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Conformal automorphisms of algebraic surfaces and algebraic curves in the complex projective space

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ABSTRACT

We study the automorphism group of curves and surfaces in $\mathbb{CP}^3$ with respect to the conformal group, i.e. the group of all $A \in PGL(4, \mathbb{C})$ commuting with the anti-holomorphic involution j defined by $j((z_0 : z_1 : z_2 : z_3)) = (-\bar{z}_1 : \bar{z}_0 : \bar{z}_3 : -\bar{z}_2)$. For some singular surfaces we check when this group is finite. Among the singular surfaces we handle there are:
(1) certain cones;
(2) surfaces X containing no line and with $j(X) \neq X$;
(3) surfaces containing only finitely many, k, twistor lines with $k \geq 3$.
In many cases the proofs need results on conformal automorphisms of singular curves.

1. Introduction

Let $G(2, 4)$ denote the Grassmannian of all 2-dimensional linear subspaces of $\mathbb{C}^4$, i.e. the Grassmannian of all 1-dimension (projective) linear subspace of $\mathbb{CP}^3$. The linear group $GL(4, \mathbb{C})$ acts linearly on $\mathbb{C}^4$ and hence it induces a holomorphic action on $G(2, 4)$. The quotient group $PGL(4, \mathbb{C}) = GL(4, \mathbb{C})/\mathbb{C}^*\mathrm{Id}_{4\times 4}$ acts effectively and 2-transitively on $\mathbb{CP}^3$. Let $\mathbb{H} = \mathbb{R}+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ be the non-commutative field of quaternions. Identify $\mathbb{C}^4$ with $\mathbb{H}^2$. Left multiplication by j induces an $\mathbb{R}$-linear map $j : \mathbb{C}^4 \to \mathbb{C}^4$ given by the formula

$$j(z_0, z_1, z_2, z_3) = (-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2). \tag{1}$$

[1–9]. By the formula (1) we have $j^2 = -\mathrm{Id}_{\mathbb{C}^4}$. The map $j : \mathbb{C}^4 \to \mathbb{C}^4$ induces a map $j : \mathbb{CP}^3 \to \mathbb{CP}^3$. Formula (1) immediately shows that $j : \mathbb{CP}^3 \to \mathbb{CP}^3$ is a fixed-point-free anti-holomorphic involution. Thus for each set $S \subseteq \mathbb{CP}^3$ we have defined the set $j(S)$. We always have $j(j(S)) = S$. Using the explicit formula for the map j it is easy to check that if S is a line, then $j(S)$ is a line. Since $j(j(S)) = S$, we see that j induces an anti-holomorphic involution $G(2, 4) \to G(2, 4)$ [8, Section 2]. Its fixed points are called *twistor lines*, i.e. a line L is a twistor line if and only if $j(L) = L$. Since $G(2, 4)$ is compact and the map $G(2, 4) \to G(2, 4)$ is an anti-holomorphic involution, the set of all twistor lines is a compact 4-dimensional manifold. This manifold is diffeomorphic to S^4; more precisely this manifold is identified with $\mathbb{HP}^1$ [8, Section 2]. Identifying $\mathbb{C}^4$ with $\mathbb{H}^2$ the quotient map map $\mathbb{H}^2 \setminus \{0\} \to \mathbb{HP}^1$ factors through the surjection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}^3$ and hence it induces a submersion

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$\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$ called the twistor fibration. The fibers of π are exactly the twistor lines. Let G be the conformal group of S^4, i.e. the closed subgroup of $GL(4, \mathbb{C})$ formed by the matrices A which commute with j, i.e. by all matrices

$$\begin{pmatrix} a_1 & -\bar{a}_2 & b_1 & -\bar{b}_2 \\ a_2 & \bar{a}_1 & b_2 & \bar{b}_1 \\ c_1 & -\bar{c}_2 & d_1 & -\bar{d}_2 \\ c_2 & \bar{c}_1 & d_2 & \bar{d}_1 \end{pmatrix}, \tag{2}$$

where all the entries are complex numbers (see [3, Section 2]). The group G acts on $\mathbb{CP}^3$ (as any subgroup of $GL(4, \mathbb{C})$), but this action is not effective: any non-zero multiple of the identity matrix $\mathrm{Id}_{4\times 4}$ acts as the identity on $\mathbb{CP}^3$. To get an effective action on $\mathbb{CP}^3$ we use the groups $\mathbb{G} := G/\mathbb{R}^*\mathrm{Id}_{4\times 4}$ and $PGL(4, \mathbb{C}) = GL(4, \mathbb{C})/\mathbb{C}^*\mathrm{Id}_{4\times 4}$. The groups G and $\mathbb{G}$ are real Lie groups. For any closed set $X \subset \mathbb{CP}^3$ the stabilizer $\mathrm{Stab}_{\mathbb{G}}(X)$ of X is the set of all $g \in \mathbb{G}$ such that $g(X) = X$. If X contains at least 5 points no 4 of them coplanar, then any $g \in PGL(4, \mathbb{C})$ fixing each point of X is the identity. Thus in this case $\mathrm{Stab}_{\mathbb{G}}(X)$ is the set of all conformal symmetries of X, up to the quotient $G \to \mathbb{G}$. We will say that $\mathrm{Stab}_{\mathbb{G}}(X)$ is the conformal automorphism group of X. Since $\mathrm{Stab}_{\mathbb{G}}(X) \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$, the group $\mathrm{Stab}_{\mathbb{G}}(X)$ is finite if $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ is finite. In particular $\mathrm{Stab}_{\mathbb{G}}(X)$ is finite if X is a smooth surface of degree $d > 2$ (Remark 4). Since the conformal automorphism group of a smooth quadric surface was computed in [7, Section 4], for smooth surfaces the only interesting open question is to describe the maximal integer α_d of all $|\mathrm{Stab}_{\mathbb{G}}(X)|$ for X a smooth surface of degree d and to compute the structure of the group $\mathrm{Stab}_{\mathbb{G}}(X)$ when its cardinality is high (e.g. to give an upper bound for the order of its cyclic subgroups). We have no non-trivial result on this problem. We just point out that as in the refinements of Riemann–Hurwitz upper bound for the biholomorphic automorphisms of complex curves of genus at least 2 it may be useful to consider separately cyclic subgroups and abelian subgroup and then apply a classical result of C. Jordan (Remark 1). This strategy was used in [10] to give upper bounds in the case of smooth hypersurfaces in any complex projective space.

We prove the following result.

Theorem 1. *Let $X \subset \mathbb{CP}^3$ be an integral degree $d > 1$ surface containing at least 3 twistor lines, but containing only finitely many twistor lines. The stabilizer $\mathrm{Stab}_{\mathbb{G}}(X)$ of X in $\mathbb{G}$ is infinite if and only if there is an integer $t > 0$ and a conformal change of coordinates such that in the new system of coordinates for all monomials $z_0^{k_0}z_1^{k_1}z_2^{k_2}z_3^{k_3}$ appearing with non-zero coefficient in X we have $k_0 + k_2 = t$ and $k_1 + k_3 = d - t$. Moreover, if $\mathrm{Stab}_{\mathbb{G}}(X)$ is infinite, its connected component of the identity has finite index, it is isomorphic to S^1 as a topological group and its Zariski closure in $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ is isomorphic to $\mathbb{C}^*$ as a complex algebraic group.*

We give examples of surfaces X as Theorem 1 (Example 2). The proof of Theorem 1 shows that, up to a change of coordinates, these are the only examples.

We recall that in the classification of complex affine algebraic groups a torus is a complex algebraic group isomorphic to $(\mathbb{C}^*)^s$ for some integers $s \geq 1$. We prove some results on cones (Proposition 2 and Corollaries 1 and 2) and the following result.

Theorem 2. *Let $X \subset \mathbb{CP}^3$ be an integral surface containing no line and such that $j(X) \neq X$. Assume that $\mathrm{Stab}_{\mathbb{G}}(X)$ is not finite. Let H be the connected component of the identity of the Zariski closure of $\mathrm{Stab}_{\mathbb{G}}(X)$ in $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$. Then $H \cong \mathbb{C}^*$.*

We point out that the assumption that X contains no line implies $\deg(X) \geq 4$. We collect in Example 1 many examples of surfaces X with $\mathrm{Stab}_{\mathbb{G}}(X)$ and with the connected components of the identity of $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ isomorphic to $\mathbb{C}^*$.

In Section 2 we consider the case of curves $Y \subset \mathbb{CP}^3$. For all integers $d > 0$ and $g \geq 0$ such that there is a smooth, connected and non-degenerate curve $Y \subset \mathbb{CP}^3$ with degree d and genus g, let $\alpha_{d,g}$ be the maximal cardinality of the integer $|\mathrm{Stab}_{\mathbb{G}}(Y)|$ among all smooth, connected and non-degenerate curves $Y \subset \mathbb{CP}^3$ with degree d and genus g. We obviously need $d \geq 3$. For a list of all pairs (d, g) which may occur see [11] or [12]. If we drop the assumption that Y is non-degenerate (but instead of $|\mathrm{Stab}_{\mathbb{G}}(Y)|$ we use the cardinality of the image of $\mathrm{Stab}_{\mathbb{G}}(Y)$ in the group $\mathrm{Aut}(Y)$), we call $\alpha'_{d,g}$ the corresponding integer. The integer $\alpha'_{d,g}$ is defined for all $d > 0$, but for $d = 1,2$ it is only defined $\alpha'_{1,0}$, while if $d \geq 3$ either $\alpha'_{d,g} = \alpha_{d,g}$ or $g = (d-1)(d-2)/2$ by the genus formula for smooth plane curves. For $g = 0,1$ a priori it may be $\alpha'_{d,g} = +\infty$, but we always have $\alpha'_{d,1} < +\infty$ (Lemma 3). To get a good upper bound for the integers $\alpha_{d,g}$ one should use a theorem of C. Jordan to reduce our task to the study of finite abelian subgroups of $PGL(r + 1, \mathbb{C})$ (Remark 1). This theorem was used in [10] for linear automorphisms of hypersurfaces in any $\mathbb{CP}^r$. If $g \geq 2$ we have $\alpha_{d,g} \leq 84(g - 1)$ (with strict inequality for several g) by Hurwitz' upper bound for the automorphism group of a smooth curve of genus $g \geq 2$ and the classification of all curves achieving this bound; notice that for all g there are smooth genus g curves with at least $8(g - 1)$ automorphisms [13,14]. If we restrict the curves $Y \subset \mathbb{CP}^3$ to smooth and connected curves Y with $j(Y) = Y$ we get the definition of the integers $\beta_{d,g}$ and $\beta'_{d,g}$. However $\beta'_{d,g}$ is defined only for non-degenerate curves, except the case $(d, g) = (1, 0)$ corresponding to the twistor lines, because $j(M) \neq M$ for every plane $M \subset \mathbb{CP}^3$. The integer (or $+\infty$ for $g = 0$) $\beta_{d,g}$ is well-defined if and only if there is a smooth, connected and non-degenerate curve $Y \subset \mathbb{CP}^3$ with degree d and genus g. Since we do not have a description of all pairs (d, g) for which $\beta_{d,g}$ is defined we pose the following question.

Question 1. For which pairs (d, g) there is a smooth, connected and non-degenerate curve $Y \subset \mathbb{CP}^3$ with degree d and genus g with $j(Y) = Y$? When $g > 0$ and $\beta_{d,g}$ is defined, what is $\beta_{d,g}/\alpha_{d,g}$? For a fixed integer $g \geq 2$ what $\limsup_{d\to+\infty} \beta_{d,g}/\alpha_{d,g}$ and $\liminf_{d\to+\infty} \beta_{d,g}/\alpha_{d,g}$ are ?

In many proofs about surfaces we use results on the conformal automorphism group of singular (and often reducible) curves. Let $X \subset \mathbb{CP}^3$ be a reduced curve, i.e. a curve without multiple components, without isolated points and without embedded points. We allow the case in which X is reducible. It is easy to check that the groups $\mathrm{Stab}_{\mathbb{G}}(X)$ and $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ are finite if at least one irreducible component of X is not rational, i.e. its normalization has not genus 0, and it spans $\mathbb{CP}^3$ (Lemma 4). When $j(X) = X$ and $\mathrm{Stab}_{\mathbb{G}}(X)$ is infinite, we get a stronger result (Lemma 5).

The following remark explains the difference (in the holomorphic category) between isomorphisms as abstract complex compact varieties and automorphisms induced by $PGL(4,\mathbb{C})$. For its use for the projective automorphisms of smooth hypersurfaces, see [10].

Remark 1. C. Jordan proved that for all integers $n > 0$ there is an integer $f(n)$ such that every finite group $H \subset GL(n,\mathbb{C})$ has an abelian subgroup with index at most $f(n)$. Moreover, we may take $f(n) = \lfloor(\sqrt{8n}+1)^{2n^2} - (\sqrt{8n}-1)^{2n^2}\rfloor$ [15, Theorem 9.6]. The optimal $f(n)$ (call it again $f(n)$) was computed in [16, Theorems A and B] together with the classification of the groups achieving the bound. We have $f(2) = f(3) = 60$ and $f(4) = 7200$. For $n \geq 71$ we have $f(n) = n!$.

2. Curves

We first explain why for each closed algebraic subscheme X of $\mathbb{CP}^3$ there is a natural scheme-structure on the support of $j(X)$ preserving dimensions, degrees and multiplicities for the irreducible components of X_{red} and all other projective invariants of X. This is obvious in the real algebraic (or the real analytic) category, because j is a real algebraic isomorphism, but we need a proof in the complex algebraic category. We need to find a homogeneous ideal (which will be the homogeneous ideal of $j(X)$) associated to the homogeneous ideal of X. For any $z \in \mathbb{C}^4$ and any $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$ set

$$z^\alpha = z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}.$$

For any $\alpha \in \mathbb{N}^4$ we write $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Consider the complex vector space $H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$ of all $f \in \mathbb{C}[z_0, z_1, z_2, z_3]$ homogeneous of degree d. The complex vector space $H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$ has dimension $\binom{d+3}{3}$. Consider the map $\hat{\jmath} : H^0(\mathcal{O}_{\mathbb{CP}^3}(d)) \to H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$ defined in the following way

$$H^0(\mathcal{O}_{\mathbb{CP}^3}(d)) \ni f = \sum_{|\alpha|=d} c_\alpha z^\alpha \mapsto \hat{\jmath}(f) = \sum_{|\alpha|=d} \hat{c}_\alpha z^\alpha,$$

where $\hat{c}_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = (-1)^{\alpha_0+\alpha_2}\overline{c}_{\alpha_1,\alpha_0,\alpha_3,\alpha_2}$.

Note that $\hat{\jmath}$ is $\mathbb{R}$-linear and that $\hat{\jmath}^2(f) = (-1)^d f$. For any $z \in \mathbb{C}^4$ we have $f(j(z)) = (-1)^d \overline{\hat{\jmath}(f)(z)}$. We say that $f = \sum_\alpha c_\alpha z^\alpha$ is j-invariant if and only if there is a constant $a \in \mathbb{C} \setminus \{0\}$ such that $\hat{\jmath}(f) = af$, i.e. if and only if the map $\hat{\jmath}$ fixes the line $\mathbb{C}f$. If $q = (z_0 : z_1 : z_2 : z_3) \in \mathbb{CP}^3$ and $f \in H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$ we have $f(q) = 0$ if and only if $\hat{\jmath}(f)(j(q)) = 0$. The homogeneous ideal of $j(X)$ is generated by the polynomials $\hat{\jmath}(f)$ with f a homogeneous polynomial vanishing on X.

Lemma 1. *Take $g \in PGL(4,\mathbb{C})$, $g \neq \mathrm{Id}$, and a reduced curve $E \subset \mathbb{CP}^3$ such that $g(q) = q$ for all $q \in E$.*

(1) *If E spans $\mathbb{CP}^3$, then E is a union of 2 disjoint lines.*
(2) *If either $g \in \mathbb{G}$ or $g(q) = q$ for all $q \in j(E)$, then either E is a twistor line or $E \cup j(E) = L \cup j(L)$ with L a non-twistor line.*

Proof. The union F of 2 disjoint lines is the only reduced curve spanning $\mathbb{CP}^3$ and not containing 5 points in linearly general position, i.e. no 4 of them are coplanar. Thus we get the first assertion of the lemma. Let U be the linear span of E. Since E contains at least 5 points in linearly general position in U (unless E is a disjoint union of 2 lines, a case we excluded in the statement) and $g(q) = q$ for all $q \in E$, we have $g(q) = q$ for all $q \in U$. In particular if E spans $\mathbb{CP}^3$ and E is not the union of 2 disjoint lines, then g is the identity map, a contradiction. Now assume that g is not the identity map and that E is not a line. We get that U is a plane. If $g \in \mathbb{G}$, g commutes with j and hence it fixes each point of the plane $j(U)$. If $g \notin \mathbb{G}$ by assumption it fixes each point of $j(D)$ and hence of $j(U)$. Since $j(U) \neq U$, g fixes pointwise a set spanning $\mathbb{CP}^3$ and then g is the identity: a contradiction. □

Lemma 2. *We have $\alpha'_{1,0} = \beta'_{1,0} = +\infty$.*

Proof. It is sufficient to prove that $\beta'_{1,0} = +\infty$. All the twistor lines are conformally equivalent. Take the twistor line $L = \{z_2 = z_3 = 0\}$. Use the images in $PGL(4,\mathbb{C})$ of the matrices $A = (a_{ij})$ with $a_{22} = a_{33} = 1$ and $a_{ij} = 0$ if either $i = 0{,}1$ and $j = 2{,}3$ or $i = 2$ and $j = 0, 1, 3$ or $i = 3$ and $j = 0, 1, 2$. □

Lemma 3. *Let $Y \subset \mathbb{CP}^3$ be an integral and non-degenerate degree $d \geq 4$ curve such that its normalization is an elliptic curve. Then $|\mathrm{Stab}_{PGL(4,\mathbb{C})}(Y)| \leq 2^d!$.*

Proof. Let $v : D \to Y$ be the normalization map. Set $\mathcal{L} := v^*(\mathcal{O}_Y(1))$. By assumption D is a smooth elliptic curve and $\mathcal{L}$ is a degree d line bundle. Let $u : D \to \mathbb{CP}^{d-1}$ be the linearly normal embedding of D induced by the complete linear system $|\mathcal{L}|$. Every $g \in \mathrm{Stab}_{PGL(4,\mathbb{C})}(Y)$ induces $g' \in PGL(d,\mathbb{C})$ such that $g'(u(D)) = u(D)$ and the map $g \mapsto g'$ is injective. Call

$H \subset PGL(d-1, \mathbb{C})$ the image of $\mathrm{Stab}_{PGL(4,\mathbb{C})}(D)$ by the map $g \mapsto g'$. Let $S \subset u(D)$ be the set of all flexes of $u(D)$, i.e. the set of all $q \in u(D)$ such that the hyperplane of $\mathbb{CP}^{d-1}$ has order of contact $\geq d$ with $u(D)$ at q. Since $\deg(u(D)) = d$, S is the set of all $u(o)$, $o \in D$, such that $do \in |\mathcal{L}|$. Thus $|S| = 2^d$. Since $u(D)$ is a non-degenerate and of degree d, any $d+1$ points of $u(D)$ span $\mathbb{CP}^{d-1}$. We get that any $h \in PGL(d, \mathbb{C})$ with $h(q) = q$ for all $q \in S$ is the identity. Since the notion of flex is a projective one, we have $f(S) = S$ for every $f \in PGL(d, \mathbb{C})$ such that $f(u(D)) = u(D)$. Thus $|H| \leq 2^d!$. □

Lemma 4. *Let $X \subset \mathbb{CP}^3$ be a reduced curve spanning $\mathbb{CP}^3$ such that at least one irreducible component E of X is not rational, i.e. its normalization has no genus 0. Then $\mathrm{Stab}_{\mathbb{G}}(X)$ is finite. If E is non-degenerate, then $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ is finite.*

Proof. First assume that E is non-degenerate. In this case it is sufficient to prove that the group $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ is finite. Assume that this is not the case. Since $PGL(4, \mathbb{C})$ is a complex algebraic group, its connected component H of the identity has positive dimension. We claim that there is a subgroup H' of H isomorphic either to the multiplicative group $\mathbb{C}^*$ or to the additive group $\mathbb{C}_a$ (i.e. $\mathbb{C}$ with the addition as group multiplication). We recall that either $H \cong \mathbb{C}^*$ and there is $g \in SL(4, \mathbb{C})$ such that gHg^{-1} is diagonal (the semisimple case) or $H \cong \mathbb{C}_a$ and there is $g \in SL(4, \mathbb{C})$ such that gHg^{-1} is in upper triangular forms with 1's in the diagonal (the unipotent case); see [17, page 261] or [18, 11.1]) for the existence of a Borel subgroup, so that H is solvable, [17, Cor. 3 at page 110] for the fact that H is abelian (being minimal and contained in a solvable connected group), [17, Theorem 4.7 at page 156] or [19, Theorem 15.5] for the decomposition into a multiplicative and a unipotent part for commutative groups and then for the existence of g see [17, Theorem at page 158] for the unipotent case and [17, page 155] for the multiplicative case. Since H' is connected and X has only finitely many irreducible components, H' sends each irreducible component into itself. Let E be an irreducible component of X which is not rational (if any) and let $E' \to E$ the normalization map. The action of H' on E lift to an action of H' on E' by the universal property of the normalization (or because E' and E have the same field $\mathbb{C}(E')$ of rational functions and any $\mathbb{C}$-automorphism of the field $\mathbb{C}(E')$ induces an automorphism of the smooth and connected projective curve E'). If E' has genus ≥ 2, then $\mathrm{Aut}(E')$ is finite and hence the connected group H' fixes each point of E' and hence of E, contradicting Lemma 1. If E' is an elliptic curve we use Lemma 3.

Now assume that E is degenerate and that $\mathrm{Stab}_{\mathbb{G}}(X)$ is infinite. Since E is not rational, it is not a line and thus E spans a plane M. Since E is not a line, Lemma 1 shows that the image of G_0 in $\mathrm{Aut}(E)$ is infinite. This is false if the normalization of E has genus ≥ 2, because $\mathrm{Aut}(E) \leq 84(g-1)$ by Hurwitz' bound for the automorphisms of compact Riemann surfaces [13,14]. If the normalization of E is an elliptic curve we use Lemma 3. □

Remark 2. Let $D \subset \mathbb{CP}^3$ be an integral and non-degenerate curve such that the algebraic group $\mathrm{Stab}_{PGL(4,\mathbb{C})}(D)$ is not finite, i.e. $\dim \mathrm{Stab}_{PGL(4,\mathbb{C})}(D) > 0$ and let $u : D' \to D$ be the normalization map. Let $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(D)$ be a minimal connected subgroup. Either $H \cong \mathbb{C}^*$ (the multiplicative group) or $H \cong \mathbb{C}$ with the addition as its group structure (the additive group) (see the quotations in the proof of Lemma 4). Since H is a one-dimensional affine connected algebraic group acting non-trivially on the smooth and connected compact complex curve D', H acts on D' with an open orbit Hb, $b \in D'$, and hence D' has genus 0, i.e. D is rational. We have $|D' \setminus Hb| = 2$ in the multiplicative case, because $\mathbb{CP}^1 \setminus S$ is hyperbolic for every finite set S with $|S| \geq 3$, while $\mathbb{C}$ and $\mathbb{C}^*$ are not even homeomorphic. For the same reasons we have $|D' \setminus Hb| = 1$ in the additive case. Thus (setting $a := u(b)$) we have $|D \setminus Da| \leq 2$ in the multiplicative case and $|D \setminus Du| \leq 1$ in the additive case. The orbit Ha is obviously contained in the smooth part of D. Note that $|D \setminus Da| = 2$ if and only if H is multiplicative and u is injective.

Lemma 5. *Let $D \subset \mathbb{CP}^3$ be an integral and non-degenerate curve such that $\mathrm{Stab}_{\mathbb{G}}(D)$ is infinite and $j(D) = D$. Let $u : D' \to D$ denote the normalization map. Fix a minimal non-trivial and connected subgroup G_0 of $\mathrm{Stab}_{\mathbb{G}}(D)$ and call $H \subseteq \mathrm{Stab}_{\mathbb{G}}(D)$ the Zariski closure of G_0. Then $H \cong \mathbb{C}^*$ and there are a smooth $a \in D$, $b \in D'$ with $u(b) = a$ such that $Hb \cong H$, $Ha \cong H$, $|D' \setminus Hb| = |D \setminus Ha| = 2$ and the 2 points of $D \setminus Ha$ are interchanged by j.*

Proof. We know by Example 2 that D is rational and that either $H' = \mathbb{C}^*$ or $H' = \mathbb{C}$ with the additive structure. In each of these two cases we distinguish the possibilities for $|D \setminus Da|$ and $|D' \setminus Db|$; we exclude the case $H = \mathbb{C}$, because H is a Zariski closure of a subgroup of the conformal group and hence j permutes the set $D \setminus Da$ (note that j has no fixed point). □

Remark 3. Let $A \subset \mathbb{CP}^3$ be a complete intersection curve, say the complete intersection of a surface of degree x and a surface of degree y; we allow the case in which A has multiple components. Set $B := A_{\mathrm{red}}$. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^3}(-x-y) \to \mathcal{O}_{\mathbb{CP}^3}(-x) \oplus \mathcal{O}_{\mathbb{CP}^3}(-y) \to \mathcal{I}_A \to 0 \tag{3}$$

Since $h^i(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(z)) = 0$ for $i = 1,2$ and all $z \in \mathbb{Z}$, formula (3) implies $h^1(\mathbb{CP}^3, \mathcal{I}_A) = 0$. Thus $h^0(A, \mathcal{O}_A) = 1$. Hence B is connected.

Proposition 1. *For infinitely many integers d there is an integral and non-degenerate degree d curve $Y \subset \mathbb{CP}^3$ such that the connected component H of the identity of $\mathrm{Stab}_{PGL(4,\mathbb{C})}(Y)$ is isomorphic to $\mathbb{C}^*$ and its intersection with $\mathbb{G}$ is isomorphic to S^1 (resp. $\mathbb{R}^*$). In the former case $\mathrm{Stab}_{\mathbb{G}}(Y)$ has finite cyclic subgroups of any order. In the latter case $\mathbb{G} \cap H$ has only $\{-1, 1\}$ as its element with finite order.*

Proof. Take the maximal torus $\mathbb{D} \cong (\mathbb{C}^*)^4$ of $GL(4, \mathbb{C})$ formed by the diagonal matrices. Call $\psi_h : \mathbb{C}^* \to \mathbb{D}$, $h = 0, 1, 2, 3$, the homomorphism sending $z \in \mathbb{C}^*$ to the diagonal matrix $A = (a_{ij})$ with $a_{ij} = 0$ if $i \neq j$, $a_{ii} = 1$ if $i \neq h$ and $a_{hh} = z$. For any $(n_0, n_1, n_2, n_3) \in \mathbb{Z}^4$ we have a homomorphism $\varphi = \psi_0^{n_0}\psi_1^{n_1}\psi_2^{n_2}\psi_3^{n_3} : \mathbb{C}^* \to (\mathbb{C}^*)^4$. If $n_1 = -n_0$ and $n_3 = -n_2$, then $\varphi(z) \in G$ for all $z \in S^1$. If $n_0 = n_1$ and $n_2 = n_3$, then $\varphi(z) \in G$ for all $z \in \mathbb{R}^*$. In both cases we take $n_3 = 1$, call H_{n_1} the corresponding image and take as rational curve the H_{n_1}-orbits of general points of $\mathbb{CP}^3$. $\square$

3. Algebraic surfaces

Remark 4. For any integer $d \geq 4$ the group $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ is the full group of all biholomorphic automorphisms of X [20, Theorem 2], it is always finite and it is trivial for a general degree d surface $X \subset \mathbb{CP}^3$ [20,21]. By [22] for every integer $d \geq 3$ there is a smooth degree d surface X defined over $\mathbb{R}$ and with trivial $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$.

Example 1. Fix any integral and non-degenerate curve $Y \subset \mathbb{CP}^3$ such that the group $\mathrm{Stab}_{PGL(4,\mathbb{C})}(Y)$ is finite. Fix a subgroup $H \subset PGL(4, \mathbb{C})$ such that $H \cong \mathbb{C}^*$ and $H \cap \mathbb{G}$ has positive dimension. For instance we may take as H the elements of $PGL(4, \mathbb{C})$ induced by the diagonal matrices $A = (a_{ij})$ such that $a_{ij} = 0$ for all $i \neq j$, $a_{22} = a_{33} = 1$ and $a_{00} = a_{11} \neq 0$; in this case $H \cap \mathbb{G} \cong \mathbb{R}^*$, because $H \cap \mathbb{G}$ is induced by the matrices with $a_{11} = a_{00} = \overline{a}_{00}$. Taking H induced by the matrices $A = (a_{ij})$ with $a_{ij} = 0$ for all $i \neq j$, $a_{22} = a_{33} = 1$ and $a_{11} = a_{00}^{-1}$ we get an example with $H \cap \mathbb{G} \cong S^1$. Let now $X \subset \mathbb{CP}^3$ be the closure of the orbit of Y by the action of H. X is an integral and non-degenerate surface. Since $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$, we have $H \cap \mathbb{G} \subseteq \mathrm{Stab}_{\mathbb{G}}(X)$. Conversely, take any integral and non-degenerate surface such that $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ (and hence $H \cap \mathbb{G} \subseteq \mathrm{Stab}_{\mathbb{G}}(X)$). Let Y be the complete intersection of X with a general surface of high degree. The construction just explained gives the surface X, because for a general $q \in Y$ the H-orbit Hq is contained in X.

The following result gives a class of integral but singular surfaces X for which $\mathrm{Stab}_{\mathbb{G}}(X)$ is much smaller than $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$. For smooth quadrics see the classification in [7].

Proposition 2. *Let $X \subset \mathbb{CP}^3$ be an integral cone of degree $d > 1$.*

(a) There is a subgroup $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ such that $H \cong \mathbb{C}^*$ and $H \cap \mathbb{G}$ is the identity.

(b) Assume $d \geq 3$ and call o the vertex of X and R the twistor line of $\mathbb{CP}^3$ containing o. Take a plane $M \subset \mathbb{CP}^3$ with $o \notin M$ and set $Y := X \cap M$. Let $\mathrm{Lin}(Y, o')$ be the set of all automorphisms of Y induced by a linear automorphism of M fixing o'. If $\mathrm{Lin}(Y, o')$ is finite, then $\mathrm{Stab}_{\mathbb{G}}(X)$ is finite.

Proof. Since $d > 1$ and X is integral, the vertex of X is a unique point, o. Since $\mathbb{G}$ acts transitively on $\mathbb{CP}^3$, we may assume $o = (1 : 0 : 0 : 0)$. In this system of homogeneous coordinates we have $R = \{z_2 = z_3 = 0\}$ and $X = \{f(z_1, z_2, z_3) = 0\}$ with $f(z_1, z_2, z_3)$ a degree d homogeneous polynomial. Thus $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ contains the images in $PGL(4, \mathbb{C})$ of all matrices $A = (a_{ij})$ with $a_{ij} = 0$ for all $i \neq j$, $a_{11} = a_{22} = a_{33} = 1$ and $a_{00} \in \mathbb{C}^*$. These images form a subgroup $H \cong \mathbb{C}^*$. By (2) we see that $A \in G$ if and only if $a_{00} = 1$, i.e. $A = \mathrm{Id}_{4\times 4}$, concluding the proof of part (a).

Now we prove part (b). Take $A = (a_{ij}) \in G$ inducing an element of $\mathrm{Stab}_{\mathbb{G}}(X)$. Since o is the unique vertex of X, $(1, 0, 0, 0)$ is an eigenvector of A. Thus $a_{01} = a_{02} = a_{03} = 0$. Since $A \in G$, formula (2) gives $a_{11} = a_{00}$ and $a_{10} = a_{12} = a_{13} = 0$. All sections of X by planes not containing o are irreducible and projectively equivalent. Thus it is sufficient to prove part (b) when $M = \{z_0 = 0\}$ and so $o' = (0 : 1 : 0 : 0)$. With this assumption the 3×3 submatrix B of A corresponding to the last 3 rows and columns induces an element of $\mathrm{Lin}(Y, o')$. Thus we only have finitely many possible B's. By (2) we also have finitely many possibilities for $a_{02} = \overline{a}_{13}$ and for $a_{03} = -\overline{a}_{12}$. $\square$

Corollary 1. *Let X be an integral cone of degree $d \geq 4$ with the vertex as its only singular point. Then $\mathrm{Stab}_{\mathbb{G}}(X)$ is finite.*

Proof. With the set-up of Proposition 2 Y is a smooth plane curve of degree $d \geq 4$. Since Y has genus $g = (d-1)(d-2)/2$, it has only finitely many automorphisms. $\square$

Corollary 2. *Let $X \subset \mathbb{CP}^3$ be the general integral cone of degree $d \geq 4$. Then $\mathrm{Stab}_{\mathbb{G}}(X) = \{\mathrm{Id}\}$.*

Proof. Use that a general plane curve of degree $d \geq 4$ has no holomorphic automorphism. $\square$

Proof of Theorem 1. Every linear algebraic group has only finitely many connected components. Thus $\dim H > 0$ and $H \cap \mathrm{Stab}_{\mathbb{G}}(X)$ is infinite. Let L, R, D be 3 of the twistor lines of X. Every conformal transformation sends a twistor line into a twistor line. Since X contains only finitely many twistor lines, there is a subgroup G_1 of finite index of $\mathrm{Stab}_{\mathbb{G}}(X)$ such that $g(L) = L$, $g(R) = R$ and $g(D) = 0$. Note that H is the connected component of the Zariski closure in $\mathrm{Stab}_{PGL(4,\mathbb{C})}(X)$ of G_1. Since $G_1 \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(L) \cap \mathrm{Stab}_{PGL(4,\mathbb{C})}(R) \cap \mathrm{Stab}_{PGL(4,\mathbb{C})}(D)$ and the latter is a closed subgroup of $PGL(4, \mathbb{C})$, we have $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(L) \cap \mathrm{Stab}_{PGL(4,\mathbb{C})}(R) \cap \mathrm{Stab}_{PGL(4,\mathbb{C})}(D)$. Since G acts 2-transitively on S^4, up to a conformal transformation (i.e. up to take $h(X)$ instead of X for some $h \in G$), we may assume $L = \{z_0 = z_1 = 0\}$ and $R = \{z_2 = z_3 = 0\}$. Take any $g \in G_1$ and call $A = (a_{ij}) \in GL(4, \mathbb{C})$, $0 \leq i \leq 3$, $0 \leq j \leq 3$, a matrix representing g. Since $g(L) = 0$, we have $a_{ij} = 0$ for $i = 0,1$ and $j = 2,3$. Since $g(R) = R$, we have $a_{ij} = 0$ for $i = 2,3$ and $j = 0,1$. Now we use D. Since L, D, R are twistor lines, they are disjoint. Thus there is a unique quadric surface Q containing $L\cup R\cup D$ and this quadric is smooth. Since Q contains 3 twistor lines, the conformal classification of smooth quadric surfaces gives that Q is a real smooth quadric [7, Theorem 1.11]. Since G_1 send $L\cup R\cup D$ into itself and Q is the unique quadric containing $L\cup R\cup D$, we have $G_1 \subseteq \mathrm{Stab}_{GL(4,\mathbb{C})}(Q)$. Call E the scheme-theoretic intersection of X and Q and F the set-theoretic intersection. Since X and Q are integral and $X \neq Q$, E is a complete intersection curve of degree $2d$ and F is a curve, union of all the irreducible components of E, without their multiplicities. Every complete intersection curve is connected (Remark 3). Thus F is connected. Call $|\mathcal{O}_Q(1,0)|$ and $|\mathcal{O}_Q(0,1)|$ the 2 rulings of Q, with $L \in |\mathcal{O}_Q(1,0)|$ and hence $D, R \in |\mathcal{O}_Q(1,0)|$. $E \subset Q$ is a curve of bidegree (d,d) and hence $F \in |\mathcal{O}_Q(a,b)|$ for some a, b at most d. Since $L\cup R\cup D \subseteq F$, we have $a \geq 3$. Write $F = (L\cup R\cup D)\cup K$ with $K \in |\mathcal{O}_Q(a,b)|$. Since g stabilizes X, Q and $L\cup R\cup D$, it stabilizes E and hence F. Thus g stabilizes K. Since F is connected and without multiple components, $S_L := L\cap K$, $S_R := R\cap K$ and $S_D := D\cap K$ are finite non-empty sets and they are stabilized by each element of G_0. Fix $o \in L\cap K$, $q \in R\cap K$ and $q' \in D\cap K$. The points o, q and q' are distinct, because any 2 twistor lines are disjoint. Up to a conformal transformation involving only z_0 and z_1 (resp. z_2 and z_3) we may assume $o = (1:0:0:0)$ (resp. $q = (0:0:1:0)$). Note that $j(o) = (0:1:0:0)$ and $j(q) = (0:0:0:1)$. Since every conformal transformation commutes with j and G_1 fixes the points o and q, each element of G_1 fixes the points o, $j(o)$, q and $j(q)$. Thus A is a diagonal matrix and o, $j(o)$, q and $j(q)$ represent linearly independent eigenvectors of A. Using L and D we get that q' and $j(q')$ are different eigenvectors. Hence A has some multiple eigenvalue. Since $g \in G$, we have $a_{11} = \overline{a}_{00}$ and $a_{33} = \overline{a}_{22}$. We need to prove that $a_{ii} = a_{00}$ for all i and hence that A is a real multiple of the identity $\mathrm{Id}_{4\times 4}$, unless X is as in the exceptional case. Since $A \in G$, formula (2) shows that it is sufficient to prove that A has an eigenvalue with eigenspace of dimension at least 3. Since $L\cap D = R\cap D = \emptyset$ neither q' nor $j(q')$ are in $L\cup R$ and hence at least one of the eigenvalues for o or $j(o)$ must be equal to an eigenvalue of q' and $j(q')$.

Suppose for instance that any non-zero vector associated to q' is in the eigenspace spanned by o and q (we may reduce to this case exchanging the names of q and $j(q)$ and of q' and $j(q')$). We get $a_{00} = a_{22}$ and so $a_{11} = \overline{a}_{00} = \overline{a}_{22} = a_{33}$. We get that the Zariski closure H' in $GL(4,\mathbb{C})$ of this set of matrices is the set of all diagonal matrices $B = (b_{ij})$ with $b_{11} = b_{33}$ and $b_{00} = b_{22}$.

Let $f \in \mathbb{C}[z_0, z_1, z_2, z_3]$ be an equation of X in the new system of coordinates with $o = (1:0:0:0)$, $j(o) = (0:1:0:0)$, $q = (0:0:1:0)$ and $j(q) = (0:0:0:1)$. Let $\mathcal{S}$ be the set of all monomials of f appearing with non-zero coefficient. The matrix A sends the monomial $z_0^{k_0}z_1^{k_1}z_2^{k_2}z_3^{k_3}$ to $a_{00}^{k_0+k_2}\overline{a}_{00}^{k_1+k_3}z_0^{k_0}z_1^{k_1}z_2^{k_2}z_3^{k_3}$. Note that $k_1 + k_3 = d - k_0 - k_1$. Thus Af is a multiple of f for every $a_{00} \in \mathbb{C}^*$ if and only if there is an integer t with $0 \leq t \leq d$ such that $k_0 + k_2 = t$ for all $z_0^{k_0}z_1^{k_1}z_2^{k_2}z_3^{k_3} \in \mathcal{S}$. Since X is irreducible, we have $t \neq 0$ and $t \neq d$. We also get the last part of the theorem (starting with the word "Moreover") using the quotient maps $G \to \mathbb{G}$ and $GL(4,\mathbb{C}) \to PGL(4,\mathbb{C})$. $\square$

Example 2. Assume $d \geq 3$ and fix an integer t such that $3 \leq t \leq d-3$. Take $o = (1:0:0:0)$ and $q = (0:0:1:0)$ and so $j(o) = (0:1:0:0)$ and $j(q) = (0:0:0:1)$. Let L be the line spanned by $\{o, j(o)\}$, R the line spanned by $\{q, j(q)\}$ and L_1 the line spanned by $\{o, q\}$. Fix $q' \in L_1 \setminus \{o, q\}$. Let D be the line spanned by $\{q', j(q')\}$. The lines L, R and D are 3 different twistor lines. For simplicity we take $q' = (1:0:1:0)$ and hence $j(q') = (0:1:0:1)$ and $R = \{z_0 - z_2 = z_1 - z_3 = 0\}$. For all integers $x \geq y \geq 0$ let $\mathcal{S}_{x,y}$ be the set of all monomials $z_0^{k_0}z_1^{k_1}z_2^{k_2}z_3^{k_3}$ such that $k_0 + k_2 = y$ and $k_1 + k_2 = x - y$. Let $\mathcal{A}_{x,y}$ be the set of all $\mathbb{C}$-linear combinations of elements of $\mathcal{S}_{x,y}$. Since $|\mathcal{S}_{x,y}| = (y+1)(x-y+1)$, $\mathcal{A}_{x,y}$ is a $\mathbb{C}$-vector space of dimension $(x+1)(x-1+1)$. Let $\mathcal{A}_{d,t}(-L-R-D)$ be the set of all surfaces $X \subset \mathbb{CP}^3$ containing $L\cup D\cup R$ and with equation in $\mathcal{A}_{d,t}$. Since $h^0(\mathbb{CP}^3, \mathcal{I}_{L\cup D\cup R}(x)) = \binom{x+3}{3} - 9$ for all $x \geq 2$, the set $\mathcal{A}_{d,t}(-L-R-D)$ is a projective space of dimension at least $(t+1)(d-t+1) - 10$. Take a general $X \in \mathcal{A}_{x,t}(-L-R-D)$. To give an example for Theorem 1 it is sufficient to prove that X is irreducible. By the second Bertini's theorem [23, part 4 of Theorem 6.3] it is sufficient to prove that the linear system $|\mathcal{A}_{d,t}(-L-R-D)|$ has no base components and the rational map γ induced by $|\mathcal{A}_{d,t}(-L-R-D)|$ has not a curve as its image. We have $L = \{z_2 = z_3 = 0\}$ and $R = \{z_0 = z_1 = 0\}$. Since $R = \{z_0 - z_2 = z_1 - z_3 = 0\}$, $z_0, z_2 \in \mathcal{A}_{1,1}$ and $z_1, z_3 \in \mathcal{A}_{1,0}$, the curve $L\cup D\cup R$ is contained in the zero-locus of the product of 2 elements of $\mathcal{A}_{1,1}$ and 2 elements of $\mathcal{A}_{1,0}$. For all $x > y > 0$ the linear system $|\mathcal{A}_{x,y}|$ has no base points and maps $\mathbb{CP}^3$ onto a 3-dimensional variety. Since $t \geq 3$ and $d - t \geq 3$, we get that $|\mathcal{A}_{d,t}(-L-R-D)|$ has 3-dimensional image and as possible base components only hyperplanes $\{z_i = 0\}$. We immediately check that no such hyperplane is a base component.

Lemma 6. *Let $X \subset \mathbb{CP}^3$ be an integral degree $d > 2$ surface such that $j(X) \neq X$. Set $T := X \cap j(X)$ and $F := T_{\mathrm{red}}$. The group $\mathrm{Stab}_{\mathbb{G}}(X)$ is finite, unless there are a rational component $D \subseteq F$ such that $j(D) = D$ and $o \in D$, such that $D \setminus \{o, j(o)\} \cong \mathbb{C}^*$, and o and $j(o)$ are stabilized by the connected component of $\mathrm{Stab}_{\mathbb{G}}(X)$ containing the identity.*

Proof. Let G_1 be the connected component of the identity of $\mathrm{Stab}_{\mathbb{G}}(X)$ and let G_2 be the Zariski closure of G_1 in $\mathrm{Stab}_{\mathbb{G}}(X)$. Since $X \neq j(X)$, T is a complete intersection curve of degree d^2 (perhaps with multiple components) and hence F is a reduced curve. Since $T = X \cap j(X)$, we have $j(F) = F$. Since every conformal map commutes with j, $G_1 \subseteq \mathrm{Stab}_{\mathbb{G}}(F)$ and hence $G_2 \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(F)$. Since G_1 is connected, it send each irreducible component of it into itself and hence G_2 does the same. Let $H \subset G_2$ be the Zariski closure of a minimal non-trivial connected subgroup of G_1. By

Lemma 5 we have $H \cong \mathbb{C}^*$ and there is at least one irreducible component, E, of F on which H acts with an open orbit Ha with $E \setminus \mathbb{C}^* = \{o, j(o)\}$. $\square$

Remark 5. Let H be a connected complex linear algebraic group such that $H \neq \{\mathrm{Id}\}$, i.e. $\dim H > 0$. By [24, Lemma 6.10] and the definition of reductive group and radical [18, 11.21] H is reductive with a torus as its radical and every connected solvable subgroup of H is a torus if and only if there is no subgroup of H isomorphic to the additive group $\mathbb{C}_a$ (i.e. $\mathbb{C}$ with the addition as its group operation).

Proof of Theorem 2. Since each element of G commutes with j, we have $\mathrm{Stab}_{\mathbb{G}}(X) \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(j(X))$. Since $\mathrm{Stab}_{PGL(4,\mathbb{C})}(j(X))$ is closed in $PGL(4,\mathbb{C})$ in the Zariski topology and G_2 is Zariski dense in H, we have $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(j(X))$. Set $T := X \cap j(X)$ and $F := T_{\mathrm{red}}$. Since H acts on X and $j(X)$, it acts on T and F. Since H is connected, we have $H \subseteq \mathrm{Stab}_{PGL(4,\mathbb{C})}(F_i)$ for every irreducible component of F.

(a) In this step we prove that H contains no subgroup isomorphic to $\mathbb{C}_a$. Assume, by contradiction, that H has a subgroup $H' \cong \mathbb{C}_a$.

Fix any irreducible component D of F. Either H' fixes each point of D or there are $a, b \in D$ such that $D = \{b\} \sqcup H'a$ with $H' \cong \mathbb{C}_a$. By Lemma 1 the former case may occur only if D is a line, which we excluded. Hence for each irreducible component D there is a unique $b \in D$ such that $H'b = b$ (Lemma 5). Call S' the set of all $x \in F$ such that $H'x = H'$. We just say that $S' \neq \emptyset$ and that each irreducible component of F meets S' at a unique point. For each $x \in S'$ let F_x be the union of the irreducible components of F containing x. Since any two different orbits of H' are disjoint, we get that two different irreducible components of F either are disjoint or they meet at a unique point of S', say x, and so they are contained in $F(x)$. Thus if $x, x' \in S'$ we have $F(x) \cap F(x') = \emptyset$. Since F is connected (Remark 3) we get $|S'| = 1$. However, since $H' \subseteq H$, $j(F) = F$ and $H \cap \mathbb{G}$ is Zariski dense in H, we see that $j(S') = S'$. Since j has no fixed point, we get a contradiction.

(b) Now we conclude the proof that $H \cong \mathbb{C}^*$. By Remark 5 step (a) proves that H is reductive with a torus as its radical and every connected solvable subgroup of H is a torus. Since H is connected and it has no subgroup isomorphic to $\mathbb{C}_a$ (step (a)), by the classification of connected and one-dimensional linear algebraic groups [24, Theorem 2.6.6] it is sufficient to prove $\dim H = 1$. We saw in step (a) that every irreducible component D of F is rational and that the natural map $H \to \mathrm{Stab}_{PGL(4,\mathbb{C})}(D)$ is injective. Thus H is isomorphic to a subgroup of $PGL(2,\mathbb{C})$. Since $\dim PGL(2,\mathbb{C}) = 3$ and $PGL(2,\mathbb{C})$ contains a subgroup isomorphic to $\mathbb{C}_a$ (the set of all 2×2 strictly upper triangular matrices) we get $\dim H \leq 2$. The classification of all simple groups gives that every semi-simple algebraic group has dimension at least 3. Thus H is solvable. Since H contains no subgroup isomorphic to $\mathbb{C}_a$ (step (a1)), H is a torus [24, Lemma 6.10]. All maximal tori of a semisimple group like $PGL(2,\mathbb{C})$ are conjugate [18, Corollary 11.3] and a maximal torus of $PGL(2,\mathbb{C})$ has dimension 1, because any two commuting semisimple element of $SL(2,\mathbb{C})$ may be simultaneously diagonalized. □

If $X \subset \mathbb{CP}^3$ is an integral degree d surface with $j(X) \neq X$, then $T := X \cap j(X)$ is a complete intersection curve (perhaps with multiple components or not irreducible). Obviously $j(T) = T$ and $\mathrm{Stab}_{\mathbb{G}}(X) \subseteq \mathrm{Stab}_{\mathbb{G}}(T)$. The following lemma shows that the converse holds and gives a tool to find many surfaces X with $j(X) \neq X$ and $T = X \cap j(X)$.

Lemma 7. *Let $T \subset \mathbb{CP}^3$ be the complete intersection of 2 degree d surfaces (we allow the case in which T has multiple components). Then there is a degree d surface $X \subset \mathbb{CP}^3$ without multiple components such that $j(X)$ and X have no common components and $T = X \cap j(X)$.*

Proof. Since T is a complete intersection of 2 degree 2 surfaces, Bezout's theorem gives $h^0(\mathbb{CP}^3, \mathcal{I}_T(d)) = 2$ and that T is the base-locus of the linear system $|\mathcal{I}_T(d)|$. Thus T is the scheme-theoretic intersection of any 2 elements of $|\mathcal{I}_T(d)|$. Let X be a general element of $|\mathcal{I}_T(d)|$. Since T is the base-locus of $|\mathcal{I}_T(d)|$, Bertini's theorem ([23, part b of Theorem 6.3] or [25, III.10.9]) implies that X is smooth outside T. Thus X has no multiple component. Since $j(T) = T$, the anti-holomorphic involution j induces an anti-holomorphic involution $\gamma : |\mathcal{I}_T(d)| \to |\mathcal{I}_T(d)|$. □

It is usually very easy to check that a complete intersection curve, even a reducible one, is not a complete intersection of 2 surfaces, at least one of them being reducible.

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