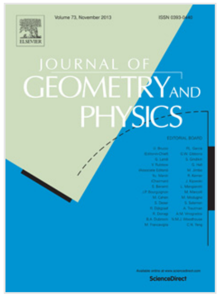

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On rational Frobenius manifolds of rank three with symmetries

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Abstract

We study Frobenius manifolds of rank three and dimension one that are related to submanifolds of certain Frobenius manifolds arising in mirror symmetry of elliptic orbifolds. We classify such Frobenius manifolds that are defined over an arbitrary field $\mathbb{K} \subset \mathbb{C}$ via the theory of modular forms. By an arithmetic property of an elliptic curve $\mathcal{E}_\tau$ defined over $\mathbb{K}$ associated to such a Frobenius manifold, it is proved that there are only two such Frobenius manifolds defined over $\mathbb{C}$ satisfying a certain symmetry assumption and thirteen Frobenius manifolds defined over $\mathbb{Q}$ satisfying a weak symmetry assumption on the potential.

Keywords: Frobenius manifolds, Singularity theory.
2000 MSC: 53D45, 32S30, 11G05.

Introduction

The notion of a Frobenius manifold was introduced by Boris Dubrovin in the 90s (cf. [1]) as the mathematical axiomatization of a 2D topological conformal field theory. A special class of Frobenius manifolds is given by certain structures on the base space of the universal unfolding of a hypersurface singularity.

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These structures were introduced in the early 80s by Kyoji Saito (cf. [3] for an introduction to this theory) and called at that time Saito's flat structures.

Actually, Saito found a richer structure than his flat structure, consisting of the filtered de Rham cohomology with the Gauß–Manin connection, higher residue pairings and a primitive form [4]. Unlike the general setting of a Frobenius manifold it has much more geometric data coming naturally from singularity theory. It is also now generalized as a so-called non-commutative Hodge theory by [5] which will be a necessary tool to understand the classical mirror symmetry (isomorphism of Frobenius manifolds) via a Kontsevich's homological mirror symmetry.

It is a very important problem to study some arithmetic aspect of a Saito structure with a geometric origin such as singularity theory. However, it is quite difficult at this moment. Therefore we start our consideration from the larger setting of Frobenius manifolds.

Namely we define particular $\mathrm{GL}(2,\mathbb{C})$-action on the Frobenius manifolds of the rank 3 and dimension 1. This action corresponds to the change of the primitive form of the simple elliptic singularities. More precisely, we shall define the Frobenius manifold $M^{(\tau_0,\omega_0)}$ of rank three and dimension one obtained by acting with a certain element $A^{(\tau_0,\omega_0)} \in \mathrm{GL}(2,\mathbb{C})$ depending on $\tau_0 \in \mathbb{H}$, $\omega_0 \in \mathbb{C}\backslash\{0\}$ (see Subsection 2.5) on the "basic" Frobenius manifold M^∞ (see Proposition 2.9 for the definition of M^∞). The Frobenius manifold M^∞ itself is connected to the Gromov-Witten Frobenius structures of the orbifold projective lines $\mathbb{P}^1_{2,2,2,2}$, $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$. These orbifold projective lines provide Calabi-Yau/Landau-Ginzburg mirror symmetry for simple elliptic singularities, what involves choosing the primitive form at the so-called *large complex structure limit* (LCSL for brevity). Therefore we can consider M^∞ as corresponding to the primitive form choice at the LCSL.

The general context of the global mirror symmetry requires existence of the so-called orbifolded Landau-Ginzburg A-model that is the Frobenius manifold, associated to the pair - singularity and a symmetry group of it. The systematic approach to this problem was given in [2] and is now called FJRW-theory. However it appears to be very hard to compute.

Looking for the Frobenius manifold that could potentially serve an orbifolded Landau-Ginzburg A-model it is natural to assume it to have some special properties. Namely to be defined over $\mathbb{Q}$ and have some "symmetries". By the global mirror symmetry assumption the orbifolded Landau-Ginzburg A-model should also correspond to some primitive form choice. This motivates our classification of the Frobenius manifolds $M^{(\tau_0,\omega_0)}$ defined over the field $\mathbb{K} \subset \mathbb{C}$ and also having "symmetries".

Results

Let $\mathbb{K} \subset \mathbb{C}$ be a field. We say that a Frobenius manifold M is *defined over* $\mathbb{K}$ if there exist flat coordinates $t_1, \ldots, t_\mu$ in which the Frobenius potential of M belongs to $\mathbb{K}\{t_1, \ldots, t_\mu\}$ and is defined at the point $t_1 = \cdots = t_\mu = 0$.

We associate the elliptic curve $\mathcal{E}_{\tau_0}$ with the modulus τ_0 to $M^{(\tau_0,\omega_0)}$. The first theorem of this paper states several criterion of the Frobenius manifold $M^{(\tau_0,\omega_0)}$ to be defined over $\mathbb{K}$. The criterion are given in terms of the values of the modular forms at the point $\tau_0 \in \mathbb{H}$.

In what follows we translate some properties of the elliptic curve $\mathcal{E}_{\tau_0}$ into special properties of the Frobenius manifold $M^{(\tau_0,\omega_0)}$. Considering the $\mathrm{SL}(2,\mathbb{R})$-action on $M^{(\tau_0,\omega_0)}$ we define the property of the Frobenius manifold $M^{(\tau_0,\omega_0)}$ to be "symmetric" and "weakly symmetric" (Definition 4.2). Namely we call $M^{(\tau_0,\omega_0)}$ symmetric if its potential is preserved by the action of some $A \in \mathrm{SL}(2,\mathbb{R})$ and weakly symmetric if its potential is rescaled by the action of A.

In the second theorem of this paper we show that the Frobenius manifold $M^{(\tau_0,\omega_0)}$ has a "symmetry" if and only if τ_0 is in the $\mathrm{SL}(2,\mathbb{Z})$ orbit of $\sqrt{-1}$ or $\exp(2\pi\sqrt{-1}/3)$ and the Frobenius manifold $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ has a "weak symmetry" if and only if $\mathcal{E}_{\tau_0}$ is isomorphic to one of 13 elliptic curves listed in Corollary 4.3.

Organization of the paper

After recalling some basic definitions and terminologies in Section 1, we shall study a rational structure on $M^{(\tau_0,\omega_0)}$. The $\mathrm{GL}(2,\mathbb{C})$-action and in particular $A^{(\tau_0,\omega_0)}$-action are defined in Section 2. In Section 3 we shall prove the first theorem of this paper and also give two natural examples of $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$. Section 4 is devoted to the study of the symmetries of $M^{(\tau_0,\omega_0)}$. It contains the second theorem of this paper. Finally, some useful data are given in the Appendix.

1. Preliminaries

1.1. Frobenius manifolds

We give some basic properties of a Frobenius manifold [1]. Let us recall the equivalent definition taken from Saito-Takahashi [3].

Definition. Let $M = (M, \mathcal{O}_M)$ be a connected complex manifold of dimension μ whose holomorphic tangent sheaf and cotangent sheaf are denoted by $\mathcal{T}_M$ and Ω^1_M respectively and let d be a complex number.

A *Frobenius structure of rank μ and dimension d on M* is a tuple $(\eta, \circ, e, E)$, where η is a non-degenerate $\mathcal{O}_M$-symmetric bilinear form on $\mathcal{T}_M$, $\circ$ is an $\mathcal{O}_M$-bilinear product on $\mathcal{T}_M$, defining an associative and commutative $\mathcal{O}_M$-algebra structure with a unit e, and E is a holomorphic vector field on M, called the Euler vector field, which are subject to the following axioms:

1. The product $\circ$ is self-adjoint with respect to η: that is,
$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M.$$
2. The Levi–Civita connection $\nabla\!\!\!/ : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ with respect to η is flat: that is,
$$[\nabla\!\!\!/_\delta, \nabla\!\!\!/_{\delta'}] = \nabla\!\!\!/_{[\delta,\delta']}, \quad \delta, \delta' \in \mathcal{T}_M.$$
3. The tensor $C : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ defined by $C_\delta \delta' := \delta \circ \delta'$, $(\delta, \delta' \in \mathcal{T}_M)$ is flat: that is,
$$\nabla\!\!\!/ C = 0.$$
4. The unit element e of the $\circ$-algebra is a $\nabla\!\!\!/$-flat holomorphic vector field: that is,
$$\nabla\!\!\!/ e = 0.$$
5. The metric η and the product $\circ$ are homogeneous of degree $2-d$ $(d \in \mathbb{C})$ and 1 respectively with respect to the Lie derivative Lie_E of the Euler vector field E: that is,
$$Lie_E(\eta) = (2-d)\eta, \quad Lie_E(\circ) = \circ.$$

A manifold M equipped with a Frobenius structure $(\eta, \circ, e, E)$ is called a *Frobenius manifold*.

From now on in this section, we shall always denote by M a Frobenius manifold. We expose some basic properties of Frobenius manifolds without their proofs.

Let us consider the space of horizontal sections of the connection $\nabla\!\!\!/$:
$$\mathcal{T}_M^f := \{\delta \in \mathcal{T}_M \mid \nabla\!\!\!/_{\delta'} \delta = 0 \text{ for all } \delta' \in \mathcal{T}_M\}$$
which is a local system of rank μ on M such that the metric η takes a constant value on $\mathcal{T}_M^f$. Namely, we have
$$\eta(\delta, \delta') \in \mathbb{C}, \quad \delta, \delta' \in \mathcal{T}_M^f.$$

Proposition 1.1. *At each point of M, there exist local coordinates $(t_1, \ldots, t_\mu)$, called flat coordinates, such that $e = \partial_1$, $\mathcal{T}_M^f$ is spanned by $\partial_1, \ldots, \partial_\mu$ and $\eta(\partial_i, \partial_j) \in \mathbb{C}$ for all $i, j = 1, \ldots, \mu$, where we denote $\partial/\partial t_i$ by ∂_i.*

The axiom $\nabla\!\!\!/ C = 0$ implies the following:

Proposition 1.2. *At each point of M, there exists a local holomorphic function $\mathcal{F}$, called Frobenius potential, satisfying*

$$\eta(\partial_i \circ \partial_j, \partial_k) = \eta(\partial_i, \partial_j \circ \partial_k) = \partial_i\partial_j\partial_k\mathcal{F}, \quad i,j,k = 1,\dots,\mu,$$

for any system of flat coordinates. In particular, one has

$$\eta_{ij} := \eta(\partial_i, \partial_j) = \partial_1\partial_i\partial_j\mathcal{F}.$$

The product structure on $\mathcal{T}_M$ is described locally by $\mathcal{F}$ as

$$\partial_i \circ \partial_j = \sum_{k=1}^{\mu} c_{ij}^k \partial_k \quad i,j = 1,\cdots,\mu,$$

$$c_{ij}^k := \sum_{l=1}^{\mu} \eta^{kl}\partial_i\partial_j\partial_l\mathcal{F}, \quad (\eta^{ij}) = (\eta_{ij})^{-1}, \quad i,j,k = 1,\cdots,\mu.$$

In what follows we rely on the following proposition proved by B. Dubrovin:

Proposition 1.3 (cf. Lemma 1.2 in [1])**.** *Locally a Frobenius manifold with the diagonalizable ∇E is described by its potential and vice versa.*

We finish this subsection by introducing the notion of a Frobenius manifold defined over a field.

Definition. Let $\mathbb{K} \subset \mathbb{C}$ be a field. We say that a Frobenius manifold M is *defined over* $\mathbb{K}$ if there exist flat coordinates $t_1,\dots,t_\mu$ such that the Frobenius potential $\mathcal{F}$ belongs to $\mathbb{K}\{t_1,\dots,t_\mu\}$ and is defined at the point $t_1 = \cdots = t_\mu = 0$.

1.2. Eisenstein series

Throughout this paper, we denote by $\mathbb{H}$ the complex upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Recall the following famous facts on Eisenstein series.

Proposition 1.4. *Let $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$ be the Eisenstein series defined by*

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

$$E_4(\tau) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$E_6(\tau) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ and $q = \exp(2\pi\sqrt{-1}\tau)$.

1. *For any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, *we have*

$$E_2(\tau) = \frac{1}{(c\tau+d)^2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{6c}{\pi\sqrt{-1}(c\tau+d)}, \tag{2a}$$

$$E_4(\tau) = \frac{1}{(c\tau+d)^4} E_4\left(\frac{a\tau+b}{c\tau+d}\right), \tag{2b}$$

$$E_6(\tau) = \frac{1}{(c\tau+d)^6} E_6\left(\frac{a\tau+b}{c\tau+d}\right). \tag{2c}$$

2. *The derivatives of the Eisenstein series satisfy the following identities due to Ramanujan*:

$$\frac{1}{2\pi\sqrt{-1}} \frac{dE_2(\tau)}{d\tau} = \frac{1}{12}\left(E_2(\tau)^2 - E_4(\tau)\right), \tag{3a}$$

$$\frac{1}{2\pi\sqrt{-1}} \frac{dE_4(\tau)}{d\tau} = \frac{1}{3}\left(E_2(\tau)E_4(\tau) - E_6(\tau)\right), \tag{3b}$$

$$\frac{1}{2\pi\sqrt{-1}} \frac{dE_6(\tau)}{d\tau} = \frac{1}{2}\left(E_2(\tau)E_6(\tau) - E_4(\tau)^2\right). \tag{3c}$$

We shall also consider the complex-valued real-analytic function $E_2^*(\tau)$ on $\mathbb{H}$ defined by

$$E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)},$$

which is a so-called almost holomorphic modular form of weight two since we have the following.

Proposition 1.5. *We have*

$$E_2^*(\tau) = \frac{1}{(c\tau+d)^2} E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) \quad \text{for any} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Proof. The formula

$$\left(\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right)^{-1} = \frac{|c\tau+d|^2}{\mathrm{Im}(\tau)}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \tag{4}$$

yields the statement. □

In general, an almost holomorphic modular form is defined as follows.

Definition. A polynomial $f(\tau)$ in $\mathrm{Im}(\tau)^{-1}$ over the ring of holomorphic functions on $\mathbb{H}$ satisfying

$$f(\tau) = \frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

is called an *almost holomorphic modular form* of weight k.

Proposition 1.6 (cf. Paragraph 5.1. in [6])**.** *Let $f(\tau)$ be an almost holomorphic modular form of weight k. Then the almost holomorphic derivative of $f(\tau)$ defined by*

$$\partial_k f(\tau) := \frac{1}{2\pi\sqrt{-1}} \frac{\partial f(\tau)}{\partial \tau} - \frac{k}{4\pi \mathrm{Im}(\tau)} f(\tau), \tag{5}$$

is an almost holomorphic modular form of weight $k+2$.

Proof. One can check this directly by using the equations (2a) and (4). We briefly explain for the reader's convenience the modularity property of $\partial_2 E_2^*(\tau)$. We have:

$$\begin{aligned} \partial_2 E_2^* &= \frac{1}{12}\left(E_2(\tau)^2 - E_4(\tau)\right) - \frac{3}{4\pi(\mathrm{Im}(\tau))^2} - \frac{1}{2\pi\mathrm{Im}(\tau)} E_2^*(\tau). \\ &= \frac{1}{12}\left(E_2(\tau)^2 - \frac{6E_2(\tau)}{\pi\mathrm{Im}(\tau)} + \frac{9}{(\pi\mathrm{Im}(\tau))^2}\right) - \frac{1}{12}E_4(\tau). \\ &= \frac{1}{12}E_2^*(\tau)^2 - \frac{1}{12}E_4(\tau). \end{aligned}$$

Due to the modularity properties of E_4 and E_2^* the proposition follows. □

In what follows we will drop the subscript k in the derivative keeping in mind that it is always fixed as we are given a modular form of weight k to differentiate. We will use the notation ∂^p meaning:

$$\partial^p g := \partial_{k+2(p-1)} \dots \partial_k g,$$

for g - an almost holomorphic modular form of weight k.

Proposition 1.7. *We have*

$$\partial E_2^*(\tau) = \frac{1}{12}\left(E_2^*(\tau)^2 - E_4(\tau)\right),$$

$$\partial^2 E_2^*(\tau) = \frac{1}{36}\left(E_6(\tau) - \frac{3}{2}E_2^*(\tau)E_4(\tau) + \frac{1}{2}E_2^*(\tau)^2\right).$$

Proof. This follows from direct calculations using the equations (3). □

1.3. Elliptic curves

We have a family of elliptic curves parameterized by $\mathbb{H}$:

$$\pi : \mathcal{E} := \{(x, y, \tau) \in \mathbb{C}^2 \times \mathbb{H} \mid y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\} \longrightarrow \mathbb{H},$$

where

$$g_2(\tau) := \frac{4\pi^4}{3} E_4(\tau), \quad g_3(\tau) := \frac{8\pi^6}{27} E_6(\tau). \tag{7}$$

Denote by $\mathcal{E}_{\tau_0}$ the fiber of π over a point $\tau_0 \in \mathbb{H}$.

Definition. Let $\mathbb{K} \subset \mathbb{C}$ be a field. Choose a point $\tau_0 \in \mathbb{H}$. We say that an elliptic curve $\mathcal{E}_{\tau_0}$ is *defined over* $\mathbb{K}$ if there exist $g_2, g_3 \in \mathbb{K}$ such that the algebraic variety

$$E_{g_2,g_3} := \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\}$$

is isomorphic to $\mathcal{E}_{\tau_0}$.

2. Frobenius manifolds of rank three and dimension one

From now on, we shall consider a Frobenius manifold M of rank three and dimension one with flat coordinates t_1, t_2, t satisfying the following conditions:

- The unit vector field e is given by $\frac{\partial}{\partial t_1}$.
- The Euler vector field E is given by $E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{2} t_2 \frac{\partial}{\partial t_2}$.
- The Frobenius potential $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{1}{2} t_1^2 t + t_1 t_2^2 + t_2^4 f(t) \tag{8}$$

  where $f(t)$ is a holomorphic function in t on an open domain in $\mathbb{C}$.

Our particular interest is attached to the concrete Frobenius manifolds obtained from the "basic" rank 3 Frobenius manifold by the certain group action. However in order to introduce it we have to develop the general theory of the $\mathrm{GL}(2, \mathbb{C})$ group action on the space of rank 3 Frobenius manifolds as above. Unlike the heavy machinery of the Givental's action on the space of Frobenius manifolds this $\mathrm{GL}(2, \mathbb{C})$-action is defined in the easy and straightforward way and therefore deserves to be studied separately.

2.1. Solutions of the WDVV equation

Proposition 2.1. *The WDVV equation is equivalent to the following differential equation.*

$$\frac{d^3 f(t)}{dt^3} = -24 f(t) \frac{d^2 f(t)}{dt^2} + 36 \left(\frac{df(t)}{dt} \right)^2 . \tag{9}$$

Proof. This is obtained by a straightforward calculation. □

Remark 2.2. Put

$$\gamma(t) := -4 f(t).$$

By a straightforward calculation, it turns out that the holomorphic function $\gamma(t)$ satisfies the following differential equation

$$\frac{d^3 \gamma(t)}{dt^3} = 6 \frac{d^2 \gamma(t)}{dt^2} \gamma(t) - 9 \left(\frac{d\gamma(t)}{dt} \right)^2 . \tag{10}$$

The differential equation (10) is classically known as Chazy's equation.

Proposition 2.3. *Suppose that $f(t)$ is a convergent power series in t given as $f(t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$, then the differential equation (9) is equivalent to the following recursion relation*:

$$c_{n+3} = \sum_{a=0}^{n} \binom{n}{a} \left(-24 c_a c_{n-a+2} + 36 c_{a+1} c_{n-a+1} \right) . \tag{11}$$

In particular, we have

$$c_3 = -24 c_2 c_0 + 36 c_1^2 .$$

Proof. This is also obtained by a straightforward calculation. □

Therefore, the first three coefficients c_0, c_1 and c_2 are enough to determine all the coefficients $c_n, n \geq 3$ due to the recursion relation (11).

2.2. $\mathrm{GL}(2, \mathbb{C})$*-action on the set of Frobenius structures*

Proposition 2.4. *Suppose that a holomorphic function $f(t)$ on a domain in $\mathbb{C}$ is a solution of the differential equation (9). For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$, define a holomorphic function $f^A(t)$ on a suitable domain in $\mathbb{C}$ as*

$$f^A(t) := \frac{\det(A)}{(ct+d)^2} f \left(\frac{at+b}{ct+d} \right) + \frac{c}{2(ct+d)} . \tag{12}$$

Then $f^A(t)$ becomes a solution of the differential equation (9).

Proof. This is obtained by a straightforward calculation. □

It is important to note that this $\mathrm{GL}(2,\mathbb{C})$-action is the inverse action of the $\mathrm{GL}(2,\mathbb{C})$-action on the set of solutions of the WDVV equations for the potential (8) given in Appendix B in [1]. Indeed, we have the following.

Proposition 2.5. *Consider Dubrovin's inversion I of the Frobenius manifold defined as follows*:

$$\widehat{t}_1 := t_1 + \frac{1}{4}\frac{t_2^2}{t},\ \widehat{t}_2 := \frac{t_2}{t},\ \widehat{t} := -\frac{1}{t},$$
$$\widehat{\mathcal{F}}(\widehat{t}) := \frac{1}{t^2}\left[\mathcal{F}(t) - t_1^2 t - \frac{1}{4}t_1 t_2^2\right].$$

Then, the new Frobenius manifold given by the new flat coordinates $\widehat{t}_1, \widehat{t}_2, \widehat{t}$ together with the new Frobenius potential $\widehat{\mathcal{F}}$ coincides with the one associated to the solution $f^A(t)$ of (9) *with* $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. Some calculations yield the statement. □

2.3. $\mathrm{GL}(2,\mathbb{C})$*-orbit of constant solutions*

The differential equation (9) obviously has constant solutions. Therefore, we have the following.

Proposition 2.6. *For any $e \in \mathbb{C}$ and any point $[c : d] \in \mathbb{P}^1$, the meromorphic function on $\mathbb{C}$*

$$f(t) := \frac{e}{(ct+d)^2} + \frac{c}{2(ct+d)} \tag{13}$$

is a solution of the differential equation (9).

Proof. This is clear. □

Definition. We will call the solution $f(t)$ as above constant solution of the equation (9).

Corollary 2.7. *If $f(t)$ is holomorphic at $t = 0$ and belongs to the* $\mathrm{GL}(2,\mathbb{C})$*-orbit of constant solutions, then there exist $\alpha, \beta \in \mathbb{C}$ such that*

$$f(t) := \frac{\alpha}{(1+\beta t)^2} + \frac{\beta}{2(1+\beta t)}. \tag{14}$$

Proof. One can set $\alpha := e/d^2$ and $\beta := c/d$ in (13) since d cannot be zero. □

The Taylor expansion at $t=0$ of $f(t)$ in the equation (14) is given by

$$f(t) = c_0(\alpha,\beta) + c_1(\alpha,\beta)t + c_2(\alpha,\beta)\frac{t^2}{2} + \dots,$$

$$c_0(\alpha,\beta) = \alpha + \frac{\beta}{2}, \quad c_1(\alpha,\beta) = -2\alpha\beta - \frac{1}{2}\beta^2, \quad c_2(\alpha,\beta) = 6\alpha\beta^2 + \beta^3.$$

For some $c_0, c_1, c_2 \in \mathbb{C}$, consider the cubic curve in $\mathbb{C}^2$ defined by

$$y^2 = 4x^3 - 12c_0x^2 - 6c_1x - \frac{c_2}{2}.$$

Note that if $c_i = c_i(\alpha,\beta)$ for $i = 1,2,3$, then the cubic curve is singular since we have

$$4x^3 - 12c_0(\alpha,\beta)x^2 - 6c_1(\alpha,\beta)x - \frac{c_2(\alpha,\beta)}{2} = \frac{1}{2}(2x - 6\alpha - \beta)(2x-\beta)^2.$$

Proposition 2.8. *Suppose that $c_0, c_1, c_2 \in \mathbb{C}$ satisfy the equation*

$$32(c_1 + 2c_0^2)^3 - (c_2 + 12c_1c_0 + 16c_0^3)^2 = 0.$$

Then there exist $\alpha(c_0,c_1,c_2), \beta(c_0,c_1,c_2) \in \mathbb{C}$ such that

$$\begin{aligned}
c_0 =& \alpha(c_0,c_1,c_2) + \frac{\beta(c_0,c_1,c_2)}{2},\\
c_1 =& -2\alpha(c_0,c_1,c_2)\beta(c_0,c_1,c_2) - \frac{1}{2}\beta(c_0,c_1,c_2)^2,\\
c_2 =& 6\alpha(c_0,c_1,c_2)\beta(c_0,c_1,c_2)^2 + \beta(c_0,c_1,c_2)^3,
\end{aligned}$$

and the unique solution $f(t)$ of the differential equation (9) *holomorphic at $t=0$ satisfying*

$$f(0) = c_0, \quad \frac{df}{dt}(0) = c_1, \quad \frac{d^2f}{dt^2}(0) = c_2$$

is given by

$$f(t) := \frac{\alpha(c_0,c_1,c_2)}{(1+\beta(c_0,c_1,c_2)t)^2} + \frac{\beta(c_0,c_1,c_2)}{2(1+\beta(c_0,c_1,c_2)t)}.$$

Proof. Consider the system of PDE's called Halphen's system of equations:

$$\begin{cases}
\frac{d}{dt}(X_2(t) + X_3(t)) = 2X_2(t)X_3(t),\\
\frac{d}{dt}(X_3(t) + X_4(t)) = 2X_3(t)X_4(t),\\
\frac{d}{dt}(X_4(t) + X_2(t)) = 2X_4(t)X_2(t),
\end{cases}$$

One can check that the function defined by $f(t) := -\frac{1}{6}(X_2(t) + X_3(t) + X_4(t))$ is a solution of the equation (9). Consider the third order equation in x:

$$4x^3 - 12f(t)x^2 - 6f'(t)x - \frac{f''(t)}{2} = 0,$$

where t is considered as a parameter. Let $\{x_k(t)\}$ be the triplet of solutions of a this third order equation. By straightforward computations one checks that the unordered triplet $\{-2X_k(t)\}$ is equal to the triplet $\{x_k(t)\}$.

The discriminant Δ^Q of the third order equation at $t=0$ is equal to $32(c_1 + 2c_0^2)^3 - (c_2 + 12c_1c_0 + 16c_0^3)^2 = 0$. In this case it is easy to solve Halphen's system to get:

$$X_3(t) = X_4(t) = -\frac{\beta}{1+\beta t}, \quad X_2(t) = -\frac{\beta}{1+\beta t} - 6\frac{\alpha}{(1+\beta t)^2}.$$

Hence the function $f(t)$ is of the right form and solves equation (9). □

2.4. Special solution

Under the change of variables $t = 2\pi\sqrt{-1}\tau$ the equation (9) transforms to:

$$\frac{d^3 f(\tau)}{d\tau^3} = -48\pi\sqrt{-1}\frac{d^2 f(\tau)}{d\tau^2} f(\tau) + 72\pi\sqrt{-1}\left(\frac{df(\tau)}{d\tau}\right)^2. \tag{16}$$

Proposition 2.9. *The holomorphic function $f^\infty(\tau)$ defined on $\mathbb{H}$ by:*

$$f^\infty(\tau) := -\frac{1}{24}E_2(\tau) \tag{17}$$

satisfies the differential equation (16). *Therefore, the holomorphic function $\mathcal{F}^\infty$ on $M^\infty := \mathbb{C}^2 \times \mathbb{H}$ given by*

$$\mathcal{F}^\infty = \frac{1}{2}t_1^2(2\pi\sqrt{-1}\tau) + t_1t_2^2 + t_2^4 f^\infty(\tau)$$

defines on M^∞ a Frobenius structure of rank three and dimension one.

Proof. This follows from a direct calculation by the use of the equations (3). □

Remark 2.10. It is a well-known consequence of the equations (3) that the function $\frac{\pi\sqrt{-1}}{3}E_2(\tau)$ satisfies the Chazy equation (10) with $t=\tau$ (cf. Appendix C in [1]).

Proposition 2.11. *The holomorphic function $2\pi\sqrt{-1}f^\infty(\tau)$ is invariant under the* $\mathrm{SL}(2,\mathbb{Z})$*-action* (12).

Proof. This follows from a direct calculation using the modular property (2a) of $E_2(\tau)$. □

2.5. Choice of a primitive form and a $\mathrm{GL}(2,\mathbb{C})$*-action*

Consider the Frobenius manifold $M^{D_4^{(1,1)}}$ constructed from the invariant theory of the elliptic Weyl group of type $D_4^{(1,1)}$. The Frobenius manifold M^∞ could be considered as a Frobenius submanifold in $M^{D_4^{(1,1)}}$, which is connected to the GW Frobenius structures of the orbifold projective lines $\mathbb{P}^1_{2,2,2,2}$, $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$. These orbifold projective lines provide Calabi-Yau/Landau-Ginzburg mirror symmetry for simple elliptic singularities. The Frobenius manifold structure associated to $D_4^{(1,1)}$ varies according to a choice of the vector which is identified with a cycle in the homology group of the elliptic curve (see also Example 2 in Section 3.3 in [4]). Based on this observation we consider M^∞ as corresponding to the primitive form choice at the LCSL. We introduce a special class of $\mathrm{GL}(2,\mathbb{C})$-actions on the function $2\pi\sqrt{-1}f^\infty(\tau)$ motivated by this.

Consider a free abelian group $H_\mathbb{Z}$ generated by two letters α,β

$$H_\mathbb{Z} := \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$$

equipped with a symplectic form $(-,-)$ such that $(\alpha,\beta)=1$. We can identify $H_\mathbb{Z}$ with the homology group $H_1(\mathcal{E}_{\sqrt{-1}},\mathbb{Z})$ of an elliptic curve $\mathcal{E}_{\sqrt{-1}}$, the fiber at $\sqrt{-1}\in\mathbb{H}$ of the family of elliptic curves $\pi:\mathcal{E}\longrightarrow\mathbb{H}$ (see Subsection 1.3). Then

$$H^*_\mathbb{C} := (H_\mathbb{C})^* := (H_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{C})^* = \mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee,$$

where $\{\alpha^\vee,\beta^\vee\}$ is the dual basis of $\{\alpha,\beta\}$, can be identified with the cohomology group $H^1(\mathcal{E}_{\sqrt{-1}},\mathbb{Z})$. In particular, the relative holomorphic volume form $\Omega\in\Gamma(\mathbb{H},\Omega^1_{\mathcal{E}/\mathbb{H}})$ is described in terms of $\alpha^\vee,\beta^\vee$ as

$$\Omega = x(\tau)\left(\alpha^\vee+\tau\beta^\vee\right)$$

for some nowhere vanishing holomorphic function $x(\tau)$ on $\mathbb{H}$.

The relative holomorphic volume form $\zeta^\infty=\alpha^\vee+\tau\beta^\vee$ is, very roughly speaking, the *primitive form* associated to the choice of the vector $\alpha\in H_\mathbb{C}$, which satisfies

$$\int_\alpha \zeta^\infty = 1 \text{ and } \int_\beta \zeta^\infty = \tau,$$

and gives the Frobenius structure M^∞. There is a systematic way to obtain a primitive form by the use of the canonical opposite filtration to the Hodge filtration corresponding to a point $\tau_0\in\mathbb{H}$ as follows.

Proposition 2.12. *For $\tau_0\in\mathbb{H}$ and $\omega_0\in\mathbb{C}\backslash\{0\}$, there exists a unique relative holomorphic volume form $\zeta\in\Gamma(\mathbb{H},\Omega^1_{\mathcal{E}/\mathbb{H}})$ such that*

$$\int_{\alpha'}\zeta = 1,\quad \alpha' := \frac{1}{\omega_0(\bar\tau_0-\tau_0)}\left(\bar\tau_0\alpha-\beta\right).$$

Proof. Some calculation yields

$$\zeta = \omega_0 \frac{\bar{\tau}_0 - \tau_0}{\bar{\tau}_0 - \tau} \left(\alpha^\vee + \tau\beta^\vee\right).$$

□

This holomorphic volume form ζ is the primitive form uniquely determined by the choice of the vector $\alpha' \in H_{\mathbb{C}}$. We first fix $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$ so that we have

$$\int_\alpha \zeta = \omega_0 \text{ and } \int_\beta \zeta = \omega_0\tau_0 \quad \text{at } \tau = \tau_0.$$

Next we choose $\beta' \in H_{\mathbb{C}}$ so that $\int_{\beta'} \zeta = 0$ at $\tau = \tau_0$ and $(\alpha', \beta') = 1$. It is easy to see that

$$\beta' := -\omega_0 \left(\tau_0\alpha - \beta\right).$$

As the flat coordinate $2\pi\sqrt{-1}\tau$ of the Frobenius manifold M^∞ associated to the primitive form ζ^∞, define the coordinate $t(\tau)$ by the period

$$\frac{t(\tau)}{2\pi\sqrt{-1}} := \int_{\beta'} \zeta = 2\sqrt{-1}\omega_0^2 \mathrm{Im}(\tau_0) \frac{\tau_0 - \tau}{\bar{\tau}_0 - \tau}.$$

This motivates the following $\mathrm{GL}(2,\mathbb{C})$-action $A^{(\tau_0,\omega_0)}$ and the Frobenius manifold $M^{(\tau_0,\omega_0)}$.

Definition. Choose $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$.

1. Define a holomorphic function $f^{(\tau_0,\omega_0)}(t)$ on $\{t \in \mathbb{C} \mid |t| < |-4\pi\omega_0^2\mathrm{Im}(\tau_0)|\}$ applying the $\mathrm{GL}(2,\mathbb{C})$-action (12) specified by

$$A^{(\tau_0,\omega_0)} := \begin{pmatrix} \dfrac{\bar{\tau}_0}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0\tau_0 \\ \dfrac{1}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0 \end{pmatrix}$$

   to the function $2\pi\sqrt{-1}f^\infty(\tau)$.
2. Define complex numbers $c_i(\tau_0,\omega_0)$, $i \in \mathbb{Z}_{\geq 0}$, by the coefficients of the Taylor expansion of $f^{(\tau_0,\omega_0)}(t)$ at $t = 0$:

$$f^{(\tau_0,\omega_0)}(t) = \sum_{n=0}^{\infty} \frac{c_n(\tau_0,\omega_0)}{n!} t^n.$$

3. Denote by $M^{(\tau_0,\omega_0)} := \mathbb{C}^2 \times \{t \in \mathbb{C} \mid |t| < |-4\pi\omega_0^2\mathrm{Im}(\tau_0)|\}$ the Frobenius manifold given by the Frobenius potential

$$\mathcal{F}^{(\tau_0,\omega_0)} = \frac{1}{2}t_1^2 t + t_1 t_2^2 + t_2^4 f^{(\tau_0,\omega_0)}(t).$$

3. Frobenius manifolds $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{K}$ via modular forms

The essential technique dealing with the $\mathrm{GL}(2,\mathbb{C})$-action is provided by the theory of modular forms. We use it to give a complete classification of the Frobenius manifolds $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{K}\subset\mathbb{C}$ (recall the definition given in Section 1).

3.1. Classification of $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{K}$

Theorem 3.1. *Let $\mathbb{K}\subset\mathbb{C}$ be a field. Let $\tau_0\in\mathbb{H}$ and $\omega_0\in\mathbb{C}\backslash\{0\}$. The following are equivalent*:

1. *The Frobenius manifold $M^{(\tau_0,\omega_0)}$ is defined over $\mathbb{K}$.*
2. *All the coefficients of $f^{(\tau_0,\omega_0)}(t)$ series expansion are in $\mathbb{K}$.*
3. *We have*
$$E_2^*(\tau_0)\in\mathbb{K}\omega_0^2,\quad E_4(\tau_0)\in\mathbb{K}\omega_0^4,\quad E_6(\tau_0)\in\mathbb{K}\omega_0^6.$$
4. *Let ∂ be the almost holomorphic derivative defined by* (5)*. We have*
$$-\frac{1}{24}E_2^*(\tau_0)\in\mathbb{K}\omega_0^2,\quad -\frac{1}{24}\partial E_2^*(\tau_0)\in\mathbb{K}\omega_0^4,\quad -\frac{1}{24}\partial^2E_2^*(\tau_0)\in\mathbb{K}\omega_0^6.$$
5. *We have*
$$E_2^*(\tau_0)\in\mathbb{K}\omega_0^2,\quad \mathcal{E}_{\tau_0}\ \textit{is defined over}\ \mathbb{K}.$$

Proof. By definition, the Frobenius manifold $M^{(\tau_0,\omega_0)}$ is defined over $\mathbb{K}$ if and only if there are flat coordinates $t_1,\widetilde{t}_2,\widetilde{t}$ such that the Frobenius potential is given by

$$\mathcal{F}^{(\tau_0,\omega_0)}=\frac{1}{2}\eta_1t_1^2\widetilde{t}+\eta_2t_1\widetilde{t}_2^2+\widetilde{t}_2^4\widetilde{f}(\widetilde{t})\quad\text{for some } \eta_1,\eta_2\in\mathbb{K}\text{ and }\widetilde{f}(\widetilde{t})\in\mathbb{K}\{t\}.$$

However, this immediately implies that $t_2^2=\eta_2\widetilde{t}_2^2$, $t=\eta_1\widetilde{t}$ and $f^{(\tau_0,\omega_0)}(t)=\eta_2^{-2}\widetilde{f}(\widetilde{t})$, and hence the equivalence between the conditions (i) and (ii).

Due to the recursion relation (11) to get (iii) it is enough to check that $c_i(\tau_0,\omega_0)\in\mathbb{K}$ for $2\geq i\geq 0$. By definition of $f^\infty(\tau)$ (see (17)), we have

$$f^{(\tau_0,\omega_0)}(t)=-\frac{(4\pi\omega_0^2\mathrm{Im}(\tau_0))^2}{24\omega_0^2(t+4\pi\omega_0^2\mathrm{Im}(\tau_0))^2}E_2\left(\frac{\bar{\tau}_0t+4\pi\omega_0^2\mathrm{Im}(\tau_0)\tau_0}{t+4\pi\omega_0^2\mathrm{Im}(\tau_0)}\right)+\frac{1}{2(t+4\pi\omega_0^2\mathrm{Im}(\tau_0))}.$$

Setting $t=0$, we get

$$c_0(\tau_0,\omega_0)=-\frac{1}{24\omega_0^2}E_2(\tau_0)+\frac{1}{8\pi\omega_0^2\mathrm{Im}(\tau_0)}=-\frac{1}{24\omega_0^2}\left(E_2(\tau_0)-\frac{3}{\pi\mathrm{Im}(\tau_0)}\right).$$

Using the formula (3), we compute the derivative of $f^{(\tau_0,\omega_0)}(t)$ at $t=0$ and we obtain

$$
\begin{aligned}
c_1(\tau_0,\omega_0) =& \frac{1}{12\omega_0^2(4\pi\omega_0^2\mathrm{Im}(\tau_0))}E_2(\tau_0) - \frac{1}{288\omega_0^4}\left(E_2(\tau_0)^2 - E_4(\tau_0)\right) - \frac{1}{2(4\pi\omega_0^2\mathrm{Im}(\tau_0))^2} \\
=& -2\left(-\frac{1}{24\omega_0^2}E_2(\tau_0) + \frac{1}{8\pi\omega_0^2\mathrm{Im}(\tau_0)}\right)^2 + \frac{1}{288\omega_0^4}E_4(\tau_0) \\
=& -2c_0(\tau_0,\omega_0)^2 + \frac{1}{288\omega_0^4}E_4(\tau_0).
\end{aligned}
$$

In a similar way, after some calculations, we get

$$c_2(\tau_0,\omega_0) = -\frac{1}{864\omega_0^6}E_6(\tau_0) - 12c_0(\tau_0,\omega_0)c_1(\tau_0,\omega_0) - 16c_0(\tau_0,\omega_0)^3.$$

To summarize, we obtain

$$E_2^*(\tau_0) = -24c_0(\tau_0,\omega_0)\omega_0^2, \tag{18a}$$

$$E_4(\tau_0) = 288\left(c_1(\tau_0,\omega_0) + 2c_0(\tau_0,\omega_0)^2\right)\omega_0^4, \tag{18b}$$

$$E_6(\tau_0) = -864\left(c_2(\tau_0,\omega_0) + 12c_0(\tau_0,\omega_0)c_1(\tau_0,\omega_0) + 16c_0(\tau_0,\omega_0)^3\right)\omega_0^6. \tag{18c}$$

Equivalently, we have

$$E_2^*(\tau_0) = -24c_0(\tau_0,\omega_0)\omega_0^2, \tag{19a}$$

$$\partial E_2^*(\tau_0) = -24c_1(\tau_0,\omega_0)\omega_0^4, \tag{19b}$$

$$\partial^2 E_2^*(\tau_0) = -24c_2(\tau_0,\omega_0)\omega_0^6. \tag{19c}$$

This proves the theorem. □

3.2. Examples

Proposition 3.2 (cf. Lemma 3.2 in [8]). *The equation*

$$E_2^*(\tau) = 0 \tag{20}$$

holds if and only if $\tau \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ *or* $\tau \in \mathrm{SL}(2,\mathbb{Z})\rho$ *where* $\rho := \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$.

The values of the Eisenstein series at $\tau = \sqrt{-1}$ are

$$E_2(\sqrt{-1}) = \frac{3}{\pi},\ E_4(\sqrt{-1}) = 3\frac{\Gamma\left(\frac{1}{4}\right)^8}{64\pi^6},\ E_6(\sqrt{-1}) = 0. \tag{21}$$

If

$$\omega_0 \in \mathbb{Q}\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\pi^{\frac{3}{2}}}$$

then $c_0(\sqrt{-1},\omega_0)=c_2(\sqrt{-1},\omega_0)=0$ and $c_1(\sqrt{-1},\omega_0)\in\mathbb{Q}$.

The values of the Eisenstein series at $\tau=\rho$ are

$$E_2(\rho)=\frac{2\sqrt{3}}{\pi},\ E_4(\rho)=0,\ E_6(\rho)=\frac{27}{2}\frac{\Gamma\left(\frac{1}{3}\right)^{18}}{2^8\pi^{12}}. \tag{22}$$

If

$$\omega_0\in\mathbb{Q}\frac{\Gamma\left(\frac{1}{3}\right)^3}{4\pi^2}$$

then $c_0(\rho,\omega_0)=c_1(\rho,\omega_0)=0$ and $c_2(\rho,\omega_0)\in\mathbb{Q}$.

4. SL-action on the set of Frobenius manifolds $M^{(\tau_0,\omega_0)}$

Let $A:=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be an element of $\mathrm{SL}(2,\mathbb{R})$. The correspondence

$$\tau_0\mapsto\tau_1:=\frac{a\tau_0+b}{c\tau_0+d},\quad \omega_0\mapsto\omega_1:=(c\tau_0+d)\omega_0$$

defines a $\mathrm{SL}(2,\mathbb{R})$-action on the set $\{(\tau_0,\omega_0)\mid \tau_0\in\mathbb{H},\ \omega_0\in\mathbb{C}\setminus\{0\}\}$. This is exactly the $\mathrm{SL}(2,\mathbb{R})$-action induced by (12) since

$$A\begin{pmatrix}\dfrac{\bar\tau_0}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0\tau_0\\ \dfrac{1}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0\end{pmatrix}=\begin{pmatrix}\dfrac{(a\bar\tau_0+b)}{4\pi\omega_0\mathrm{Im}(\tau_0)} & (a\tau_0+b)\omega_0\\ \dfrac{(c\bar\tau_0+d)}{4\pi\omega_0\mathrm{Im}(\tau_0)} & (c\tau_0+d)\omega_0\end{pmatrix}=\begin{pmatrix}\dfrac{\bar\tau_1}{4\pi\omega_1\mathrm{Im}(\tau_1)} & \omega_1\tau_1\\ \dfrac{1}{4\pi\omega_1\mathrm{Im}(\tau_1)} & \omega_1\end{pmatrix}.$$

4.1. $\mathrm{SL}(2,\mathbb{Z})$*-action*

The equations (18) yield the following.

Proposition 4.1. *Let $\tau_0,\tau_1\in\mathbb{H}$ and $\omega_0,\omega_1\in\mathbb{C}\backslash\{0\}$. The following are equivalent:*

1. *There is an isomorphism of Frobenius manifolds $M^{(\tau_0,\omega_0)}\cong M^{(\tau_1,\omega_1)}$.*
2. *The equality $f^{(\tau_0,\omega_0)}(t)=f^{(\tau_1,\omega_1)}(t)$ holds.*
3. *There exists an element $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})$ such that*

$$\tau_1=\frac{a\tau_0+b}{c\tau_0+d},\quad \omega_1^k=(c\tau_0+d)^k\omega_0^k,$$

where $k=4$ if $\tau_0\in\mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$, $k=6$ if $\tau_0\in\mathrm{SL}(2,\mathbb{Z})\rho$ and $k=2$ otherwise.

Proof. It is almost clear that condition (i) is equivalent to (ii). By the equations (18), condition (ii) is equivalent to the equations

$$\frac{E_2^*(\tau_0)}{\omega_0^2} = \frac{E_2^*(\tau_1)}{\omega_1^2}, \quad \frac{E_4(\tau_0)}{\omega_0^4} = \frac{E_4(\tau_1)}{\omega_1^4}, \quad \frac{E_6(\tau_0)}{\omega_0^6} = \frac{E_6(\tau_1)}{\omega_1^6}. \tag{23}$$

The equations (23) imply that $j(\tau_0) = j(\tau_1)$ and hence

$$\tau_1 = \frac{a\tau_0 + b}{c\tau_0 + d}, \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{24}$$

Therefore we obtain

$$\frac{E_2^*(\tau_0)}{\omega_0^2} = \frac{(c\tau_0 + d)^2 E_2^*(\tau_0)}{\omega_1^2}, \frac{E_4(\tau_0)}{\omega_0^4} = \frac{(c\tau_0 + d)^4 E_4(\tau_0)}{\omega_1^4}, \frac{E_6(\tau_0)}{\omega_0^6} = \frac{(c\tau_0 + d)^6 E_6(\tau_0)}{\omega_1^6},$$

which implies, by the use of (20), (21) and (22), $\omega_1^k = (c\tau_0 + d)^k \omega_0^k$ where $k = 4$ if $\tau_0 \in \mathrm{SL}(2, \mathbb{Z})\sqrt{-1}$, $k = 6$ if $\tau_0 \in \mathrm{SL}(2, \mathbb{Z})\rho$ and $k = 2$ otherwise.

It is easy to show that the condition (iii) yields the equations (23). The proposition is proved. □

4.2. $\mathrm{SL}(2, \mathbb{Q})$*-action and complex multiplication*

Definition. An elliptic curve $\mathcal{E}_\tau$ is said to have complex multiplication if its modulus τ is imaginary quadratic. Namely $\tau \in \mathbb{Q}(\sqrt{-D})$ for a positive integer D.

A profound result of the theory of elliptic curves is that elliptic curves over $\mathbb{Q}$ with complex multiplication are easily classified:

Theorem 4.2 (cf. Paragraph II.2 in [9])**.** *Up to isomorphism there are only 13 elliptic curves defined over $\mathbb{Q}$ that have complex multiplication.*

We give the list of the Weierstrass models of these elliptic curves in the Appendix.

Corollary 4.3. *The modulus τ_0 of the elliptic curve $\mathcal{E}_{\tau_0}$ with complex multiplication defined over $\mathbb{Q}$ is in the* $\mathrm{SL}(2, \mathbb{C})$ *orbit of one of:*

$$\sqrt{-D}, \quad D \in \{1, 2, 3, 4, 7\},$$

or

$$\frac{-1 + \sqrt{-D}}{2}, \quad D \in \{3, 7, 11, 19, 27, 43, 67, 163\}.$$

Imaginary quadratic $\tau_0 \in \mathbb{C}$ are amazing from the point of view of the theory of modular forms too:

Proposition 4.4 (cf. Theorem A1 in [8]). *Let $\tau \in \mathbb{C}$ be imaginary quadratic and $\tau \notin \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$. Then we have:*

$$\frac{E_2^*(\tau)E_4(\tau)}{E_6(\tau)} \in \mathbb{Q}(j(\tau)),$$

where $j(\tau)$ is the value of the j-invariant of the elliptic curve $\mathcal{E}_\tau$.

We build up the connection between the elliptic curve $\mathcal{E}_{\tau_0}$ having some special properties and the Frobenius manifold $M^{(\tau_0,\omega_0)}$ by introducing the property of the latter one to have symmetry and weak symmetry.

Definition. Let $\tau_0 \in \mathbb{H}$, $\omega_0 \in \mathbb{C}\backslash\{0\}$.

1. The Frobenius manifold $M^{(\tau_0,\omega_0)}$ is said to *have a symmetry* if there exists an element $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \{1\}$ such that

$$\left(f^{(\tau_0,\omega_0)}\right)^A(t) = f^{(\tau_0,\omega_0)}(t).$$

2. The Frobenius manifold $M^{(\tau_0,\omega_0)}$ is said to *have a weak symmetry* if there exists an element $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \{1,-1\}$ such that

$$\left(f^{(\tau_0,\omega_0)}\right)^A(t) = f^{(\tau_0,\omega_0')}(t) \quad \text{for some } \omega_0' \in \mathbb{C}\backslash\{0\}.$$

Remark 4.5. It is important to note that weak symmetry is not a symmetry of the Frobenius manifold unless $\omega_0 = \omega_0'$, because the corresponding A-action relates different points in the space of all Frobenius manifolds of rank three.

These two properties allows us to get the classification result that is rather unusual for the Frobenius manifolds theory.

Theorem 4.6. *Let $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$.*

1. *The Frobenius manifold $M^{(\tau_0,\omega_0)}$ has a symmetry if and only if τ_0 is in the $\mathrm{SL}(2,\mathbb{Z})$ orbit of $\sqrt{-1}$ or ρ.*
2. *The Frobenius manifold $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ has a weak symmetry if and only if τ_0 is from the list given in Corollary 4.3.*

Proof. From Proposition 4.1 $M^{(\tau_0,\omega_0)}$ has a symmetry if and only if $\frac{a\tau_0+b}{c\tau_0+d} = \tau_0$ and

$$\omega_0^4 = (c\tau_0+d)^4\omega_0^4 \quad \text{for} \quad \tau_0 \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1},$$

or

$$\omega_0^6 = (c\tau_0+d)^6\omega_0^6 \quad \text{for} \quad \tau_0 \in \mathrm{SL}(2,\mathbb{Z})\rho,$$

or otherwise
$$\omega_0^2 = (c\tau_0 + d)^2\omega_0^2.$$
The last equation is satisfied if and only if $(c\tau_0 + d)^2 = 1$. It has no solutions for $\tau_0 \in \mathbb{H}$ and $c, d \in \mathbb{Z}$. It is an easy exercise to show that there is a suitable $A \in \mathrm{SL}(2,\mathbb{Z})$ solving the first two equations. This proves (i).

Let $M^{(\tau_0,\omega_0)}$ be defined over $\mathbb{Q}$ and have a weak symmetry. By Theorem 3.1 the elliptic curve $\mathcal{E}_{\tau_0}$ is defined over $\mathbb{Q}$.

Due to Proposition 4.1 we have $\frac{a\tau_0+b}{c\tau_0+d} = \tau_0$. It is an easy exercise to show that τ_0 satisfies a quadratic equation with negative discriminant. Hence the elliptic curve $\mathcal{E}_{\tau_0}$ has complex multiplication. From Proposition 4.2 we know that there are only 13 such τ_0 up to the $\mathrm{SL}(2,\mathbb{Z})$-action. Hence τ_0 is from the given list.

Assume that τ_0 is the modulus of one of the elliptic curves from this list. From the rationality assumption on the elliptic curve $\mathcal{E}_{\tau_0}$ we have $j(\tau_0) \in \mathbb{Q}$. The case of $\tau_0 = \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ was treated in Example 3.2 and we can apply Proposition 4.4. Its statement reads:
$$\frac{E_2^*(\tau_0)E_4(\tau_0)}{E_6(\tau_0)} \in \mathbb{Q}.$$

At the same time, since the elliptic curve is defined over $\mathbb{Q}$, there exists $a \in \mathbb{C}\backslash\{0\}$ such that:
$$a^2 g_2(\tau_0) \in \mathbb{Q}, \quad a^3 g_3(\tau_0) \in \mathbb{Q}.$$

From the equations (7) we have:
$$a^2\pi^4 E_4(\tau_0) = a^2 g_2(\tau_0)\frac{3}{4} \in \mathbb{Q}, \quad a^3\pi^6 E_6(\tau_0) = a^3 g_3(\tau_0)\frac{27}{8} \in \mathbb{Q}.$$

We conclude:
$$a\pi^2 E_2^*(\tau_0) \in \mathbb{Q}.$$
Summing up:
$$E_2^*(\tau_0) \in \mathbb{Q}(a\pi^2)^{-1}, \quad E_4(\tau_0) \in \mathbb{Q}(a\pi^2)^{-2}, \quad E_6(\tau_0) \in \mathbb{Q}(a\pi^2)^{-3}.$$
Taking $\omega_0^2 := (a\pi^2)^{-1}$ we get $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ because of Theorem 3.1. $\square$

Note that $\mathcal{E}_{\tau_0}$ for $\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ and $\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\rho$ are only elliptic curves with non-trivial automorphisms.

Remark 4.7. We can rephrase Theorem 4.6 (i) above as: a Frobenius manifold $M^{(\tau_0,\omega_0)}$ has a symmetry if and only if $\mathcal{E}_{\tau_0}$ has non-trivial automorphisms.

Appendix

In the following table we give Weierstrass models and modulus of 13 elliptic curves defined over $\mathbb{Q}$ with complex multiplication.

Modulus τ	Weierstrass equation	j-invariant	Δ_E
$(-1+\sqrt{-3})/2$	$y^2 = 4x^3 + 1$	0	3^3
$\sqrt{-3}$	$y^2 = 4x^3 - 60x + 88$	$2^4 3^3 5^3$	$2^8 3^3$
$(-1+3\sqrt{-3})/2$	$y^2 = 4x^3 - 120x + 253$	$-2^{15} 3 5^3$	3^5
$\sqrt{-1}$	$y^2 = 4x^3 + 4x$	$2^6 3^3$	2^5
$2\sqrt{-1}$	$y^2 = 4x^3 - 44x + 64$	$2^3 3^3 11^3$	2^9
$(-1+\sqrt{-7})/2$	$y^2 = 4x^3 - \frac{35}{4}x - \frac{49}{8}$	$-3^3 5^3$	7^3
$\sqrt{-7}$	$y^2 = 4x^3 - 2380x + 22344$	$3^3 5^3 17^3$	$2^{12} 7^3$
$\sqrt{-2}$	$y^2 = 4x^3 - 120x + 224$	$2^6 5^3$	2^9
$(-1+\sqrt{-11})/2$	$y^2 = 4x^3 - \frac{88}{3}x - \frac{847}{27}$	-2^{15}	11^3
$(-1+\sqrt{-19})/2$	$y^2 = 4x^3 - 152x + 361$	$-2^{15} 3^3$	19^3
$(-1+\sqrt{-43})/2$	$y^2 = 4x^3 - 3440x + 38829$	$-2^{18} 3^3 5^3$	43^3
$(-1+\sqrt{-67})/2$	$y^2 = 4x^3 - 29480x + 974113$	$-2^{15} 3^3 5^3 11^3$	67^3
$(-1+\sqrt{-163})/2$	$y^2 = 4x^3 - 8697680x + 4936546769$	$-2^{18} 3^3 5^3 23^3 29^3$	163^3

Table 1: 13 elliptic curves over $\mathbb{Q}$ with complex multiplication.

Introduce the notation:

$$\psi(\tau) := \frac{3E_2^*(\tau)E_4(\tau)}{2E_6(\tau)}.$$

The values of this function where computed in [8]. Using g_2, g_3 given by the Weierstrass forms of the previous table we compute $c_0(\tau,\omega)$, $c_1(\tau,\omega)$, $c_2(\tau,\omega)$ for some choice of ω:

Modulus τ	$\psi(\tau)$	$c_0(\tau,\omega)$	$c_1(\tau,\omega)$	$c_2(\tau,\omega)$
$(-1+\sqrt{-3})/2$	0	0	0	$-1/256$
$\sqrt{-3}$	$15/22$	$1/16$	$-21/128$	$-115/512$
$(-1+3\sqrt{-3})/2$	$240/253$	$1/8$	$-11/32$	$-129/256$
$\sqrt{-1}$	∞	0	$1/96$	0
$2\sqrt{-1}$	$11/14$	$1/16$	$-47/384$	$-67/512$
$(-1+\sqrt{-7})/2$	$5/14$	$-1/64$	$-143/6144$	$643/32768$
$\sqrt{-7}$	$255/266$	$9/16$	$-2623/384$	$-22539/512$
$\sqrt{-2}$	$15/28$	$1/48$	$-41/1152$	$-109/4608$
$(-1+\sqrt{-11})/2$	$48/77$	$1/24$	$-23/288$	$-193/2304$
$(-1+\sqrt{-19})/2$	$16/19$	$1/8$	$-41/96$	$-205/256$
$(-1+\sqrt{-43})/2$	$320/301$	$3/4$	$-121/12$	$-17325/256$
$(-1+\sqrt{-67})/2$	$16720/14539$	$19/8$	$-8453/96$	$-386557/256$
$(-1+\sqrt{-163})/2$	$38632640/30285563$	$181/4$	$-80236/3$	$-1598234897/256$

Table 2: $c_0(\tau,\omega)$, $c_1(\tau,\omega)$, $c_2(\tau,\omega)$ for elliptic curves from Table 1.

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