or symmetries of Lagrangians and Lie symmetries or symmetries of the corresponding Euler–Lagrangian equations (in general symmetries of differential equations (DEs)) play an important role in finding solutions of DEs [1]. They can be used to reduce the order of ordinary differential equations (ODEs). In the case of partial differential equations (PDEs) they reduce the number of independent variables [2]. They are also used for the linearization of non-linear DEs [3–6]. Among these two type of symmetries, the Noether symmetries are more useful (for variational problems only) in the sense that they give double reduction of DEs. From the physical point of view Noether symmetries are also important as they yield conservation laws via Noether theorem [7].

Conservation laws or conserved quantities are extensively studied in the literature (e.g. [8–16]). Recently, it was proved that there exist new conserved quantities only for spaces of zero curvature or having a section of zero curvature. For completeness, we must know the number and form of these quantities. Zero curvature sections have their own importance, e.g., in quantum gravity, string theory [17] and foliation of spacetimes [18–20]. It has been seen that an n-dimensional space of zero curvature admits $n+2$ new conserved quantities, which are discussed in [21]. It was conjectured that spaces with an m-dimensional section of zero curvature admit m new conserved quantities and the corresponding Noether symmetries have the form $s\frac{\partial}{\partial x_i}$; $i = 1, 2, \ldots, m$. Here it is proved as a theorem.

On the other hand, spacetime symmetries have their own importance in General Relativity (GR) [22]. Spacetime symmetries, e.g. *isometries* or *Killing vectors (KVs)*, *conformal Killing vectors* (*CKVs*), *homothetic vectors* (*HVs*) and *curvature collineations* (*CCs*) are extensively used in the classification of spacetimes [23–28]. They are also used for finding exact solutions of Einstein Field Equations [29]. Among these spacetime symmetries, the set of KVs is the basic one in the sense that this set is always contained in the set of all other types of spacetime symmetries, e.g. CKVs, HVs, CCs, etc. (for details

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see [30]). The algebra of KVs form a subalgebra of the symmetries of the geodesic equations (Euler–Lagrange equations) of the underlying spaces [31]. It is also known that the set of Noether symmetries always contained in the set of symmetries of the corresponding Euler–Lagrange equations [32]. This gives rise to the question how the Noether symmetries are related with the spacetime symmetries, e.g. CKVs, HVs, etc.

For the Minkowski spacetime, an answer to the above question is already known. Minkowski spacetime which is flat and hence conformally flat, admits 15 CKVs [33]. A Lagrangian for the geodesic equations for this spacetime admits 17 Noether symmetries which properly contains the 15 CKVs [34]. Since KVs and HVs form subalgebras of the CKVs, for the Minkowski spacetime all these spacetime symmetries are contained in the set of 17 Noether symmetries. On the basis of this result, it was conjectured, that the CKVs form a subalgebra of the symmetries of the Lagrangian that minimizes the arc length for any spacetime [34]. A counter example of a cylindrically symmetric static spacetime with non-zero curvature was constructed for which the set of symmetries of the Lagrangian for the geodesic equations only contain the KVs and not the HVs and CKVs. Hence the conjecture was proved false [35]. The Lagrangian for the geodesic equation depends on the metric tensor and not on its conformal structure; therefore it seems reasonable that the Lagrangian may only admit the symmetries of the metric tensor i.e. KVs and not the CKVs. Since for conformally flat spacetimes the metric is transformed conformally, one can expect that the Lagrangian for geodesic equations may admit the CKVs for conformally flat spacetimes. With this a question comes to mind: "Is the above-mentioned conjecture (regarding CKVs and Noether symmetries) true for conformally flat spacetime?". Here an answer is given to this question.

The plan of the paper is as follows. In the next section, mathematical formalism to be used is given. In Section 3, the conjecture about the conserved quantities for spaces with a section of zero curvature is proved. The example of a conformally flat spacetime is discussed in Section 4. A summary and discussion are given in the last section.

2. Preliminaries

The vector field $\mathbf{X}$ is known as a CKV if the following condition holds [30]

$$£_{\mathbf{X}} g_{\mu\nu} = \psi g_{\mu\nu}, \tag{1}$$

where $\psi = \psi(x^\sigma)$ is a conformal factor and £ denotes the Lie derivative operator. If $\psi_{,\sigma} = 0$, then $\mathbf{X}$ is known as HV field and KV field if $\psi = 0$, where the comma denotes the partial derivative. If we replace in (1) the metric tensor $g_{\mu\nu}$ with the Riemann curvature tensor $R^{\mu}_{\nu\lambda\sigma}$ and put $\psi = 0$, then the vector field $\mathbf{X}$ is known as a CC. Here all the indices μ, ν, λ and σ run from 0 to 3.

Noether point symmetries are defined as follows. Consider a vector field [1]

$$\mathbf{X} = \xi(s, x^\mu)\frac{\partial}{\partial s} + \eta^\nu(s, x^\mu)\frac{\partial}{\partial x^\nu}, \tag{2}$$

where s, the arc length parameter, is the independent variable and x^μ is the dependent variable. The first prolongation of the above vector field, is

$$\mathbf{X}^{[1]} = \mathbf{X} + (\eta^\nu_{,s} + \eta^\nu_{,\mu}\dot{x}^\mu - \xi_{,s}\dot{x}^\nu - \xi_{,\mu}\dot{x}^\mu\dot{x}^\nu)\frac{\partial}{\partial \dot{x}^\nu}. \tag{3}$$

Then $\mathbf{X}$ is a Noether point symmetry of the Lagrangian

$$L(s, x^\mu, \dot{x}^\mu) = g_{\mu\nu}(x^\sigma)\dot{x}^\mu\dot{x}^\nu, \tag{4}$$

if there exists a gauge function, $A(s, x^\mu)$, such that

$$\mathbf{X}^{[1]}L + (D_s\xi)L = D_sA, \tag{5}$$

where

$$D_s = \frac{\partial}{\partial s} + \dot{x}^\mu\frac{\partial}{\partial x^\mu}, \tag{6}$$

and "·" denotes differentiation with respect to s.

The significance of Noether symmetries is clear from the following theorem [1].

Theorem 1. *If $\mathbf{X}$ is a Noether point symmetry corresponding to a Lagrangian $L(s, x^\mu, \dot{x}^\mu)$ of a second-order ODE $\ddot{x}^\mu = g(s, x, \dot{x}^\mu)$, then*

$$T = \xi L + (\eta^\mu - \dot{x}^\mu\xi)\frac{\partial L}{\partial \dot{x}^\mu} - A, \tag{7}$$

is a first integral (conserved quantity) of the ODE associated with $\mathbf{X}$*.*

A spacetime is said to be conformally flat if all the components of the Weyl tensor

$$C^{\mu\nu}_{\lambda\sigma} = R^{\mu\nu}_{\lambda\sigma} - 2\delta^{[\mu}_{[\lambda}R^{\nu]}_{\sigma]} + \frac{1}{3}\delta^{\mu}_{[\lambda}\delta^{\nu}_{\sigma]}R, \tag{8}$$

are equal to zero. Here $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. From (8) it is trivially evident that every flat spacetimes is conformally flat but the converse is not true in general.

3. Conserved quantities for spaces with a section of zero curvature

The classification of spherically symmetric, plane symmetric and cylindrically symmetric spacetimes according to their KVs [23,36] shows that the spacetimes may have zero curvature, section of zero curvature or non-zero curvature everywhere. Conserved quantities for the first and the last cases were discussed in [21] and the conjecture given there, about the conserved quantities of spaces having a section of zero curvature, is proved now.

Theorem 2. *Spaces with an m-dimensional section of zero curvature admit m new conserved quantities and the corresponding Noether symmetries have the form* $s\frac{\partial}{\partial x_i}$, $i = 1, 2, \ldots, m$.

Proof. To find the total number of new Noether symmetries appearing for spaces having an m-dimensional section of zero curvature, one may divide these spaces into two parts; one containing the whole section of zero curvature and the other having non-zero curvature everywhere. It has been proved in Theorem 2 of [21] that the Noether symmetries appearing in the former part (i.e. having non-zero curvature) can only be the isometries and the symmetry $\mathbf{Y}_0 = \frac{\partial}{\partial s}$. In the later part, there are $\frac{1}{2}(m^2+3m+6)$ Noether symmetries among which $\frac{m}{2}(m-1)$ are rotations, m are translations, one $\mathbf{Y}_0 = \frac{\partial}{\partial s}$ corresponds to the Lagrangian and the remaining $m+2$ are

$$\mathbf{Y}_1 = s^2\frac{\partial}{\partial s} + sx^i\frac{\partial}{\partial x^i}, \quad \text{with gauge term } A = \sum_{i=1}^{m}(x^i)^2 \tag{9}$$

$$\mathbf{Y}_2 = 2s\frac{\partial}{\partial s} + x^i\frac{\partial}{\partial x^i}, \quad \text{with gauge term } A = 0, \tag{10}$$

$$\mathbf{Y}_i = s\frac{\partial}{\partial x^i}, \quad \text{with gauge term } A = 2x^i;\ i = 1, 2, \ldots, m. \tag{11}$$

The first two symmetries have the scaling of all (m) coordinates on the part of zero curvature. (If we transform these symmetries into spherical coordinates then they give scaling of radial parameter, [34]). When we combine the two parts these two Noether symmetries will disappear and m symmetries along with the isometries of both parts will be left.

Notice that these m symmetries given by (11) come from the flat part of the space and they give us the required new i.e. other than the Lagrangian and those corresponding to the isometries, conserved quantities

$$T^i = s\dot{x}^i - x^i. \tag{12}$$

They are expressed as

$$x^i = \frac{1}{2}sp^i_{\mathbf{x}} - T^i, \tag{13}$$

with x^i-intercepts of the trajectories whose slopes are $\frac{1}{2}p^i_{\mathbf{x}} = \dot{x}^i$. □

4. Noether symmetries of conformally flat spacetimes

In this section we construct an example from GR, i.e. of a plane symmetric static spacetime to illustrate the above theorem. The general line element of plane symmetric static spacetime is given by

$$ds^2 = e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2), \tag{14}$$

where ν, and μ are arbitrary functions of x. This spacetime admits four KVs in general, which form the Lie group $SO(2) \otimes_s \mathbb{R}^2 \otimes \mathbb{R}$, (where $\otimes_s$ and $\otimes$ denote semi-direct and direct product respectively) with generators [29]

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_1 = \frac{\partial}{\partial y}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial z}, \qquad \mathbf{X}_3 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}. \tag{15}$$

Here we choose the anti-Einstein spacetime for which the metric coefficients in (14) are defined as [36]

$$e^{\nu} = 1, \qquad e^{\mu} = e^{x/a}, \tag{16}$$

where a is a constant having dimensions of length. For this spacetime the non-zero components of the Riemann curvature tensor are

$$R^1_{212} = -\frac{e^{(x/a)}}{4a^2} = R^1_{313} = R^2_{323}. \tag{17}$$

For a diagonal and static (plane, spherically and cylindrically symmetric) spacetime there are six independent non-zero components of the Riemann curvature tensor [37] i.e. R^i_{0i0}, R^1_{j1j} and R^2_{323}, where $i = 1, 2, 3$ and $j = 2, 3$. For the anti-Einstein spacetime the temporal part of the curvature tensor is zero i.e. $R^i_{0i0} = 0$. Hence there is a one-dimensional section of zero curvature.

This spacetime admit 7 KVs which form the Lie group $SO(1,3)\otimes\mathbb{R}$, with symmetry generators other than given in (15)

$$\mathbf{X}_4 = y\frac{\partial}{\partial x} - \left[\frac{1}{2a}(y^2 - z^2) - a\mathrm{e}^{(x/a)}\right]\frac{\partial}{\partial y} - \frac{1}{a}yz\frac{\partial}{\partial z}, \tag{18}$$

$$\mathbf{X}_5 = z\frac{\partial}{\partial x} - \left[\frac{1}{2a}(z^2 - y^2) - a\mathrm{e}^{(x/a)}\right]\frac{\partial}{\partial z} - \frac{1}{a}yz\frac{\partial}{\partial y}, \tag{19}$$

$$\mathbf{X}_6 = \frac{\partial}{\partial x} - \frac{1}{a}\left[y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right]. \tag{20}$$

There is no proper HV for this spacetime and CCs are infinite [26]. Since anti-Einstein spacetimes are conformally flat they admit 15 CKVs. Note that the choice (16) is not the only possible choice that annihilate the Weyl tensor. In GR, the stress-energy tensor $T_{\mu\nu}$ acts as a source of spacetime curvature, and is the current density associated with gauge transformations of gravity which are general curvilinear coordinate transformations. The non-zero components of stress-energy tensor for the spacetime (14) with the choice (16) are

$$T_{00} = -3T_{11} = \frac{3}{4a^2}, \qquad T_{22} = T_{33} = \frac{\mathrm{e}^x/a}{4a^2}. \tag{21}$$

For detailed relation of stress-energy tensor and Noether symmetries, one can see e.g. [38,39].

The Lagrangian that minimize arc length in the above anti-Einstein spacetime is

$$L = \dot{t}^2 - \dot{x}^2 - \mathrm{e}^{(x/a)}(\dot{y}^2 + \dot{z}^2). \tag{22}$$

This Lagrangian admits nine Noether symmetry generators. Seven of these are the KVs given by (15) and (18)–(20). The other two generators are

$$\mathbf{Y}_0 = \frac{\partial}{\partial s}, \qquad \mathbf{Y}_1 = s\frac{\partial}{\partial t}. \tag{23}$$

Here the gauge function comes out

$$A = c_0 + 2c_1 t, \tag{24}$$

where c_0 and c_1 are constants of integration. The generator $\mathbf{Y}_0$ always exists for a Lagrangian for the geodesic equations [6]. Since there is one section of zero curvature for the above discussed spacetime, we get one new conservation law corresponding to the generator $\mathbf{Y}_1$

$$T = s\dot{t} - t. \tag{25}$$

It is known that corresponding to $\mathbf{X}_0$ the conserved quantity is energy [40], i.e.

$$E = \frac{\partial L}{\partial \dot{t}} = 2\dot{t}. \tag{26}$$

Therefore the new conserved quantity corresponding to $\mathbf{Y}_1$ can be expressed as

$$t = \frac{1}{2}sE - T, \tag{27}$$

with t-intercept of the trajectory whose slope is $\frac{E}{2}$.

Beside, we also see that for the above spacetime only the KVs are the symmetries of the Lagrangian for the geodesic equations and not the HVs or CKVs. This point is further discussed in the next section.

5. Summary and discussion

The conjecture about the conserved quantities for spaces with a section of zero curvature stated in [21], is proved here as a theorem. The m symmetry generators given by (11), disappear if the gauge function becomes a constant. Then an example of conformally flat spacetime with one-dimensional section of zero curvature is constructed which illustrated the theorem. For the Lagrangian which minimize the arc length in this spacetime we obtained one new symmetry generator $s\frac{\partial}{\partial t}$ along with the KVs and the trivial symmetry generator $\frac{\partial}{\partial s}$. We calculated the new conserved quantity corresponding to this new generator given by (27). Another example having two-dimensional section of zero curvature is discussed in [21].

From the example discussed here we also got some new insights about the relationship between the Noether symmetries and spacetime symmetries. The conformally flat spacetime discussed in Section 4 admits 7 KVs and 15 CKVs. The Lagrangian for the geodesic equation in this spacetime admits nine symmetry generators which includes the 7 KVs given in (15) and (18)–(20) and the other two symmetry generators are given in (23). Here we see that the symmetry algebra of the Lagrangian for the geodesic equation in this conformally flat spacetime only contains the KVs and not the CKVs. Hence the conjecture stated in [34] which was proved false in [35] for non-flat and non-conformally flat spacetimes is also false in the case of conformally flat (non-flat) spacetimes.

It was claimed that the Lagrangian for conformally transformed Friedman model admits 17 Noether symmetries [41]. Unfortunately this claim is not true. The metric considered there was not conformally transformed Friedman model but was the Minkowski metric in another coordinate representation. One can easily check that the metric taken there can be transformed to Minkowski by the coordinate transformation $\tau = 3t^{(1/3)}$, for which we already know that the Lagrangian admits 17 Noether symmetries [34]. Thus the above-mentioned claim does not provide an example of non-trivial conformally flat (i.e. non-flat conformally flat) spacetime for which the Lagrangian admits the maximam number 17 Noether symmetries.

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