



Tamed symplectic forms and generalized geometry[☆]



Nicola Enrietti^a, Anna Fino^{a,*}, Gueo Grantcharov^b

^a Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

^b Department of Mathematics and Statistics, Florida International University, 11200 S.W. 8th Street, Miami, FL 33199, USA

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ABSTRACT

We show that symplectic forms taming complex structures on compact manifolds are related to special types of almost generalized Kähler structures. By considering the commutator Q of the two associated almost complex structures $J_{\pm}$, we prove that if either the manifold is 4-dimensional or the distribution $\text{Im } Q$ is involutive, then the manifold can be expressed locally as a disjoint union of twisted Poisson leaves.

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1. Introduction

Let (M, Ω) be a compact symplectic manifold of dimension $2n$. An almost complex structure J on M is said to be *tamed* by Ω if

$$\Omega(JX, X) > 0$$

for any non-zero vector field X on M . When J is a complex structure (i.e. J is integrable) and Ω is tamed by J , the pair (Ω, J) has also been called a *Hermitian-symplectic structure* in [1]. Although any symplectic structure always admits tamed almost complex structures, it is still an open problem to find an example of a compact complex manifold admitting a Hermitian-symplectic structure, but no Kähler structures. From [1,2] there exist no examples in dimension 4. Moreover, the study of tamed symplectic structures in dimension 4 is related to a more general conjecture of Donaldson (see for instance [2–4]), i.e. if given an almost complex structure J on a compact 4-manifold which is tamed by a symplectic form, there exists a symplectic form compatible with J .

In [5] it was shown that symplectic forms taming complex structures on compact manifolds are strictly related to Hermitian metrics having the fundamental form $\partial\bar{\partial}$ -closed (called strong Kähler with torsion or simply SKT metrics), and some negative results on compact quotients of Lie groups by discrete subgroups were also given. In particular, if M is a

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* Corresponding author. Tel.: +39 0116702886; fax: +39 0116702878.

E-mail addresses: nicola.enrietti@unito.it (N. Enrietti), annamaria.fino@unito.it, anna.fino@gmail.com (A. Fino), grantchg@fiu.edu (G. Grantcharov).

nilmanifold (not a torus) endowed with an invariant complex structure J , then (M, J) does not admit any symplectic form taming J [5].

The generalized complex structures [6–8] contain, as particular cases, the complex and symplectic structures, so it is a natural question to see if there is any geometrical interpretation of the tamed condition in terms of generalized complex structures. There is an associated notion of almost generalized Kähler structure which consists of a pair of commuting almost generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on the bundle $TM \oplus T^*M$. By Gualtieri [6] an almost generalized Kähler structure is equivalent to an almost bi-Hermitian structure (g, J_+, J_-) together with a 2-form b . The almost generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is integrable if and only if J_+, J_- are integrable and there is a 2-form b such that

$$J_+d\omega_+ = -J_-d\omega_- = db,$$

where $\omega_{\pm}(\cdot, \cdot) = g(\cdot, J_{\pm}\cdot)$ denotes the fundamental form associated to the almost Hermitian structure $(J_{\pm}, g)$ and we use the convention

$$J_{\pm}d\omega_{\pm}(\cdot, \cdot, \cdot) = d\omega(J_{\pm}\cdot, J_{\pm}\cdot, J_{\pm}\cdot).$$

This bi-Hermitian structure appeared in the physics literature as long ago as 1984 [9] as a target space for the supersymmetric σ -model.

The purpose of this paper is twofold—to consider the generalized geometry approach to tamed symplectic forms and to study the related Poisson and twisted Poisson structures. In the first part of the paper (Section 2) we give an interpretation of a symplectic form Ω taming a complex structure J as a special case of an almost generalized Kähler structure. More precisely, we consider an almost bi-Hermitian structure (J_+, J_-, g, b) defined naturally by the symplectic form Ω taming J . This is done by twisting the complex structure $J_+ = J$ with the symplectic form Ω considered as an isomorphism between TM and T^*M . Then the twisted operator $J_- = -\Omega^{-1}J^*\Omega$ defines an almost complex structure, $g = \Omega^{1,1}J$ is a Riemannian metric g compatible with J and b is the 2-form given by $-\Omega^{2,0+0,2}J$. In general, J_- is not integrable, and if it is integrable then one obtains a generalized Kähler structure. In Section 2 we will prove that the bi-Hermitian structure $(g, J_+ = J, J_-, b)$ is equivalent to an almost generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that $\mathcal{J}_1 = \mathcal{J}_{\Omega}$, i.e. is the generalized complex structure induced by the symplectic form Ω (a similar result was proven in [10] when J_- is integrable). An almost generalized Kähler structure defined in this way will be called *tamed*.

Moreover, we will determine an explicit relation between the torsion 3-form of the Bismut connection associated to (J_+, g) and the 3-form $J_-d\omega_-$, where ω_- denotes the fundamental 2-form associated to the almost Hermitian structure (J_-, g) .

According to [6,11] there are strong relations between generalized Kähler and Poisson structures. Indeed, given a generalized Kähler structure $(J_{\pm}, g, b)$, if we consider the $(2, 0) + (0, 2)$ -form with respect to both $J_{\pm}$ given by

$$S(X, Y) = g(QX, Y),$$

where $Q = [J_+, J_-]$, then its $(0, 2)$ part can be identified with a holomorphic Poisson structure for either complex structure J_+ or J_- [11].

The second part of the paper (Sections 3 and 4) is devoted to the study of twisted Poisson structures on tamed almost generalized Kähler manifolds, such as obtained from tamed symplectic structures. In particular, if J_- is not integrable, we obtain (Theorem 4.6) an obstruction for the bivector induced by the $(2, 0)$ -part of the commutator of the two complex structures $J_{\pm}$ to be holomorphic. Twisted Poisson manifolds considered by Severa and Weinstein [12] are a natural generalization of the Poisson structures which also arise in string theory. A pair (π, ϕ) , where ϕ is a bivector field and ϕ is a closed 3-form, is called a *twisted Poisson structure* if it satisfies the equation

$$[\pi, \pi]_s = \frac{1}{2}\pi^{\#}\phi,$$

where $[\cdot, \cdot]_s$ denotes the Schouten–Nijenhuis bracket and $\pi^{\#}$ is the vector bundle homomorphism $T^*M \rightarrow TM$ induced by π . In the case $\phi = 0$ one recovers the usual notions of Poisson tensor and Poisson manifold. In [10] it was shown that twisted Poisson structures are in one-to-one correspondence with Dirac structures on split exact Courant algebroids which are T^*M -projectable. They also are the structures on the target space of Poisson sigma models with Wess–Zumino term [13].

Let (M, g) be a Riemannian manifold and $Q : TM \rightarrow TM$ a skew-symmetric endomorphism of the tangent bundle of M , i.e. such that $g(QX, Y) = -g(X, QY)$, for every X and Y . In Section 3 we show that if $\text{Im}(Q)$ is an involutive distribution, then it is integrable to a generalized foliation, i.e. M can be expressed locally as a disjoint union of embedded submanifolds (the leaves) such that at any point $p \in M$, the tangent space to the leaf through p is precisely $\text{Im}(Q)(p)$. Moreover, every leaf has a twisted Poisson structure. If additionally the rank of Q is constant, then M itself carries a twisted Poisson structure.

In Section 4 we apply the previous results with $Q = [J_+, J_-]$ to investigate the existence of twisted Poisson structures on a tamed almost generalized Kähler structure when J_- is not integrable. We prove in particular that if either the manifold is 4-dimensional or the distribution $\text{Im}(Q)$ is involutive, then $\text{Im}(Q)$ induces a foliation on the M such that on every leaf L we have two twisted Poisson structures induced by the projections on L of the endomorphisms Q and $J_+ - J_-$. Moreover, it turns out that the $(2, 0)$ -part $\tilde{Q}^{(2,0)}$ of the associated bivector $\tilde{Q}$ is holomorphic if and only if the $(3, 0)$ part of $\Omega(X, N_-(Y, Z))$

vanishes. So, in particular, for a complex surface (M, J) endowed with a symplectic form Ω that tames J , the $(2, 0)$ -part $\bar{Q}^{(2,0)}$ of the bivector $\bar{Q}$ is holomorphic.

We note that many of the results in Section 4 can be interpreted in terms of complex Courant algebroids but we use the more traditional geometric methods.

2. Generalized Kähler structures and symplectic structures

Let (M, J, g) be a Hermitian manifold and let $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ be its fundamental 2-form. We say that the metric g is *strong Kähler with torsion* (or simply *SKT*) if the fundamental 2-form ω is $\partial\bar{\partial}$ -closed, i.e. $\partial\bar{\partial}\omega = 0$. The study of SKT metrics is strictly related with Hermitian connections. Any Hermitian manifold (M, J, g) admits a unique linear connection ∇^B , called *Bismut connection*, such that $\nabla^B g = \nabla^B J = 0$ and the torsion 3-form $c(X, Y, Z) = g(X, T^B(Y, Z))$ is skew-symmetric [14,15]. This connection is also called Kähler with torsion, or simply KT, and we say that it is *strong* if $dc = 0$. It is easy to prove that the torsion 3-form c of ∇^B is a multiple of $Jd\omega$, so $dc = 0$ if and only if the J -Hermitian metric g is SKT. The theory of SKT manifolds has been recently developed in terms of generalized complex geometry in [16] by studying Lie algebroids and differential operators associated to SKT structures.

SKT structures are deeply connected with symplectic forms taming complex structures. We say that a symplectic form Ω on a complex manifold (M, J) *tames* J if it satisfies

$$\Omega(JX, X) > 0$$

for every non-zero vector field X on M . In [5] it was proved that a symplectic form taming J is equivalent to an SKT metric such that $\partial\omega = \bar{\partial}\beta$ for a ∂ -closed $(2, 0)$ -form β . In particular, this implies that the torsion 3-form of the Bismut connection c is exact, since $c = d(\beta + \bar{\beta})$.

SKT metrics with exact torsion 3-form have also been studied because of their relations with generalized complex and Kähler geometry [6]. An *almost generalized complex structure* on M is an endomorphism $\mathcal{J}$ of $TM \oplus T^*M$ that satisfies the condition $\mathcal{J}^2 = -id$ and is orthogonal with respect to the natural inner product of $TM \oplus T^*M$. The $+i$ -eigenbundle L of $\mathcal{J}$ is a maximal isotropic subbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$: if L is closed under the Courant bracket, we say that $\mathcal{J}$ is *integrable*.

A pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized (almost) complex structures such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on the bundle $TM \oplus T^*M$ is called *generalized (almost) Kähler structure*.

Any almost generalized Kähler structure induces a quadruple (g, b, J_+, J_-) , where g is a Riemannian metric, b is a 2-form and $J_{\pm}$ are almost Hermitian structures on (M, g) . Conversely, for any quadruple (g, b, J_+, J_-) we can define an almost generalized Kähler structure by

$$\begin{aligned} \mathcal{J}_1 &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} J_- - J_+ & \omega_-^{-1} + \omega_+^{-1} \\ -(\omega_- + \omega_+) & -(J_-^* - J_+^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \\ \mathcal{J}_2 &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} J_- + J_+ & \omega_-^{-1} - \omega_+^{-1} \\ -(\omega_- - \omega_+) & -(J_-^* + J_+^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \end{aligned} \quad (1)$$

Note that $\mathcal{J}_1, \mathcal{J}_2$ are integrable if and only if

- (i) J_+, J_- are integrable;
- (ii) $J_+d\omega_+ = -J_-d\omega_- = db$, i.e. g is SKT with respect to $J_{\pm}$ and both the torsion 3-forms of the Bismut connections $\nabla_{\pm}^B$ are exact.

Note that a symplectic form Ω on a manifold M induces the generalized complex structure

$$\mathcal{J}_{\Omega} = \begin{pmatrix} 0 & \Omega^{-1} \\ -\Omega & 0 \end{pmatrix},$$

which is integrable since $d\Omega = 0$. The following proposition allows us to see symplectic structures taming complex structures as particular cases of (almost) generalized Kähler structures.

Proposition 2.1. *Let (M, Ω) be a symplectic manifold. Giving an almost generalized Kähler structure on M such that one of the generalized complex structures is induced by Ω is equivalent to assigning an almost complex structure J such that Ω tames J , i.e. $\Omega(JX, X) > 0$ for any vector field $X \neq 0$.*

Moreover, the almost generalized Kähler structure is integrable if and only if the almost complex structures J and $-\Omega^{-1}J_+^\Omega$ are integrable.*

Proof. We consider an almost generalized Kähler structure in the form $(\mathcal{J}_1 = \mathcal{J}_{\Omega}, \mathcal{J}_2)$, and by imposing that $\mathcal{J}_1 = \mathcal{J}_{\Omega}$ in Eqs. (1) we find the conditions on the bi-Hermitian structure $(g, b, J_{\pm})$. The $(1, 2)$ element of the matrix representing $\mathcal{J}_1$ gives $-\frac{1}{2}(\omega_-^{-1} + \omega_+^{-1}) = \Omega^{-1}$, i.e.

$$g = \frac{1}{2}\Omega(J_+ + J_-). \quad (2)$$

Moreover, by applying this condition to the $(1, 1)$ -component we obtain

$$b = -\frac{1}{2}\Omega(J_+ - J_-). \quad (3)$$

Note that $g \mp b = \Omega J_{\pm}$, so by using that g is symmetric and b skew-symmetric

$$J_- = -\Omega^{-1}J_+^*\Omega. \quad (4)$$

Now we can prove that Ω tames J_+ , since

$$\Omega J_+ = \Omega^{1,1}J_+ = \frac{1}{2}(\Omega J_+ - J_+^*\Omega) = g > 0.$$

We still have to prove the integrability conditions: an almost generalized Kähler structure is integrable if and only if J_+, J_- are integrable and $J_+d\omega_+ = -J_-d\omega_- = db$. However, $b = \mp\Omega_{\pm}^{2,0+0,2}J_{\pm}$, i.e.

$$b_{\pm}^{2,0} = \mp i\Omega_{\pm}^{2,0} \quad \text{and} \quad b_{\pm}^{0,2} = \pm i\Omega_{\pm}^{0,2},$$

where $b_{\pm}^{2,0}$ is the $(2, 0)$ -part of b with respect to J_+ . Then, if J_+, J_- are integrable the condition $J_+d\omega_+ = -J_-d\omega_- = db$ is necessarily satisfied. $\square$

The bi-Hermitian structure $(g, J_{\pm}, b)$ associated to $(\mathcal{J}_{\Omega}, \mathcal{J}_2)$ was also considered in [10].

Remark 2.2. By a direct computation, if $\mathcal{J}_1 = \mathcal{J}_{\Omega}$ then the second generalized complex structure $\mathcal{J}_2$ can be written as

$$\mathcal{J}_2 = \begin{pmatrix} -2(J_+ + J_-)^{-1} & -\Omega^{-1}(J_+^* - J_-^*)(J_+^* + J_-^*)^{-1} \\ -\Omega(J_+ - J_-)(J_+ + J_-)^{-1} & +2(J_+^* + J_-^*)^{-1} \end{pmatrix}.$$

Moreover, by applying Eqs. (6.4) and (6.5) of [6] we have that

$$\begin{aligned} L^+ &= L \cap L_{\Omega} = \{X + i\Omega X \mid X \in T_{\mathbb{C}}^{1,0}\}, \\ L^- &= L \cap \bar{L}_{\Omega} = \{X - i\Omega X \mid \Omega X \in (T_{\mathbb{C}}^*)^{1,0}\}, \end{aligned}$$

where L_{Ω}, L denote the maximal isotropic subspaces associated to $\mathcal{J}_{\Omega}, \mathcal{J}_2$.

When both J_+ and J_- are integrable, the condition $J_{\pm}d\omega_{\pm} = \pm db$ holds. In the following proposition we measure the failure of this condition in the non-integrable case.

Proposition 2.3. If $(J_{\pm}, g, b)$ is the almost bi-Hermitian structure induced by an almost complex structure J taming a symplectic form Ω , then

$$\begin{aligned} -J_+d\omega_+(X, Y, Z) + db(X, Y, Z) &= \frac{1}{2}\sigma_{XYZ}[\Omega(J_-X, N_+(Y, Z))], \\ -J_-d\omega_-(X, Y, Z) - db(X, Y, Z) &= \frac{1}{2}\sigma_{XYZ}[\Omega(J_+X, N_-(Y, Z))], \end{aligned}$$

where by $N_{\pm}$ we denote the Nijenhuis tensor of $J_{\pm}$ given by

$$N_{\pm}(X, Y) = [X, Y] - [J_{\pm}X, J_{\pm}Y] + J_{\pm}[J_{\pm}X, Y] + J_{\pm}[X, J_{\pm}Y]$$

and by σ_{XYZ} we mean the sum over cyclic permutations of X, Y and Z .

Proof. We reduce to the proof of the second condition (the first is similar). Using the relations

$$g = \frac{1}{2}\Omega(J_+ + J_-), \quad d\Omega = 0, \quad \Omega J_{\pm} = -J_{\mp}^*\Omega,$$

we obtain

$$\begin{aligned} J_-d\omega_-(X, Y, Z) &= d\omega_-(J_-X, J_-Y, J_-Z) \\ &= \sigma_{XYZ}[J_-X \omega_-(J_-Y, J_-Z) + \omega_-(J_-X, [J_-Y, J_-Z])] \\ &= \frac{1}{2}\sigma_{XYZ}[J_-X \Omega((J_+ + J_-)Y, J_-Z) + \Omega((J_+ + J_-)X, [J_-Y, J_-Z])] \\ &= \frac{1}{2}\sigma_{XYZ}[J_-X \Omega(J_-Y, J_-Z) + \Omega(J_-X, [J_-Y, J_-Z]) + J_-X \Omega(J_+Y, J_-Z) + \Omega(J_+X, [J_-Y, J_-Z])] \\ &= \frac{1}{2}\sigma_{XYZ}[J_-X \Omega(Y, Z) + \Omega(J_+X, [J_-Y, J_-Z])]. \end{aligned}$$

Moreover,

$$\begin{aligned}
 db(X, Y, Z) &= \sigma_{XYZ}[X b(Y, Z) + b(X, [Y, Z])] \\
 &= -\frac{1}{2}\sigma_{XYZ}[X \Omega((J_+ - J_-)Y, Z) + \Omega((J_+ - J_-)X, [Y, Z])] \\
 &= -\frac{1}{2}\sigma_{XYZ}[X \Omega(J_+Y, Z) + \Omega(J_+X, [Y, Z]) - X \Omega(J_-Y, Z) - \Omega(J_-X, [Y, Z])] \\
 &= -\frac{1}{2}\sigma_{XYZ}[-X \Omega(Y, J_-Z) + \Omega(J_+X, [Y, Z]) - X \Omega(J_-Y, Z) - \Omega(J_-X, [Y, Z])].
 \end{aligned}$$

So

$$\begin{aligned}
 -J_-d\omega_-(X, Y, Z) - db(X, Y, Z) &= \frac{1}{2}\sigma_{XYZ}[-J_-X \Omega(Y, Z) - \Omega(J_+X, [J_-Y, J_-Z]) - X \Omega(Y, J_-Z) \\
 &\quad + \Omega(J_+X, [Y, Z]) - X \Omega(J_-Y, Z) - \Omega(J_-X, [Y, Z])].
 \end{aligned}$$

Note that since $d\Omega = 0$

$$\begin{aligned}
 \sigma_{XYZ}[-J_-X \Omega(Y, Z) - X \Omega(Y, J_-Z) - X \Omega(J_-Y, Z) - \Omega(J_-X, [Y, Z])] \\
 &= \sigma_{XYZ}[\Omega(X, [Y, J_-Z]) + \Omega(X, [J_-Y, Z])] \\
 &= \sigma_{XYZ}[\Omega(J_+X, J_-[Y, J_-Z] + J_-[J_-Y, Z])].
 \end{aligned}$$

Then

$$-J_-d\omega_-(X, Y, Z) - db(X, Y, Z) = \frac{1}{2}\sigma_{XYZ} \Omega(J_+X, [Y, Z] - [J_-Y, J_-Z] + J_-[Y, J_-Z] + J_-[J_-Y, Z]),$$

which concludes the proof. $\square$

In the following, we will focus our attention on a particular case.

Definition 2.4. Let M be a $2n$ -dimensional manifold. An almost generalized Kähler structure on M is called *tamed* if it is induced as in Proposition 2.1 by an integrable complex structure J and a symplectic form Ω taming J .

Applying Proposition 2.3 we have immediately that a tamed almost generalized Kähler structure satisfies

$$J_+d\omega_+ = db, \quad (J_+d\omega_+ + J_-d\omega_-)(X, Y, Z) = -\frac{1}{2}\sigma_{XYZ}[\Omega(J_+X, N_-(Y, Z))]. \quad (5)$$

The following example shows that the almost complex structure $J_- = -\Omega^{-1}J^*\Omega$ is not always integrable, even in the compact 4-dimensional case.

Example 2.5 (*A Compact 4-Dimensional Example*). Consider the hyper-elliptic surface that can be viewed as a compact quotient by a lattice of the solvable Lie group with structure equations

$$\begin{cases} de^1 = e^{24}, \\ de^2 = -e^{14}, \\ de^3 = de^4 = 0, \end{cases}$$

endowed with the invariant complex structure J_+ defined by $J_+e_1 = -e_2, J_+e_3 = -e_4$ and the invariant symplectic form $\Omega = e^{12} + e^{24} + e^{34}$. Note that J_+ is integrable and Ω tames J_+ , but the almost complex structure $J_- = -\Omega^{-1}J_+^*\Omega$ defined by

$$J_-(e_1) = -e_2 + e_3, \quad J_-(e_2) = -e_4, \quad J_-(e_3) = -e_1 - e_4, \quad J_-(e_4) = e_2$$

is not integrable.

By Cavalcanti [17, Theorem 3.2], for every generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ with one generalized complex structure defined by a globally defined nowhere vanishing closed spinor, the DGAs $(\Omega^*(\bar{L}_i), d_{L_i})$ are formal for $i = 1, 2$. In particular, when $\mathcal{J}_1$ is induced by a symplectic structure, $(\Omega^*(\bar{L}_1), d_{L_1})$ is isomorphic to $(\Omega^*(M), d)$ and as a consequence of [17, Theorem 3.2] we have the following.

Corollary 2.6. Let (M, J) be a complex manifold. If it admits a symplectic form Ω taming J and such that the almost complex structure $-\Omega^{-1}J^*\Omega$ is integrable, then M is formal.

3. Twisted Poisson structures on Riemannian manifolds

We recall the following.

Definition 3.1. A bivector field π on a manifold M is called a *twisted Poisson structure* if there exists a closed 3-form ϕ such that

$$[\pi, \pi]_s = \frac{1}{2} \Lambda^3 \pi^\# \phi,$$

where $\pi^\# : T^*M \rightarrow TM$ is the vector bundle homomorphism induced by π and $[\cdot, \cdot]_s$ is the Schouten bracket on bivector fields.

Let (M, g) be a Riemannian manifold and $Q : TM \rightarrow TM$ a skew-symmetric endomorphism of the tangent bundle of M , i.e. such that $g(QX, Y) = -g(X, QY)$, for every X and Y . Denote with S the 2-form corresponding to Q as $S(X, Y) = g(QX, Y)$, and define the bivector field $\tilde{Q}$ as $\tilde{Q} = \#_2^{-1} S$, where $\# : TM \rightarrow T^*M$ is the isomorphism induced by the metric g . Our aim is to understand under which conditions $\tilde{Q}$ is a twisted Poisson structure.

Lemma 3.2. Let (M, g) be a Riemannian manifold and $Q : TM \rightarrow TM$ a skew-symmetric endomorphism of the tangent bundle of M . Then the bivector $\tilde{Q}$ satisfies

$$[\tilde{Q}, \tilde{Q}] = -2 \#_3^{-1} (\sigma_{XYZ} g((\nabla_{QX}^{LC} Y), Z)).$$

Proof. First, note that $\nabla_X^{LC} S(Y, Z) = g((\nabla_X^{LC} Q)Y, Z)$. Using [18], $[\tilde{Q}, \tilde{Q}] = \#_3^{-1} \{2\delta_g S \wedge S - \delta_g(S \wedge S)\}$, where δ_g is the co-differential $\delta_g = - * d *$ with respect to g . For δ_g we have the following formula: $\delta_g \alpha(X_1, \dots, X_{n-1}) = - \sum_i (\nabla_{E_i}^{LC} \alpha)(E_i, X_1, \dots, X_{n-1})$ for any n -form α and orthonormal frame $(E_1, \dots, E_{\dim(M)})$. Then, a direct calculation shows that

$$-2\delta_g S \wedge S(X, Y, Z) = 2 \sum_i \{g((\nabla_{E_i}^{LC} Q)E_i, X)g(QY, Z) + g((\nabla_{E_i}^{LC} Q)E_i, Y)g(QZ, X) + g((\nabla_{E_i}^{LC} Q)E_i, Z)g(QX, Y)\}.$$

Also

$$-\delta_g(S \wedge S)(X, Y, Z) = \sum_i g(\nabla_{E_i}^{LC} (S \wedge S))(E_i, X, Y, Z) = 2 \sum_i ((\nabla_{E_i}^{LC} S) \wedge S)(E_i, X, Y, Z).$$

When expanded, the last term becomes

$$2\{g((\nabla_{E_i}^{LC} Q)E_i, X)g(QY, Z) + g((\nabla_{E_i}^{LC} Q)E_i, Y)g(QZ, X) + g((\nabla_{E_i}^{LC} Q)E_i, Z)g(QX, Y) \\ + g((\nabla_{E_i}^{LC} Q)Y, Z)g(QE_i, X) + g((\nabla_{E_i}^{LC} Q)Z, X)g(QE_i, Y) + g((\nabla_{E_i}^{LC} Q)X, Y)g(QE_i, Z)\}.$$

So after we take the difference we obtain

$$(2\delta_g S \wedge S - \delta_g(S \wedge S))(X, Y, Z) = +2 \sum_i \{g((\nabla_{E_i}^{LC} Q)Y, Z)g(QE_i, X) + g((\nabla_{E_i}^{LC} Q)Z, X)g(QE_i, Y) \\ + g((\nabla_{E_i}^{LC} Q)X, Y)g(QE_i, Z)\}.$$

But

$$\sum_i g((\nabla_{E_i}^{LC} Q)X, Y)g(QE_i, Z) = \sum_i g((\nabla_{g(QE_i, Z)E_i}^{LC} Q)X, Y) = -g((\nabla_{QZ}^{LC} Q)X, Y). \quad \square$$

Proposition 3.3. Let (M, g) be a Riemannian manifold and $Q : TM \rightarrow TM$ a skew-symmetric endomorphism of the tangent bundle. Then

$$(\#_3[\tilde{Q}, \tilde{Q}]_s)(X, Y, Z) = -2\{QX(g(Y, QZ)) + QY(g(Z, QX)) + QZ(g(X, QY)) \\ + g(X, [QY, QZ]) + g(Y, [QZ, QX]) + g(Z, [QX, QY])\}. \quad (6)$$

Proof. By Lemma 3.2, we need to compute $g((\nabla_{QX}^{LC} Q)Y, Z)$. Since $\nabla^{LC} g = 0$ we obtain

$$g((\nabla_{QX}^{LC} Q)Y, Z) = g(\nabla_{QX}^{LC} QY, Z) - g(Q(\nabla_{QX}^{LC} Y), Z) \\ = g(\nabla_{QX}^{LC} Y, QZ) + g(\nabla_{QX}^{LC} QY, Z) \\ = QX(g(Y, QZ)) - g(Y, \nabla_{QX}^{LC} QZ) + g(\nabla_{QX}^{LC} QY, Z).$$

Then

$$\begin{aligned}(\#_3[Q, Q]_s)(X, Y, Z) &= QX(g(Y, QZ)) - g(Y, \nabla_X^{LC} QZ) + g(\nabla_X^{LC} QY, Z) \\ &\quad QY(g(Z, QX)) - g(Z, \nabla_Y^{LC} QX) + g(\nabla_Y^{LC} QZ, X) \\ &\quad QZ(g(X, QY)) - g(X, \nabla_Z^{LC} QY) + g(\nabla_Z^{LC} QX, Y).\end{aligned}$$

But $g(\nabla_Y^{LC} QZ, X) - g(X, \nabla_Z^{LC} QY) = g(X, \nabla_Y^{LC} QZ - \nabla_Z^{LC} QY) = g(X, [QY, QZ])$ since the Levi-Civita connection has no torsion, so we have the thesis. $\square$

$\tilde{Q}$ is a twisted Poisson structure if there exists a closed 3-form ϕ such that

$$[\tilde{Q}, \tilde{Q}]_s = \frac{1}{2} \Lambda^3 Q^\# \phi.$$

Note that for any 1-form α we have $\tilde{Q}\alpha = Q^\#^{-1}\alpha$, so

$$(\Lambda^3 \tilde{Q}^\# \phi)(\alpha, \beta, \gamma) = \phi(Q^\#^{-1}\alpha, Q^\#^{-1}\beta, Q^\#^{-1}\gamma).$$

Then $\frac{1}{2} \Lambda^3 Q^\# \phi = \frac{1}{2} \phi(QX, QY, QZ)$, so in view of (6) $\tilde{Q}$ is a twisted Poisson structure if and only if there exists a closed 3-form ϕ such that

$$\phi(QX, QY, QZ) = -4\sigma_{XYZ}(QX(g(Y, QZ)) + g(X, [QY, QZ])).$$

Remark 3.4. If we suppose that Q is invertible, i.e. it has maximal rank at any point, then Q^{-1} is well defined, so $\tilde{Q}$ is non-degenerate and we can define its inverse $q(X, Y) = g(Q^{-1}X, Y)$. Moreover,

$$\begin{aligned}\phi(X, Y, Z) &= -4\{\sigma_{XYZ}(X(g(Q^{-1}Y, Z))) + \sigma_{XYZ}(g(Q^{-1}X, [Y, Z]))\} \\ &= -4dq(X, Y, Z),\end{aligned}$$

so Q is a twisted Poisson structure. Note that it is Poisson if and only if $dq = 0$, i.e. q is a symplectic form. Moreover, if on a complex manifold (M, J) we choose $Q = J$, then $q = \omega$ and we obtain

$$[\omega^{-1}, \omega^{-1}]_s = \#_3^{-1}(2Jd\omega). \quad (7)$$

The endomorphism $Q : TM \rightarrow TM$ in general does not have constant rank and it defines two distributions (in the sense of [19]) given by $\text{Im}(Q)$ and $\text{Ker}(Q)$, which are orthogonal since $g(QX, Y) = -g(X, QY)$. We recall that in general for a distribution $\mathcal{D}$ on M , also called a “generalized distribution”, we mean a mapping $\mathcal{D}$ that assigns to every point $p \in M$ a linear subspace $\mathcal{D}(p)$ of the tangent space $T_p M$ without requiring the dimension of $\mathcal{D}(p)$ to be constant.

A foliated manifold M is one that can be expressed as a disjoint union of subsets called leaves. A leaf is a connected submanifold (injective immersion) $L \subset M$ such that any point $p \in L$ has a neighborhood $U \subset M$ where the connected component of p in $L \cap U$ is an embedded submanifold of M . In a usual foliation all the leaves have the same dimension, while a generalized foliation allows the dimension of the leaves to vary. In [19] Sussmann described necessary and sufficient conditions for a distribution $\mathcal{D}$ to be integrable into a generalized foliation (i.e. for every point $p \in M$ the tangent space to the leaf through p coincides with $\mathcal{D}(p)$).

A distribution $\mathcal{D}$ is involutive if for any two vector fields $X, Y \in \mathcal{D}$ their commutator $[X, Y] \in \mathcal{D}$. If a distribution $\mathcal{D}$ is regular, i.e. if $\dim \mathcal{D}(p)$ is constant, then by the classical Frobenius Theorem the distribution $\mathcal{D}$ is involutive if and only if it is integrable. In general (i.e., when $\mathcal{D}$ has “singularities”), the condition that $\mathcal{D}$ is involutive is necessary but not sufficient for the integrability of $\mathcal{D}$. However, as pointed out in [6], distributions locally of finite type satisfy a Frobenius-like result even if they are not regular in general. We say that a smooth distribution is *locally of finite type* if for every $p \in M$ there are smooth vector fields $X_1, \dots, X_n \in \mathcal{D}$ such that

$$\mathcal{D}(p) = \text{span}\{X_1(p), \dots, X_n(p)\}.$$

By Sussmann [19, Theorem 8.1] a smooth distribution $\mathcal{D}$ locally of finite type is integrable to a generalized foliation if and only if it is involutive.

Since $Q : TM \rightarrow TM$ is a bundle map of TM and $\text{Im}(Q)$ is the image of a smooth bundle map, then $\text{Im}(Q)$ is spanned by the smooth vector fields $Y = Q(X)$ and hence it is a smooth distribution. Furthermore, for any point $p \in M$ we may choose a local basis of sections $X_1, \dots, X_n$ of TM in some neighborhood U of p . Then $Q(X_1), \dots, Q(X_n)$ certainly span $\text{Im}(Q)$ on U , so $\text{Im}(Q)$ is locally of finite type and is integrable if and only if $[\text{Im}(Q), \text{Im}(Q)] \subset \text{Im}(Q)$. Moreover, note that the function $\text{rank}(Q)$ is lower semi-continuous.

Theorem 3.5. *Let (M, g) be a Riemannian manifold and $Q : TM \rightarrow TM$ a skew-symmetric endomorphism of the tangent bundle of M . If $\text{Im}(Q)$ is an involutive distribution, then it is integrable to a generalized foliation, i.e. M can be expressed locally as a disjoint union of embedded submanifolds (the leaves) such that at any point $p \in M$, the tangent space to the leaf through p is precisely $\text{Im}(Q)(p)$ and the dimension of the leaf is a lower semi-continuous function on the manifold. Moreover, every leaf has a twisted Poisson structure.*

Proof. The first part of the proposition follows by Sussmann [19, Theorem 8.1] and by using that $\text{Im}(Q)$ is involutive, i.e. $[\text{Im}(Q), \text{Im}(Q)] \subset \text{Im}(Q)$. Moreover, since on every leaf L the endomorphism Q is non-degenerate, as noted in Remark 3.4 $\tilde{Q}_L = \tilde{Q}|_L$ satisfy

$$[\tilde{Q}_L, \tilde{Q}_L]_s = \frac{1}{2} \Lambda^3 Q_L^\# dq_L,$$

where $q_L(X, Y) = g(Q_L^{-1}X, Y)$. $\square$

As a consequence we have the following.

Corollary 3.6. Let (M, g) be a Riemannian manifold and $Q : TM \rightarrow TM$ a skew-symmetric endomorphism of the tangent bundle of M . If $\text{Im}(Q)$ is an involutive distribution of constant rank, then $\tilde{Q}$ is a twisted Poisson structure.

Proof. For every point $p \in M$ we have the orthogonal decomposition $T_p M = \text{Im}(Q)(p) \oplus \text{Ker}(Q)(p)$. As a consequence of Theorem 3.5 the 3-form ϕ

$$\phi(X, Y, Z) = \begin{cases} (\#_3[\tilde{Q}, \tilde{Q}]_s)(Q^{-1}X, Q^{-1}Y, Q^{-1}Z) & \text{if } X, Y, Z \in \text{Im}(Q) \\ 0 & \text{otherwise} \end{cases}$$

is well defined. If one considers the 2-form

$$q(X, Y) = \begin{cases} g(Q^{-1}X, Y) & X, Y \in \text{Im}(Q), \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi = -4dq$ and so ϕ is closed and

$$\phi(QX, QY, QZ) = -4\sigma_{XYZ} (QX(g(Y, QZ)) + g(X, [QY, QZ])). \quad \square$$

4. Twisted Poisson structures on tamed almost generalized Kähler structures

Generalized complex and Kähler structures are deeply related with Poisson structures. If $\mathcal{J}$ is a generalized complex structure on M , the bivector $\beta = \pi_{TM}(\mathcal{J}|_{T^*M})$ is a Poisson structure (see [20]). Moreover, let $(M, J_\pm, g, b)$ be a generalized Kähler manifold. Then the endomorphism of the tangent bundle $Q = [J_+, J_-]$ is skew-symmetric and satisfies $QJ_\pm = -J_\pm Q$, so the induced 2-form S has type $(2, 0) + (0, 2)$ with respect to both $J_\pm$. In [11] Hitchin proved that the $(2, 0)$ -part σ_+ (with respect to J_+) of the bivector $\tilde{Q} = \#_2^{-1}S$ is a holomorphic Poisson structure, i.e. $\bar{\partial}\sigma_+ = 0$ and $[\sigma_+, \sigma_+]_s = 0$. In particular, this implies that the bivector $\tilde{Q} = \#_2^{-1}S$ is a Poisson structure.

Now, let $(\mathcal{J}_\Omega, \mathcal{J}_2)$ be a tamed almost generalized Kähler structure on M induced by (Ω, J) . If $J_- = -\Omega^{-1}J^*\Omega$ is integrable, both $\mathcal{J}_\Omega$ and $\mathcal{J}_2$ induce Poisson structures, which are given by

$$\beta^1 = \omega_+^{-1} + \omega_-^{-1} = 2\Omega^{-1} \quad \text{and} \quad \beta^2 = \omega_+^{-1} - \omega_-^{-1}$$

(note that the first one is simply the Poisson structure induced by the symplectic form Ω). Moreover, the $(2, 0)$ -part of the bivector $\tilde{Q}$ induced by $Q = [J_+, J_-]$ is a holomorphic Poisson structure.

The goal of this section is to use the results obtained in Section 3 to investigate the existence of twisted Poisson structures on a tamed almost generalized Kähler structure when J_- is not integrable. First, we consider the structures defined by $\pi_{TM}(\mathcal{J}|_{T^*M})$ on the generalized complex structures $(\mathcal{J}_\Omega, \mathcal{J}_2)$.

Lemma 4.1. Given a tamed almost generalized Kähler manifold $(M, J_\pm, g, b)$, then β^1 is a Poisson structure and

$$[\beta^2, \beta^2]_s = 4\#_3^{-1}(J_+d\omega_+ + J_-d\omega_-). \quad (8)$$

In particular, β^2 is a Poisson structure if and only if $J_-d\omega_- = -J_+d\omega_+$, i.e. if (5) vanishes.

Proof. For any almost Hermitian manifold (M, J, g) the associated fundamental 2-form ω satisfies the relation (7), as also proved in [21]. Applying this result, we obtain

$$[\omega_+^{-1}, \omega_+^{-1}]_s = \#_3^{-1}(2J_+d\omega_+), \quad [\omega_-^{-1}, \omega_-^{-1}]_s = \#_3^{-1}(2J_-d\omega_-).$$

Moreover, by using that $\beta^1 = 2\Omega^{-1}$ is a Poisson structure, we find that

$$[\omega_+^{-1}, \omega_-^{-1}]_s = -\#_3^{-1}(J_+d\omega_+ + J_-d\omega_-),$$

and then (8). $\square$

Proposition 4.2. Let $(\mathcal{J}_\Omega, \mathcal{J}_2)$ be a tamed almost generalized Kähler structure on M induced by a Hermitian symplectic structure (Ω, J) . Then the bivector β^2 is a twisted Poisson structure if and only if there is a closed 3-form ϕ such that $J_-d\omega_- + J_+d\omega_+ = \frac{1}{8}(J_+ - J_-)\phi$.

Proof. Note that $\beta^2\alpha = (J_+ - J_-)\#^{-1}\alpha$ for any $\alpha \in T^*M$, so the twisted Poisson condition becomes

$$[\beta^2, \beta^2]_s = \frac{1}{2}\#_3^{-1}(J_+ - J_-)\phi$$

for some closed 3-form ϕ . Then the proof follows by Lemma 4.1. $\square$

Now, let us consider the endomorphism $Q = [J_+, J_-] : TM \rightarrow TM$. In the integrable case, the $(2, 0)$ -part of the bivector $\tilde{Q}$ induced by Q as in Section 3 is a holomorphic Poisson structure. However, when J_- is not integrable, we can apply Theorem 3.5 to obtain conditions for the existence of a “local” twisted Poisson structure.

Proposition 4.3. *Let $(\mathcal{J}_\Omega, \mathcal{J}_2)$ be a tamed almost generalized Kähler structure on M induced by a Hermitian symplectic structure (Ω, J) . If $\text{Im}(Q)$ is an involutive distribution, then it is integrable to a generalized foliation and on every leaf L the bivectors $\tilde{Q}_L = \tilde{Q}|_L$ and $\beta_L^2 = \beta^2|_L$ are twisted Poisson structures.*

Proof. The only non-trivial part is to prove that β_L^2 is a twisted Poisson structure on every leaf of the foliation induced by $\text{Im}(Q)$. Since $\beta^2\alpha = (J_+ - J_-)\#^{-1}\alpha$ for any $\alpha \in T^*M$, β^2 is the bivector induced as in Section 3 by the endomorphism $J_+ - J_-$. Moreover, since $Q = (J_+ - J_-)(J_+ + J_-)$ and $J_+ + J_-$ is invertible, $J_+ - J_-$ is non-degenerate if and only if Q is non-degenerate. Then, on every leaf L the restriction $(J_+ - J_-)|_L$ is non-degenerate, so as in the proof of Theorem 3.5

$$[\beta_L^2, \beta_L^2]_s = \frac{1}{2}\Lambda^3(J_+ + J_-)|_L^\# dp_L,$$

where $p_L(X, Y) = g|_L((J_+ - J_-)|_L^{-1}X, Y)$ for any vector fields X, Y on L . $\square$

If J_- is integrable, in [22] it has been shown that the spaces orthogonal to the kernels of $(J_+ \pm J_-)$ and of the commutator $[J_+, J_-]$ (which coincide with the images of these operators since they are skew-symmetric) are always integrable as generalized distributions and thus they are in particular involutive. As remarked in [21], this property follows also from the fact that the spaces orthogonal to the kernels of $(J_+ \pm J_-)$ correspond to the symplectic leaves of $\omega_+^{-1} + \omega_-^{-1}$ and $\omega_+^{-1} - \omega_-^{-1}$ respectively.

In the following proposition we compute $[\tilde{Q}, \tilde{Q}]_s$ for a tamed almost generalized Kähler structure.

Proposition 4.4. *Let $(\mathcal{J}_\Omega, \mathcal{J}_2)$ be a tamed almost generalized Kähler structure on M induced by a Hermitian symplectic structure (Ω, J) . Then*

$$\begin{aligned} \#_3[\tilde{Q}, \tilde{Q}]_s(X, Y, Z) = & -\sigma_{X,Y,Z}[(N_- + 3bN_-)(QX, J_-Y, J_+Z) + (N_- + 3bN_-)(QX, J_+Y, J_-Z) \\ & - J_+d\omega_+(QX, J_-Y, J_+Z) - J_+d\omega_+(QX, J_+Y, J_-Z) - J_+d\omega_+(QX, Y, J_+J_-Z) \\ & - J_+d\omega_+(QX, J_+J_-Y, Z) + J_-d\omega_-(QX, J_-Y, J_+Z) + J_-d\omega_-(QX, Y, J_-J_+Z) \\ & + J_-d\omega_-(QX, J_-J_+Y, Z) + J_-d\omega_-(QX, J_+Y, J_-Z)], \end{aligned} \quad (9)$$

where $N_-(X, Y, Z) = g(X, N_-(Y, Z))$ is the 3-tensor associated to the Nijenhuis tensor of J_- and bN_- is the skew-symmetric part of N_- defined by

$$(bN_-)(X, Y, Z) = \frac{1}{3}[N_-(X, Y, Z) + N_-(Y, Z, X) + N_-(Z, X, Y)].$$

Proof. By Lemma 3.2, we need to compute $g((\nabla_{QX}^{LC}Q)Y, Z)$. In the proof we use the Bismut connections $\nabla^\pm$ of the (almost) complex structures $J_\pm$, which satisfy $\nabla^\pm J_\pm = 0$ and following [15] are defined by

$$\begin{aligned} g(\nabla_X^+ Y, Z) &= g(\nabla_X^{LC} Y, Z) - \frac{1}{2}J_+d\omega_+(X, Y, Z), \\ g(\nabla_X^- Y, Z) &= g(\nabla_X^{LC} Y, Z) - \frac{1}{4}(N_- + 3bN_-)(X, Y, Z) - \frac{1}{2}J_-d\omega_-(X, Y, Z). \end{aligned}$$

Using these definitions,

$$\begin{aligned} g((\nabla_{QX}^{LC}Q)Y, Z) &= -g(\nabla_{QX}^{LC}Y, QZ) - g(\nabla_{QX}^{LC}QY, Z) \\ &= -g(\nabla_{QX}^+ Y, QZ) - g(\nabla_{QX}^+ QY, Z) - \frac{1}{2}J_+d\omega_+(QX, Y, QZ) - \frac{1}{2}J_+d\omega_+(QX, QY, Z) \\ &= -g((\nabla_{QX}^+ J_-)Y, J_+Z) - g((\nabla_{QX}^+ J_-)J_+Y, Z) - \frac{1}{2}J_+d\omega_+(QX, Y, QZ) - \frac{1}{2}J_+d\omega_+(QX, QY, Z). \end{aligned}$$

By definition of ∇^+ and ∇^- we obtain that

$$g((\nabla^+ J_-)Y, Z) = -\frac{1}{2}(N_- + 3bN_-)(X, J_-Y, Z) + \frac{1}{2}(J_+d\omega_+ - J_-d\omega_-)(X, Y, J_-Z) + \frac{1}{2}(J_+d\omega_+ - J_-d\omega_-)(X, J_-Y, Z).$$

Using this condition in the previous equation, we obtain the thesis. $\square$

Remark 4.5. We know that when J_- is integrable, $\tilde{Q}$ is a Poisson structure, i.e. $[\tilde{Q}, \tilde{Q}]_s = 0$. Indeed, if J_- is integrable $N_- = 0$ and $J_- d\omega_- = -J_+ d\omega_+$, so $J_+ d\omega_+$ has type $(2, 1) + (1, 2)$ with respect to both complex structures, then (9) becomes

$$\begin{aligned} \#_3[\tilde{Q}, \tilde{Q}]_s(X, Y, Z) = & \sigma_{X,Y,Z}[2J_+ d\omega_+(QX, J_+Y, J_-Z) + 2J_+ d\omega_+(QX, J_-Y, J_+Z) \\ & - J_+ d\omega_+(QX, PY, Z) + J_+ d\omega_+(QX, Y, PZ)], \end{aligned}$$

where $P = J_+ J_- + J_- J_+$, and arguing as in [23] we obtain $[\tilde{Q}, \tilde{Q}]_s = 0$.

In [11] it was shown that for a generalized Kähler structure (g, J_+, J_-, b) the skew form $g(QX, Y)$ for the bi-Hermitian metric is of type $(2, 0) + (0, 2)$ and defines a holomorphic Poisson structure for either complex structure J_+ or J_- . For a tamed almost generalized Kähler structure we can prove the following theorem.

Theorem 4.6. Let $(\mathcal{J}_\Omega, \mathcal{J}_2)$ be a tamed almost generalized Kähler structure on M induced by a Hermitian symplectic structure (Ω, J) . Then the $(2, 0)$ -part $\tilde{Q}^{(2,0)}$ of the bivector $\tilde{Q}$ is holomorphic if and only if $\psi^{(3,0)} = 0$, where

$$\psi(X, Y, Z) = \Omega(X, N_-(Y, Z)).$$

Proof. Let D^+ be the Chern connection of (g, J_+) . Then

$$g(D_X^+ Y, Z) = g(\nabla_X^{LC} Y, Z) + \frac{1}{2} d\omega_+(J_+ X, Y, Z),$$

where $\omega_+(X, Y) = g(X, J_+ Y)$ is the fundamental 2-form of (J_+, g) . From (5) we have

$$J_+ d\omega_+ = db, \quad (J_+ d\omega_+ + J_- d\omega_-)(X, Y, Z) = -\frac{1}{2} \sigma_{XYZ}[\Omega(J_+ X, N_-(Y, Z))].$$

By Gauduchon [15, Proposition 1] we have the following formula for the covariant derivative of J_- :

$$\begin{aligned} 2g((\nabla_X^{LC} J_-)(Y), Z) = & -d\omega_-(X, Y, Z) + d\omega_-(X, J_-Y, J_-Z) - g(J_-X, N_-(Y, Z)) \\ = & -(d\omega_-)^+(X, Y, Z) + (d\omega_-)^+(X, J_-Y, J_-Z) \\ & - \frac{1}{2} (g(J_-X, N_-(Y, Z)) + g(J_-Y, N_-(X, Z)) - g(J_-Z, N_-(X, Y))), \end{aligned}$$

where $(d\omega_-)^+$ denotes the $(2, 1) + (1, 2)$ -component of the 3-form $d\omega_-$ with respect to J_- . In particular,

$$(d\omega_-)^+(X, Y, Z) - (d\omega_-)^+(X, J_-Y, J_-Z) = (d\omega_-)^+(J_-X, Y, J_-Z) + (d\omega_-)^+(J_-X, J_-Y, Z)$$

(see [15, formula (1.3.6)]). Similarly, from $(d\omega_-)^{(2,1)} = 0$ follows

$$(d\omega_-)^-(X - iJ_-X, Y + iJ_-Y, Z + iJ_-Z) = 0$$

and

$$(d\omega_-)^-(X, Y, Z) - (d\omega_-)^-(X, J_-Y, J_-Z) = -(d\omega_-)^-(J_-X, Y, J_-Z) - (d\omega_-)^-(J_-X, J_-Y, Z),$$

where by $(d\omega_-)^-$ we denote the $(3, 0) + (0, 3)$ -component of the 3-form $d\omega_-$ with respect to J_- . From above

$$\begin{aligned} 2g((\nabla_X^{LC} J_-)(Y), Z) = & -(d\omega_-)^+(X, Y, Z) + (d\omega_-)^+(X, J_-Y, J_-Z) \\ & - (d\omega_-)^-(X, Y, Z) + (d\omega_-)^-(X, J_-Y, J_-Z) - g(J_-X, N_-(Y, Z)) \\ = & -(d\omega_-)^+(J_-X, Y, J_-Z) - (d\omega_-)^+(J_-X, J_-Y, Z) \\ & + (d\omega_-)^-(J_-X, Y, J_-Z) + (d\omega_-)^-(J_-X, J_-Y, Z) - g(J_-X, N_-(Y, Z)) \\ = & -d\omega_-(J_-X, Y, J_-Z) - d\omega_-(J_-X, J_-Y, Z) \\ & + 2(d\omega_-)^-(J_-X, Y, J_-Z) + 2(d\omega_-)^-(J_-X, J_-Y, Z) - g(J_-X, N_-(Y, Z)) \\ = & -d\omega_-(J_-X, Y, J_-Z) - d\omega_-(J_-X, J_-Y, Z) \\ & - 2(d\omega_-)^-(X, Y, Z) + 2(d\omega_-)^-(X, J_-Y, J_-Z) - g(J_-X, N_-(Y, Z)) \\ = & -d\omega_-(J_-X, Y, J_-Z) - d\omega_-(J_-X, J_-Y, Z) + 2\phi_1 + g(J_-X, N_-(Y, Z)), \end{aligned}$$

where

$$\begin{aligned} \phi_1(X, Y, Z) = & -(d\omega_-)^-(X, Y, Z) + (d\omega_-)^-(X, J_-Y, J_-Z) - g(J_-X, N_-(Y, Z)) \\ = & -\frac{1}{2} \left(g(J_-X, N_-(Y, Z)) - \frac{1}{2} g(J_-Y, N_-(X, Z)) + \frac{1}{2} g(J_-Z, N_-(X, Y)) \right). \end{aligned}$$

Denote

$$\begin{aligned}\psi_1(X, Y, Z) &= -\frac{1}{2}\sigma_{XYZ}[\Omega(J_+X, N_-(Y, Z))], & \phi_2(X, Y, Z) &= 2\phi_1(X, Y, Z) + g(J_-X, N_-(Y, Z)), \\ \psi_2(X, Y, Z) &= \psi_1(X, J_-Y, Z) + \psi_1(X, YJ_-Z), & \psi &= \phi_2 + \psi_2.\end{aligned}$$

Then from [23]:

$$\begin{aligned}2g((D_X^+J_-)(Y), Z) &= 2g(D_X^+J_-Y, Z) + 2g(D_X^+Y, J_-Z) \\ &= 2g((\nabla_X^{LC}J_-)(Y), Z) + d\omega_+(J_+X, J_-Y, Z) + d\omega_+(J_+X, Y, J_-Z) \\ &= -d\omega_-(J_-X, Y, J_-Z) - d\omega_-(J_-X, J_-Y, Z) + \phi_2(X, Y, Z) \\ &\quad + d\omega_+(J_+X, J_-Y, Z) + d\omega_+(J_+X, Y, J_-Z) \\ &= -d^-\omega_-(X, J_-Y, Z) - d^-\omega_-(X, Y, J_-Z) \\ &\quad - d^+\omega_+(X, J_+J_-Y, J_+Z) - d^+\omega_+(X, J_+Y, J_+J_-Z) + \phi_2(X, Y, Z)\end{aligned}$$

and since $d^\pm\omega_\pm(X, Y, Z) = -d\omega_\pm(J_\pm X, J_\pm Y, J_\pm Z) = -J_\pm d\omega_\pm(X, Y, Z)$ Eq. (6) from [23] becomes

$$\begin{aligned}2g((D_X^+J_-)(Y), Z) &= +d^+\omega_+(X, J_-Y, Z) + d^+\omega_+(X, Y, J_-Z) \\ &\quad - d^+\omega_+(X, J_+J_-Y, J_+Z) - d^+\omega_+(X, J_+Y, J_+J_-Z) + \psi(X, Y, Z).\end{aligned}\quad (10)$$

Because J_+ is integrable again we have

$$d^+\omega_+(A, B, C) = d^+\omega_+(J_+A, J_+B, C) + d^+\omega_+(J_+A, B, J_+C) + d^+\omega_+(A, J_+B, J_+C).$$

So (7) from [23] becomes

$$\begin{aligned}2g((D_X^+J_-)(Y), Z) &= +d^+\omega_+(J_+X, J_-Y, J_+Z) + d^+\omega_+(J_+X, J_+J_-Y, Z) \\ &\quad + d^+\omega_+(J_+X, Y, J_+J_-Z) + d^+\omega_+(J_+X, J_+Y, J_-Z) + \psi(X, Y, Z).\end{aligned}\quad (11)$$

Thus, for $Q = [J_+, J_-]$ we get

$$\begin{aligned}2g((D_X^+Q)(Y), Z) - 2g((D_{J_+X}^+Q)(Y), J_+Z) &= -2g((D_X^+J_-)(Y), J_+Z) - 2g((D_X^+J_-)(J_+Y), Z) \\ &\quad - 2g((D_{J_+X}^+J_-)(Y), Z) + 2g((D_{J_+X}^+J_-)(J_+Y), J_+Z).\end{aligned}$$

Then, after applying (10) to the first and the second terms, and (11) to the third and the fourth terms, we easily get the last and most important identity (8) from [23] in the form of

$$2g((D_X^+Q)(Y), Z) - 2g((D_{J_+X}^+Q)(Y), J_+Z) = \psi(J_+X, J_+Y, J_+Z) - \psi(J_+X, Y, Z) - \psi(X, J_+Y, Z) - \psi(X, Y, J_+Z).$$

From here we notice that

$$\begin{aligned}\psi(X - iJ_+X, Y - iJ_+Y, Z - iJ_+Z) &= \psi(X, Y, Z) - \psi(J_+X, J_+Y, Z) - \psi(J_+X, Y, J_+Z) - \psi(X, J_+Y, J_+Z) \\ &\quad + i(\psi(J_+X, J_+Y, J_+Z) - \psi(J_+X, Y, Z) - \psi(X, J_+Y, Z) - \psi(X, Y, J_+Z)).\end{aligned}$$

Then, we proceed further similarly to obtain

$$\begin{aligned}g(D_{X+iJ_+X}^+\tilde{Q}^{(2,0)}, Y \wedge Z) &= 2(D_{X+iJ_+X}^+S^{(0,2)})(Y, Z) \\ &= [g((D_X^+Q)(Y), Z) + ig((D_X^+Q)(Y), J_+Z)] + i[g((D_{J_+X}^+Q)(Y), Z) + ig((D_{J_+X}^+Q)(Y), J_+Z)] \\ &= [g((D_X^+Q)(Y), Z) - g((D_{J_+X}^+Q)(Y), J_+Z)] + i[g((D_X^+Q)(Y), J_+Z) + g((D_{J_+X}^+Q)(Y), Z)],\end{aligned}$$

which leads to

$$2(D_{X+iJ_+X}^+S^{(0,2)})(Y, Z) = \psi(X - iJ_+X, Y - iJ_+Y, Z - iJ_+Z) = \psi^{(0,3)}(X, Y, Z).\quad (12)$$

So $\tilde{Q}^{(2,0)}$ is holomorphic iff $\psi^{(3,0)} = 0$.

Now we compute ψ . By recalling that $g = \frac{1}{2}\Omega(J_+ + J_-)$, $g(J_-X, N_-(Y, Z)) = g(X, N_-(J_-Y, Z)) = g(X, N_-(Y, J_-Z))$ and $\Omega(J_+X, Y) = -\Omega(X, J_-Y)$ we obtain that

$$\begin{aligned}\phi_2(X, Y, Z) &= 2\phi_1(X, Y, Z) + g(J_-X, N_-Y, Z) = g(J_-Y, N_-(Z, X)) + g(J_-Z, N_-(X, Y)) \\ &\quad \times \frac{1}{2}[\Omega(J_+J_-Y, N_-(Z, X)) + \Omega(J_+J_-Z, N_-(X, Y)) - \Omega(Y, N_-(Z, X)) - \Omega(Z, N_-(X, Y))].\end{aligned}$$

Moreover,

$$\begin{aligned}\psi_2(X, Y, Z) &= \psi_1(X, J_-Y, Z) + \psi_1(X, Y, J_-Z) \\ &= \frac{1}{2}[-\Omega(J_+X, N_-(J_-Y, Z)) - \Omega(J_+J_-Y, N_-(Z, X)) - \Omega(J_+Z, N_-(X, J_-Y)) \\ &\quad - \Omega(J_+X, N_-(Y, J_-Z)) - \Omega(J_+Y, N_-(J_-Z, X)) - \Omega(J_+J_-Z, N_-(X, Y))] \\ &= \frac{1}{2}[2\Omega(X, N_-(Y, Z)) - \Omega(J_+J_-Y, N_-(Z, X)) \\ &\quad + \Omega(Z, N_-(X, Y)) + \Omega(Y, N_-(Z, X)) - \Omega(J_+J_-Z, N_-(X, Y))],\end{aligned}$$

so

$$\psi(X, Y, Z) = \phi_2(X, Y, Z) + \psi_2(X, Y, Z) = \Omega(X, N_-(Y, Z)). \quad \square$$

Remark 4.7. The $(3, 0)$ -part of ψ can be written as

$$\begin{aligned}\psi^{(3,0)}(X, Y, Z) &= \psi(X, Y, Z) - \psi(J_+X, J_+Y, Z) - \psi(J_+X, Y, J_+Z) - \psi(X, J_+Y, J_+Z) \\ &\quad + i(\psi(J_+X, J_+Y, J_+Z) - \psi(J_+X, Y, Z) - \psi(X, J_+Y, Z) - \psi(X, Y, J_+Z)) \\ &= \Omega(X, N_-(Y, Z) + J_-N_-(J_+Y, Z) + J_-N_-(Y, J_+Z) - N_-(J_+Y, J_+Z)) \\ &\quad - i\Omega(J_+X, N_-(Y, Z) + J_-N_-(J_+Y, Z) + J_-N_-(Y, J_+Z) - N_-(J_+Y, J_+Z)).\end{aligned}$$

Thus $\psi^{(3,0)} = 0$ if and only if the tensor

$$\mathfrak{N}(Y, Z) = N_-(Y, Z) + J_-N_-(J_+Y, Z) + J_-N_-(Y, J_+Z) - N_-(J_+Y, J_+Z)$$

vanishes.

Example 4.8 (*A Compact 6-Dimensional Example*). Now we consider an example in dimension greater than 4. Let $\mathfrak{g}$ be the unimodular solvable 6-dimensional Lie algebra with structure equations

$$\begin{cases} de^1 = e^{26}, \\ de^2 = -e^{16}, \\ de^3 = e^{46}, \\ de^4 = -e^{36}, \\ de^5 = de^6 = 0, \end{cases}$$

together with the complex structure J_+ defined by $J_+e_1 = -e_2, J_+e_3 = -e_4, J_+e_5 = -e_6$ and the symplectic form $\Omega = e^{12} + e^{34} + e^{56} + e^{16}$. Then (J_+, Ω) is a Hermitian symplectic structure, but the almost complex structure $J_- = -\Omega^{-1}J_+^*\Omega$ defined by

$$J_-e_1 = -e_6, \quad J_-e_2 = e_1 - e_5, \quad J_-e_3 = -e_4, \quad J_-e_4 = e_3, \quad J_-e_5 = e_2 - e_6, \quad J_-e_6 = e_1$$

is not integrable. Moreover, the almost bi-Hermitian metric is

$$g = \frac{1}{2}(\Omega J_+ - J_+^* \Omega) = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

with fundamental 2-forms

$$\omega_+ = e^{12} + e^{34} + e^{56} + \frac{1}{2}(e^{16} - e^{25}), \quad \omega_- = e^{34} + \frac{1}{2}(e^{12} + e^{56} + e^{25}),$$

and

$$J_+d\omega_+ = \frac{1}{2}e^{256}, \quad J_-d\omega_- = -\frac{1}{2}e^{126}.$$

Evaluating $\tilde{Q} = \frac{1}{2}[J_+, J_-]g^{-1} = (J_+ - J_-)\Omega^{-1}$ we obtain that

$$\tilde{Q}e^1 = -e_5, \quad \tilde{Q}e^2 = e_6, \quad \tilde{Q}e^3 = 0, \quad \tilde{Q}e^4 = 0, \quad \tilde{Q}e^5 = e_1, \quad \tilde{Q}e^6 = -e_2,$$

considering $\tilde{Q}$ as a morphism $T^*M \rightarrow TM$. $\tilde{Q}$ is skew-symmetric, so we can see it as $\tilde{Q} = e_{51} + e_{26} \in \Lambda^2 TM$; moreover, $\tilde{Q}(J\alpha, J\beta) = -\tilde{Q}(\alpha, \beta)$, so $\tilde{Q} \in \Lambda^2 T^{1,0} \oplus \Lambda^2 T^{0,1}$. We want to know if the bivector $\tilde{Q}$ is a Poisson structure, i.e. $[\tilde{Q}, \tilde{Q}]_S = 0$. We compute

$$[\tilde{Q}, \tilde{Q}]_S = 2 e_{126},$$

so Q is not a Poisson structure. However, $\tilde{Q}$ is a twisted Poisson structure, indeed

$$e_{126} = -\Lambda^3 \tilde{Q}^\#(e^{256}),$$

and $d e^{256} = 0$. Indeed, since Q is given by

$$Qe_1 = -e_1 + 2e_5, \quad Qe_2 = e_2 - 2e_6, \quad Qe_3 = 0, \quad Qe_4 = 0, \quad Qe_5 = -2e_1 + e_5, \quad Qe_6 = 2e_2 - e_6,$$

$\text{Im}(Q) = \langle e_1, e_2, e_5, e_6 \rangle$, and $[X, Y] \in \langle e_1, e_2 \rangle \subset \text{Im}(Q)$ for any $X, Y \in \text{Im}(Q)$.

To conclude the example, we prove that the $(2, 0)$ -part of $\tilde{Q}$ is not holomorphic. Indeed, by computing ψ we obtain

$$\psi = e^5 \otimes (e^{12} - e^{25} + e^{56}) + e^6 \otimes (e^{26} - e^{15}),$$

so the $(3, 0)$ -part of ψ becomes

$$\psi^{3,0} = 3(e^6 + ie^5) \otimes (e^{26} - e^{15} + ie^{25} + ie^{16}),$$

which is not zero, so the $(2, 0)$ -part $\tilde{Q}^{(2,0)}$ of the bivector $\tilde{Q}$ is not holomorphic. This example arises in fact as a deformation of the symplectic form of a Kähler structure. Indeed, for the given complex structure and the family of symplectic forms $e^{12} + e^{34} + e^{56} + te^{16}$, for $t = 0$ we have a Kähler structure and for $t \neq 0$ the symplectic form is no longer of type $(1, 1)$. The 3-form twist is $H = d(e^{15})$, which is globally defined since Q has constant rank.

4.1. Complex surfaces

Here we consider the case of 4-dimensional manifolds. When (M, J) is a complex surface we have some additional properties. For instance, we have no $(3, 0)$ - or $(0, 3)$ -forms, and if M is compact, we know that if Ω is a symplectic form taming J then J admits a Kähler metric [2]. Thanks to these peculiarities, we prove stronger versions of Propositions 4.3 and 4.4.

Proposition 4.9. *Let (M, J) be a complex surface and Ω a symplectic form that tames J . Then the distribution $\text{Im } Q$ induced by $Q = [J_+, J_-]$ is involutive.*

Proof. Note that $\text{rank}(\text{Im } Q_p) = \text{rank}(\tilde{Q}_p)$ for every $p \in M$, and $\tilde{Q}$ is a $(2, 0) + (0, 2)$ bivector. Since $\dim M = 4$, the space of $(2, 0) + (0, 2)$ bivectors is locally generated by

$$\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \right),$$

where $(\frac{\partial}{\partial x^i})$ is a local frame of TM such that $J \frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^2}$ and $J \frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^4}$. So for every point either $\tilde{Q}$ is zero or it is non-degenerate, i.e. $\text{rank}(\tilde{Q}) = 4$, and the same holds for $\text{Im } Q$. $\square$

Thus, on a 4-dimensional manifold $\text{Im } Q_p$ either coincides with $T_p M$ or it is $\{0\}$. Since $\text{Im } Q$ is a distribution locally of finite type, the rank is a lower semicontinuous function, i.e. for every point we have an open neighborhood where the rank cannot decrease. Moreover, by Theorem 3.5 $\text{Im } Q$ is integrable, so the leaves of the foliation are either

- M except some points p_i
or
- the points p_i .

Proposition 4.10. *Let (M, J) be a complex surface and Ω a symplectic form that tames J , then*

$$\#_3^{-1}[\tilde{Q}, \tilde{Q}](X, Y, Z) = (\sigma_{XYZ}(-N_-(QX, J_-Y, J_+Z) - N_-(QX, J_+Y, J_-Z) - (J_+d\omega_+ + J_-d\omega_-)(QX, Y, QZ))) . \quad (13)$$

Proof. In dimension 4 we have that any 3-form is of type $(1, 2) + (2, 1)$ with respect to J_+ and J_- , thus

$$\sigma_{XYZ}J_+d\omega_+(A, B, C) = \sigma_{X,Y,Z}J_+d\omega_+(J_+A, J_+B, C) = \sigma_{XYZ}J_+d\omega_+(J_-A, J_-B, C)$$

and $bN_- = 0$. Therefore, by a direct computation we obtain

$$\begin{aligned} \sigma_{XYZ} - J_+d\omega_+(QX, J_-Y, J_+Z) - J_+d\omega_+(QX, J_+Y, J_-Z) - J_+d\omega_+(QX, J_+J_-Y, Z) \\ - J_+d\omega_+(QX, Y, J_+J_-Z) = \sigma_{XYZ}J_+d\omega_+(QX, Y, QZ). \end{aligned}$$

Similarly

$$\begin{aligned} \sigma_{XYZ}J_-d\omega_-(QX, J_-Y, J_+Z) + J_-d\omega_-(QX, J_+Y, J_-Z) + J_-d\omega_-(QX, J_-J_+Y, Z) \\ + J_-d\omega_-(QX, Y, J_-J_+Z) = -\sigma_{XYZ}J_-d\omega_-(QX, QY, Z) = \sigma_{XYZ}J_-d\omega_-(QX, Y, QZ), \end{aligned}$$

and (13) follows. $\square$

Finally, as a consequence of Theorem 4.6 we have the following.

Corollary 4.11. *Let (M, J) be a complex surface and Ω a symplectic form that tames J . If $\Omega^{2,0} \neq 0$, then the $(2, 0)$ -part $\tilde{Q}^{(2,0)}$ of the bivector $\tilde{Q}$ defines a non-vanishing holomorphic Poisson structure.*

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