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Left invariant pseudo-Riemannian metrics on solvable Lie groups

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ABSTRACT

In this paper, we consider a special class of solvable Lie groups such that for any x, y in its Lie algebra, $[x, y]$ is a linear combination of x and$y$. For the convenience, we call such a Lie group a LCS Lie group. We investigate non-trivial m-quasi-Einstein metrics on pseudo-Riemannian LCS Lie group. We proved that although there exists only trivial Ricci soliton on pseudo-Riemannian LCS Lie group, any left invariant pseudo-Riemannian metric on LCS Lie group is non-trivial m-quasi-Einstein metric. Moreover, non-trivial m-quasi-Einstein metric is shrinking, expanding, or steady if LCS Lie group has positive, negative, or zero constant sectional curvature respectively. In particular, any left invariant Riemannian or Lorentzian metric on LCS Lie group is non-trivial m-quasi-Einstein metric.

1. Introduction

In [12], Milnor considered a special class of solvable Lie groups such that for any x, y in its Lie algebra, $[x, y]$ is a linear combination of x and y. For the convenience, we call such a Lie group a LCS Lie group. It was proved that every left-invariant Riemannian metric on such a Lie group is of constant negative sectional curvature. Then, Nomizu proved that every left-invariant Lorentzian metric on a LCS Lie group is also of constant sectional curvature. However, relying on the choice of left invariant Lorentz metrics, the sign of the constant sectional curvature may be positive, negative, or zero [13]. Later, Albuquerque generalized this result to semi-Riemannian case [1]. In [8], Guediri proved that if G is a LCS Lie group, then every left-invariant Lorentzian metric on G is geodesically incomplete. Note also that if μ is a left invariant pseudo-Riemannian metric on a LCS Lie group, then μ is flat if and only if the restriction of $\mu(e)$ to $[\mathfrak{g}, \mathfrak{g}]$ is degenerate; see [10]. Recently, the third author and S. Deng have investigated harmonicity properties of vector fields on Lorentzian LCS Lie groups [16].

On the other hand, a natural extension of the Ricci tensor is the m-Bakry–Emery Ricci tensor

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df, \tag{1.1}$$

where $0 < m \leq \infty$, f is a smooth function on M^n, and $\nabla^2 f$ stands for the Hessian form. For an arbitrary vector field X on M^n, Barros and Ribeiro [2] and Limoncu [11] extended m-Bakry–Emery Ricci tensor independently as follows:

$$Ric_X^m = Ric + \frac{1}{2}\mathfrak{L}_X g - \frac{1}{m} X^* \otimes X^* \tag{1.2}$$

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where $\mathfrak{L}_X g$ denotes the Lie derivative on M^n and X^* denotes the canonical 1-form associated with X. (M^n, g) is called an *m-quasi-Einstein metric* if there exist a vector field $X \in \mathfrak{X}(M^n)$ and constants m and λ such that

$$Ric_X^m = \lambda g. \tag{1.3}$$

An m-quasi-Einstein metric is called *trivial* when $X \equiv 0$. The triviality definition is equivalent to say that M^n is an Einstein manifold. When $m = \infty$, Eq. (1.3) reduces to a *Ricci soliton*, for more details see [4]. And an m-quasi-Einstein metric is called *expanding, steady* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. If m is a positive integer and X is a gradient vector field, the condition corresponds to a wrapped product Einstein metric, for more details see [5].

Although compact homogeneous Ricci solitons are Einstein (see [9,14,15]), compact homogeneous m-quasi-Einstein metrics are not necessarily Einstein for m finite [3]. It was proved that every compact simple Lie group admits non-trivial m-quasi-Einstein Lorentzian metrics and most of them admit infinitely many metrics [7]. Moreover, infinitely many non-trivial m-quasi-Einstein metrics were constructed on solvable quadratic Lie group $G(n)$ for m finite [6].

In this paper, we investigate non-trivial m-quasi-Einstein metrics on pseudo-Riemannian LCS Lie group. This paper is organized as the following: In Section 2, we state some basic facts on pseudo-Riemannian LCS Lie group and present some results on the structure of the Lie algebras of n-dimensional pseudo-Riemannian LCS Lie groups. Pseudo-Riemannian LCS Lie groups are classified into *type A* and *type B*, depending on the induced metric on the commutative ideal $\mathfrak{u}$ (see Definition 2.1). In Sections 3 and 4, we investigate non-trivial m-quasi-Einstein metrics on pseudo-Riemannian LCS Lie group of type A and of type B respectively. We proved that although there exists only trivial Ricci soliton on pseudo-Riemannian LCS Lie group, any left invariant pseudo-Riemannian metric on LCS Lie group is non-trivial m-quasi-Einstein metric. Moreover, non-trivial m-quasi-Einstein metric is shrinking, expanding, or steady if LCS Lie group has positive, negative, or zero constant sectional curvature respectively.

2. Preliminaries

A non-commutative Lie group G is said to belong to the class $\mathcal{F}$ if its Lie algebras $\mathfrak{g}$ has the property that $[x, y]$ is a linear combination of x and y, for any $x, y \in \mathfrak{g}$.

In [12], it is shown that $G \in \mathcal{F}$ if and only if there exist a commutative ideal $\mathfrak{u}$ of codimension 1 and an element $b \notin \mathfrak{u}$ such that $[b, x] = x$ for every $x \in \mathfrak{u}$. Furthermore, $G \in \mathcal{F}$ if and only if every left invariant Riemannian metric on G has sectional curvature of constant sign.

In [13], K. Nomizu proved that every left-invariant Lorentz metric on such a Lie group is also of constant sectional curvature. However, depending on the choice of left invariant Lorentz metric, the sign of the constant sectional curvature may be positive, negative, or zero.

Let G be a n-dimensional non-commutative Lie group belonging to $\mathcal{F}$ (i.e. G is a LCS Lie group), endowed with a left invariant pseudo-Riemannian metric. We take a commutative ideal $\mathfrak{u}$ of codimension 1 and an element $b \notin \mathfrak{u}$ such that $[b, x] = x$ for every $x \in \mathfrak{u}$. Let $\langle , \rangle$ be the pseudo-Riemannian metric in Lie algebra $\mathfrak{g}$ induced by a given left-invariant pseudo-Riemannian metric on G. Let us denote by $(p, n-p)$ the signature $(-, \ldots, -, +, \ldots, +)$ of a metric with p minus signs. Throughout this paper, we consider LCS Lie group G endowed with a left invariant pseudo-Riemannian metric of signature $(p, n-p)$.

Definition 2.1. A pseudo-Riemannian LCS Lie group G is said to be of *type A* if the induced metric is nondegenerate when it is restricted to the commutative ideal $\mathfrak{u}$ of codimension 1 in Lie algebra $\mathfrak{g}$ of G. A pseudo-Riemannian LCS Lie group G is said to be of *type B* if the induced metric is degenerate when it is restricted to the commutative ideal $\mathfrak{u}$ of codimension 1 in Lie algebra $\mathfrak{g}$ of G.

Type A

If $\mathfrak{u}$ is nondegenerate, then there exists a vector b' such that $\langle b', \mathfrak{u}\rangle = 0$ and $\mathfrak{g} = \{b'\} + \mathfrak{u}$(direct sum). Writing $b' = \lambda b + \mathfrak{u}_0$, with some $\lambda \neq 0$ and $\mathfrak{u}_0 \in \mathfrak{u}$. Then we have

$$[b', v] = \lambda v, \forall v \in \mathfrak{u}.$$

Now we may take b'/λ and rename it b. Then we can obtain the following equations:

$$\mathfrak{g} = \mathbb{R}b \oplus \mathfrak{u}, \langle b, \mathfrak{u}\rangle = 0, [b, x] = x, \forall x \in \mathfrak{u}.$$

In this case, a Lie group G is said to be of *type* A_1 if b is timelike; and G is said to be of *type* A_2 if b is spacelike.

type A_1: b is timelike, i.e., $\langle b, b\rangle = -\alpha^2$, where $\alpha > 0$. Set $b = \alpha e_1$ and fix a pseudo-orthonormal basis $\{e_2, e_3, \ldots, e_n\}$ of $\mathfrak{u}$, $\varepsilon_i = -1, 2 \le i \le p$, $\varepsilon_j = 1, p+1 \le j \le n$. Then we have the following identities:

$$\begin{aligned} &[e_1, e_i] = \frac{1}{\alpha} e_i, \quad i = 2, 3, \ldots, n, \quad \langle e_j, e_j\rangle = -1, \quad j = 1, 2, \ldots, p, \\ &\langle e_k, e_k\rangle = 1, \quad k = p+1, \ldots, n. \end{aligned} \tag{2.1}$$

type A_2: b is spacelike, i.e, $\langle b, b\rangle = \alpha^2 > 0$, where $\alpha > 0$. Similarly as above, fix a pseudo-orthonormal basis $\{e_1, e_2, \dots, e_{n-1}\}$ of $\mathfrak{u}$, $\varepsilon_i = -1$, $1 \le i \le p$, $\varepsilon_j = 1$, $p+1 \le j \le n-1$, and set $b = \alpha e_n$. Then we have

$$\begin{aligned}
&[e_n, e_i] = \frac{1}{\alpha} e_i, \quad i = 1, 2, \dots, n-1, \quad \langle e_j, e_j\rangle = -1, \quad j = 1, 2, \dots, p,\\
&\langle e_k, e_k\rangle = 1, \quad k = p+1, \dots, n.
\end{aligned} \tag{2.2}$$

Type B

If $\mathfrak{u}$ is degenerate, we have the following result.

Lemma 2.2. *If $\mathfrak{u}$ is degenerate, $\mathfrak{u}$ contains a lightlike vector c and an $(n-2)$-dimensional subspace $\mathfrak{u}_1$ on which the metric is non-degenerate such that $\mathfrak{u} = \mathbb{R}c + \mathfrak{u}_1$ (direct sum) and $\langle c, \mathfrak{u}_1\rangle = 0$.*

Proof. Since $\mathfrak{u}$ is degenerate, there exists a vector c such that $\langle c, x\rangle = 0$, $\forall x \in \mathfrak{u}$. In particular, c is a lightlike vector. And there is an $(n-2)$-dimensional subspace $\mathfrak{u}_1$ such that $\mathfrak{u} = \mathbb{R}c + \mathfrak{u}_1$ (direct sum) and $\langle c, \mathfrak{u}_1\rangle = 0$. Now we assert that the induced metric on $\mathfrak{u}_1$ is non-degenerate. Otherwise, we can suppose that there are some lightlike vectors $c, d, \dots, q$ such that $\mathfrak{u} = \mathbb{R}c + \mathbb{R}d + \cdots + \mathbb{R}q + \mathfrak{u}_2$ (orthogonal direct-sum), and the induced metric on $\mathfrak{u}_2$ is non-degenerate, then the orthogonal complement $\mathfrak{u}_2^\perp$ is also non-degenerate. However, the subspace $\mathfrak{u}_2^\perp$ which contains $\dim \mathfrak{u}_2^\perp - 1$ linear-independent orthogonal lightlike vectors is degenerate. This is contradiction. So the induced metric on $\mathfrak{u}_1$ is non-degenerate. $\square$

By Lemma 2.2, we know $\mathfrak{u} = \mathbb{R}c + \mathfrak{u}_1$ (direct sum) and $\langle c, \mathfrak{u}_1\rangle = 0$. In the orthogonal complement $\mathfrak{u}_1^\perp$ of $\mathfrak{u}_1$ in $\mathfrak{g}$, we can find a vector b' such that

$$\langle b', b'\rangle = 0, \quad \langle b', c\rangle = -1.$$

Writing $b' = \lambda b + \mathfrak{u}_0$, with some $\lambda \neq 0$ and $\mathfrak{u}_0 \in \mathfrak{u}$. Then we have

$$[b', v] = \lambda v, \forall v \in \mathfrak{u}.$$

Now if we denote b'/λ and λc by b, c, then we have the following identities:

$$\begin{aligned}
&\mathfrak{g} = \mathbb{R}b \oplus \mathbb{R}c \oplus \mathfrak{u}_1, \quad \mathfrak{u} = \mathbb{R}c \oplus \mathfrak{u}_1;\\
&\langle b, b\rangle = \langle c, c\rangle = 0, \quad \langle b, c\rangle = -1, \quad \langle b, \mathfrak{u}_1\rangle = \langle c, \mathfrak{u}_1\rangle = 0;\\
&[b, c] = c, \quad [b, y] = y, \quad \forall y \in \mathfrak{u}_1.
\end{aligned}$$

Fix a pseudo-orthonormal basis $\{e_1, e_2, \dots, e_{n-2}\}$ of $\mathfrak{u}_1$, $\varepsilon_i = -1$, $1 \le i \le p-1$, $\varepsilon_j = 1$, $p \le j \le n-2$, and set

$$e_{n-1} = \frac{\sqrt{2}}{2}(b - c), \quad e_n = \frac{\sqrt{2}}{2}(b + c).$$

Then we have the following identities:

$$\begin{aligned}
&[e_{n-1}, e_i] = [e_n, e_i] = \frac{\sqrt{2}}{2} e_i, \quad i = 1, 2, \dots, n-2,\\
&[e_{n-1}, e_n] = \frac{\sqrt{2}}{2}(e_n - e_{n-1}),\\
&\langle e_i, e_i\rangle = -1, \quad 1 \le i \le p-1,\\
&\langle e_j, e_j\rangle = 1, \quad p \le j \le n-2,\\
&\langle e_{n-1}, e_{n-1}\rangle = 1, \quad \langle e_n, e_n\rangle = -1.
\end{aligned} \tag{2.3}$$

3. *m*-quasi-Einstein metric on pseudo-Riemannian LCS Lie group: *type A*

Consider an n-dimensional connected simply connected pseudo-Riemannian Lie group G of *type* A_1. Using (2.1) and the well-known Koszul formula, one can determine the Levi-Civita connection as follows:

$$\begin{aligned}
&\nabla_{e_i} e_1 = -\frac{1}{\alpha} e_i, \quad 2 \le i \le n, \quad \nabla_{e_j} e_j = \frac{1}{\alpha} e_1, \quad 2 \le j \le p,\\
&\nabla_{e_k} e_k = -\frac{1}{\alpha} e_1, \quad p+1 \le k \le n.
\end{aligned} \tag{3.1}$$

with $\nabla_{e_i} e_j = 0$ in other cases.

Given a left invariant vector field $V = \sum_{i=1}^n K_i e_i$, we have

$$\nabla_{e_1} V = 0, \quad \nabla_{e_j} V = \frac{1}{\alpha}(K_j e_1 - K_1 e_j), \quad 2 \le j \le p,$$

$$\nabla_{e_j} V = -\frac{1}{\alpha}(K_j e_1 + K_1 e_j), \quad p+1 \le j \le n. \tag{3.2}$$

Using the identities $R(e_i, e_j) = \nabla_{[e_i,e_j]} - \nabla_{e_i}\nabla_{e_j} + \nabla_{e_j}\nabla_{e_i}$, we have

$$\begin{aligned}
&R(e_1, e_i)e_1 = -\frac{1}{\alpha^2}e_i, \quad 2 \le i \le n, \quad R(e_1, e_j)e_j = \frac{1}{\alpha^2}e_1, \quad 2 \le j \le p,\\
&R(e_1, e_k)e_k = -\frac{1}{\alpha^2}e_1, \quad p+1 \le k \le n,\\
&R(e_j, e_k)e_j = -\frac{1}{\alpha^2}e_k, \quad 2 \le k \le n, \quad 2 \le j \le p,\\
&R(e_j, e_k)e_j = \frac{1}{\alpha^2}e_k, \quad 2 \le k \le n, \quad p+1 \le j \le n.
\end{aligned}$$

From the above equations and the section curvature formula

$$K(X, Y) = \frac{\langle R(X,\ Y)X, Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2},$$

it is easy to see that in this case, it has constant section curvature $\frac{1}{\alpha^2}$. Then applying the Ricci tensor formula $\rho(X, Y) = \sum_{i=1}^{n} \varepsilon_i g(R(X, e_i)Y, e_i)$, we get non-vanishing equations:

$$\rho_{jj} = -\frac{n-1}{\alpha^2}, 1 \le j \le p, \quad \rho_{kk} = \frac{n-1}{\alpha^2}, p+1 \le k \le n. \tag{3.3}$$

By the identity $(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$, we have

$$\begin{aligned}
&(L_X g)(e_i, e_i) = \frac{2}{\alpha}K_1, 2 \le i \le p, \ \ (L_X g)(e_j, e_j) = -\frac{2}{\alpha}K_1, p+1 \le j \le n,\\
&(L_X g)(e_i, e_1) = (L_X g)(e_1, e_i) = -\frac{1}{\alpha}K_i, 2 \le i \le p,\\
&(L_X g)(e_j, e_1) = (L_X g)(e_1, e_j) = \frac{1}{\alpha}K_j, p+1 \le j \le n.
\end{aligned} \tag{3.4}$$

with $(L_X g)(e_i, e_j) = 0$ in other cases.

A vector field V is called a *geodesic* vector field if $\nabla_V V = 0$, and is called a *Killing* vector field if $L_V g = 0$, where L denotes the Lie derivative. Recall that X is a Killing vector field if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathfrak{X}(M)$. A vector field V is called a *parallel* vector field if $\nabla_X V = 0$ for all $X \in \mathfrak{X}(M)$. It is obvious that parallel vector fields are both geodesic vector fields and Killing vector fields.

By direct calculation, we obtain

$$\nabla_V V = \frac{1}{\alpha}\sum_{j=2}^{p} K_j(K_j e_1 - K_1 e_j) - \frac{1}{\alpha}\sum_{j=p+1}^{n} K_j(K_j e_1 + K_1 e_j).$$

And we have the following.

Proposition 3.1. *A left invariant vector field $V = \sum_{i=1}^{n} K_i e_i$ on the n-dimensional pseudo-Riemannian LCS Lie group G of type A_1 is a geodesic vector field if and only if*

(i) $K_1 = 0$ *and V is lightlike vector field;*
(ii) $V = K_1 e_1$.

On the other hand, a left invariant vector field $V \neq 0$ on G is neither parallel vector field nor Killing vector field.

Remark 3.2. When $p = 1$, it is a Lorentzian metric. In this case, a left invariant vector field $V = \sum_{i=1}^{n} K_i e_i$ is a geodesic vector field if and only if $V = K_1 e_1$.

And notice $X^* \otimes X^*(e_i, e_j) = \varepsilon_i \varepsilon_j K_i K_j$, where $\varepsilon_i = -1$, $1 \le i \le p$, $\varepsilon_j = 1$, $p+1 \le j \le n$. Now we can prove

Theorem 3.3. *Any left-invariant pseudo-Riemannian metric on the n-dimensional LCS Lie group G of type A_1 is a trivial Ricci soliton but a shrinking non-trivial m-quasi-Einstein metric.*

Proof. By (3.3), (3.4) and (1.3), we have

$$\begin{cases} \dfrac{n-1}{\alpha^2} + \dfrac{1}{m}K_1^2 = \lambda, \\ \dfrac{K_i}{2\alpha} + \dfrac{1}{m}K_iK_1 = 0, \quad i = 2, 3, \dots, n, \\ -\dfrac{1}{m}K_iK_j = 0, \quad i \neq j, \quad i, j = 2, 3, \dots, n, \\ -\dfrac{n-1}{\alpha^2} + \dfrac{1}{\alpha}K_1 - \dfrac{1}{m}K_i^2 = -\lambda, \quad 2 \leq i \leq p, \\ \dfrac{n-1}{\alpha^2} - \dfrac{1}{\alpha}K_1 - \dfrac{1}{m}K_i^2 = \lambda, \quad p+1 \leq i \leq n. \end{cases}$$

If $0 < m < \infty$, from the third and fourth equations, we have $K_i = 0$, $2 \leq i \leq p$. From the third and fifth equations, we have $K_j = 0$, $p+1 \leq j \leq n$. Then from the first and fifth equations, we obtain $K_1 = -\frac{m}{\alpha}$. So we have

$$X = -\frac{m}{\alpha}e_n, \quad \lambda = \frac{m+n-1}{\alpha^2} > 0$$

If $m = \infty$, from the second equation, we get: $K_i = 0$, $2 \leq i \leq n$, then from the first and fifth equations, we have $K_1 = 0$, i.e. it is trivial. Hence, theorem holds. □

Next we consider n-dimensional simply connected Lie groups G of *type* A_2. By (2.2), we have:

$$\nabla_{e_i}e_n = -\frac{1}{\alpha}e_i, \quad i = 1, 2, \dots, n-1,$$
$$\nabla_{e_i}e_i = -\frac{1}{\alpha}e_n,\ 1 \leq i \leq p, \quad \nabla_{e_j}e_j = \frac{1}{\alpha}e_n, \quad p+1 \leq j \leq n-1, \tag{3.5}$$

with $\nabla_{e_i}e_j = 0$ in other cases.

For a left invariant vector field $V = \sum_{i=1}^n K_ie_i$, we have

$$\nabla_{e_j}V = -\frac{1}{\alpha}(K_je_n + K_ne_j), \quad 1 \leq j \leq p,$$
$$\nabla_{e_j}V = \frac{1}{\alpha}(K_je_n - K_ne_j), \quad p+1 \leq j \leq n-1, \quad \nabla_{e_n}V = 0. \tag{3.6}$$

By some calculations, we have:

$$R(e_i, e_j)e_i = \frac{1}{\alpha^2}e_j, \quad 1 \leq i \leq p, \quad 1 \leq j \leq n-1,$$
$$R(e_j, e_k)e_j = -\frac{1}{\alpha^2}e_k, \quad p+1 \leq j \leq n-1, \quad 1 \leq k \leq n-1,$$
$$R(e_n, e_i)e_n = -\frac{1}{\alpha^2}e_i, \quad 1 \leq i \leq n-1, \quad R(e_n, e_j)e_j = -\frac{1}{\alpha^2}e_n, \quad 1 \leq j \leq p,$$
$$R(e_n, e_k)e_k = \frac{1}{\alpha^2}e_n, \quad p+1 \leq k \leq n-1.$$

From the above equations, it is easy to see that in this case, it has constant section curvature $-\frac{1}{\alpha^2}$. And we have

$$\rho_{jj} = \frac{n-1}{\alpha^2},\ 1 \leq j \leq p, \quad \rho_{kk} = -\frac{n-1}{\alpha^2},\ p+1 \leq k \leq n, \tag{3.7}$$

with $\rho_{ij} = 0$ in other cases.

By the identity $(L_Xg)(Y, Z) = g(\nabla_YX, Z) + g(Y, \nabla_ZX)$, we have

$$(L_Xg)(e_i, e_i) = \frac{2}{\alpha}K_n,\ 1 \leq i \leq p, \quad (L_Xg)(e_j, e_j) = -\frac{2}{\alpha}K_n,\ p+1 \leq j \leq n-1,$$
$$(L_Xg)(e_i, e_n) = (L_Xg)(e_n, e_i) = -\frac{1}{\alpha}K_i,\ 1 \leq i \leq p,$$
$$(L_Xg)(e_j, e_n) = (L_Xg)(e_n, e_j) = \frac{1}{\alpha}K_j,\ p+1 \leq j \leq n-1. \tag{3.8}$$

with $(L_Xg)(e_i, e_j) = 0$ in other cases.

The proof of the following result is easy and will be omitted.

Proposition 3.4. *A left invariant vector field $V = \sum_{i=1}^n K_ie_i$ on the n-dimensional pseudo-Riemannian LCS Lie group G of type A_2 is a geodesic vector field if and only if one of the following conditions holds:*

(i) $K_n = 0$ *and* V *is lightlike vector field;*
(ii) $V = K_n e_n$.

Moreover, a left invariant vector field $V \neq 0$ *is neither a parallel vector field nor a Killing vector field.*

Remark 3.5. When $p = 0$, i.e., in the Riemannian case, a left invariant vector field $V = \sum_{i=1}^{n} K_i e_i$ is a geodesic vector field if and only if $V = K_n e_n$.

And notice $X^* \otimes X^*(e_i, e_j) = \varepsilon_i \varepsilon_j K_i K_j$, where $\varepsilon_i = -1, 1 \le i \le p, \varepsilon_j = 1, p+1 \le j \le n$. Now we have the following theorem

Theorem 3.6. *Any left-invariant pseudo-Riemannian metric on the n-dimensional LCS Lie group G of type* A_2 *is a trivial Ricci soliton but an expanding non-trivial m-quasi-Einstein metric.*

Proof. By (3.7), (3.8) and (1.3), we have

$$\begin{cases} \dfrac{n-1}{\alpha^2} + \dfrac{1}{\alpha} K_n - \dfrac{1}{m} K_i^2 = -\lambda, \quad 1 \le i \le p, \\ -\dfrac{n-1}{\alpha^2} - \dfrac{1}{\alpha} K_n - \dfrac{1}{m} K_i^2 = \lambda, \quad p+1 \le i \le n-1, \\ -\dfrac{n-1}{\alpha^2} - \dfrac{1}{m} K_n^2 = \lambda, \\ \frac{K_i}{2\alpha} - \frac{1}{m} K_i K_n = 0, \quad 1 \le i \le n-1, \\ -\dfrac{1}{m} K_i K_j = 0, \quad i \neq j, \quad 1 \le i, j \le n-1. \end{cases}$$

If $0 < m < \infty$, from the first and second equations, we have $K_i = 0, 1 \le i \le n-1$. Then from the second and third equations, we obtain $K_n = \frac{m}{\alpha}$. So we have

$$X = \frac{m}{\alpha} e_n, \quad \lambda = -\frac{m+n-1}{\alpha^2} < 0$$

If $m = \infty$, from the fourth equation, we get: $K_i = 0, 1 \le i \le n-1$, then from the second and third equations, we have $K_n = 0$, i.e. it is trivial. Hence, theorem holds. □

4. *m*-quasi-Einstein metric on pseudo-Riemannian LCS Lie group: *type B*

Consider an n-dimensional simply connected pseudo-Riemannian Lie group G of the *type B*. From 2.3, we obtain

$$\begin{aligned} &\nabla_{e_i} e_i = \frac{\sqrt{2}}{2}(e_n - e_{n-1}), \quad 1 \le i \le p-1, \\ &\nabla_{e_j} e_j = \frac{\sqrt{2}}{2}(e_{n-1} - e_n), \quad p \le j \le n-2, \\ &\nabla_{e_j} e_{n-1} = \nabla_{e_j} e_n = -\frac{\sqrt{2}}{2} e_j, \quad 1 \le j \le n-2, \\ &\nabla_{e_{n-1}} e_{n-1} = \nabla_{e_n} e_{n-1} = -\frac{\sqrt{2}}{2} e_n, \quad \nabla_{e_{n-1}} e_n = \nabla_{e_n} e_n = -\frac{\sqrt{2}}{2} e_{n-1}, \end{aligned} \tag{4.1}$$

with $\nabla_{e_i} e_j = 0$ in all other cases. For an arbitrary left invariant vector field $V = \sum_{i=1}^{n} K_i e_i$, we have

$$\begin{aligned} &\nabla_{e_i} V = \frac{\sqrt{2}}{2} K_i (e_n - e_{n-1}) - \frac{\sqrt{2}}{2}(K_{n-1} + K_n) e_i, \quad 1 \le i \le p-1, \\ &\nabla_{e_j} V = \frac{\sqrt{2}}{2} K_j (e_{n-1} - e_n) - \frac{\sqrt{2}}{2}(K_{n-1} + K_n) e_j, \quad p \le j \le n-2, \\ &\nabla_{e_{n-1}} V = \nabla_{e_n} V = -\frac{\sqrt{2}}{2}(K_{n-1} e_n + K_n e_{n-1}). \end{aligned} \tag{4.2}$$

It is easy to see G is flat in this case, hence G is Ricci flat. And by the identity $(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$, we have

$$\begin{aligned} &(L_X g)(e_i, e_i) = \sqrt{2}(K_{n-1} + K_n), \quad 1 \le i \le p-1, \\ &(L_X g)(e_j, e_j) = -\sqrt{2}(K_{n-1} + K_n), \quad p \le j \le n-2, \\ &(L_X g)(e_{n-1}, e_{n-1}) = -\sqrt{2} K_n, \quad (L_X g)(e_n, e_n) = \sqrt{2} K_{n-1}, \end{aligned}$$

$$
\begin{aligned}
&(L_Xg)(e_n, e_{n-1}) = (L_Xg)(e_{n-1}, e_n) = \frac{\sqrt{2}}{2}(K_{n-1} - K_n),\\
&(L_Xg)(e_i, e_{n-1}) = (L_Xg)(e_{n-1}, e_i) = -\frac{\sqrt{2}}{2}K_i, \quad 1 \le i \le p-1,\\
&(L_Xg)(e_j, e_{n-1}) = (L_Xg)(e_{n-1}, e_j) = \frac{\sqrt{2}}{2}K_j, \quad p \le j \le n-2,\\
&(L_Xg)(e_i, e_n) = (L_Xg)(e_n, e_i) = -\frac{\sqrt{2}}{2}K_i, \quad 1 \le i \le p-1,\\
&(L_Xg)(e_j, e_n) = (L_Xg)(e_n, e_j) = \frac{\sqrt{2}}{2}K_j, \quad p \le j \le n-2.
\end{aligned}
\tag{4.3}
$$

with $(L_Xg)(e_i, e_j) = 0$ in other cases. i.e. Regarding to geodesic and Killing vector fields, we have the following.

Proposition 4.1. *A left invariant vector field $V = \sum_{i=1}^{n} K_i e_i$ on the n-dimensional pseudo-Riemannian LCS Lie group G of type B is a geodesic vector field if and only if $K_{n-1} + K_n = 0$ and V is lightlike vector field. A left invariant vector field $V \neq 0$ is neither a parallel vector field nor a Killing vector field.*

And notice $X^* \otimes X^*(e_i, e_j) = \varepsilon_i \varepsilon_j K_i K_j$, where $\varepsilon_i = -1$, $1 \le i \le p-1$, $\varepsilon_j = 1$, $p \le j \le n-1$, $\varepsilon_n = -1$. Now we can prove

Theorem 4.2. *Any left-invariant pseudo-Riemannian metric on the n-dimensional LCS Lie group G of type B is a trivial Ricci soliton but a steady non-trivial m-quasi-Einstein metric.*

Proof. By (4.3) and (1.3), we have

$$
\begin{cases}
\frac{\sqrt{2}}{2}(K_{n-1} + K_n) - \frac{1}{m}K_i^2 = -\lambda, \quad 1 \le i \le p-1,\\
-\frac{\sqrt{2}}{2}(K_{n-1} + K_n) - \frac{1}{m}K_j^2 = \lambda, \quad p \le j \le n-2,\\
-\frac{\sqrt{2}}{2}K_n - \frac{1}{m}K_{n-1}^2 = \lambda,\\
\frac{\sqrt{2}}{2}K_{n-1} - \frac{1}{m}K_n^2 = -\lambda\\
-\frac{1}{m}K_iK_j = 0, \quad i \neq j, \quad 1 \le i, j \le n-2,\\
-\frac{\sqrt{2}}{4}K_i + \frac{1}{m}K_iK_{n-1} = 0, \quad 1 \le i \le p-1,\\
-\frac{\sqrt{2}}{4}K_j - \frac{1}{m}K_jK_n = 0, \quad 1 \le j \le p-1,\\
\frac{\sqrt{2}}{4}K_i - \frac{1}{m}K_iK_{n-1} = 0, \quad p \le i \le n-2,\\
\frac{\sqrt{2}}{4}K_j + \frac{1}{m}K_jK_n = 0, \quad p \le j \le n-2,\\
\frac{\sqrt{2}}{4}(K_{n-1} - K_n) + \frac{1}{m}K_{n-1}K_n = 0.
\end{cases}
$$

If $0 < m < \infty$, from the first and second equations, we can obtain: $K_i = 0$, $1 \le i \le n-2$. Then the above system of equations can reduce to

$$
\begin{cases}
-\frac{\sqrt{2}}{2}(K_{n-1} + K_n) = \lambda,\\
-\frac{\sqrt{2}}{2}K_n - \frac{1}{m}K_{n-1}^2 = \lambda,\\
\frac{\sqrt{2}}{2}K_{n-1} - \frac{1}{m}K_n^2 = -\lambda\\
\frac{\sqrt{2}}{4}(K_{n-1} - K_n) + \frac{1}{m}K_{n-1}K_n = 0.
\end{cases}
$$

Then from the first and second equations, we have

$$\frac{1}{m}K_{n-1}^2 = \frac{\sqrt{2}}{2}K_{n-1}.$$

If $K_{n-1} = 0$, from the last equation, we have $K_n = 0$. It is a trivial solution. So we have $K_{n-1} = \frac{\sqrt{2}}{2}m$. Similarly, we can obtain $K_n = -\frac{\sqrt{2}}{2}m$. So

$$X = \frac{\sqrt{2}}{2}m(e_{n-1} - e_n), \quad \lambda = 0.$$

If $m = \infty$, it is easy to see $K_i = 0$, $1 \le i \le n-2$. And from the last equation, we have $K_{n-1} = K_n$. From the third and fourth equations, we get: $\lambda = -\frac{\sqrt{2}}{2}K_n = -\frac{\sqrt{2}}{2}K_{n-1}$. Then from the first equation, we obtain $\lambda = -\sqrt{2}K_n$. So $K_{n-1} = K_n = 0$, it is trivial solution. Hence, theorem holds. □

Combining Theorems 3.3, 3.6 and 4.2, we obtain the following

Proposition 4.3. *Any left-invariant pseudo-Riemannian metric on the n-dimensional LCS Lie group G is a trivial Ricci soliton but a non-trivial m-quasi-Einstein metric.*

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