



# Two classes of stacked central configurations for the spatial $2n + 1$ -body problem: Nested regular polyhedra plus one



Xia Su<sup>a,b,\*</sup>, Chunhua Deng<sup>a</sup>

<sup>a</sup> Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China

<sup>b</sup> College of Science, Hohai University, Nanjing 210098, China

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## ABSTRACT

In this paper we consider  $2n$  mass points located at the vertices of two nested regular polyhedra with the same number of vertices and the  $(2n + 1)$ th mass located at the geometrical center of the nested regular polyhedra. We show the existence of central configurations for any given mass ratios and the size ratio of nested polyhedra.

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## 1. Introduction and main results

The classical  $n$ -body problem concerns with the motion of  $n$  mass points moving in space according to the Newtonian law:

$$m_i \ddot{x}_i = \sum_{k=1, k \neq i}^n \frac{m_k m_i (x_k - x_i)}{r_{ki}^3}, \quad i = 1, \dots, n. \quad (1.1)$$

Here the gravitational constant equals to one,  $x_i \in \mathbb{R}^d$  ( $1 \leq d \leq 3$ ) is the position of mass  $m_i > 0$  and  $r_{ki} = |x_k - x_i|$  is the Euclidean distance between  $x_k$  and  $x_i$ .

Let  $M = m_1 + \dots + m_n$  be the total mass and

$$c = \frac{1}{M} (m_1 x_1 + \dots + m_n x_n)$$

be the center of mass of the configuration  $x = (x_1, \dots, x_n)$ . The space of configuration is defined by

$$X = \{x \in (\mathbb{R}^d)^n : c = 0, x_i \neq x_j \text{ for all } i \neq j\}.$$

A configuration  $x = (x_1, \dots, x_n) \in X$  is called a *central configuration* if there exists some positive constant  $\lambda$ , called the multiplier, such that

$$-\lambda x_i = \sum_{j=1, j \neq i}^n \frac{m_j (x_j - x_i)}{r_{ij}^3}, \quad i = 1, \dots, n. \quad (1.2)$$

\* Corresponding author at: Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China. Tel.: +86 15152359290.  
E-mail address: [sxdch2004@aliyun.com](mailto:sxdch2004@aliyun.com) (X. Su).

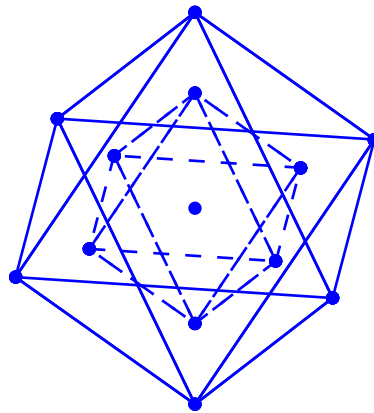


Fig. 1. Nested regular octahedra plus one.

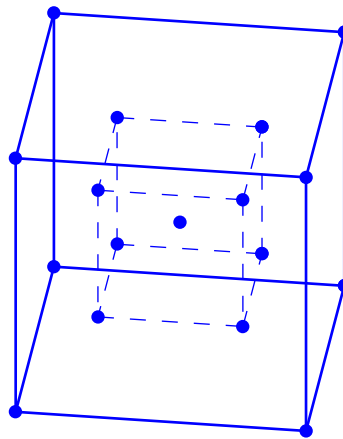


Fig. 2. Nested regular cube plus one.

It is easy to see that a central configuration remains a central configuration after a rotation in  $\mathbb{R}^d$  and a scalar multiplication. More precisely, let  $A \in SO(d)$  and  $a > 0$ , if  $x = (x_1, \dots, x_n)$  is a central configuration, so are  $Ax = (Ax_1, \dots, Ax_n)$  and  $ax = (ax_1, \dots, ax_n)$ .

Two central configurations are said to be equivalent if one can be transformed to the other by a scalar multiplication and a rotation. In this paper, when we say a central configuration, we mean a class of central configurations as defined by the above equivalence relation.

The study of central configuration goes back to Euler and Lagrange. For  $n = 3$ , it is a classical result that there are three collinear, called Euler, central configurations and one equilateral triangular, called Lagrange, central configurations. For  $n = 4$ , Moulton [1] proved that there is exactly one collinear central configuration for each arrangement of the mass points on the line.

There are several reasons why central configurations are of special importance in the study of the  $n$ -body problem; see [2–4] for details.

A *stacked central configuration* is a central configuration in which a proper subset of the  $n$  bodies is already in a central configuration. This class of central configuration of 5-body problem was introduced by Hampton in [5]. The work of [5] was complemented by Llibre in [6,7].

In this paper we are interested in spatial central configurations, that is  $d = 3$ . Zhang and Zhou [8] showed the existence of double pyramidal central configurations of  $N + 2$ -body problem. The authors [9–11] provided some examples of stacked central configurations for the spatial 7-body problem.

The authors [12] showed the existence of spatial central configurations of nested regular polyhedra. In this paper we consider  $2n$  masses located at the vertices of two nested regular polyhedra with the same number of vertices and the  $(2n + 1)$ th mass located at the geometrical center of the nested regular polyhedra (see Figs. 1–2). The case  $n = 4$  can be found in [13]. The stacked nested regular octahedra(cube) plus one central configurations are characterized in the following sections.

Without loss of generality, we can assume that the masses in the inner polyhedra are  $m_1, \dots, m_n$ , the masses in the outer polyhedra are  $\tilde{m}_1, \dots, \tilde{m}_n$  and the mass at the geometrical center is  $m_0$ , and the corresponding position vectors are  $x_1, \dots, x_n, \tilde{x}_1, \dots, \tilde{x}_n, x_0$ .

For the spatial central configurations, Eqs. (1.2) are equivalent to the following equations:

$$-\lambda x_i = \sum_{j=1, j \neq i}^n \frac{m_j(x_j - x_i)}{|x_j - x_i|^3} + \sum_{j=1}^n \frac{\tilde{m}_j(\tilde{x}_j - x_i)}{|\tilde{x}_j - x_i|^3} + \frac{m_0(x_0 - x_i)}{|x_0 - x_i|^3}, \quad i = 1, \dots, n. \quad (1.3)$$

$$-\lambda \tilde{x}_i = \sum_{j=1}^n \frac{m_j(x_j - \tilde{x}_i)}{|x_j - \tilde{x}_i|^3} + \sum_{j=1, j \neq i}^n \frac{\tilde{m}_j(\tilde{x}_j - \tilde{x}_i)}{|\tilde{x}_j - \tilde{x}_i|^3} + \frac{m_0(x_0 - \tilde{x}_i)}{|x_0 - \tilde{x}_i|^3}, \quad i = 1, \dots, n. \quad (1.4)$$

$$-\lambda x_0 = \sum_{j=1}^n \frac{m_j(x_j - x_0)}{|x_j - x_0|^3} + \sum_{j=1}^n \frac{\tilde{m}_j(\tilde{x}_j - x_0)}{|\tilde{x}_j - x_0|^3}. \quad (1.5)$$

The main results of this paper are as follows.

**Theorem 1.** We consider  $2n$  masses located at the vertices of two nested regular polyhedra with  $n$  vertices, where  $n = 6$  and  $8$  and the  $(2n + 1)$ th mass located at the geometrical center of the nested regular polyhedra; the mass points form a central configuration. Then

1. the masses in the inner polyhedra  $m_1, \dots, m_n$  are equal;
2. the masses in the outer polyhedra  $\tilde{m}_1, \dots, \tilde{m}_n$  are equal.

Let  $m$  and  $\tilde{m}$  be the ratios of the masses in the inner polyhedra to the mass at the geometrical center and the masses in the outer polyhedra to the mass at the geometrical center, respectively, and  $\rho$  be the ratio of the sizes of the inner and the outer polyhedra. Then we have the following theorem.

**Theorem 2.** If the  $2n + 1$  bodies form a central configuration, then for any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \neq 1$ .

**Remark 3.** Theorems 1 and 2 can be extended to spatial central configurations of  $2n + 1$ -body problem for  $n = 12$  and  $n = 20$ , that is, nested icosahedra [12] plus one and nested dodecahedra [12] plus one.

The proofs of the theorems are given in the next sections.

## 2. Nested regular octahedra plus one

In this section we study the spatial central configurations of 13-body problem when the masses are located at the vertices of two nested regular octahedra and the geometrical center of the nested octahedra. Taking conveniently we can assume that the mass at the geometrical center equals to one.

Without loss of generality, we can assume that

$$\begin{aligned} x_1 &= (1, 0, 0), & x_2 &= (0, 1, 0), & x_3 &= (-1, 0, 0), \\ x_4 &= (0, -1, 0), & x_5 &= (0, 0, 1), & x_6 &= (0, 0, -1), \\ \tilde{x}_1 &= (\rho, 0, 0), & \tilde{x}_2 &= (0, \rho, 0), & \tilde{x}_3 &= (-\rho, 0, 0), \\ \tilde{x}_4 &= (0, -\rho, 0), & \tilde{x}_5 &= (0, 0, \rho), & \tilde{x}_6 &= (0, 0, -\rho), \\ x_0 &= (0, 0, 0). \end{aligned} \quad (2.1)$$

where  $\rho$  be the ratio of the sizes of the nested octahedra and  $\rho \neq 1$ .

**Proposition 2.1.** Consider the spatial configurations according to Fig. 1; in order that the thirteen mass points are in a central configuration, the following statements are necessary:

1. the masses  $m_1, m_2, m_3, m_4, m_5$  and  $m_6$  are equal;
2. the masses  $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4, \tilde{m}_5$  and  $\tilde{m}_6$  are equal.

**Proof.** By our assumptions, the center of mass of the configuration is at the origin, we have the following equations:

$$\begin{aligned} (m_1 - m_3) + \rho(\tilde{m}_1 - \tilde{m}_3) &= 0, \\ (m_2 - m_4) + \rho(\tilde{m}_2 - \tilde{m}_4) &= 0, \\ (m_5 - m_6) + \rho(\tilde{m}_5 - \tilde{m}_6) &= 0. \end{aligned} \quad (2.2)$$

From Eqs. (1.5), we obtain the equations

$$\begin{aligned}(m_1 - m_3) + \frac{1}{\rho^2}(\tilde{m}_1 - \tilde{m}_3) &= 0, \\(m_2 - m_4) + \frac{1}{\rho^2}(\tilde{m}_2 - \tilde{m}_4) &= 0, \\(m_5 - m_6) + \frac{1}{\rho^2}(\tilde{m}_5 - \tilde{m}_6) &= 0.\end{aligned}\tag{2.3}$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 - m_3, \tilde{m}_1 - \tilde{m}_3$  (or  $m_2 - m_4, \tilde{m}_2 - \tilde{m}_4; m_5 - m_6, \tilde{m}_5 - \tilde{m}_6$ ) is

$$\begin{vmatrix} 1 & \rho \\ 1 & \frac{1}{\rho^2} \end{vmatrix} = \frac{1 - \rho^3}{\rho^2} \neq 0$$

as  $\rho \neq 1$ , so Eqs. (2.2) and (2.3) hold if and only if  $m_1 = m_3, \tilde{m}_1 = \tilde{m}_3$  ( $m_2 = m_4, \tilde{m}_2 = \tilde{m}_4; m_5 = m_6, \tilde{m}_5 = \tilde{m}_6$ ).

Let  $\bar{x}, \bar{y}, \bar{z}$  be the unit vectors in the x-direction, y-direction and z-direction, respectively.  $(1.3)_k$  denotes the  $k$ th equation in (1.3) in hereafter, and  $(1.3)_k$  taking the inner product by  $\bar{x}$  is denoted by  $(1.3)_k \cdot \bar{x}$ .

Substituting  $m_1 = m_3, m_2 = m_4, m_5 = m_6$  and  $\tilde{m}_1 = \tilde{m}_3, \tilde{m}_2 = \tilde{m}_4, \tilde{m}_5 = \tilde{m}_6$  into Eqs. (1.3) and (1.4).  $(1.3)_1 \cdot \bar{x}$  minus  $(1.3)_2 \cdot \bar{y}$ , we have

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)(m_1 - m_2) + \left(\frac{2}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} - \frac{1}{(\rho+1)^2}\right)(\tilde{m}_1 - \tilde{m}_2) = 0.\tag{2.4}$$

$(1.4)_1 \cdot \bar{x}$  minus  $(1.4)_2 \cdot \bar{y}$ , we have

$$\left(\frac{2\rho}{(\sqrt{1+\rho^2})^3} + \frac{1-\rho}{|\rho-1|^3} - \frac{1}{(\rho+1)^2}\right)(m_1 - m_2) + \left(\frac{1}{\sqrt{2}\rho^2} - \frac{1}{4\rho^2}\right)(\tilde{m}_1 - \tilde{m}_2) = 0.\tag{2.5}$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 - m_2, \tilde{m}_1 - \tilde{m}_2$  is

$$\begin{vmatrix} \frac{1}{\sqrt{2}} - \frac{1}{4} & \frac{2}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} - \frac{1}{(\rho+1)^2} \\ \frac{2\rho}{(\sqrt{1+\rho^2})^3} + \frac{1-\rho}{|\rho-1|^3} - \frac{1}{(\rho+1)^2} & \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right) \frac{1}{\rho^2} \end{vmatrix} > 0$$

as  $\rho \neq 1$ , so Eqs. (2.4) and (2.5) hold if and only if  $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$ .

$(1.3)_2 \cdot \bar{y}$  minus  $(1.3)_5 \cdot \bar{z}$ , we have

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)(m_2 - m_5) + \left(\frac{2}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} - \frac{1}{(\rho+1)^2}\right)(\tilde{m}_2 - \tilde{m}_5) = 0.\tag{2.6}$$

$(1.4)_2 \cdot \bar{y}$  minus  $(1.4)_5 \cdot \bar{z}$ , we have

$$\left(\frac{2\rho}{(\sqrt{1+\rho^2})^3} + \frac{1-\rho}{|\rho-1|^3} - \frac{1}{(\rho+1)^2}\right)(m_2 - m_5) + \left(\frac{1}{\sqrt{2}\rho^2} - \frac{1}{4\rho^2}\right)(\tilde{m}_2 - \tilde{m}_5) = 0.\tag{2.7}$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_2 - m_5, \tilde{m}_2 - \tilde{m}_5$  is

$$\begin{vmatrix} \frac{1}{\sqrt{2}} - \frac{1}{4} & \frac{2}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} - \frac{1}{(\rho+1)^2} \\ \frac{2\rho}{(\sqrt{1+\rho^2})^3} + \frac{1-\rho}{|\rho-1|^3} - \frac{1}{(\rho+1)^2} & \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right) \frac{1}{\rho^2} \end{vmatrix} > 0$$

as  $\rho \neq 1$ , so Eqs. (2.6) and (2.7) hold if and only if  $m_2 = m_5, \tilde{m}_2 = \tilde{m}_5$ . This completes the proof.

**Proposition 2.2.** Consider the masses  $m_i = m$ ,  $\tilde{m}_i = \tilde{m}$  for  $i = 1, \dots, 6$  at the vertices of the nested regular octahedron which have position vectors according to assumption (2.1). Then the following statements hold.

1. The mass ratios  $m$ ,  $\tilde{m}$  and the size ratio  $\rho$  of the nested polyhedra satisfy the equation

$$\left( \frac{4\rho}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} + \frac{1}{(\rho+1)^2} - \frac{\rho}{4} - \sqrt{2}\rho \right) m + \left( \frac{\sqrt{2}}{\rho^2} + \frac{1}{4\rho^2} - \frac{4\rho}{(\sqrt{1+\rho^2})^3} - \frac{\rho}{(\rho+1)^2} + \frac{\rho(\rho-1)}{|\rho-1|^3} \right) \tilde{m} + \frac{1}{\rho^2} - \rho = 0.$$

2. For any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \neq 1$ .

**Proof.** Substituting  $m_1 = \dots = m_6 = m$  and  $\tilde{m}_1 = \dots = \tilde{m}_6 = \tilde{m}$  into Eqs. (1.3) and (1.4), we may obtain the following equivalent equations:

$$\left( \sqrt{2} + \frac{1}{4} \right) m + \left( \frac{4}{(\sqrt{1+\rho^2})^3} - \frac{\rho-1}{|\rho-1|^3} + \frac{1}{(\rho+1)^2} \right) \tilde{m} + 1 = \lambda. \quad (2.8)$$

$$\left( \frac{4\rho}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} + \frac{1}{(\rho+1)^2} \right) m + \left( \frac{\sqrt{2}}{\rho^2} + \frac{1}{4\rho^2} \right) \tilde{m} + \frac{1}{\rho^2} = \lambda\rho. \quad (2.9)$$

From (2.8) and (2.9), eliminating  $\lambda$  we have  $f(\rho) = 0$ , where

$$f(\rho) = \left( \frac{4\rho}{(\sqrt{1+\rho^2})^3} + \frac{\rho-1}{|\rho-1|^3} + \frac{1}{(\rho+1)^2} - \frac{\rho}{4} - \sqrt{2}\rho \right) m + \left( \frac{\sqrt{2}}{\rho^2} + \frac{1}{4\rho^2} - \frac{4\rho}{(\sqrt{1+\rho^2})^3} - \frac{\rho}{(\rho+1)^2} + \frac{\rho(\rho-1)}{|\rho-1|^3} \right) \tilde{m} + \frac{1}{\rho^2} - \rho.$$

So statement 1 of Proposition 2.1 is proved.

$f(\rho)$  is a continuous function in  $\rho \in (0, 1)$  and for any given positive mass ratios  $m$  and  $\tilde{m}$ ,  $f(\rho) \rightarrow +\infty$  as  $\rho \rightarrow 0^+$  and  $f(\rho) \rightarrow -\infty$  as  $\rho \rightarrow 1^-$ .

$$f'(\rho) = \left( \frac{4-8\rho^2}{(\sqrt{1+\rho^2})^5} - \frac{4+12\rho^2}{(1-\rho^2)^3} - \frac{1}{4} - \sqrt{2} \right) m - \left( \frac{2\sqrt{2}}{\rho^3} + \frac{1}{2\rho^3} + \frac{4-8\rho^2}{(\sqrt{1+\rho^2})^5} + \frac{2\rho^4+12\rho^2+2}{(1-\rho^2)^3} \right) \tilde{m} - \left( \frac{2}{\rho^3} + 1 \right)$$

is negative as  $\rho \in (0, 1)$ . Thus for any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \in (0, 1)$ .

$f(\rho)$  is a continuous function in  $\rho \in (1, +\infty)$  and for any given positive mass ratios  $m$  and  $\tilde{m}$ ,  $f(\rho) \rightarrow +\infty$  as  $\rho \rightarrow 1^+$  and  $f(\rho) \rightarrow -\infty$  as  $\rho \rightarrow +\infty$ :

$$f'(\rho) = \left( \frac{4-8\rho^2}{(\sqrt{1+\rho^2})^5} - \frac{4\rho(3+\rho^2)}{(\rho^2-1)^3} - \frac{1}{4} - \sqrt{2} \right) m - \left( \frac{2\sqrt{2}}{\rho^3} + \frac{1}{2\rho^3} + \frac{4-8\rho^2}{(\sqrt{1+\rho^2})^5} + \frac{8\rho(1+\rho^2)}{(\rho^2-1)^3} \right) \tilde{m} - \left( \frac{2}{\rho^3} + 1 \right)$$

is negative as  $\rho \in (1, +\infty)$ . Thus for any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \in (1, +\infty)$ . This completes the proof.

### 3. Nested regular cube plus one

In this section we study the spatial central configurations of 17-body problem when the masses are located at the vertices of two nested regular cubes and the geometrical center of the nested cube. Taking conveniently we can assume that the mass at the geometrical center equals to one.

Without loss of generality, we can assume that

$$\begin{aligned} x_1 &= (2, 0, -\sqrt{2}), & x_2 &= (0, 2, -\sqrt{2}), & x_3 &= (-2, 0, -\sqrt{2}), & x_4 &= (0, -2, -\sqrt{2}), \\ x_5 &= (2, 0, \sqrt{2}), & x_6 &= (0, 2, \sqrt{2}), & x_7 &= (-2, 0, \sqrt{2}), & x_8 &= (0, -2, \sqrt{2}), \\ \tilde{x}_1 &= (2\rho, 0, -\sqrt{2}\rho), & \tilde{x}_2 &= (0, 2\rho, -\sqrt{2}\rho), & \tilde{x}_3 &= (-2\rho, 0, -\sqrt{2}\rho), & \tilde{x}_4 &= (0, -2\rho, -\sqrt{2}\rho), \\ \tilde{x}_5 &= (2\rho, 0, \sqrt{2}\rho), & \tilde{x}_6 &= (0, 2\rho, \sqrt{2}\rho), & \tilde{x}_7 &= (-2\rho, 0, \sqrt{2}\rho), & \tilde{x}_8 &= (0, -2\rho, \sqrt{2}\rho), \\ x_0 &= (0, 0, 0). \end{aligned} \quad (3.1)$$

where  $\rho$  be the ratio of the sizes of the nested cube and  $\rho \neq 1$ .

**Proposition 3.1.** Consider the spatial configurations according to Fig. 2; in order that the seventeen mass points are in a central configuration, the following statements are necessary:

1. The masses  $m_1, m_2, m_3, m_4, m_5, m_6, m_7$  and  $m_8$  are equal;
2. The masses  $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4, \tilde{m}_5, \tilde{m}_6, \tilde{m}_7$  and  $\tilde{m}_8$  are equal.

**Proof.** By our assumptions, the center of the mass of the configuration is at the origin, we have the following equations:

$$\begin{aligned} (m_1 - m_3 + m_5 - m_7) + \rho(\tilde{m}_1 - \tilde{m}_3 + \tilde{m}_5 - \tilde{m}_7) &= 0, \\ (m_2 - m_4 + m_6 - m_8) + \rho(\tilde{m}_2 - \tilde{m}_4 + \tilde{m}_6 - \tilde{m}_8) &= 0. \end{aligned} \quad (3.2)$$

From Eqs. (1.5), we obtain the equations

$$\begin{aligned} (m_1 - m_3 + m_5 - m_7) + \frac{1}{\rho^2}(\tilde{m}_1 - \tilde{m}_3 + \tilde{m}_5 - \tilde{m}_7) &= 0, \\ (m_2 - m_4 + m_6 - m_8) + \frac{1}{\rho^2}(\tilde{m}_2 - \tilde{m}_4 + \tilde{m}_6 - \tilde{m}_8) &= 0. \end{aligned} \quad (3.3)$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 - m_3 + m_5 - m_7, \tilde{m}_1 - \tilde{m}_3 + \tilde{m}_5 - \tilde{m}_7$  (or  $m_2 - m_4 + m_6 - m_8, \tilde{m}_2 - \tilde{m}_4 + \tilde{m}_6 - \tilde{m}_8$ ) is

$$\begin{vmatrix} 1 & \rho \\ 1 & \frac{1}{\rho^2} \end{vmatrix} = \frac{1 - \rho^3}{\rho^2} \neq 0$$

as  $\rho \neq 1$ , so Eqs. (3.2) and (3.3) hold if and only if

$$m_1 - m_3 + m_5 - m_7 = 0, \quad \tilde{m}_1 - \tilde{m}_3 + \tilde{m}_5 - \tilde{m}_7 = 0. \quad (3.4)$$

$$m_2 - m_4 + m_6 - m_8 = 0, \quad \tilde{m}_2 - \tilde{m}_4 + \tilde{m}_6 - \tilde{m}_8 = 0. \quad (3.5)$$

Taking the inner product of (1.3) for  $i = 1$  and 3 by  $\bar{x}$ , and then adding them together, we have

$$\begin{aligned} \frac{1}{16}(m_1 - m_3) + \frac{1}{12\sqrt{6}}(m_5 - m_7) + \left( \frac{2(\rho + 1)}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} + \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} \right) (\tilde{m}_1 - \tilde{m}_3) \\ + \left( \frac{2(\rho - 1)}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} + \frac{1}{3\sqrt{6}(\rho + 1)^2} \right) (\tilde{m}_5 - \tilde{m}_7) = 0. \end{aligned} \quad (3.6)$$

Taking the inner product of (1.3) for  $i = 2$  by  $\bar{x}$ , we have

$$\frac{1}{8\sqrt{2}}(m_1 - m_3) + \frac{1}{32}(m_5 - m_7) + \frac{2\rho}{(\sqrt{6}\rho^2 - 4\rho + 6)^3}(\tilde{m}_1 - \tilde{m}_3) + \frac{2\rho}{(\sqrt{6}\rho^2 + 4\rho + 6)^3}(\tilde{m}_5 - \tilde{m}_7) = 0. \quad (3.7)$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 - m_3, m_5 - m_7, \tilde{m}_1 - \tilde{m}_3, \tilde{m}_5 - \tilde{m}_7$  is

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{1}{16} & \frac{1}{12\sqrt{6}} & \frac{2(\rho + 1)}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} + \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} & \frac{2(\rho - 1)}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} + \frac{1}{3\sqrt{6}(\rho + 1)^2} \\ \frac{1}{8\sqrt{2}} & \frac{1}{32} & \frac{2\rho}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} & \frac{2\rho}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} \end{vmatrix} \neq 0$$

as  $\rho \neq 1$ , so Eqs. (3.4), (3.6) and (3.7) hold if and only if  $m_1 = m_3, m_5 = m_7, \tilde{m}_1 = \tilde{m}_3, \tilde{m}_5 = \tilde{m}_7$ .

Taking the inner product of (1.3) for  $i = 2$  and 4 by  $\bar{y}$ , and then adding them together, we have

$$\begin{aligned} & \frac{1}{16}(m_2 - m_4) + \frac{1}{12\sqrt{6}}(m_6 - m_8) + \left( \frac{2(\rho + 1)}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} + \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} \right) (\tilde{m}_2 - \tilde{m}_4) \\ & + \left( \frac{2(\rho - 1)}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} + \frac{1}{3\sqrt{6}(\rho + 1)^2} \right) (\tilde{m}_6 - \tilde{m}_8) = 0. \end{aligned} \quad (3.8)$$

Taking the inner product of (1.3) for  $i = 1$  by  $\bar{y}$ , we have

$$\frac{1}{8\sqrt{2}}(m_2 - m_4) + \frac{1}{32}(m_6 - m_8) + \frac{2\rho}{(\sqrt{6}\rho^2 - 4\rho + 6)^3}(\tilde{m}_2 - \tilde{m}_4) + \frac{2\rho}{(\sqrt{6}\rho^2 + 4\rho + 6)^3}(\tilde{m}_6 - \tilde{m}_8) = 0. \quad (3.9)$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_2 - m_4$ ,  $m_6 - m_8$ ,  $\tilde{m}_2 - \tilde{m}_4$ ,  $\tilde{m}_6 - \tilde{m}_8$  is

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{1}{16} & \frac{1}{12\sqrt{6}} & \frac{2(\rho + 1)}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} + \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} & \frac{2(\rho - 1)}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} + \frac{1}{3\sqrt{6}(\rho + 1)^2} \\ \frac{1}{8\sqrt{2}} & \frac{1}{32} & \frac{2\rho}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} & \frac{2\rho}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} \end{vmatrix} \neq 0$$

as  $\rho \neq 1$ , so Eqs. (3.5), (3.8) and (3.9) hold if and only if  $m_2 = m_4$ ,  $m_6 = m_8$ ,  $\tilde{m}_2 = \tilde{m}_4$ ,  $\tilde{m}_6 = \tilde{m}_8$ .

Substituting  $m_1 = m_3$ ,  $m_2 = m_4$ ,  $m_5 = m_7$ ,  $m_6 = m_8$  and  $\tilde{m}_1 = \tilde{m}_3$ ,  $\tilde{m}_2 = \tilde{m}_4$ ,  $\tilde{m}_5 = \tilde{m}_7$ ,  $\tilde{m}_6 = \tilde{m}_8$  into Eqs. (1.3)–(1.5), and the center of the mass of the configuration is at the origin, we obtain the equations

$$\begin{aligned} (m_1 + m_2 - m_5 - m_6) + \rho(\tilde{m}_1 + \tilde{m}_2 - \tilde{m}_5 - \tilde{m}_6) &= 0, \\ (m_1 + m_2 - m_5 - m_6) + \frac{1}{\rho^2}(\tilde{m}_1 + \tilde{m}_2 - \tilde{m}_5 - \tilde{m}_6) &= 0. \end{aligned} \quad (3.10)$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 + m_2 - m_5 - m_6$ ,  $\tilde{m}_1 + \tilde{m}_2 - \tilde{m}_5 - \tilde{m}_6$  is

$$\begin{vmatrix} 1 & \rho \\ 1 & \frac{1}{\rho^2} \end{vmatrix} = \frac{1 - \rho^3}{\rho^2} \neq 0$$

as  $\rho \neq 1$ , so Eq. (3.10) holds if and only if

$$\begin{aligned} m_1 + m_2 - m_5 - m_6 &= 0, \\ \tilde{m}_1 + \tilde{m}_2 - \tilde{m}_5 - \tilde{m}_6 &= 0. \end{aligned} \quad (3.11)$$

(1.3)<sub>1</sub> ·  $\bar{x}$  minus (1.3)<sub>5</sub> ·  $\bar{x}$ , we have

$$\begin{aligned} & \left( \frac{1}{12\sqrt{6}} - \frac{1}{16} \right) (m_1 - m_5) + \left( \frac{1}{16} - \frac{1}{4\sqrt{2}} \right) (m_2 - m_6) \\ & + \left( \frac{1}{3\sqrt{6}(\rho + 1)^2} + \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} - \frac{2(\rho + 1)}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} - \frac{2(\rho - 1)}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} \right) (\tilde{m}_1 - \tilde{m}_5) \\ & + \left( \frac{4}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} - \frac{4}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} \right) (\tilde{m}_2 - \tilde{m}_6) = 0. \end{aligned} \quad (3.12)$$

(1.4)<sub>1</sub> ·  $\bar{x}$  minus (1.4)<sub>5</sub> ·  $\bar{x}$ , we have

$$\left( \frac{1}{3\sqrt{6}(\rho + 1)^2} + \frac{1 - \rho}{3\sqrt{6}|\rho - 1|^3} - \frac{2(\rho + 1)}{(\sqrt{6}\rho^2 + 4\rho + 6)^3} - \frac{2(1 - \rho)}{(\sqrt{6}\rho^2 - 4\rho + 6)^3} \right) (m_1 - m_5)$$

$$\begin{aligned}
& + \left( \frac{4\rho}{(\sqrt{6\rho^2 + 4\rho + 6})^3} - \frac{4\rho}{(\sqrt{6\rho^2 - 4\rho + 6})^3} \right) (m_2 - m_6) \\
& + \left( \frac{1}{12\sqrt{6\rho^2}} - \frac{1}{16\rho^2} \right) (\tilde{m}_1 - \tilde{m}_5) + \left( \frac{1}{16\rho^2} - \frac{1}{4\sqrt{2}\rho^2} \right) (\tilde{m}_2 - \tilde{m}_6) = 0.
\end{aligned} \quad (3.13)$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 - m_5, m_2 - m_6, \tilde{m}_1 - \tilde{m}_5, \tilde{m}_2 - \tilde{m}_6$  is negative as  $\rho \neq 1$ , so Eqs. (3.11)–(3.13) hold if and only if  $m_1 = m_5, m_2 = m_6, \tilde{m}_1 = \tilde{m}_5, \tilde{m}_2 = \tilde{m}_6$ .

Substituting  $m_1 = m_3 = m_5 = m_7, m_2 = m_4 = m_6 = m_8$  and  $\tilde{m}_1 = \tilde{m}_3 = \tilde{m}_5 = \tilde{m}_7, \tilde{m}_2 = \tilde{m}_4 = \tilde{m}_6 = \tilde{m}_8$  into Eqs. (1.3) and (1.4), (1.3)<sub>1</sub> ·  $\bar{x}$  minus (1.3)<sub>2</sub> ·  $\bar{y}$ , we have

$$\begin{aligned}
& \left( \frac{1}{4\sqrt{2}} - \frac{1}{12\sqrt{6}} \right) (m_1 - m_2) + \left( \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} - \frac{1}{3\sqrt{6}(\rho + 1)^2} \right. \\
& \left. + \frac{2(\rho + 1)}{(\sqrt{6\rho^2 - 4\rho + 6})^3} - \frac{2(\rho - 1)}{(\sqrt{6\rho^2 + 4\rho + 6})^3} \right) (\tilde{m}_1 - \tilde{m}_2) = 0
\end{aligned} \quad (3.14)$$

(1.4)<sub>1</sub> ·  $\bar{x}$  minus (1.4)<sub>2</sub> ·  $\bar{y}$ , we have

$$\begin{aligned}
& \left( \frac{2(\rho + 1)}{(\sqrt{6\rho^2 - 4\rho + 6})^3} + \frac{2(\rho - 1)}{(\sqrt{6\rho^2 + 4\rho + 6})^3} - \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} - \frac{1}{3\sqrt{6}(\rho + 1)^2} \right) (m_1 - m_2) \\
& + \left( \frac{1}{4\sqrt{2}\rho^2} - \frac{1}{12\sqrt{6}\rho^2} \right) (\tilde{m}_1 - \tilde{m}_2) = 0.
\end{aligned} \quad (3.15)$$

The determinant of the matrix of the coefficients of the homogeneous linear system in the variables  $m_1 - m_2, \tilde{m}_1 - \tilde{m}_2$  is positive as  $\rho \neq 1$ , so Eqs. (3.14) and (3.15) hold if and only if  $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$ . This completes the proof.

**Proposition 3.2.** Consider the masses  $m_i = m, \tilde{m}_i = \tilde{m}$  for  $i = 1, \dots, 8$  at the vertices of the nested regular cube which have position vectors according to assumption (3.1). Then the following statements hold.

1. The mass ratios  $m$  and  $\tilde{m}$  and the size ratio  $\rho$  of the nested cube satisfy the equation

$$\begin{aligned}
& \left( \frac{\rho}{8} + \frac{\rho}{4\sqrt{2}} + \frac{\rho}{12\sqrt{6}} - \frac{1}{3\sqrt{6}(1 + \rho)^2} - \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} - \frac{2(3\rho - 1)}{(\sqrt{6\rho^2 - 4\rho + 6})^3} - \frac{2(3\rho + 1)}{(\sqrt{6\rho^2 + 4\rho + 6})^3} \right) m \\
& + \left( \frac{\rho}{3\sqrt{6}(\rho + 1)^2} - \frac{\rho(\rho - 1)}{3\sqrt{6}|\rho - 1|^3} + \frac{2\rho(3 - \rho)}{(6\rho^2 - 4\rho + 6)^3} + \frac{2\rho(3 + \rho)}{(\sqrt{6\rho^2 + 4\rho + 6})^3} - \frac{1}{8\rho^2} \right. \\
& \left. - \frac{1}{4\sqrt{2}\rho^2} - \frac{1}{12\sqrt{6}\rho^2} \right) \tilde{m} + \frac{\rho}{3\sqrt{6}} - \frac{1}{3\sqrt{6}\rho^2} = 0.
\end{aligned}$$

2. For any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \neq 1$ .

**Proof.** Substituting  $m_1 = \dots = m_8 = m$  and  $\tilde{m}_1 = \dots = \tilde{m}_8 = \tilde{m}$  into Eqs. (1.3) and (1.4), we may obtain the following equivalent equations:

$$\begin{aligned}
& \left( \frac{1}{8} + \frac{1}{4\sqrt{2}} + \frac{1}{12\sqrt{6}} \right) m + \left( \frac{2(\rho + 3)}{(\sqrt{6\rho^2 + 4\rho + 6})^3} + \frac{2(3 - \rho)}{(6\rho^2 - 4\rho + 6)^3} \right. \\
& \left. - \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} + \frac{1}{3\sqrt{6}(\rho + 1)^2} \right) \tilde{m} + \frac{1}{3\sqrt{6}} = 2\lambda.
\end{aligned} \quad (3.16)$$

$$\begin{aligned}
& \left( \frac{2(3\rho + 1)}{(\sqrt{6\rho^2 + 4\rho + 6})^3} + \frac{2(3\rho - 1)}{(6\rho^2 - 4\rho + 6)^3} + \frac{\rho - 1}{3\sqrt{6}|\rho - 1|^3} + \frac{1}{3\sqrt{6}(\rho + 1)^2} \right) m \\
& + \left( \frac{1}{8\rho^2} + \frac{1}{4\sqrt{2}\rho^2} + \frac{1}{12\sqrt{6}\rho^2} \right) \tilde{m} + \frac{1}{3\sqrt{6}\rho^2} = 2\lambda\rho.
\end{aligned} \quad (3.17)$$



From (3.16) and (3.17), eliminating  $\lambda$  we have  $f(\rho) = 0$ , where

$$f(\rho) = \left( \frac{\rho}{8} + \frac{\rho}{4\sqrt{2}} + \frac{\rho}{12\sqrt{6}} - \frac{1}{3\sqrt{6}(\rho+1)^2} - \frac{\rho-1}{3\sqrt{6}|\rho-1|^3} - \frac{2(3\rho-1)}{(\sqrt{6\rho^2-4\rho+6})^3} - \frac{2(3\rho+1)}{(\sqrt{6\rho^2+4\rho+6})^3} \right) m \\ + \left( \frac{\rho}{3\sqrt{6}(\rho+1)^2} - \frac{\rho(\rho-1)}{3\sqrt{6}|\rho-1|^3} + \frac{2\rho(3-\rho)}{(6\rho^2-4\rho+6)^3} + \frac{2\rho(3+\rho)}{(\sqrt{6\rho^2+4\rho+6})^3} - \frac{1}{8\rho^2} \right. \\ \left. - \frac{1}{4\sqrt{2}\rho^2} - \frac{1}{12\sqrt{6}\rho^2} \right) \tilde{m} + \frac{\rho}{3\sqrt{6}} - \frac{1}{3\sqrt{6}\rho^2}.$$

So statement 1 of Proposition 3.1 is proved.

$f(\rho)$  is a continuous function in  $\rho \in (0, 1)$  and for any given positive mass ratios  $m$  and  $\tilde{m}$ ,  $f(\rho) \rightarrow -\infty$  as  $\rho \rightarrow 0^+$  and  $f(\rho) \rightarrow +\infty$  as  $\rho \rightarrow 1^-$ :

$$f'(\rho) = \left( \frac{1}{8} + \frac{1}{4\sqrt{2}} + \frac{1}{12\sqrt{6}} + \frac{4(3\rho^2+1)}{3\sqrt{6}(1-\rho^2)^3} + \frac{24(3\rho+1)(\rho-1)}{(\sqrt{6\rho^2-4\rho+6})^5} + \frac{24(3\rho-1)(\rho+1)}{(\sqrt{6\rho^2+4\rho+6})^5} \right) m \\ + \left( \frac{2(\rho^4+6\rho^2+1)}{3\sqrt{6}(1-\rho^2)^3} + \frac{4(3\rho^3-17\rho^2-3\rho+9)}{(\sqrt{6\rho^2-4\rho+6})^5} - \frac{4(3\rho^3+17\rho^2-3\rho-9)}{(\sqrt{6\rho^2+4\rho+6})^5} + \frac{1}{4\rho^3} \right. \\ \left. + \frac{1}{2\sqrt{2}\rho^3} + \frac{1}{6\sqrt{6}\rho^3} \right) \tilde{m} + \left( \frac{1}{3\sqrt{6}} + \frac{2}{3\sqrt{6}\rho^3} \right)$$

is positive as  $\rho \in (0, 1)$ . Thus for any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \in (0, 1)$ .

$f(\rho)$  is a continuous function in  $\rho \in (1, +\infty)$  and for any given positive  $m$  and  $\tilde{m}$ ,  $f(\rho) \rightarrow -\infty$  as  $\rho \rightarrow 1^+$  and  $f(\rho) \rightarrow +\infty$  as  $\rho \rightarrow +\infty$ :

$$f'(\rho) = \left( \frac{1}{8} + \frac{1}{4\sqrt{2}} + \frac{1}{12\sqrt{6}} + \frac{4\rho(\rho^2+3)}{3\sqrt{6}(\rho^2-1)^3} + \frac{24(3\rho+1)(\rho-1)}{(\sqrt{6\rho^2-4\rho+6})^5} + \frac{24(3\rho-1)(\rho+1)}{(\sqrt{6\rho^2+4\rho+6})^5} \right) m \\ + \left( \frac{8\rho(\rho^2+1)}{3\sqrt{6}(\rho^2-1)^3} + \frac{4(3\rho^3-17\rho^2-3\rho+9)}{(\sqrt{6\rho^2-4\rho+6})^5} - \frac{4(3\rho^3+17\rho^2-3\rho-9)}{(\sqrt{6\rho^2+4\rho+6})^5} + \frac{1}{4\rho^3} \right. \\ \left. + \frac{1}{2\sqrt{2}\rho^3} + \frac{1}{6\sqrt{6}\rho^3} \right) \tilde{m} + \left( \frac{1}{3\sqrt{6}} + \frac{2}{3\sqrt{6}\rho^3} \right)$$

is positive as  $\rho \in (1, +\infty)$ . Thus for any given mass ratios  $m$  and  $\tilde{m}$ , there exists only one central configuration corresponding to the radius ratio  $\rho \in (1, +\infty)$ . This completes the proof.

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