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On anticommutative algebras for which $[R_a, R_b]$ is a derivation

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ABSTRACT

We study anticommutative algebras with the property that commutator of any two multiplications is a derivation.

0. Introduction

Consider the following property of an (nonassociative) algebra: the commutator of two (say, right) multiplications is a derivation of an algebra. Commutative algebras with this property were studied in the literature under the names "Lie triple algebras" and "almost Jordan algebras": see [5,6,10,12,20], and references therein. Jordan algebras are properly contained in this class.

It appears only natural to consider then the anticommutative analog: that is, anticommutative algebras in which the commutator of any two multiplications is a derivation. Such algebras are dubbed, for no better term, as CD algebras. It turns out that the variety of CD algebras lies between Lie algebras and binary Lie algebras, both inclusions being strict, so it seems a class of algebras worth studying.

Earlier, low-dimensional nilpotent CD algebras were classified in [2,3,13], and [14], and here we continue the study of CD algebras.

1. Notation, conventions, preliminary facts

We consider (nonassociative) algebras over the ground field K, which is assumed to be arbitrary of characteristic $\neq 2, 3$. *All algebras and varieties of algebras are assumed to be anticommutative, without explicitly mentioning it*. Thus, left and right multiplications differ only by sign, left and right ideals coincide, etc.

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The multiplication in algebras will be always denoted by juxtaposition, with the exception of Section 5, where we deal with Lie algebras with multiplication denoted customarily by brackets $[\cdot,\cdot]$.

Let A be an algebra. For an element $a \in A$, $R_a : A \to A$ denotes the linear map of right multiplication on a: $R_a(x) = xa$. $\mathrm{Der}(A)$ denotes the Lie algebra of derivations of A, and $\mathfrak{gl}(A)$ denotes the Lie algebra of all linear maps $A \to A$, subject to the usual commutator $[f,g] = g \circ f - f \circ g$ (we always assume action from the right, so $g \circ f(a) = g(f(a))$). For any algebra A, any $a \in A$, and any $D \in \mathrm{Der}(A)$, the following identity in $\mathfrak{gl}(A)$ holds:

$$[D, R_a] = R_{D(a)}. \tag{1}$$

$Z(A)$ denotes the center of A, i.e., the set of elements $z \in A$ such that $zA = Az = 0$; this is obviously an ideal of A.

The *Jacobiator* $J(x,y,z)$ of elements $x,y,z \in A$ is defined as

$$J(x,y,z) = (xy)z + (zx)y + (yz)x.$$

Thus, an algebra is a Lie algebra if and only if the Jacobiator on it is identically zero.

An algebra A is called a *CD algebra* if it satisfies the property that for any $a, b \in A$, the commutator $[R_a, R_b]$ is a derivation of A. This condition can be written as a homogeneous identity of degree 4 comprising 6 monomials:

$$((xy)a)b - ((xy)b)a - ((xa)b)y + ((xb)a)y + ((ya)b)x - ((yb)a)x = 0. \tag{2}$$

2. Identities and non-identities

Let us compare the variety of CD algebras with the other known varieties: Lie, binary Lie, Malcev, and Sagle (for binary Lie and Malcev algebras, see, for example, [3,4,8,17,18] and references therein, and for Sagle algebras, see [7] and references therein). Let us briefly recall their definitions. An algebra is called *binary Lie* if it satisfies the identity

$$J(xy,x,y) = 0.$$

Taking into account anticommutativity, the latter identity is equivalent to

$$((xy)x)y = ((xy)y)x. \tag{3}$$

This is also equivalent to the condition that any 2-generated subalgebra is Lie.

An algebra is called *Malcev*, if it satisfies the identity

$$J(x,y,xz) = J(x,y,z)x;$$

and an algebra is called *Sagle*, if it satisfies the identity

$$J(x,y,z)w = J(w,z,xy) + J(w,y,zx) + J(w,x,yz). \tag{4}$$

There are the following well known strict inclusions between these varieties:

$$\begin{array}{lll} & & \text{Malcev} \;\subset\; \text{Binary Lie} \\ & \rotatebox{45}{$\subset$} & \\ \text{Lie} & & \\ & \rotatebox{-45}{$\subset$} & \\ & & \text{Sagle} \end{array}$$

How CD algebras fit into the picture? We are going to prove that

$$\begin{array}{lll} & & \text{Binary Lie} \\ & \rotatebox{45}{$\subset$} & \\ \text{Lie} \;\subset\; \text{Malcev} \cap \text{Sagle} \;\subset\; \text{CD} & & \\ & \rotatebox{-45}{$\subset$} & \\ & & \text{Almost Lie} \end{array} \tag{5}$$

where all inclusions are, again, strict, and, moreover,

$$\begin{gathered} \text{Malcev} \cap \text{CD} = \text{Sagle} \cap \text{CD} = \text{Malcev} \cap \text{Sagle} \\ \text{Binary Lie} \cap \text{Almost Lie} = \text{CD} \end{gathered} \tag{6}$$

(the graphically inclined reader may wish to draw Venn diagrams representing all this).

Here the variety of "Almost Lie" (anticommutative) algebras is defined by the identity[1]

$$J(x,y,z)w = 0. \tag{7}$$

1 There are other meanings of "almost Lie" one can encounter in the literature, but since neither of them seems to be widespread and accepted, and since almost Lie algebras in our sense are already mentioned in a number of related papers – for example [13] and [14] – we choose to keep this terminology.

Lemma 1. *The variety* Malcev ∩ Sagle *coincides with the variety of almost Lie algebras which additionally satisfy the identity*

$$J(x, y, zw) = 0. \tag{8}$$

Proof. As proved in [17, Lemma 2.10], any Malcev algebra satisfies the identity

$$3J(y, z, wx) = J(x, y, z)w - J(y, z, w)x - 2J(z, w, x)y + 2J(w, x, y)z. \tag{9}$$

Expressing via this identity all the summands on the right-hand side of (4), which are of the form $J(\bullet, \bullet, \bullet\bullet)$, through terms of the form $J(\bullet, \bullet, \bullet)\bullet$, we get the identity (7), i.e., any algebra which is both Malcev and Sagle, is almost Lie. Then (9) implies that any algebra which is both Malcev and Sagle, satisfies also the identity (8).

Conversely, any algebra satisfying both (7) and (8) is, obviously, both Malcev and Sagle. □

Lemma 2. *Binary Lie* ∩ Almost Lie ⊆ CD.

Proof. As proved in [18, §3], any binary Lie algebra satisfies the identity

$$3(J(wx, y, z) + J(yz, w, x)) = -J(x, y, z)w + J(y, z, w)x - J(z, w, x)y + J(w, x, y)z.$$

If the algebra is simultaneously almost Lie, the right-hand side of this identity vanishes, and we are left with the identity

$$J(wx, y, z) + J(yz, w, x) = 0. \tag{10}$$

Using the last identity and anticommutativity, we get

$$\begin{aligned} &((xz)w)y + ((yw)z)x \\ &= J(xz, w, y) + ((xz)y)w - (wy)(xz) + J(yw, z, x) + ((yw)x)z - (zx)(yw) \\ &= ((xz)y)w + ((yw)x)z \end{aligned} \tag{11}$$

for any elements x, y, z, w of an algebra which is simultaneously binary Lie and almost Lie.

Now transform the left-hand side of (2):

$$\begin{aligned} &((xy)a)b - ((xy)b)a - ((xa)b)y + ((xb)a)y + ((ya)b)x - ((yb)a)x \\ &= ((xy)a)b - ((xy)b)a - ((xa)y)b - ((yb)x)a + ((xb)y)a + ((ya)x)b \\ &= J(x, y, a)b - J(x, y, b)a = 0, \end{aligned}$$

where the first equality is obtained by applying the identity (11) twice, to the pairs formed by the 3rd and 6th summands, and by the 4th and 5th summands. □

Proposition 1. *All inclusions in the diagram (5) do indeed take place.*

Proof. "Lie ⊂ Malcev ∩ Sagle": Obvious. To see that the inclusion is strict (we do not need this in what follows, but doing this for completeness), consider the free anticommutative algebra freely generated by elements x, y, z, such that any product of any 4 elements vanishes. This 9-dimensional nilpotent algebra satisfies any identity of degree 4, in particular, it is both Malcev and Sagle, but, obviously, not Lie, as $J(x, y, z) \neq 0$.

"Malcev ∩ Sagle ⊂ CD": By Lemma 1,

$$\text{Malcev} \cap \text{Sagle} \subseteq \text{Binary Lie} \cap \text{Almost Lie},$$

and then apply Lemma 2.

To show that the inclusion is strict, on can take, for example, the one-parametric family of nilpotent algebras $\mathbf{B}_{6,1}^{\alpha}$ from [3] (see Theorems 3 and 10 there). These are 6-dimensional algebras with the basis $\{e_i\}_{i=1,\dots,6}$ and multiplication table

$$e_1e_2 = e_4, \quad e_1e_3 = e_5, \quad e_2e_3 = \alpha e_6, \quad e_4e_5 = e_6,$$

where $\alpha \in K$; these algebras are CD, but not Malcev.

"CD ⊂ Binary Lie": As noted in [3], substituting $a = x$ and $b = y$ in (2) yields (3). It is not difficult to find examples of binary Lie algebras which are not CD; for nilpotent such algebras, see, again, [3]. Another nice example is a 7-dimensional simple Malcev algebra (what follows from Proposition 3 below).

"CD ⊂ Almost Lie": Let A be a CD algebra. Write the Jacobi identity for elements of $\mathfrak{gl}(A)$:

$$[[R_x, R_y], R_z] + [[R_z, R_x], R_y] + [[R_y, R_z], R_x] = 0 \tag{12}$$

for any $a, b, c \in A$.

Write the identity (1) for the derivation $[R_x, R_y]$:

$$[[R_x, R_y], R_z] = R_{[R_x,R_y](z)}$$

for any $x, y, z \in A$. Taking the last identity into account, the identity (12) can be rewritten as

$$[R_x, R_y](z) + [R_z, R_x](y) + [R_y, R_z](x) \in Z(A).$$

Due to anticommutativity of A and the fact that the characteristic of the ground field is different from 2, the left-hand side in the last inclusion is nothing but the Jacobiator $J(x, y, z)$, whence the statement.

Examples of almost Lie algebras which are not CD can be constructed by considering central extensions of Lie algebras in the suitable variety; this is deferred to Section 5.[2] □

Proposition 2. *All equalities in (6) do indeed take place.*

Proof. "Binary Lie ∩ Almost Lie = CD": The inclusion "CD ⊆ Binary Lie ∩ Almost Lie" follows from (5), and the inverse inclusion is proved in Lemma 2.

"Malcev ∩ CD = Malcev ∩ Sagle": By Lemma 1, the variety Malcev ∩ Sagle coincides with the variety of almost Lie algebras which, additionally, satisfy the identity (8).

On the other hand, by just proved,

$$\text{Malcev} \cap \text{CD} = \text{Malcev} \cap \text{Binary Lie} \cap \text{Almost Lie} = \text{Malcev} \cap \text{Almost Lie}.$$

But by (9), any Malcev algebra which is simultaneously almost Lie, satisfies (8), what shows that

$$\text{Malcev} \cap \text{Almost Lie} = \text{Malcev} \cap \text{Sagle}.$$

"Sagle ∩ CD = Malcev ∩ Sagle": The inclusion Malcev ∩ Sagle ⊆ Sagle ∩ CD follows from (5), so let us prove the inverse inclusion.

By the already proved, we have

$$\text{Sagle} \cap \text{CD} = \text{Sagle} \cap \text{Binary Lie} \cap \text{Almost Lie}.$$

The algebra which is simultaneously binary Lie and almost Lie, satisfies (10), and by (4), the algebra which is simultaneously Sagle and almost Lie, satisfies the identity

$$J(w, z, xy) + J(w, y, zx) + J(w, x, yz) = 0.$$

Permuting in the last identity z and w, and using (10), we get

$$-J(w, z, xy) + J(w, x, yz) + J(w, y, zx) = 0.$$

The last two identities yield the identity (8), and thus, by Lemma 1,

$$\text{Sagle} \cap \text{Binary Lie} \cap \text{Almost Lie} \subseteq \text{Malcev} \cap \text{Sagle}. \quad \square$$

Perhaps, the most important among all these multiple relations is the inclusion

$$\text{CD} \subset \text{Almost Lie} \tag{13}$$

which shows that, after all, CD algebras are not that far from the Lie ones.

An immediate corollary of this inclusion is

Proposition 3. *For any CD algebra A, the quotient $A/Z(A)$ is a Lie algebra. In particular, any centerless (and, in particular, simple) CD algebra is a Lie algebra.*

Thus, CD algebras are, essentially, central extensions, in the suitable variety, of Lie algebras. As central extensions should be described by second degree cohomology, this suggests that there should be a "CD cohomology", extending the usual Chevalley–Eilenberg cohomology, responsible for such central extensions. And indeed, such cohomology is constructed in Section 4.

It is natural to ask whether any simple binary Lie algebra is Malcev (and hence is either Lie, or is a 7-dimensional simple Malcev algebra). For finite-dimensional algebras over a field of characteristic zero the question was answered in affirmative in [8], while the cases of positive characteristic, and of infinite-dimensional algebras remain open (see, e.g., [1, Problems 2.33 and 3.87] and [4, p. 263]). Proposition 3 shows that in the narrower class of CD algebras the answer is also affirmative.

Note also that Proposition 3 implies that CD non-Lie algebras of dimension ≤ 5 listed in [2] actually exhaust *all* CD non-Lie algebras in those dimensions.

2 An alternative method would be to use Albert [22] to construct explicitly the multiplication table of a suitable finite-dimensional homomorphic image of a free almost Lie algebra, and then verify in some other general-purpose computer algebra system like GAP, that this homomorphic image is not CD. A similar procedure – for another set of identities – is described more thoroughly in [21, §4].

3. Further relations with Lie algebras

Let A be a CD algebra. Consider the subspace $R(A)$, spanned by all maps R_a, $a \in A$, and the subalgebra $\mathrm{Der}(A)$ in the Lie algebra $\mathfrak{gl}(A)$ of all linear maps on A (subject to the usual commutator of linear maps), and consider their formal direct sum $R(A) \oplus \mathrm{Der}(A)$ ("formal", as they may intersect, so the sum $R(A) + \mathrm{Der}(A)$, considered as the linear subspace of $\mathfrak{gl}(A)$, is not necessarily direct; more on that below). As A is a CD algebra, we have $[R(A), R(A)] \subseteq \mathrm{Der}(A)$. Moreover, because of (1), it holds $[\mathrm{Der}(A), R(A)] \subseteq R(A)$. Thus $R(A) \oplus \mathrm{Der}(A)$ is a Lie algebra with respect to the usual commutator, actually a semidirect sum with $\mathrm{Der}(A)$ acting on $R(A)$.

This construction is completely analogous to those in the commutative case (cf., e.g., [5, Definition II.1.4]), and similar to the construction of the structure algebra of a Jordan algebra used in the Kantor–Koecher–Tits construction (cf., e.g., [11, Chapter VIII, §4]); the construction of the holomorph of a Lie algebra (i.e., the semidirect sum $L \oplus \mathrm{Der}(L)$ for a Lie algebra L) is somewhat similar in spirit, but different, as in the holomorph we have $[L, L] \subseteq L$.

This construction can be modified in several ways. For example, instead of the linear space $R(A)$ we may consider the Lie multiplication algebra $M(A)$ (again subject to the commutator). As $M(A)$ is generated by $R(A)$, the commutation relations in the Lie algebra $R(A) \oplus \mathrm{Der}(A)$ can serve as defining relations in the Lie algebra $M(A) \oplus \mathrm{Der}(A)$.

Another possibility is to consider not formal, but "real" direct sum, i.e., the Lie subalgebra of $\mathfrak{gl}(A)$, spanned by $R(A)$ and $\mathrm{Der}(A)$. To consider this variant more thoroughly, define the *Lie center* of A, denoted by $LZ(A)$, as the set of all elements $z \in A$ such that $J(a, b, z) = 0$ for any $a, b \in A$. Obviously, the Lie center is always a vector subspace of A, and $LZ(A) = A$ if and only if A is a Lie algebra. In a sense, it serves as a measure of "non-Lieness" of an algebra.

In what follows, it will be convenient to identify elements of A with the corresponding multiplications in $R(A)$, up to the center. Namely, the kernel of the linear map $A \to R(A)$, $a \mapsto R_a$ coincides with $Z(A)$; hence we have the isomorphism of vector spaces

$$A/Z(A) \xrightarrow{\sim} R(A). \tag{14}$$

Due to Proposition 3, this is also an isomorphism of Lie algebras.

Lemma 3. *For any CD algebra A, there is isomorphism of Lie algebras*

$$LZ(A)/Z(A) \simeq R(A) \cap \mathrm{Der}(A).$$

Proof. It is obvious that $Z(A) \subseteq LZ(A)$. The condition $z \in LZ(A)$ is equivalent, taking into account the anticommutativity of A, to the condition $R_z \in \mathrm{Der}(A)$. Hence the image of $LZ(A)$ under the isomorphism (14) coincides with $R(A) \cap \mathrm{Der}(A)$. □

Lemma 4. *For any CD algebra A, $LZ(A)$ is an ideal of A.*

Proof. Let $z \in LZ(A)$, and $x \in A$. By Lemma 3, we have $R_z \in \mathrm{Der}(A)$, and then by (1) we have $[R_z, R_x] = R_{R_z(x)} = R_{xz}$. Then, since A is a CD algebra, $R_{xz} \in \mathrm{Der}(A)$, and hence, again by Lemma 3, $xz \in LZ(A)$. □

Thus, identifying $R(A)$ with $A/Z(A)$ via isomorphism (14), the intersection between $R(A)$ and $\mathrm{Der}(A)$ is identified with $LZ(A)/Z(A)$ by Lemma 4, and (generally, non-direct) sum $R(A) + \mathrm{Der}(A)$ can be identified with the direct sum $A/LZ(A) \oplus \mathrm{Der}(A)$, the Lie subalgebra of $\mathfrak{gl}(A)$.

4. "Naive" cohomology

The goal of this section is to construct cohomology theory of CD algebras. The standard approach to construct cohomology in a variety of algebras is an operadic one: provided that the corresponding operad $\mathcal{P}$ is quadratic and Koszul, there is a small explicit cochain complex built out of the Koszul dual cooperad $\mathcal{P}^{!^{\text{c}}}$. However, this works only for quadratic Koszul operads. The operad defined by the identity (2) is cubic. The standard trick, employed, for example, in the case of the Jordan operad, is to pass to triple systems; the corresponding category of triple systems should be equivalent (say, in representation-theoretic sense) to the initial category of binary algebras, and the corresponding operad will be (ternary) quadratic. However, it is not immediately clear which triple systems should correspond to CD algebras, and whether the corresponding operad will be Koszul (we believe it will be not).

Thus we rely on the "naive" approach to cohomology. Under this, we mean an attempt to construct (the beginning of) the corresponding cochain complex by utilizing the low-degree structural interpretations of cohomology in the given variety: derivations, central extension, deformations, etc. We take cohomology of Lie algebras as a model.

For that, we need first to define what is a module over a CD algebra is. We follow a nowadays standard approach which goes back to Eilenberg (see, for example, [11, Chapter II, §5]). Namely, for an algebra A in a given variety, a vector space M with an A-action on it, is declared a module over A, if the semidirect sum $A \oplus M$, where multiplication between elements of A and M is determined by the given action, and multiplication on M is zero, belongs to the same variety. According to this approach, a vector space M with a (left) action of a CD algebra A, denoted by $a \bullet m$, is called a *module over A*, if the following equality holds:

$$(xy)a \bullet m + a \bullet ((xy) \bullet m) - x \bullet ((ya) \bullet m) + y \bullet ((xa) \bullet m) - x \bullet (a \bullet (y \bullet m)) + y \bullet (a \bullet (x \bullet m)) = 0.$$

for any $x, y, a \in A$ and $m \in V$.

As in the Lie case, A, considered as a (left) module over itself, is called the *adjoint module*.

As we are interested in central extensions and deformations of CD algebras, we start with the second cohomology. The second cohomology is interpreted as equivalence classes of square-zero extensions. Namely, let A be a CD algebra and M an A-module, and consider the CD algebra structure on the direct sum of vector space $A \oplus M$, where multiplication on A is given by the formula $x * y = xy + \varphi(x, y)$ for some bilinear map $\varphi : A \times A \to M$, multiplication between A and M is given by action of A on M, and multiplication on M is zero. Then φ is skew-symmetric, and

$$\begin{aligned}
&\varphi((xy)a, b) - \varphi((xy)b, a) - \varphi((xa)b, y) + \varphi((xb)a, y) + \varphi((ya)b, x) - \varphi((yb)a, x) \\
&\quad + a \bullet \varphi(xy, b) - b \bullet \varphi(xy, a) - x \bullet \varphi(ya, b) + x \bullet \varphi(yb, a) + y \bullet \varphi(xa, b) - y \bullet \varphi(xb, a) \\
&\quad - a \bullet (b \bullet \varphi(x, y)) + b \bullet (a \bullet \varphi(x, y)) - x \bullet (a \bullet \varphi(y, b)) + x \bullet (b \bullet \varphi(y, a)) - y \bullet (b \bullet \varphi(x, a)) + y \bullet (a \bullet \varphi(x, b)) \\
&\quad = 0
\end{aligned} \tag{15}$$

for any $x, y, a, b \in A$.

The usual notion of equivalence of square-zero extensions $0 \to M \to \cdot \to A \to 0$ leads to the notion of trivial, or split, extension, which corresponds to a cocycle of the form

$$\varphi(x, y) = \psi(xy) - x \bullet \psi(y) + y \bullet \psi(x) \tag{16}$$

for any $x, y \in A$ and some linear map $\psi : A \to M$. Thus the right-hand side of this equality suggests the definition of the first order cocycles, what confirms the standard interpretation of the first cohomology as outer derivations.

The inner derivations of A, according to the general approach devised by Schafer (see, for example, [19, Chapter II, §3]), are defined as derivations lying in the Lie multiplication algebra $M(A)$. Since $[R_a, R_b] \in \mathrm{Der}(A)$ and due to (1), $M(A)$ is linearly spanned by linear maps of the form R_a and $[R_a, R_b]$ for $a, b \in A$. On the other hand, by Lemma 3, R_a is a derivation of A if and only if $a \in LZ(A)$. Thus any inner derivation of A is of the form $R_z + \sum_i [R_{a_i}, R_{b_i}]$ for some $z \in LZ(A)$ and $a_i, b_i \in A$.

Another interpretation of the second cohomology with coefficients in the adjoint module is equivalence classes of infinitesimal deformations of an algebra. Thus, following nowadays standard approach by Gerstenhaber, for a CD algebra A consider a deformed algebra over the ring $K[[t]]$, with multiplication

$$x * y = xy + \varphi_1(x, y)t + \varphi_2(x, y)t^2 + \dots$$

That the deformed algebra is also CD algebra, is equivalent to an (infinite) series of equalities, obtained by collecting coefficients by powers of t in the CD identity (2) for the multiplication $*$. The zeroth of these equalities (coefficients by t^0) coincides with the CD identity for the original multiplication in A, and thus gives nothing new. The first of these equalities (coefficients by t^1) is obtained by ignoring all terms with powers of t higher than 1, and thus is equivalent to the CD identity in the algebra $A \oplus At$, where At is the adjoint module over A with trivial multiplication. Hence it is equivalent to the identity (15) with $M = A$ and $\bullet$ being multiplication in the algebra. The further equalities (coefficients by t^2 and higher degrees) involve Massey brackets of φ_i's which can be interpreted as obstructions of prolongations of an infinitesimal deformation to a global one, and lie in the third cohomology of A with coefficients in the adjoint module. But as it leads to cumbersome formulae, and our primary interest is in the second cohomology, we will not pursue this further.

The condition of triviality of such deformation, i.e., the equivalence to the initial algebra with respect to a homomorphism of the form

$$\psi(x) = x + \psi_1(x)t + \psi_2(x)t^2 + \dots$$

leads to an infinite series of equalities, the first of which is

$$\varphi_1(x, y) = \psi_1(xy) - \psi_1(x)y - x\psi_1(y),$$

what is the partial case of (16) in the case of the adjoint module.

Putting all this together, we define the initial terms of the cochain complex associated to a CD algebra A and an A-module M:

$$0 \to LZ(M) \oplus (A \otimes M) \xrightarrow{\mathrm{d}^0} \mathrm{C}^1(A, M) \xrightarrow{\mathrm{d}^1} \mathrm{C}^2(A, M) \xrightarrow{\mathrm{d}^2} \mathrm{C}^4(A, M). \tag{17}$$

Here $LZ(M)$ is a "Lie center" of M defined as

$$LZ(M) = \{m \in M \mid xy \bullet m - x \bullet (y \bullet m) + y \bullet (x \bullet m) = 0 \text{ for any } x, y \in M\},$$

and $\mathrm{C}^n(A, M)$ for $n \geq 1$ is a linear space of skew-symmetric linear maps $\underbrace{A \times \dots \times A}_{n} \to M$. The differentials are defined as follows:

$$\mathrm{d}^0(m)(b) = b \bullet m$$

for $b \in A$, $m \in LZ(M)$,

$$\mathrm{d}^0(a \otimes m)(b) = a \bullet (b \bullet m) - b \bullet (a \bullet m)$$

for $a, b \in A$, $m \in M$. The "$LZ(M)$" component of d^0 is similar to the zeroth differential in the Chevalley–Eilenberg complex computing the Lie algebra cohomology, and the "$A \otimes M$" component is similar to the zeroth differential in the complex computing cohomology of quadratic Jordan algebras defined in [16, §I.3].

Further,

$$\mathrm{d}^1(\varphi)(x, y) = \varphi(xy) - x \bullet \varphi(y) + y \bullet \varphi(x)$$

for $\varphi \in \mathrm{C}^1(A, M)$, $x, y \in A$, and

$$\begin{aligned} &\mathrm{d}^2(\varphi)(x, y, a, b) \\ &\quad = \varphi((xy)a, b) - \varphi((xy)b, a) - \varphi((xa)b, y) + \varphi((xb)a, y) + \varphi((ya)b, x) - \varphi((yb)a, x) \\ &\quad + a \bullet \varphi(xy, b) - b \bullet \varphi(xy, a) - x \bullet \varphi(ya, b) + x \bullet \varphi(yb, a) + y \bullet \varphi(xa, b) - y \bullet \varphi(xb, a) \\ &- a \bullet (b \bullet \varphi(x, y)) + b \bullet (a \bullet \varphi(x, y)) - x \bullet (a \bullet \varphi(y, b)) + x \bullet (b \bullet \varphi(y, a)) - y \bullet (b \bullet \varphi(x, a)) + y \bullet (a \bullet \varphi(x, b)) \end{aligned} \tag{18}$$

for $\varphi \in \mathrm{C}^2(A, M)$, $x, y, a, b \in A$.

The equalities $\mathrm{d}^1 \circ \mathrm{d}^0 = 0$ and $\mathrm{d}^2 \circ \mathrm{d}^1 = 0$ are verified in a straightforward, if not a bit cumbersome way, or just follow from the structural interpretations described above.

This is not the only sensible way to define cohomology of CD algebras. For example, one may argue that a proper notion of derivation in this context is the following: a linear map $D : A \to A$ such that the semidirect sum $A \oplus KD$, where multiplication between A and D is determined by action of D on A. In the variety of Lie algebras, this leads to the usual notion of derivation, but in the variety of CD algebras, this leads to what might be called a *CD derivation* of a CD algebra A: a linear map $D : A \to A$ such that

$$D((xy)a) - D(xy)a - D(xa)y + D(ya)x + (D(x)a)y - (D(y)a)x = 0$$

for any $x, y, a \in A$. An inner derivation in this context is, as in the variety of Lie algebras, just a multiplication R_a, $a \in A$. Accordingly, one may define the initial terms of the cochain complex responsible for cohomology of a CD algebra A with coefficients in an A-module M, as

$$0 \to M \xrightarrow{\mathrm{d}^0} \mathrm{C}^1(A, M) \xrightarrow{\mathrm{d}^1} \mathrm{C}^3(A, M),$$

where

$$\mathrm{d}^0(m)(x) = x \bullet m$$

for $m \in M$ and $x \in A$, and

$$\mathrm{d}^1(\varphi)(x, y, a) = \varphi((xy)a) - a \bullet \varphi(xy) - y \bullet \varphi(xa) + x \bullet \varphi(ya) + y \bullet (a \bullet \varphi(x)) - x \bullet (a \bullet \varphi(y)) \tag{19}$$

for $\varphi \in \mathrm{C}^1(A, M)$ and $x, y, a \in A$.

One cannot say which variant of these cochain complexes is "better". In that respect, the situation resembles those with cohomology of so-called mock-Lie algebras (commutative algebras satisfying the Jacobi identity), where also one cannot define a low-degree cohomology in a canonical and coherent way, basing on structural interpretations (cf. [21, §1]). However, as we are interested in central extensions and deformations, we adopt the first variant of the cochain complex, and define the second degree cohomology of a CD algebra A with coefficients in an A-module M as $\mathrm{H}^2_{CD}(A, M) = \operatorname{Ker} \mathrm{d}^2 / \operatorname{Im} \mathrm{d}^1$, where d^1 and d^2 are as in (17).

One may try to generalize the formulae for differentials above as follows. Let $n > 0$, and $\mathrm{d} : \mathrm{C}^n(A, M) \to \mathrm{C}^{n+2}(A, M)$ is given by

$$\begin{aligned} &\mathrm{d}(\varphi)(x, y, a_1, \dots, a_n) \\ &\quad = \sum_{i=1}^{n} (-1)^i \Big(\varphi((xy)a_i, a_1, \dots, \widehat{a_i}, \dots, a_n) + a_i \bullet \varphi(xy, a_1, \dots, \widehat{a_i}, \dots, a_n) \\ &\qquad - x \bullet \varphi(ya_i, a_1, \dots, \widehat{a_i}, \dots, a_n) + y \bullet \varphi(xa_i, a_1, \dots, \widehat{a_i}, \dots, a_n) \\ &\qquad - x \bullet (a_i \bullet \varphi(y, a_1, \dots, \widehat{a_i}, \dots, a_n)) + y \bullet (a_i \bullet \varphi(x, a_1, \dots, \widehat{a_i}, \dots, a_n)) \Big) \\ &\quad + \sum_{1 \le i < j \le n} (-1)^{i+j+n+1} \Big(\varphi((xa_i)a_j, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) - \varphi((xa_j)a_i, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) \\ &\qquad - \varphi((ya_i)a_j, x, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) + \varphi((ya_j)a_i, x, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) \\ &\qquad + a_i \bullet (a_j \bullet \varphi(x, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n)) - a_j \bullet (a_i \bullet \varphi(x, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n)) \Big) \end{aligned}$$

for $\varphi \in \mathrm{C}^n(A, M)$, and $x, y, a_1, \dots, a_n \in A$.

One can prove, in the absence of analogs of Cartan formulae in the Chevalley–Eilenberg complex (are there ones?), by direct verification if not without some pain, that $\mathrm{d} \circ \mathrm{d} = 0$, so we get, in fact, two complexes, which lead to what may be called "odd" and "even" CD cohomology respectively:

$$\mathrm{C}^1(A, M) \overset{\mathrm{d}}{\to} \mathrm{C}^3(A, M) \overset{\mathrm{d}}{\to} \mathrm{C}^5(A, M) \overset{\mathrm{d}}{\to} \dots$$
$$\mathrm{C}^2(A, M) \overset{\mathrm{d}}{\to} \mathrm{C}^4(A, M) \overset{\mathrm{d}}{\to} \mathrm{C}^6(A, M) \overset{\mathrm{d}}{\to} \dots$$

The differential $\mathrm{d} : \mathrm{C}^1(A, M) \to \mathrm{C}^3(A, M)$ here coincides with differential (19), and the differential $\mathrm{d} : \mathrm{C}^2(A, M) \to \mathrm{C}^4(A, M)$ coincides with differential (18).

However, we will not pursue this topic further and in the subsequent section will work exclusively with $\mathrm{H}^2_{CD}(A, M)$ as defined above.

5. Central CD extensions of Lie algebras

In this section we discuss central CD extensions, or, in other words, $\mathrm{H}^2_{CD}(L, K)$, for various Lie algebras L with coefficients in the trivial module K. According to the definition, the vector space $\mathrm{H}^2_{CD}(L, K)$ is a quotient of *CD 2-cocycles* by 2-coboundaries:

$$\mathrm{H}^2_{CD}(L, K) = \frac{\mathrm{Z}^2_{CD}(L, K)}{\mathrm{B}^2(L, K)}.$$

The space $\mathrm{Z}^2_{CD}(L, K)$ of CD 2-cocycles consists of skew-symmetric bilinear maps $\varphi : L \times L \to K$ satisfying the condition

$$\varphi([[x, y], a], b) - \varphi([[x, y], b], a) - \varphi([[x, a], b], y) + \varphi([[x, b], a], y) + \varphi([[y, a], b], x) - \varphi([[y, b], a], x) = 0 \tag{20}$$

for any $x, y, a, b \in L$, and the space of 2-coboundaries $\mathrm{B}^2(L, K)$ consists, as in the Lie case, of bilinear maps of the form $\varphi(x, y) = \psi([x, y])$ for some linear map $\psi : L \to K$.

Note that for any Lie algebra L, the usual Chevalley–Eilenberg cohomology $\mathrm{H}^2(L, K)$ is a subspace in the CD cohomology $\mathrm{H}^2_{CD}(L, K)$.

Our first goal is to obtain examples of an almost Lie algebra which is not CD, promised at the end of the proof of Proposition 1, by considering one-dimensional central extensions of a Lie algebra L. Such central extensions can be written as the vector space direct sum $L \oplus Kz$, where multiplication in L is twisted by a 2-cocycle $\varphi : L \times L \to K$, $\{x, y\} = [x, y] + \varphi(x, y)z$, and z is a central element. Such an algebra is almost Lie for any skew-symmetric φ, while it is a CD algebra if and only if φ is CD 2-cocycle. Thus any Lie algebra L whose second CD cohomology $\mathrm{H}^2_{CD}(L, K)$ is strictly larger then its second Chevalley–Eilenberg cohomology $\mathrm{H}^2(L, K)$, will lead, by extending L by any CD 2-cocycle which is not a Chevalley–Eilenberg cocycle, to an example of a CD algebra which is not Lie. Similarly, a Lie algebra L, for which $\mathrm{H}^2_{CD}(L, K)$ is strictly smaller than the space $\mathrm{C}^2(L, K)/\mathrm{B}^2(L, K)$ – which can be considered as the "2nd almost Lie cohomology" – will lead to an example of an almost Lie algebra which is not CD.

Obviously, abelian Lie algebras do not qualify for such examples, so let us look at nonabelian Lie algebras of low dimension. Elementary calculations show that the two-dimensional nonabelian Lie algebra, and the 3-dimensional nilpotent Lie algebra do not qualify either, as for these algebras

$$\mathrm{H}^2(L, K) = \mathrm{H}^2_{CD}(L, K) = \frac{\mathrm{C}^2(L, K)}{\mathrm{B}^2(L, K)}$$

(all these three spaces vanish in the case where L is two-dimensional nonabelian, and are of dimension 2 in the case where L is 3-dimensional nilpotent), but the direct sum of the two-dimensional nonabelian and the one-dimensional algebra does qualify, as for this algebra we have

$$\mathrm{H}^2(L, K) = \mathrm{H}^2_{CD}(L, K) \subsetneq \frac{\mathrm{C}^2(L, K)}{\mathrm{B}^2(L, K)}$$

(the corresponding spaces being of dimensions 1 and 2).

Further low-dimensional solvable and nilpotent Lie algebras of dimension 3 and higher could provide a plethora of such examples (including the cases where $\mathrm{H}^2(L, K)$ and $\mathrm{H}^2_{CD}(L, K)$ do not coincide).

On the other hand, for simple Lie algebras we have

Theorem ("*Second CD Whitehead lemma*")**.** *For any simple finite-dimensional Lie algebra L over a field of characteristic zero, and any finite-dimensional L-module M, $\mathrm{H}^2_{CD}(L, M) = 0$.*

Proof. This follows at once from [8, Theorem 6]. Indeed, it is proved there that any solvable (and hence abelian) extension of L in the variety of binary Lie algebras (and hence in the variety of CD algebras) splits. □

In the positive characteristic, we merely have a

Conjecture. *For any simple finite-dimensional Lie algebra L over a field of characteristic* $\neq 2, 3$, $\mathrm{H}^2_{CD}(L, K) = \mathrm{H}^2(L, K)$.

The conjecture is supported by computer calculations for algebras of small dimension.

6. Further questions

(1) How "far" a CD algebra can be from Lie algebras? To start with, describe CD algebras A with the Lie center "as small as possible", i.e., satisfying the condition $LZ(A) = Z(A)$.

(2) Which "interesting" Lie algebras can be realized as constructions described in Section 3?

(3) Study free CD algebras. Are they central extensions of free Lie algebras?

(4) For an algebra A, its minus-algebra $A^{(-)}$ is an algebra defined on the same underlying vector space A subject to multiplication given by the commutator $[x, y] = xy - yx$. If the minus-algebra of an algebra A belongs to a (necessary anticommutative) variety $\mathcal{V}$, then A is called $\mathcal{V}$-admissible. Most of the distinguished varieties of algebras have they "admissible" counterparts: thus, associative algebras are Lie-admissible, alternative algebras are Malcev-admissible, and binary Lie algebras are assocyclic-admissible, where the variety of assocyclic algebras is defined by the identity

$$(x, y, z) = (z, x, y),$$

$(x, y, z) = (xy)z - x(yz)$ being the associator of elements x, y, z.

In such situation arises the question whether each algebra in an anticommutative variety $\mathcal{V}$ is special, i.e., can be embedded into an algebra of the form $A^{(-)}$, where A is $\mathcal{V}$-admissible algebra. Thus, Lie algebras are special due to the celebrated Poincaré–Birkhoff–Witt theorem; the speciality of Malcev algebras was a long-standing problem whose negative solution was announced recently by Ivan Shestakov; and binary Lie algebras are not necessarily special either, see [4].

What would be a natural variety of CD admissible algebras? (One possible general approach in operadic language to such sort of questions is described in [15].) Would CD algebras be special with respect to that variety?

(5) Study representations of CD algebras. Does an analog of the Ado theorem hold, i.e., whether each finite-dimensional CD algebra admits a faithful finite-dimensional representation?

(6) The classical Lie theory establishes correspondence between Lie groups and Lie algebras. This correspondence has been generalized to Malcev algebras, Bol algebras, Lie triple systems, Sabinin algebras, etc. (see, for example, [9] and references therein). In these "generalized Lie" correspondences Lie groups are replaced by various kinds of analytic loops; thus, binary Lie algebras correspond to diassociative loops (i.e., the subloop generated by any two elements is a group). Which loops would correspond to CD algebras? One of the possible approaches to this question would be to find the class of loops corresponding to almost Lie algebras, and then, according to (6), take the intersection of that class with the class of diassociative loops.

(7) Let us drop the commutativity and anticommutativity conditions altogether, and consider the variety of algebras defined just by the properties that the commutator of any two left or right multiplications is a derivation (i.e., taking all possible combinations of left/right multiplications, we get 3 defining identities, see, for example, [13, §1]). Is this variety of algebras amenable to study? Low-dimensional nilpotent algebras in this variety were classified in [13] and [14].

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References

[1] V.T. Filippov, V.K. Kharchenko, I.P. Shestakov (Eds.), in: Dniester Notebook: Unsolved Problems in the Theory of Rings and Modules, fourth ed., Novosibirsk, 1993 (in Russian); Non-Associative Algebra and its Applications, in: Lect. Notes Pure Appl. Math., vol. 246, Chapman & Hall/CRC, 2006, 461–516 (English translation).

[2] H. Abdelwahab, A.J. Calderón, A. Fernández Ouaridi, Central extensions of 4-dimensional binary Lie algebras, Rock. Mt. J. Math. 50 (5) (2020) 1541–1559.

[3] H. Abdelwahab, A.J. Calderón, I. Kaygorodov, The algebraic and geometric classification of nilpotent binary Lie algebras, Int. J. Algebr. Comput. 29 (6) (2019) 1113–1129.

[4] M. Arenas, I. Shestakov, On speciality of binary-Lie algebras, J. Algebr. Appl. 10 (2) (2011) 257–268.

[5] W. Bertram, The geometry of Jordan and Lie structures, Lecture Notes in Math. 1754 (2000).

[6] A. Dzhumadil'daev, Assosymmetric algebras under Jordan product, Comm. Algebra 46 (1) (2018) 129–142.

[7] V.T. Filippov, Simple Sagle algebras, Mat. Zametki 65 (4) (1999) 607–611, (in Russian); Math. Notes 65 (4) (1999) 506–509, (English translation);

[8] A.N. Grishkov, Structure and representations of binary-Lie algebras, Izv. Akad. Nauk SSSR Ser. Mat. 44 (5) (1980) 999–1030 (in Russian); Math. USSR Izv. 17 (2) (1981) 243–269 (English translation);

[9] A.N. Grishkov, J.M. Pérez-Izquierdo, Lie's correspondence for commutative automorphic formal loops, Linear Algebra Appl. 544 (2018) 460–501

[10] I.R. Hentzel, L.A. Peresi, Almost Jordan rings, Proc. Amer. Math. Soc. 104 (2) (1988) 343–348

[11] N. Jacobson, Structure and Representations of Jordan Algebras, AMS, 1968.

[12] P. Jordan, H. Rühaak, Über einen Zusammenhang der Lie-Tripel-Algebren mit den Osborn-Algebren, Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. (3) (1969) 49–54
[13] I. Kaygorodov, M. Khrypchenko, The algebraic classification of nilpotent $\mathfrak{CD}$-algebras, Linear Multilinear Algebr. (2021) in press.
[14] I. Kaygorodov, M. Khrypchenko, The geometric classification of nilpotent $\mathfrak{CD}$-algebras, J. Algebr. Appl. (2021) in press.
[15] P.S. Kolesnikov, Commutator algebras of pre-commutative algebras, Mat. Zhurnal (Almaty) 16 (2) (2016) 145–158 (Dzhumadil'daev's Festshrift).
[16] K. McCrimmon, Representations of quadratic Jordan algebras, Trans. Amer. Math. Soc. 153 (1971) 279–305
[17] A.A. Sagle, Malcev algebras, Trans. Amer. Math. Soc. 101 (3) (1961) 426–458
[18] A.A. Sagle, On simple extended Lie algebras over fields of characteristic zero, Pacific J. Math. 15 (2) (1965) 621–648
[19] R.D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, 1966.
[20] A.V. Sidorov, Lie triple algebras, Algebr. Log. 20 (1) (1981) 101–108 (in Russian); Algebr. Log. 20 (1) (1981) 72–78 (English translation);
[21] P. Zusmanovich, Special and exceptional mock-Lie algebras, Linear Algebra Appl. 518 (2017) 79–96 arXiv:1608.05861.

SOFTWARE

[22] Albert, https://web.osu.cz/~Zusmanovich/soft/albert/.
[23] The GAP Group, GAP – Groups, algorithms, and programming, version 4.10.2, 2019, https://www.gap-system.org/.