



Super-exceptional embedding construction of the heterotic M5: Emergence of SU(2)-flavor sector

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ABSTRACT

A new super-exceptional embedding construction of the heterotic M5-brane's sigma-model was recently shown to produce, at leading order in the super-exceptional vielbein components, the super-Nambu-Goto (Green-Schwarz-type) Lagrangian for the embedding fields plus the Perry-Schwarz Lagrangian for the free abelian self-dual higher gauge field. Beyond that, further fields and interactions emerge in the model, arising from probe M2- and probe M5-brane wrapping modes. Here we classify the full super-exceptional field content and work out some of its characteristic interactions from the rich super-exceptional Lagrangian of the model. We show that $SU(2) \times U(1)$ -valued scalar and vector fields emerge from probe M2- and M5-branes wrapping the vanishing cycle in the A_1 -type singularity; together with a pair of spinor fields of $U(1)$ -hypercharge ± 1 and each transforming as $SU(2)$ iso-doublets. Then we highlight the appearance of a WZW-type term in the super-exceptional PS-Lagrangian and find that on the electromagnetic field it gives the first-order non-linear DBI-correction, while on the iso-vector scalar field it has the form characteristic of the coupling of vector mesons to pions via the Skyrme baryon current. We discuss how this is suggestive of a form of $SU(2)$ -flavor chiral hadrodynamics emerging on the single ($N = 1$) M5 brane, different from, but akin to, holographic large- N QCD.

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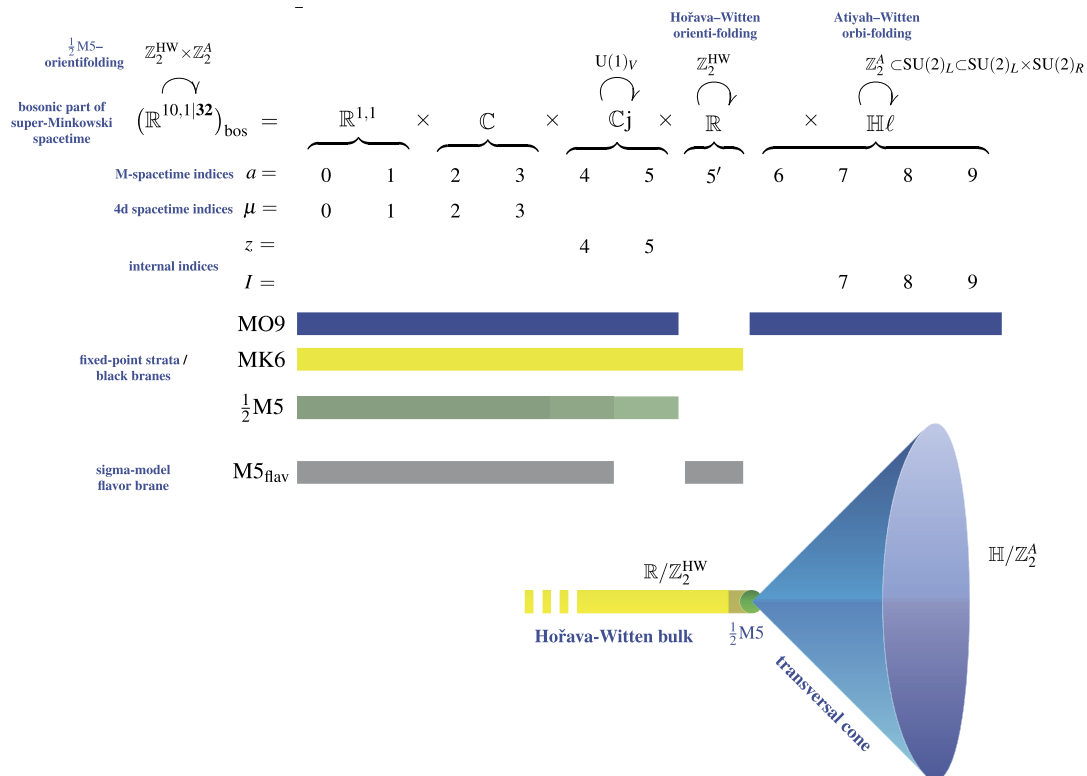
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1. Introduction

The general open problem of coincident M5-branes. It is widely appreciated that the problem of identifying/formulating the expected non-abelian higher gauge theory (“non-abelian gerbe theory” [142, p. 6, 15]) on coincident M5-branes remains open (e.g., [97, p. 77][84, p. 49][85, 6.3]). This is a key aspect of the wider open problem (e.g., [39, 6][72, p. 2][40, p. 6][100, p. 2][41, p. 330][98, 12][32, p. 2][143][42]) of formulating M-theory [40][41] itself, the non-perturbative completion of the perturbation theory formerly known as string theory [39].

M5-branes compactified to four dimensions, with probe flavor D8-branes, plausibly yield, once made rigorous, an analytic first-principles working theory of hadrons (e.g., [94][91]): the Witten-Sakai-Sugimoto model of holographic QCD [141][80][115][115] (reviewed in [109]) or its variant with flavor D4-branes [107][128]. Hence the solution of non-abelian M5-branes in M-theory could solve the *Confinement Problem* (see [61][112][111]) of quantum chromodynamics (QCD), one of the Millennium Problems [33][78].

The case of single but heterotic M5-branes. Remarkably, even the case of a single M5-brane, while commonly thought to have been understood long ago [106][126][105][19], is riddled with subtleties, indicating foundational issues not fully understood yet. One of these puzzlements is (or was) that the modern “super-embedding construction” of κ -symmetric Lagrangians defining single super p -branes [20][21][131][132] fails for M5-branes (works only for EOMs). This is noteworthy because there is a boundary case of non-abelian gauge theories on M5-branes which nominally has to do with *single* branes: the single *heterotic* M5-brane. This is the lift to heterotic M-theory (Hořava-Witten theory, see [70][140][71][103][104]) of the NS5-brane of heterotic string theory [87], and dually of the D4-brane of Type I' string theory. These branes are expected [139][58][10][11] to carry a non-abelian gauge field, specifically with gauge group $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$, understood as being the special case of N coincident M5-branes for $N = 2$, but with the two branes identified as mirror pairs of an orientifold \mathbb{Z}_2 -action. The orbi-orientifold singularity of this action, regarded as the far-horizon geometry of a solitonic brane [3, 3], is known as the $\frac{1}{2}\mathrm{NS5}$ in string theory [60, 6][35, 6][9, p. 18], or dually as the $\frac{1}{2}\mathrm{M5}$ in M-theory [73, Ex. 2.2.7][50, 4][120, 4.1]:



Expected $SU(2_f)$ flavor theory on heterotic M5. In fact, this $SU(2)$ gauge symmetry expected on single heterotic M5-branes came to be understood as being a *flavor symmetry* (e.g. [55, 4.2][101, 2.3.1]) just like the “hidden local symmetry” [117][16] (reviewed in [90, 6]) of chiral perturbation theory for confined hadrodynamics (reviewed in [94][124][23][91]), instead of a color symmetry as in the quark model of quantum chromodynamics. This is informally argued in the literature by

- (i) invoking (see [29, 2.3]) M/IIA duality along the $S^1 \subset Sp(1)_L$ -action in (1) for identifying the $\frac{1}{2}$ M5-brane configuration with a NS5 \parallel D6 \perp O8-brane configuration in Type I' string theory, and then
- (ii) appealing ([64][28], reviewed in [9, 2.1]) to open string theory for identifying the D6-branes emanating from the NS5, corresponding to the MK6-brane in (1), with flavor branes, due to their semi-infinite transverse extension.

This suggests that single but heterotic M5-branes are not just a toy example for the more general open problem of non-abelian gauge enhancement in M-theory, but, when viewed through the lens of holographic hadrodynamics, possibly the core example for making contact with phenomenology. Hence the first open problem to address is:

The specific open problem. A derivation/formulation of the (higher) $SU(2)$ -flavor gauge theory emerging on single heterotic M5-branes.

In [51] we had discussed this problem in the “topological sector”, i.e., focusing on gauge- and gravitational instanton sectors and their topological global anomaly cancellation conditions, while temporarily putting to the side local field/differential form data. There we had proven that, under the hypothesis that the M-theory C-field is charge-quantized in J-twisted Co-homotopy cohomology theory (“Hypothesis H” [118][48][49] [120][121][122], review in [125]), a topological $Sp(1) \simeq SU(2)$ gauge field sector indeed emerges in the sigma-model for single M5-branes.

Here we set out to discover the local differential form data to complete this topological picture of the non-abelian heterotic M5-brane theory, by identifying its *local field content* and its *couplings*.

Solution via super-exceptional geometry. In [50] we had demonstrated that the failure of the super-embedding approach to produce the M5-brane Lagrangian is resolved, for heterotic M5-branes, by enhancing to *super-exceptional* embeddings. This means enhancing the $\mathcal{N} = 1$, $D = 11$ -dimensional target super-spacetime $\mathbb{R}^{10,1|32}$ to the *super-exceptional spacetime* $\mathbb{R}_{\text{ex}}^{10,1|32}$ [17][46][119][50, 3] or rather to its heterotic/type-I' version $\mathbb{R}_{\text{ex}}^{9,1|16}$ [50, 4] (recalled as Definition 3 below). This has the virtue of unifying graviton and gravitino modes with M2- and M5-brane wrapping modes and with a pre-gaugino field [50, Rem. 5.4]:

$$\begin{array}{c}
 \text{Super-exceptional M-theory Minkowski spacetime } \mathbb{R}_{\text{ex}}^{10,1|32} \\
 \uparrow \text{ as a super-manifold} \\
 \text{Super-exceptional heterotic/type-I' Minkowski spacetime } \mathbb{R}_{\text{ex}}^{9,1|16} \\
 \uparrow \text{ as a } \text{Pin}^+(9,1)\text{-representation}
 \end{array}
 \begin{array}{c}
 \begin{array}{c} \text{graviton} \\ \mathbb{R}^{10,1} \end{array} \times \begin{array}{c} \text{gravitino} \\ \mathbb{R}^{0|32} \end{array} \times \begin{array}{c} \text{probe M2-brane} \\ \text{wrapping modes} \\ \wedge^2(\mathbb{R}^{10,1})^* \end{array} \times \begin{array}{c} \text{probe M5-brane} \\ \text{wrapping modes} \\ \wedge^5(\mathbb{R}^{10,1})^* \end{array} \times \begin{array}{c} \mathbb{R}^{0|32} \end{array} \\
 \hline
 \begin{array}{c} 11 \end{array} \oplus \begin{array}{c} 32 \end{array} \oplus \begin{array}{c} \wedge^9 11 \end{array} \oplus \begin{array}{c} \wedge^5 11 \end{array} \oplus \begin{array}{c} 32 \end{array} \\
 \hline
 \begin{array}{c} 10 \oplus 1 \oplus 16 \end{array} \oplus \begin{array}{c} 10_{\text{sgn}} \oplus \wedge^8 10 \oplus \wedge^5 10 \oplus \wedge^4 10 \oplus 16 \end{array} \\
 \hline
 \begin{array}{c} \text{dilaton} \end{array} \quad \begin{array}{c} \text{probe F1}_{\Omega}\text{-brane} \\ \text{wrapping modes} \end{array} \quad \begin{array}{c} \text{probe D8-brane} \\ \text{wrapping modes} \end{array} \quad \begin{array}{c} \text{probe NS5-brane} \\ \text{wrapping modes} \end{array} \quad \begin{array}{c} \text{probe D4-brane} \\ \text{wrapping modes} \end{array} \quad \begin{array}{c} \text{gaugino} \end{array}
 \end{array} \quad (2)$$

The dual M9-brane wrapping modes in the second line, anticipated in [75, p. 8–9], follow by rigorous analysis [46, 4.26] (see Proposition 4 and Remark 5 below) and lead to D4/D8-brane modes in 10d, as shown in the last line. The underlying super-symmetry algebra of this super-exceptional spacetime $\mathbb{R}_{\text{ex}}^{10,1|32}$ is the “hidden supergroup of $D = 11$ supergravity” [34][18][8], whose role or purpose had previously remained elusive. We may understand it [50, Rem. 3.9][46, 4.6][119] as that supermanifold whose real cohomology accommodates that of the classifying 2-stack of the M5-brane sigma-model [45] under Hypothesis H [49].

Result – Emergent chiral $SU(2_f)$ -theory on the heterotic M5. As a consequence of the super-exceptional enhancement (2) of target spacetime, additional worldvolume fields emerge also on the single heterotic M5-brane locus (1), originating in M2- and M5-brane wrapping modes. Here we classify this emergent super-exceptional field content by computing the representations of the residual group actions (see §3.1) on the super-exceptional vielbein fields after passage to the super-exceptional MK6-locus (Definition 6 below), reduced to 4d:

$$\begin{array}{c}
\text{Symmetry groups} \\
\text{Representation rings} \\
\text{Super-exceptional spacetimes}
\end{array}
\begin{array}{c}
\text{Spin}(10, 1) \xleftarrow{\quad t \quad} \text{Spin}(6, 1) \times \text{Spin}(4) \xleftarrow{\quad} \text{Spin}(5, 1) \times \text{SU}(2)_R \\
\text{Spin}(6, 1) \times \text{SU}(2)_L \times \text{SU}(2)_R \xleftarrow{\quad} \text{Spin}(3, 1) \times \text{Spin}(2) \times \text{SU}(2)_R \\
\text{Spin}(3, 1) \times \text{U}(1)_V \times \text{SU}(2)_R \\
\text{Rep}_{\mathbb{R}}(\text{Spin}(10, 1)) \xrightarrow[\text{fixed representation of } \mathbb{Z}_2^4 \subset \text{SU}(2)_L]{\text{pass to}} \text{Rep}_{\mathbb{R}}(\text{Spin}(6, 1) \times \text{SU}(2)_R) \xrightarrow[\text{restricted representation}]{\text{pass to}} \text{Rep}_{\mathbb{R}}(\text{Spin}(3, 1) \times \text{U}(1)_V \times \text{SU}(2)_R) \\
\mathbb{R}_{\text{ex}_s}^{10,1|\mathbf{32}} \xrightarrow[\text{localize onto brane}]{\text{super-exceptional M-theory spacetime}} (\mathbb{R}_{\text{ex}_s}^{10,1|\mathbf{32}})^{\mathbb{Z}_2^4} \xrightarrow[\text{dimensionally reduce to 4d}]{\text{super-exceptional MK6-locus}} (\mathbb{R}_{\text{ex}_s}^{10,1|\mathbf{32}})^{\mathbb{Z}_2^4}_{4d} \xrightarrow{\text{super-exceptional MK6-locus reduced to 4d}}
\end{array}
\tag{3}$$

The result is shown in the following diagram, discussed in detail in §3 below:

Here the outer \boxtimes -tensor products (35) indicate the transformation properties under the three residual groups.

$$\text{Spin}(3,1) \times \text{U}(1) \times \text{SU}(2)_R$$

$\mathbf{1} \boxtimes \mathbf{1} \boxtimes \mathbf{3}$ is an iso-vector scalar field

$\mathbf{4} \boxtimes \mathbf{1} \boxtimes \mathbf{1}$ is an iso-scalar vector field

4 \boxtimes **1** \boxtimes **3** is an iso-vector vector field

$2_{\mathbb{C}} \boxtimes_{\mathbb{C}} 1_{\mathbb{C}} \boxtimes_{\mathbb{C}} 2_{\mathbb{C}}$ is an iso-doublet spinor field

$2_{\mathbb{C}} \boxtimes 1_{\mathbb{C}}^- \boxtimes 2_{\mathbb{C}}$ carries -1 units of hypercharge

This is the kind of field content encountered in quantum hadrodynamics (e.g. [43][91]) where the iso-vector scalar field would be the pion field $\vec{\pi}$, the iso-scalar vector field would be the omega-meson ω , its iso-vector partner the rho-meson ρ , and the two hypercharged fermion iso-doublets would be baryon fields (e.g. [129]).

Indeed, we show in §4.1 that the form of the WZW-term in the super-exceptional PS-Lagrangian is that characteristic of the coupling of neutral vector mesons to triples of pions via the Skyrme baryon current [109], see §5 for further discussion.

2. The super-exceptional M5-brane model

Here we recall the relevant aspects of the super-exceptional embedding construction of the M5-brane [48] to introduce the precise setup that is analyzed in the following sections.

Remark 1 (*Super-Lie algebras*). Throughout, we use the following basic fact, see [47, 3] for review:

- (i) Finite-dimensional Lie superalgebras² are equivalently encoded in terms of their differential graded-commutative (dgc) Chevalley-Eilenberg super-algebras, known as “FDA”s in the supergravity literature:

$$\begin{array}{ccc} \text{LieSuperAlgebras} & \xrightarrow[\text{fully faithful embedding}]{\text{CE}} & \text{dgcSuperAlgebras}^{\text{op}} \\ \begin{array}{c} \text{super-} \\ \text{vector space} \\ \underbrace{(\mathfrak{g}, [-, -])}_{\text{super Lie bracket}} \end{array} & \longmapsto & \begin{array}{c} \text{super-} \\ \text{Grassmann algebra} \\ \underbrace{(\wedge^\bullet \mathfrak{g}^*, d := [-, -]^*)}_{\text{CE-differential}} \end{array} \end{array} \quad (5)$$

- (ii) Here *super-Grassman algebra* means that elements $v \in \mathfrak{g}$ of super-grading $\sigma_v \in \mathbb{Z}_2$ are dual to dgc-algebra elements $v^* \in \wedge^1 \mathfrak{g}^*$ of bidegree $(1, \sigma) \in \mathbb{N} \times \mathbb{Z}_2$. We write “ \wedge ” for the product in these super-Grassman algebras. The sign rule is such that for elements $\alpha, \beta \in \wedge^\bullet \mathfrak{g}^*$ of homogeneous bi-degrees $(n_\alpha, \sigma_\alpha), (n_\beta, \sigma_\beta) \in \mathbb{N} \times \mathbb{Z}_2$ we have

$$\alpha \wedge \beta = (-1)^{n_\alpha n_\beta + \sigma_\alpha \sigma_\beta} \beta \wedge \alpha.$$

- (iii) Generally, the Chevalley-Eilenberg algebra of a Lie superalgebra \mathfrak{g} may be identified with the de Rham algebra of left-invariant differential forms on the corresponding Lie supergroup G :

$$\begin{array}{ccc} \text{Chevalley-Eilenberg algebra} & \text{CE}(\mathfrak{g}) \xrightarrow{\simeq} & \Omega_{\text{LI}}^\bullet(G) \\ \text{of Lie superalgebra } \mathfrak{g} & & \text{de Rham algebra of} \\ & & \text{left-invariant differential forms} \\ & & \text{on Lie supergroup } G \end{array} \quad (6)$$

Example 2 (*Ordinary super-Minkowski spacetime*). For $d \in \mathbb{N}$ and $\mathbf{N} \in \text{Rep}_{\mathbb{R}}(\text{Spin}(d, 1))$ a real $\text{Spin}(d, 1)$ representation of real dimension N , the corresponding *super-Poincaré super Lie algebra* $\text{Iso}(\mathbb{R}^{d,1|\mathbf{N}})$ (“supersymmetry algebra”) is the algebra whose CE-algebra (5) is

$$\begin{array}{c} \text{CE}(\text{Iso}(\mathbb{R}^{d,1|\mathbf{N}})) \quad \left\{ \begin{array}{l} \text{...generated by} \\ \text{degree}=(1, \text{even}) \\ \underbrace{e^a, \omega^{a_1 a_2}}_{\substack{a_i \in \{0, 1, \dots, d\} \\ \alpha \in \{1, 2, \dots, N\}}} \\ \underbrace{\psi^\alpha}_{\text{degree}=(1, \text{odd})} \end{array} \right\} \quad \left| \quad \begin{array}{l} \text{...with differential on generators given by} \\ d e^a = \langle \psi \wedge \Gamma^a \cdot \psi \rangle \\ d \psi^\alpha = \frac{1}{4} \omega^{a_1 a_2} \Gamma_{a_1 a_2} \cdot \psi \\ d \omega^{a_1 a_2} = \omega^{a_1}_{a_3} \wedge \omega^{a_3}_{a_2} \end{array} \right. \quad (7) \end{array}$$

with the $\omega^{a_1 a_2}$ skew-symmetric in their indices. On the right we have the Clifford action and spinor pairing $\langle -, - \rangle$ that comes with the real Spin representation (the dot denotes matrix multiplication, hence contraction of spinor indices).

The underlying **translational super Lie algebra** $\mathbb{R}^{d,1|\mathbf{N}}$ is obtained from this by discarding the Lorentz generators $\omega^{a_1 a_2}$. The resulting CE-algebra may be identified with the de Rham algebra of the canonical super-vielbein on the $D = d + 1$, $\mathcal{N} = \dim(\mathbf{N})$ super-Minkowski spacetime, which is the Cartesian super-manifold with canonical super-coordinates $\{ \underbrace{x^a}_{\text{deg}=(0, \text{even})}, \underbrace{\theta^\alpha}_{\text{deg}=(0, \text{odd})} \}$:

$$\begin{array}{ccc} \text{CE}(\mathbb{R}^{d,1|\mathbf{N}}) & \xrightarrow{\simeq} & \Omega_{\text{LI}}^\bullet(\mathbb{R}^{d,1|\mathbf{N}}) \\ \begin{array}{c} e^a \mapsto dx^a + \langle \theta, \Gamma^a \cdot d\theta \rangle \\ \psi^\alpha \mapsto d\theta^\alpha \end{array} & & \begin{array}{c} \text{super-vielbein basis on} \\ \text{super-Minkowski spacetime} \end{array} \end{array} \quad \begin{array}{c} \text{left-invariant 1-form basis on} \\ \text{super-Minkowski spacetime} \end{array} \quad (8)$$

We consider this here specifically for $d = 3, 6, 10$, with the real $\text{Spin}(d, 1)$ representation $\mathbf{N} = \mathbf{4}, \mathbf{16}, \mathbf{32}$ given by Dirac matrices with coefficients in $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively; this is reviewed in §3.2 below.

The following is the “hidden supergroup of 11d supergravity” due to [34, 6][18], interpreted as the translational supersymmetry algebra of super-exceptional M-theory spacetime according to [17][46][119] [50, 3]:

² Our ground field is the real numbers.

Definition 3 (*Super-exceptional M-spacetime*). Regarded as a super-Lie algebra of super-translations along itself, in generalization of Example 2, the *super-exceptional Minkowski spacetime* $\mathbb{R}_{\text{ex}_s}^{10,1|32}$ (2) (for a parameter $s \in \mathbb{R} \setminus \{0\}$) has Chevalley-Eilenberg algebra (5):

$$\text{CE}(\mathbb{R}^{10,1|32}) \quad \left. \begin{array}{l} \dots \text{generated from} \\ \text{degree}=(1,\text{even}) \\ \left\{ e^{a_1}, e_{a_1 a_2}, e_{a_1 \dots a_5}, \right. \\ \left. \psi^\alpha, \eta^\alpha \right\} \\ \text{deg}=(1,\text{odd}) \end{array} \right| \begin{array}{l} \dots \text{with differential on generators given by} \\ d e^a = \langle \psi \wedge \Gamma^a \cdot \psi \rangle \\ d e_{a_1 a_2} = \frac{1}{2} \langle \psi \wedge \Gamma_{a_1 a_2} \cdot \psi \rangle \\ d e_{a_1 \dots a_5} = \frac{1}{5!} \langle \psi \wedge \Gamma_{a_1 \dots a_5} \cdot \psi \rangle \\ d \psi = 0 \\ d \eta = ((s+1)e^a \Gamma_a + e_{a_1 b_1} \Gamma^{a_1 a_2} + (1 + \frac{s}{6})e_{a_1 \dots a_5} \Gamma^{a_1 \dots a_5}) \cdot \psi \end{array} \quad (9)$$

for $a_i \in \{0, 1, 2, 3, 4, 5, 5', 6, 7, 8, 9\}$, $\alpha \in \{1, 2, \dots, 32\}$, with the generators $e_{a_1 a_2}$ and $e_{a_1 a_2 a_3 a_4 a_5}$ all skew-symmetric in their indices. Here on the right we have the \odot -Dirac matrix multiplication and spinor pairing of the 4-component octonionic realization of the **32** of $\text{Spin}(10, 1)$, reviewed in §3.2 below, and $\Gamma_{a_1 \dots a_p}$ denotes the skew-symmetrization of the Clifford algebra products $\Gamma_{a_1} \dots \Gamma_{a_p}$, as usual.

In generalization of (8), we may identify the super-exceptional vielbein (9) as left-invariant differential forms on the underlying Cartesian super-manifold of $\mathbb{R}_{\text{ex}_s}^{10,1|32}$, now given by exceptional coordinate functions (as envisioned, in the bosonic sector, in [76, 4.3])

$$\left\{ \overbrace{x^a}^{\text{ordinary bosonic}}, \overbrace{B_{a_1 a_2}, B_{a_1 \dots a_5}}^{\text{exceptional bosonic}}, \overbrace{\theta}^{\text{ordinary fermionic}}, \overbrace{\theta'}^{\text{exceptional fermionic}} \right\} \quad \text{super-exceptional coordinate functions}$$

degree=(0,even) degree=(0,odd)

as follows:

$$\begin{array}{ccc} \text{CE}(\mathbb{R}_{\text{ex}_s}^{10,1|32}) & \xrightarrow{\simeq} & \Omega_{\text{li}}^\bullet(\mathbb{R}_{\text{ex}_s}^{10,1|32}) \\ \begin{array}{l} e^a \\ e_{a_1 a_2} \\ e_{a_1 \dots a_5} \\ \psi^\alpha \\ \eta^\alpha \end{array} & \longmapsto & \begin{array}{l} dx^a + \langle \theta, \Gamma^a \cdot d\theta \rangle \\ d(B_{a_1 a_2}) + \frac{1}{2} \langle \theta, \Gamma_{a_1 a_2} \cdot d\theta \rangle \\ d(B_{a_1 \dots a_5}) + \frac{1}{5!} \langle \theta, \Gamma_{a_1 \dots a_5} \cdot d\theta \rangle \\ d\theta \\ d\theta' - ((s+1)e^a \Gamma_a + e_{a_1 b_1} \Gamma^{a_1 a_2} + (1 + \frac{s}{6})e_{a_1 \dots a_5} \Gamma^{a_1 \dots a_5}) \cdot \theta \end{array} \end{array} \quad \begin{array}{l} \text{super-exceptional} \\ \text{vielbein basis} \end{array} \quad \begin{array}{l} \text{left-invariant 1-forms on} \\ \text{super-exceptional Minkowski spacetime} \end{array} \quad (10)$$

Here the first four lines follow just as in (8). The point to notice is the last line, which follows with the same Fierz identity [34, (6.3)-(6.4)][18, (20)-(23)] that gives $dd\theta' = 0$ in (9).

Proposition 4 (*Spin representations on super-exceptional spacetime*). Regard the linear span of the generators in the list (9), as acted on by $\text{Spin}(10, 1)$ in the evident way, as³

$$\mathbb{R}^{10,1|32} \simeq_{\mathbb{R}} \mathbf{11} \oplus \wedge^2 \mathbf{11} \oplus \wedge^5 \mathbf{11} \oplus \mathbf{32} \oplus \mathbf{32} \in \text{Rep}_{\mathbb{R}}(\text{Spin}(10, 1)). \quad (11)$$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \langle e^a \rangle & \langle e_{a_1 a_2} \rangle & \langle e_{a_1 \dots a_5} \rangle & \langle \psi^\alpha \rangle & \langle \eta^\alpha \rangle \end{array}$$

(i) This gives a $\text{Spin}(10, 1)$ -action on $\text{CE}(\mathbb{R}_{\text{ex}_s}^{10,1|32})$ by dgc-superalgebra automorphisms, hence a $\text{Spin}(10, 1)$ -action on $\mathbb{R}_{\text{ex}_s}^{10,1|32}$ itself by Lie superalgebra automorphisms.

(ii) Moreover, this extends to a $\text{Pin}^+(10, 1)$ -action by automorphisms, if one lifts the $\wedge^2 \mathbf{11} \simeq_{\text{Spin}(10, 1)} \wedge^9 \mathbf{11}$ of $\text{Spin}(10, 1)$ specifically to the $\wedge^9 \mathbf{11}$ of $\text{Pin}^+(10, 1)$, hence as:

³ We recall representation-theoretic notation below in §3.1.

$$\mathbb{R}^{10,1|32} \simeq_{\mathbb{R}} \mathbf{11} \oplus \wedge^9 \mathbf{11} \oplus \wedge^5 \mathbf{11} \oplus \mathbf{32} \oplus \mathbf{32} \in \text{Rep}_{\mathbb{R}}(\text{Pin}^+(10, 1)).$$

$$\begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \langle e^a \rangle & & \langle e^{a_1 a_2 \dots e_{11}} e_{a_1 a_2} \rangle & & \langle e_{a_1 \dots a_5} \rangle & & \langle \psi^\alpha \rangle \\ & & & & & & \downarrow \\ & & & & & & \langle \eta^\alpha \rangle \end{array}$$
(12)

Proof. The first statement follows immediately from the second. The second statement follows immediately from [46, 4.26] (reproduced as [50, Lemma 3.10]) where the action of single reflection operators is given, which generate the action of $\text{Pin}^+(10, 1)$. \square

Remark 5 (Probe M9-branes). The result (12) rigorously supports the proposal [75, p. 8-9] (where Hull speaks of the “most natural interpretation”) that the summand $\wedge^2(\mathbb{R}^{10,1})^*$ in the M-theory extended super Lie algebra should, at least in part, be interpreted as the Hodge-dual incarnation of 9-brane charge.

Thus we have the following super-exceptional enhancement of the super MK6-locus [73, Thm. 4.3]:

Definition 6 ([50, 4]). The *super-exceptional MK6-locus* is the sub-Lie superalgebra inside the super-exceptional M-spacetime of Definition 3 which is fixed by the action via Proposition 4 of the subgroup $\mathbb{Z}_2^A \subset \text{SU}(2)_L \subset \text{Spin}(10, 1)$ in (3)

$$\underbrace{(\mathbb{R}^{10,1|16})^{\mathbb{Z}_2^A}}_{\substack{\text{super-exceptional} \\ \frac{1}{2}\text{M5-locus}}} \subset \underbrace{(\mathbb{R}^{10,1|32})^{\mathbb{Z}_2^A}}_{\substack{\text{super-exceptional} \\ \text{MK6-locus}}} \subset \underbrace{\mathbb{R}^{10,1|32}}_{\substack{\text{super-exceptional} \\ \text{M-theory spacetime}}} \quad (13)$$

The *super-exceptional* $\frac{1}{2}\text{M5-locus}$ is the further sub superalgebra with fermions fixed by $\mathbb{Z}_2^{\text{HW}} \subset \text{Pin}^+(10, 1)$.

Super-exceptional 3-form flux. The *raison d'être* of the super-exceptional M-spacetime (3) is that it carries a map

$$\mathbb{R}_{\text{ex}_s}^{10,1|32} \xrightarrow{i} \mathbb{R}^{10,1|32}$$

to ordinary $D = 11$ $\mathcal{N} = 1$ super-spacetime (7) and a universal 3-flux form

$$H_{\text{ex}_s} \in \text{CE}(\mathbb{R}_{\text{ex}_s}^{10,1|32}) \quad (14)$$

that solves the twisted super-form Bianchi identity

$$dH_{\text{ex}_s} = i^* G_4, \quad (15)$$

in the base case the M-theory C-field 4-flux G_4 has vanishing bosonic component

$$G_4 = G_4^{(0)} := \frac{1}{2}(\psi \wedge \Gamma_{a_1 a_2} \cdot \psi) \wedge e^{a_1} \wedge e^{a_2}, \quad (16)$$

hence in the case that it only has its bifermionic component, fixed by the torsion constraint of 11d supergravity, as shown on the right of (16). Explicitly, H_{ex_s} is a polynomial in wedge products of the super-exceptional vielbein (9) of the following form [34, 6][18, 3][50, 3.5], where the \oplus -notation indicates that we are showing only the monomials that appear, but (for readability, and since this is all we need here) not the real coefficients (which are functions of the parameter s) that multiply them:

$$\begin{aligned} \text{super-exceptional} \\ \text{3-flux form} \\ H_{\text{ex}_s} = & e_{a_1 a_2} \wedge e^{a_1} \wedge e^{a_2} \\ & \oplus e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \\ & \oplus e_{a_1 a_2} b_1 b_2 b_3 \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} b_1 b_2 b_3 \\ & \oplus e^{a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} e_{a_0 a_1 a_2 a_3 a_4} \wedge e_a \wedge e_{a_5 a_6 a_7 a_8 a_9} \\ & \oplus e^{a_1 a_2 a_3 a_4 a_5 a_6 b_1 b_2 b_3 b_4 b_5} e_{a_1 a_2 a_3 c_1 c_2} \wedge e_{a_4 a_5 a_6}^{c_1 c_2} \wedge e_{b_1 b_2 b_3 b_4 b_5} \\ & \oplus \langle \eta \wedge \Gamma_a \cdot \psi \rangle \wedge e^a \oplus \langle \eta \wedge \Gamma^{a_1 a_2} \cdot \psi \rangle \wedge e_{a_1 a_2} \oplus \langle \eta \wedge \Gamma^{a_1 a_2 a_3 a_4 a_5} \cdot \psi \rangle \wedge e_{a_1 a_2 a_3 a_4 a_5}. \end{aligned} \quad (17)$$

Super-exceptional PS-Lagrangian. From this, we found in [50, Prop. 5.9] the super-exceptional solution to the next twisted Bianchi identity

$$d\text{Lex}_s = 2\left(\frac{1}{2}H_{\text{ex}_s} \wedge i^*G_4 + i^*G_7\right)^{\text{vert}} \quad (18)$$

characterizing the gauge sector of the heterotic M5-brane Lagrangian. Still in the base case that the dual 7-flux has vanishing bosonic component

$$G_7 = \frac{1}{5!} \langle \psi \wedge \Gamma_{a_1 \dots a_5} \cdot \psi \rangle \wedge e^{a_1} \wedge \dots \wedge e^{a_5}, \quad (19)$$

hence in the case that it only has its bifermionic component fixed by the torsion constraint of 11d supergravity, as shown on the right of (19), this is given by the following *super-exceptional PS-Lagrangian* [50, (72)]:

$$\begin{aligned} \text{Super-exceptional PS-Lagrangian} \quad \mathbf{L}_{\text{ex}_s}^{\text{PS}} &= \underbrace{\left((\iota_{5'} \oplus \iota_{5'6789}) \overbrace{H_{\text{ex}_s}}^{\text{super-exceptional 3-flux form}} \right)}_{\text{contraction with super-exceptional lift of isometry } v_{5'}} \wedge \underbrace{\left(\overbrace{H_{\text{ex}_s}}^{\text{super-exceptional 3-flux form}} \wedge \underbrace{e^{5'}}_{\text{dual form to isometry } v_{5'}} \right)}_{\text{dual form to isometry } v_{5'}} \\ &= \left(\begin{aligned} &e_{a5'} \wedge e^a \\ &\oplus \varepsilon^{a_0 a_1 a_2 a_3 a_4} 5' a_5 a_6 a_7 a_8 a_9 e_{a_0 a_1 a_2 a_3 a_4} \wedge e_{a_5 a_6 a_7 a_8 a_9} \\ &\oplus e^{a5'} \wedge e_{a6789} \\ &\oplus \varepsilon^{a_0 a_1 a_2 a_3 a_4} a 5' 6789 e_{a_0 a_1 a_2 a_3 a_4} \wedge e_a \\ &\oplus \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} 5' 6789 e_{a_1 a_2 a_3 c_1 c_2} \wedge e_{a_4 a_5 a_6} c_1 c_2 \\ &\oplus \langle \eta \wedge \Gamma_{5'} \cdot \psi \rangle \end{aligned} \right) \\ &\quad \wedge \left(\begin{aligned} &e_{a_1 a_2} \wedge e^{a_1} \wedge e^{a_2} \\ &\oplus e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \\ &\oplus e_{a_1 a_2} b_1 b_2 b_3 \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} b_1 b_2 b_3 \\ &\oplus \varepsilon^{a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} e_{a_0 a_1 a_2 a_3 a_4} \wedge e_a \wedge e_{a_5 a_6 a_7 a_8 a_9} \\ &\oplus \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} b_1 b_2 b_3 b_4 b_5 e_{a_1 a_2 a_3 c_1 c_2} \wedge e_{a_4 a_5 a_6} c_1 c_2 \wedge e_{b_1 b_2 b_3 b_4 b_5} \\ &\oplus \langle \eta \wedge \Gamma_a \cdot \psi \rangle \wedge e^a \oplus \langle \eta \wedge \Gamma^{a_1 a_2} \cdot \psi \rangle \wedge e_{a_1 a_2} \oplus \langle \eta \wedge \Gamma^{a_1 a_2 a_3 a_4 a_5} \cdot \psi \rangle \wedge e_{a_1 a_2 a_3 a_4 a_5} \end{aligned} \right) \wedge e^{5'} \\ &= \underbrace{e_{\alpha_1 5'} \wedge e_{\alpha_2 \alpha_3} \wedge e^{\alpha_1} \wedge e^{\alpha_2} \wedge e^{\alpha_3} \wedge e^{5'}}_{\text{free Perry-Schwarz Lagrangian}} \oplus \underbrace{\left(e_{a5'} \wedge e^a \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \wedge e^{5'} \oplus \dots \right)}_{\text{super-exceptional correction terms}} \quad (20) \end{aligned}$$

This expression is the definition [50, (72)] spelled out, using [50, (44)] and discarding some vielbein generators with an odd number of indices $a \in \{6, 7, 8, 9\}$ in the fourth summand of the first wedge factor, since these vanish on the MK6-locus, according to Theorem 7 below.

Super-exceptional sigma-model fields. A *field configuration* of the super-exceptional sigma-model is a dgc-superalgebra homomorphism dual to a map from the super-worldvolume of a heterotic M5-brane to the super-exceptional $\frac{1}{2}$ M5-locus

$$\begin{array}{ccc} \mathcal{N} = (1,0) \text{ M5-brane super-worldvolume} & \xleftarrow{\sigma^*} & \text{super-exceptional } \frac{1}{2} \text{ M5-locus} \\ \Omega^\bullet(\mathbb{R}^{5,1|8}) & \xleftarrow{\text{super-exceptional sigma-model field}} & \text{CE}\left(\left(\mathbb{R}_{\text{ex}_s}^{9,1|16}\right)^{\mathbb{Z}_2^A}\right) \end{array} \quad (21)$$

Here the field components

$$\sigma^*(e^\alpha) := dx^\alpha + \text{fermions}, \quad \alpha \in \{0, 1, 2, 3, 4, 5'\} \quad (22)$$

are fixed by the super-embedding condition and the value of the super-exceptional PS-Lagrangian on such a field configuration is the pullback of (20) to the M5-brane worldvolume along this super-exceptional sigma-model field:

$$\mathbf{L}_{\text{ex}_s}^{\text{PS}}(\sigma) := \sigma^*(\mathbf{L}_{\text{ex}_s}^{\text{PS}}). \quad (23)$$

Recovering the free PS-Lagrangian and 4d electromagnetism. The further super-exceptional vielbein field components

$$\sigma^*(e_{\mu 5'}) = d(A_\mu) + \text{fermions}, \quad \mu \in \{0, 1, 2, 3\}, \quad (24)$$

are interpreted as those of an electromagnetic vector potential A with field strength $F := dA$ in 4-dimensional spacetime. The condition of Hodge self-duality

$$\star_6 H_3 = H_3 \quad (25)$$

of the ordinary flux 3-form

$$H_3(\sigma) := \sigma^*(e_{\alpha_1 \alpha_2} \wedge e^{\alpha_1} \wedge e^{\alpha_2}) \quad (26)$$

then enforces

$$\sigma^*(e_{\mu_1 \mu_2}) = \frac{1}{2}(\star_4 F)_{\mu_1 \mu_2} dx^4 \quad (27)$$

which is the corresponding Hodge dual field strength times dx^4 . With this, the first summand in the second line of (20), which gives [50, Prop. 5.1] the Henneaux-Teitelboim-Perry-Schwarz Lagrangian for a free self-dual 3-form field [106, (17)][67]

$$\mathbf{L}_{\text{ex}_s}^{\text{PSfree}} = e_{\alpha 5'} \wedge e^\alpha \wedge e_{\alpha_1 \alpha_2} \wedge e^{\alpha_1} \wedge e^{\alpha_2} \wedge e^{5'}, \quad (28)$$

evaluates to the Maxwell Lagrangian for source-free electromagnetism in 4d:

$$\sigma^*(\mathbf{L}_{\text{ex}_s}^{\text{PSfree}}) = (dA \wedge \star_4 dA) \wedge dx^4 \wedge dx^{5'} + \text{fermions}. \quad (29)$$

This is, at its core, the super-exceptional incarnation from [50] of the Perry-Schwarz mechanism [106][126] in the construction of the M5-brane action functional.

Beyond the free PS-Lagrangian. But the full super-exceptional Lagrangian (20) evidently has many more terms than just (28), describing a rich interacting worldvolume theory on the M5-brane. In particular, next there is a term of Wess-Zumino form

$$\mathbf{L}_{\text{ex}_s}^{\text{WZW}} := (l_{\nu 5'} \oplus \nu^{s' 6789} H_{\text{ex}_s}) \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \wedge e^{5'}. \quad (30)$$

We analyze the value of this term on super-exceptional sigma-model fields below in §4.

But first we turn now to the classification of the super-exceptional field content.

3. Classification of the field content

Here we prove the classification, shown in the big diagram (4), of the super-exceptional vielbein fields on super-exceptional M-theory spacetime (Definition 3) restricted to the super-exceptional MK6-locus (Definition 6) regarded as a representation of the residual symmetry group $\text{Spin}(3, 1) \times \text{U}(1)_V \times \text{SU}(2)_R$, according to decompositions (3).

Much of the table (4) follows from straightforward branching of exterior power representations, recalled as Example 14, and from the familiar Spin representation branching

$$\begin{array}{ccccc} \text{Rep}_{\mathbb{R}}(\text{Spin}(10, 1)) & \longrightarrow & \text{Rep}_{\mathbb{R}}(\text{Spin}(9, 1)) & \xrightarrow{(-)^{\mathbb{Z}_2^A}} & \text{Rep}_{\mathbb{R}}(\text{Spin}(5, 1)) \\ 32 & \longmapsto & 16 \oplus \overline{16} & \longmapsto & 8 \oplus \overline{8} \\ \text{recalled as:} & & \text{Remark 26} & & \text{Lemma 38} \end{array}$$

The further statements we need are captured in the following result:

Theorem 7 (Fields at A_1 -singularity). Under passage to the fixed locus (36) of the $\mathbb{Z}_2^A \subset \text{SU}(2)_L$ -action in (3) we have the following representations under the residual group actions:

$\text{Rep}_{\mathbb{R}}(\text{SU}(2)_L \times \text{SU}(2)_R) \xrightarrow{(-)^{\mathbb{Z}_2^A}} \text{Rep}_{\mathbb{R}}(\text{SU}(2)_R)$ $\wedge^2 \mathbf{4} \mapsto \mathbf{3}$ $\downarrow \wr$ $\langle e^{a_1 a_2} \rangle_{a_i \in \{6,7,8,9\}} \quad \langle \frac{1}{2} e^{6I} - \frac{1}{2} \epsilon^{IJK} e^{JK} \rangle_{I \in \{7,8,9\}}$ $\wedge^1 \mathbf{4}, \wedge^3 \mathbf{4} \mapsto 0$	discussed in §3.3; see Prop. 36
$\text{Rep}_{\mathbb{R}}(\text{Spin}(5, 1) \times \text{SU}(2)_R) \longrightarrow \text{Rep}_{\mathbb{R}}(\text{Spin}(3, 1) \times \text{U}(1)_V \times \text{SU}(2)_R)$ $\mathbf{8} \mapsto \mathbf{2}_{\mathbb{C}} \boxtimes \mathbf{1}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}}$ $\downarrow \wr$ $\mathbb{H} \quad \mathbb{C} \oplus \mathbb{C}j$ $\oplus \quad \oplus$ $\mathbb{H} \quad \mathbb{C} \oplus \mathbb{C}j$ $\downarrow \wr$ $\mathbf{8} \mapsto \bar{\mathbf{2}}_{\mathbb{C}} \boxtimes \mathbf{1}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}}$	discussed in §3.4; see Prop. 42

Remark 8 (*Real vs. complex representations*). The classification in (4) is as *real* representations, since all supersymmetry algebras (7) are based on real Spin representations (e.g. [53, 3]). Now, the two complex Weyl spinor representations $\mathbf{2}_{\mathbb{C}}, \bar{\mathbf{2}}_{\mathbb{C}}$ (see (66) below) of $\text{Spin}(3, 1)$, which are distinct as complex representations, actually become isomorphic when regarded as real representations (recalled as Lemma 22 below). But this degeneracy is lifted by the further action of $\text{U}(1)_V$: The real representations underlying their outer complex tensor product (33) with the $\mathbf{1}_{\mathbb{C}}$ of $\text{U}(1)$ (41) are *not* isomorphic even as real representations (Lemma 43): $\mathbf{2}_{\mathbb{C}} \boxtimes \mathbf{1}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}} \simeq_{\mathbb{R}} / \bar{\mathbf{2}}_{\mathbb{C}} \boxtimes \mathbf{1}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}}.$

Remark 9 (*Gauge enhancement at ADE-singularities*). A famous informal argument suggests that M2-branes on around vanishing 2-cycles inside an ADE-singularity appear as $\text{SU}(2)$ gauge fields on the transversal D-branes (e.g., [2, 3.1.2][73, Ex. 2.2.5]). Now, if we interpret:

- (a) The exceptional vielbein components $e_{a_1 a_2}$ in (4) as being charges of M2-branes stretched along the directions $v_{a_1} \wedge v_{a_2}$;
- (b) elements of $\wedge^2 \mathbf{4} = \wedge^2 \mathbb{H}$ as 2-cocycles on the transversal Euclidean conical orbifold geometry (1);
- (c) elements of the fixed locus $(\wedge^2 \mathbf{4})^{\mathbb{Z}_2^A} = \langle e^{I6} \rangle$ as the restriction of these 2-cocycles to the singular point of the $\text{SU}(2)_L$ -action, hence to their evaluation on 2-cycles that are shrunk into the singularity;

then this informal story becomes the statement of Theorem 7. Explicitly, with the identification of Lemma 33, we have:

$$\langle e^{I6} + \epsilon^{IJK} e_{IJ}, I, J \in \{7, 8, 9\} \rangle \longleftrightarrow \left\{ \begin{array}{l} \text{Charges of M2-branes wrapped on} \\ \text{vanishing 2-cycles in ADE-singularity} \end{array} \right\}.$$

In the remainder we spell out in detail the proof of Theorem 7. This becomes nicely transparent in terms of octonionic 2-component spinor representations. Since this is not as widely known as it deserves to be, we use the occasion to recall all the ingredients, such as to make the proof fully self-contained.

3.1. Background in representation theory

For reference and to fix conventions, we briefly recall some basic concepts of representation theory.

Representation ring. For G a group, we write $\text{Rep}_{\mathbb{R}}(G)$ for its *real representation ring*: elements are isomorphism classes of real-linear finite-dimensional G -representations, addition in the ring comes from direct sum of representations, and the product in the ring from the tensor product of representations.

Irreducible representation. We denote irreducible representations of G by their dimension, typeset in boldface, equipped with decorations in case there are inequivalent irreps of the same dimension. Then we write

$$k \cdot \mathbf{N} := \underbrace{\mathbf{N} \oplus \cdots \oplus \mathbf{N}}_{k \text{ summands}} \in \text{Rep}_{\mathbb{R}}(G) \quad (31)$$

for the k -fold direct sum of an irrep with itself, of total dimension $\dim_{\mathbb{R}}(k \cdot \mathbf{N}) = k \mathbf{N}$.

Exterior power representations. The representation ring is a “lambda-ring” in that for any representation $V \in \text{Rep}_{\mathbb{R}}(G)$ and $p \in \mathbb{N}$ we have the k -fold exterior power representation

$$\wedge^p V \in \text{Rep}_{\mathbb{R}}(G). \quad (32)$$

Restricted representations. For $G_1 \xrightarrow{f} G_2$ a group homomorphism there is a representation-ring homomorphism given by regarding a G_2 representation V as a G_1 -representation by acting via f :

$$\begin{array}{ccc} \text{Rep}_{\mathbb{R}}(G_2) & \xrightarrow{f^*} & \text{Rep}_{\mathbb{R}}(G_1) \\ [(g_2, v) \mapsto g_2 \cdot v] & \longmapsto & [(g_1, v) \mapsto f(g_1) \cdot v] \end{array} \quad (33)$$

Specifically when $f: H \hookrightarrow G$ is an inclusion of subgroups, then forming *restricted representations* f^* as in (33) is also called the “branching of representations” under “breaking of symmetry” from G to H , since irreps on the left will in general “branch” into direct sums of irreps on the right:

$$\begin{array}{ccc} \text{Rep}_{\mathbb{R}}(G) & \xrightarrow{\iota^*} & \text{Rep}_{\mathbb{R}}(H) \\ \mathbf{N} & \longmapsto & \mathbf{n}_1 \oplus \cdots \oplus \mathbf{n}_k \\ \text{irrep of } G & & \text{direct sum of irreps of } H \end{array} \quad (34)$$

Outer tensor product. When $G = H_1 \times H_2$ is a direct product group then the operation of forming the tensor product as G -representations of an H_1 - and an H_2 -representation, both regarded as G -representations under (33) via the projection homomorphisms $G \xrightarrow{\text{pr}_i} H_i$, is also called the *outer tensor product* of H_i -representations and denoted by a square tensor product symbol:

$$\boxtimes : \text{Rep}_{\mathbb{R}}(H_1) \times \text{Rep}_{\mathbb{R}}(H_2) \xrightarrow{\text{pr}_1^* \times \text{pr}_2^*} \text{Rep}_{\mathbb{R}}(H_1 \times H_2) \times \text{Rep}_{\mathbb{R}}(H_1 \times H_2) \xrightarrow{\otimes} \text{Rep}_{\mathbb{R}}(H_1 \times H_2). \quad (35)$$

Fixed points. Moreover, in this case when $G = H_1 \times H_2$ is a direct product group, passage to H_2 -fixed points in the representation of an $H_1 \times H_2$ -representation is an additive functor (not though a monoidal one) denoted

$$\begin{array}{ccc} (-)^{H_2} : \text{Rep}_{\mathbb{R}}(H_1 \times H_2) & \longrightarrow & \text{Rep}_{\mathbb{R}}(H_1) \\ V & \longmapsto & V^{H_2} := \{v \in V \mid h_1 \cdot v = v\}. \end{array} \quad (36)$$

Notice that the H_2 -fixed points of an outer tensor product (35) with an irreducible H_2 -representation $\mathbf{N} \in \text{Rep}_{\mathbb{R}}(H_2)$ are non-vanishing precisely if \mathbf{N} is the trivial representation:

$$\left(\underbrace{V}_{\text{any rep of } H_1} \boxtimes \underbrace{\mathbf{N}}_{\text{irrep of } H_2} \right)^{H_2} = \begin{cases} V & \text{if } \mathbf{N} = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases} \quad (37)$$

Example 10 (Pauli matrices). The fundamental complex representation of $\text{SU}(2)$

$$\mathbf{2}_{\mathbb{C}} \in \text{Rep}_{\mathbb{C}}(\text{SU}(2)), \quad \mathbf{2}_{\mathbb{C}} \in \text{Rep}_{\mathbb{C}}(\mathfrak{su}(2)) \quad (38)$$

regarded as the underlying Lie algebra representation, indicated by the same symbol in the right, has as representation matrices the Pauli matrices, which we normalize as:

$$\tau^1 := \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau^2 := \frac{i}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau^3 := \frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (39)$$

such that

$$[\tau^i, \tau^j] = \epsilon^{ij}_k \tau^k \quad (40)$$

Example 11 (Outer product with representations of $U(1)$). We denote the irreducible complex representations of $U(1)$, labeled by $n \in \mathbb{Z}$, by

$$\mathbf{1}_{\mathbb{C}}^n \in \text{Rep}_{\mathbb{C}}(U(1)) \quad \text{abbreviated for } n = \pm 1 \text{ as: } \mathbf{1}_{\mathbb{C}}^{\pm} \in \text{Rep}_{\mathbb{C}}(U(1)). \quad (41)$$

Hence their outer tensor product (35) with the Pauli matrices from Example 10

$$\mathbf{1}_{\mathbb{C}}^n \boxtimes \mathbf{2}_{\mathbb{C}} \in \text{Rep}_{\mathbb{C}}(\text{U}(1) \times \text{SU}(2))$$

with the fundamental representation (38) of $\text{SU}(2)$ has as Lie algebra representation matrices the Pauli matrices (39) and one more matrix given by

$$\tau^0 := i \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}. \quad (42)$$

Example 12 (*Spinorial representations*). For $d \in \mathbb{N}$ we write $\mathbb{R}^{d,1}$ for the real inner product space with bilinear form $\eta := \text{diag}(-1, +1, +1, \dots, +1)$. We take the corresponding Clifford algebra $\text{Cl}_{\mathbb{R}}(d, 1)$ to be the real associative algebra generated from $\{\Gamma_a\}_{a=0}^d$ subject to the relations

$$\Gamma_{a_1} \Gamma_{a_2} + \Gamma_{a_2} \Gamma_{a_1} = +2\eta_{ab}, \quad (43)$$

and we write

$$\text{Cl}_{\mathbb{R}}^{\text{ev}}(d, 1) \subset \text{Cl}_{\mathbb{R}}(d, 1) \quad \text{for the subalgebra generated from the products } \{\Gamma_{a_1} \Gamma_{a_2}\}_{a_1, a_2}. \quad (44)$$

Now if $\{R_{a_1 a_2}\}_{a_1, a_2=0}^d$ denotes the standard linear generators of the Lie algebra $\mathfrak{so}(d, 1)$, with $R_{a_1 a_2} = -R_{a_2 a_1}$ and with Lie bracket given by

$$[R_{a_1 a_2}, R_{b_1 b_2}] = \eta_{a_2 b_1} R_{a_1 b_2} - \eta_{a_1 b_1} R_{a_2 b_2} + \eta_{a_2 b_2} R_{b_1 a_1} - \eta_{a_1 b_2} R_{b_1 a_2} \quad (45)$$

then the assignment

$$\begin{aligned} \mathfrak{so}(d, 1) &\hookrightarrow \text{Cl}_{\mathbb{R}}^{\text{ev}}(d, 1) \\ R_{a_1 a_2} &\longmapsto \frac{1}{4} [\Gamma_{a_1}, \Gamma_{a_2}] = \begin{cases} \frac{1}{2} \Gamma_{a_1} \Gamma_{a_2} & | \quad a_1 \neq a_2 \\ 0 & | \quad \text{otherwise} \end{cases} \end{aligned} \quad (46)$$

constitutes a linear embedding which is a Lie algebra morphism with respect to the commutator bracket on the right (e.g. [86, Prop. 6.2]). As a consequence, every associative algebra representation of $\text{Cl}_{\mathbb{R}}^{\text{ev}}(d, 1)$ (“Clifford module”) becomes a Lie algebra representation of $\mathfrak{so}(d, 1)$, and thus a Lie group representation of the corresponding simply connected Lie group $\text{Spin}(d, 1)$, via the exponential map

$$\begin{aligned} \mathfrak{so}(d, 1) &\longrightarrow \text{Spin}(d, 1) \quad \subset \quad \text{Cl}_{\mathbb{R}}^{\text{ev}}(d, 1). \\ \alpha R_{a_1 a_2} &\longmapsto \exp(\alpha \frac{1}{2} \Gamma_{a_1} \Gamma_{a_2}) \end{aligned} \quad (47)$$

The representations obtained this way are the *spinorial* representations, in contrast to the vector representations (51) and their exterior powers (32).

Notice that for $a_1, a_2 > 0$, whence $(\Gamma_{a_1} \Gamma_{a_2})^2 = -1$, Euler’s formula (which applies in any Banach algebra) gives

$$\exp(\alpha \Gamma_{a_1} \Gamma_{a_2}) = \cos(\alpha) + \sin(\alpha) \Gamma_{a_1} \Gamma_{a_2} \in \text{Cl}_{\mathbb{R}}^{\text{ev}}(d, 1). \quad (48)$$

Therefore the exponent in (47) with the prefactor of $1/2$ from (46), is such that rotations by an angle of $\alpha = 2\pi$ are represented by

$$\exp(2\pi \frac{1}{2} \Gamma_{a_1} \Gamma_{a_2}) = -1 \quad (49)$$

and it is only rotations by $\alpha = 4\pi$ that yield the identity on spinors, reflecting the double covering

$$\text{Spin}(d, 1) \twoheadrightarrow \text{SO}(d, 1). \quad (50)$$

Example 13. For $D \in \mathbb{N}$ the vector representation of $\text{Spin}(D - 1, 1)$ (or of $\text{Spin}(D)$) is the defining representation of $\text{SO}(d, 1)$ (or $\text{SO}(D)$) via (50). This is a D -dimensional irrep, which we hence denote

$$\mathbf{D} \in \text{Rep}_{\mathbb{R}}(\text{Spin}(D - 1, 1)) \quad \text{or} \quad \mathbf{D} \in \text{Rep}_{\mathbb{R}}(\text{Spin}(D)). \quad (51)$$

Example 14 (Restriction of exterior power representation). For natural numbers $D_1 + D_2 = D$ the restriction (34) of an exterior power (32) of the vector representation \mathbf{D} of $\text{Spin}(D-1, 1)$ along the canonical inclusion $\text{Spin}(D_1-1, 1) \times \text{Spin}(D_2) \xrightarrow{\iota} \text{Spin}(D-1, 1)$ is

$$\iota^*(\wedge^k \mathbf{D}) = \bigoplus_{p \in \{0, \dots, k\}} (\wedge^p \mathbf{D}_1) \boxtimes (\wedge^{k-p} \mathbf{D}_2). \quad (52)$$

3.2. Octonionic 2-component spinors

We discuss here real Spin representations (see Example 12 for spinor conventions) in spacetime dimensions 11, 10, 7, 6 and 4 in terms of matrices with coefficients in the octonions (following [83], reviewed in [13][14][15]) which is well-adapted to the geometry of the $\frac{1}{2}$ M5-brane locus §3.3, following [73].

We will find useful the presentation of the octonions as generated from the quaternions and from one more imaginary unit ℓ . This *Dickson double construction* [37, (6)] is well-known in itself, but since the explicit minimal set of relations (54) and (55) below (highlighted in [13, 2.2]) is not as widely used (but see [73, Def. A.6][74, Def. 26]), we recall it:

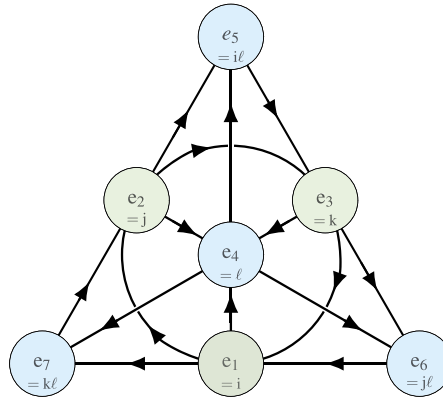
Lemma 15 (Octonions by generators-and-relations). The real star-algebra \mathbb{O} of octonions, with its star-operation (conjugation) to be denoted $(-)^*$, is equivalently that generated from the algebra of quaternions \mathbb{H} and from one more algebra element ℓ , subject to these relations, for all $q, p \in \mathbb{H}$:

$$\ell^2 = -1, \quad \ell^* = -\ell \quad (53)$$

$$q(\ell p) = \ell(q^* p), \quad (q\ell)p = (qp^*)\ell \quad (54)$$

$$(\ell q)(p\ell) = -(qp)^*. \quad (55)$$

Applied to an orthonormal basis of imaginary unit quaternions, $q \in \{i, j, k\}$, this means that \mathbb{O} is generated from the seven imaginary unit elements shown in the diagram on the right, subject to these relations: $ab = c$, $ca = b$, $bc = a$, and $ba = -c$ for every pair of consecutive arrows $a \rightarrow b \rightarrow c$ shown.



Proof. Notice that the relations (54) imply the following further relations

$$\begin{aligned} q\ell &= \ell q^* & \ell q &= q^* \ell \\ (\ell q)p &= \ell(pq) & q(p\ell) &= (pq)\ell \\ (\ell q)(\ell p) &= -pq^* & (q\ell)(p\ell) &= -p^*q \end{aligned} \quad (56)$$

$$\begin{aligned} \text{hence in particular: } \ell(\ell p) &= -p & (q\ell)\ell &= -q \\ (\ell q)\ell &= -q^* & \ell(p\ell) &= -p^*. \end{aligned}$$

Consequently, one finds the general formula for the product of any pair of octonions x_i , parametrized as $x_i = q_i + p_i \ell$ or as $x_i = q_i + \ell p_i$ (with $q_i, p_i \in \mathbb{H}$), to be, respectively:

$$\begin{aligned} (q_1 + p_1 \ell)(q_2 + p_2 \ell) &= (q_1 q_2 - p_2^* p_1) + (p_2 q_1 + p_1 q_2^*) \ell, \\ (q_1 + \ell p_1)(q_2 + \ell p_2) &= (q_1 q_2 - p_2 p_1^*) + \ell(q_1^* p_2 + q_2 p_1). \end{aligned} \quad (57)$$

This is the formula for the octonionic product according to [37, (6)] (where the version in the first line appears), reviewed in [13, 2.2] (where the isomorphic version in the second line is given). \square

Remark 16 (Division algebra inclusions and Supersymmetry breaking patterns). Any choice of octonion generators e_1, \dots, e_7 according to Lemma 15 induces algebra inclusions of, consecutively, the real numbers, the complex numbers and the quaternions into the octonions

$$\begin{array}{ccccccc} \mathbb{O} & \longleftarrow & \mathbb{H} & \longleftarrow & \mathbb{C} & \longleftarrow & \mathbb{R} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \langle 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle & & \langle 1, e_1, e_2, e_3 \rangle & & \langle 1, e_1 \rangle & & \langle 1 \rangle \end{array} \quad (58)$$

Moreover, multiplication of the linear sub-spaces corresponding to these sub-algebras with the remaining generators induces distinguished linear isomorphisms

$$\mathbb{O} \simeq_{\mathbb{R}} \mathbb{H} \oplus \mathbb{H}\ell \simeq_{\mathbb{R}} (\mathbb{C} \oplus \mathbb{C}j) \oplus (\mathbb{C} \oplus \mathbb{C}j)\ell \simeq_{\mathbb{R}} ((\mathbb{R} \oplus \mathbb{R}i) \oplus (\mathbb{R} \oplus \mathbb{R}i)j) \oplus ((\mathbb{R} \oplus \mathbb{R}i) \oplus (\mathbb{R} \oplus \mathbb{R}i)j)\ell. \quad (59)$$

This extra structure on \mathbb{O} , corresponding to the choice of an adapted linear basis according to Lemma 15, turns out to reflect the supersymmetry breaking sequences

$$\begin{array}{ccccccc} \mathbb{R}^{10,1|32} & \longrightarrow & \mathbb{R}^{6,1|16} & \longrightarrow & \mathbb{R}^{4,1|8} & \longrightarrow & \mathbb{R}^{3,1|4} \longrightarrow \mathbb{R}^{2,1|2} \\ & \searrow & & \searrow & & \searrow & \\ & \mathbb{R}^{9,1|16} & \longrightarrow & \mathbb{R}^{5,1|8} & \longrightarrow & \mathbb{R}^{3,1|4} & \longrightarrow \mathbb{R}^{2,1|2} \end{array} \quad (60)$$

This is the statement of Proposition 20 and Proposition 25 below.

An illustrative example computation with the relations (56) is the following (used below in Proposition 38):

Lemma 17 (Reversal of sign of ℓ -component by left action). The action of consecutive left multiplication by the generators e_4, e_5, e_6, e_7 from Lemma 15 on any octonion $x = q + p\ell$ ($q, p \in \mathbb{H}$) is by reversal of the sign of the ℓ -component:

$$e_4(e_5(e_6(e_7(q + p\ell)))) = q - p\ell. \quad (61)$$

Proof. Using the relations (54) and (56) and the associativity of the multiplication on quaternions, we compute as follows:

$$\begin{aligned} e_4(e_5(e_6(e_7x))) &= \ell((i\ell)((j\ell)((k\ell)x))) = \ell((i\ell)((j\ell)((kx^*)\ell))) = \ell((i\ell)((xk)j)) = \ell((i(j(kx^*))\ell)) \\ &= ((xk)j)i = (((q + p\ell)k)j)i = qkji + ((p\ell)k)j)i = qkji - (pkji)\ell = q - p\ell. \quad \square \end{aligned}$$

Notation 18 (Conjugation). In what follows, we will denote conjugation by $(-)^*$ in any of the real $*$ -algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and \mathbb{O} . For a matrix A with coefficients in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ the component-wise conjugate matrix will be denoted by A^* , while the Hermitian conjugate matrix will be denoted by $A^\dagger := {}^t A^*$.

We record the following immediate generalization of the standard Pauli matrix construction:

Lemma 19 (\mathbb{K} -Pauli matrices). Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. There is a linear isomorphism of real vector spaces equipped with quadratic forms (the color code follows the configurations in (1))

$$\begin{aligned} (\mathbb{R}^{\dim_{\mathbb{R}} \mathbb{K}+1,1}, \eta) &\xrightarrow[\simeq]{\sigma} (\text{Mat}(2, \mathbb{K})^{\text{Herm}}, -\det) \\ v = \begin{bmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \\ v^7 \\ v^8 \\ v^9 \end{bmatrix} &\mapsto \begin{bmatrix} v^0 + v^1 & v^2 \\ v^2 & v^0 - v^1 \end{bmatrix} + (v^3 e_1 + v^4 e_2 + v^5 e_3 + v^6 e_4 + v^7 e_5 + v^8 e_6 + v^9 e_7) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

(62)

from $(\dim_{\mathbb{R}} \mathbb{K} + 2)$ -dimensional Minkowski spacetime with quadratic form being its Minkowski metric $\eta = \text{diag}(-1, 1, 1, \dots, 1)$ to the vector space of 2×2 \mathbb{K} -Hermitian matrices with quadratic form being minus the determinant operation.

We denote the \mathbb{K} -Pauli matrices (62) corresponding to the coordinate basis elements as follows:

$$\sigma_a := \sigma(v_a) \quad \text{and} \quad \sigma^a := \eta^{ab} \sigma_b, \quad (63)$$

where $v_a \in \mathbb{R}^{\dim_{\mathbb{R}} \mathbb{K} + 1, 1}$ denotes the vector with components $(v_a)^b := \delta_a^b$.

The following observation is due to [83], with a streamlined review in [14]:

Proposition 20 (Real Spin representations in dimension 10, 6, 4, 3, via \mathbb{K} -Pauli matrices). Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

(i) The assignments

$$\Gamma_{a_1} \Gamma_{a_2} \mapsto \sigma^{a_1} \cdot (\sigma_{a_2} \cdot (-)) \quad \text{and} \quad \Gamma_{a_1} \Gamma_{a_2} \mapsto \sigma_{a_1} \cdot (\sigma^{a_2} \cdot (-)) \quad (64)$$

of left multiplication actions by the \mathbb{K} -Pauli matrices (63) for alternating index positions constitute representations of the even Clifford algebras (44) on the real vector space underlying \mathbb{K}^2 , hence are real algebra homomorphisms

$$\text{Cl}_{\mathbb{R}}^{\text{ev}}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1) \longrightarrow \text{End}_{\mathbb{R}}(\mathbb{K}^2) .$$

(ii) As a consequence (46) there are two real spinorial representations $2 \dim_{\mathbb{R}} \mathbb{K}$ and $\overline{2 \dim_{\mathbb{R}} \mathbb{K}}$ of $\text{Spin}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1)$, regarded as real modules of the Lie algebra $\mathfrak{so}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1)$, each isomorphic to the real vector space underlying \mathbb{K}^2 equipped, respectively, with the following action of the standard basis elements $R_{a_1 a_2}$ (45):

$$\begin{array}{ccc} \mathfrak{so}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1) \times 2 \dim_{\mathbb{R}} \mathbb{K} & \xrightarrow{\quad} & 2 \dim_{\mathbb{R}} \mathbb{K} \\ \parallel & \downarrow \cong & \downarrow \cong \\ \mathfrak{so}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1) \times \mathbb{K}^2 & \xrightarrow{\quad} & \mathbb{K}^2 \\ (R_{a_1 a_2}, \psi) & \mapsto & \frac{1}{2} \sigma^{a_1} \cdot (\sigma_{a_2} \cdot \psi) \end{array} \quad \begin{array}{ccc} \mathfrak{so}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1) \times \overline{2 \dim_{\mathbb{R}} \mathbb{K}} & \xrightarrow{\quad} & \overline{2 \dim_{\mathbb{R}} \mathbb{K}} \\ \parallel & \downarrow \cong & \downarrow \cong \\ \mathfrak{so}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1) \times \mathbb{K}^2 & \xrightarrow{\quad} & \mathbb{K}^2 \\ (R_{a_1 a_2}, \psi) & \mapsto & \frac{1}{2} \sigma_{a_1} \cdot (\sigma^{a_2} \cdot \psi) \end{array} \quad (65)$$

Remark 21 (Complex Weyl representations). For $\mathbb{K} = \mathbb{C}$, the action by \mathbb{C} -Pauli matrices in (65) is clearly \mathbb{C} -linear, so the $\text{Spin}(3, 1)$ -representations $\mathbf{4}$ and $\overline{\mathbf{4}}$ are the real representations underlying a pair of complex representations. It is manifest from (65) for $\mathbb{K} = \mathbb{C}$ that these two complex representations are the standard complex 2-dimensional Weyl Spin representations, which we denote by

$$\begin{array}{ccc} \text{left/right} & \text{complex} & \text{real} \\ \text{Weyl Spin representations} & \text{representation ring} & \text{representation ring} \\ \mathbf{2}_{\mathbb{C}}, \overline{\mathbf{2}}_{\mathbb{C}} & \in \text{Rep}_{\mathbb{C}}(\text{Spin}(3, 1)) & \longrightarrow \text{Rep}_{\mathbb{R}}(\text{Spin}(3, 1)) \end{array} \quad (66)$$

Lemma 22 (Isomorphism of real spinor irreps of $\text{Spin}(3, 1)$). The representations $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $\text{Spin}(3, 1)$ obtained from (65) for $\mathbb{K} = \mathbb{C}$ are isomorphic as real representations (not as complex representations):

$$\mathbf{4} \simeq \overline{\mathbf{4}} \in \text{Rep}_{\mathbb{R}}(\text{Spin}(3, 1)).$$

Proof. Write $\epsilon := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \text{Mat}(2, \mathbb{C})$. We claim that the \mathbb{R} -linear isomorphism of \mathbb{R} -vector spaces

$$\begin{array}{ccc} \mathbb{C}^2 \simeq_{\mathbb{R}} \mathbf{4} & \xrightarrow[\simeq]{\phi} & \overline{\mathbf{4}} \simeq_{\mathbb{R}} \mathbb{C}^2 \\ \psi & \longmapsto & \epsilon \cdot \psi^* \end{array} \quad (67)$$

is an isomorphism of real representations of $\text{Spin}(3, 1)$. To see this, use that $\epsilon \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \epsilon^{-1} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ and observe that therefore, by (63), we have $\epsilon \cdot (\sigma_{\mu})^* \cdot \epsilon^{-1} = -\sigma^{\mu}$. As complex conjugation $(-)^*: \mathbb{C} \rightarrow \mathbb{C}$ is an \mathbb{R} -algebra homomorphism we have $(A \cdot B)^* = A^* \cdot B^*$ for complex matrices (following Notation 18). Therefore:

$$\begin{aligned}\epsilon \cdot (\sigma^{\mu_1} \cdot \sigma_{\mu_2})^* \cdot \epsilon^{-1} &= (\epsilon \cdot (\sigma^{\mu_1})^* \cdot \epsilon^{-1}) \cdot (\epsilon \cdot (\sigma_{\mu_2})^* \cdot \epsilon^{-1}) \\ &= (-\sigma_{\mu_1}) \cdot (-\sigma^{\mu_2}) = \sigma_{\mu_1} \cdot \sigma^{\mu_2}.\end{aligned}\quad (68)$$

With this the claim follows:

$$\begin{aligned}\phi\left(\frac{1}{2}\sigma^{\mu_1} \cdot \sigma_{\mu_2} \cdot \psi\right) &= \epsilon \cdot \left(\frac{1}{2}\sigma^{\mu_1} \cdot \sigma_{\mu_2} \cdot \psi\right)^* = \left(\epsilon \cdot \left(\frac{1}{2}\sigma^{\mu_1} \cdot \sigma_{\mu_2}\right)^* \cdot \epsilon^{-1}\right) \cdot (\epsilon \cdot \psi^*) \\ &= \frac{1}{2}\sigma_{\mu_1} \cdot \sigma^{\mu_2} \cdot \phi(\psi). \quad \square\end{aligned}\quad (69)$$

Remark 23 (\mathbb{K} -Weyl spinors beyond the complex case). We highlight the following subtle points:

(i) The reason that the proof of Proposition 22 does not identify the two Weyl Spin-representations $2_{\mathbb{C}}, \bar{2}_{\mathbb{C}}$ (66) when regarded as complex representations is due to the complex conjugation on the right of (67), which makes ϕ a real-linear map, but not a complex-linear map.

(ii) The reason that the proof of Proposition 22 does not generalize to identify also the two real representations $\mathbf{8}$ and $\bar{\mathbf{8}}$ of $\text{Spin}(5, 1)$, nor the two real representations $\mathbf{16}$ and $\bar{\mathbf{16}}$ of $\text{Spin}(9, 1)$ given by (65) for $\mathbb{K} = \mathbb{H}$ and $\mathbb{K} = \mathbb{O}$, respectively, is that in these cases, due to the non-commutativity of the quaternions and of the octonions, equations (68) and (69) do not hold, as quaternionic and octonion conjugation is not an \mathbb{R} -algebra homomorphism but an anti-homomorphism: $(xy)^* = y^*x^*$.

Next we record the following immediate generalization of the Dirac-matrix construction:

Lemma 24 (\mathbb{K} -Dirac matrices). Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. There is a linear isomorphism of real vector spaces equipped with quadratic forms

$$\begin{aligned}(\mathbb{R}^{\dim_{\mathbb{R}} \mathbb{K} + 2, 1}, \eta) &\xrightarrow{\simeq} (\text{Mat}(2, \mathbb{K})^{\text{Herm}} \times \mathbb{R}, -\det(-) + (-)^2) \hookrightarrow \text{Mat}(4, \mathbb{K})^{\text{Herm}}, \\ \begin{bmatrix} v^0 \\ \vdots \\ v^9 \\ v^{5'} \end{bmatrix} &\longmapsto \sum_{a=0}^9 v^a \begin{bmatrix} 0 & \sigma_a \\ \sigma^a & 0 \end{bmatrix} + v^{5'} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned}\quad (70)$$

from $(\dim_{\mathbb{R}} \mathbb{K} + 3)$ -dimensional Minkowski spacetime with quadratic form being its Minkowski metric $\eta = \text{diag}(-1, 1, 1, \dots, 1)$ to the vector space of \mathbb{K} -matrices, as shown, where σ_a and σ^a are the \mathbb{K} -Pauli matrices (63) from Lemma 19.

The following observation is due to [83], with a streamlined review in [15]:

Proposition 25 (Real Spin representations in dimension 11, 7, 5, 4, via \mathbb{K} -Dirac matrices). For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the assignment

$$\begin{aligned}\text{Cl}_{\mathbb{R}}(\dim_{\mathbb{R}} + 2, 1) &\longrightarrow \text{End}_{\mathbb{R}}(\mathbb{K}^2 \oplus \mathbb{K}^2) \\ \Gamma_a &\longmapsto \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot (-) & \text{for } a = 5' \\ \begin{bmatrix} 0 & \sigma^a \\ \sigma_a & 0 \end{bmatrix} \cdot (-) & \text{otherwise} \end{cases}\end{aligned}\quad (71)$$

of left multiplication action by the \mathbb{K} -Dirac matrices (70) constitutes a real representation of the full Clifford algebra (43) on the real vector space underlying \mathbb{K}^4 .

Remark 26 (Branching of 11d spinors in 10d). The linear representation of $\text{Spin}(\dim_{\mathbb{R}} \mathbb{K} + 2, 1)$ corresponding via (46) to the Clifford representation (71) restricted to a representation of $\text{Spin}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1)$ (omitting the index $5'$) is manifestly the direct sum of the two representations in Lemma 19:

$$\begin{aligned}\text{Cl}_{\mathbb{R}}^{\text{ev}}(\dim_{\mathbb{R}} \mathbb{K} + 1, 1) &\hookrightarrow \text{Cl}_{\mathbb{R}}^{\text{ev}}(\dim_{\mathbb{R}} \mathbb{K} + 2, 1) \longrightarrow \text{End}_{\mathbb{R}}(\mathbb{K}^2 \oplus \mathbb{K}^2) \\ \Gamma_{a_1} \Gamma_{a_2} &\longmapsto \begin{bmatrix} \sigma^{a_1} \cdot (\sigma_{a_2} \cdot (-)) & 0 \\ 0 & \sigma_{a_1} \cdot (\sigma^{a_2} \cdot (-)) \end{bmatrix}\end{aligned}\quad (72)$$

In terms of \mathbb{K} -Dirac matrix calculus, the Spin-invariant spinor pairing is given as follows:

Proposition 27 (Spinor pairing). For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, let $\mathbf{N} := \mathbb{K}^2 \oplus \mathbb{K}^2$ be the $N := 4\dim_{\mathbb{R}} \mathbb{K}$ -dimensional $\text{Spin}(\dim_{\mathbb{R}} \mathbb{K} + 2, 1)$ representation from Proposition 25. Then the spinor bilinear pairing is

$$\begin{aligned} \mathbf{N} \times \mathbf{N} &\longrightarrow \mathbb{R} \\ \psi, \phi &\longmapsto \langle \psi, \phi \rangle := \text{Re}(\psi^\dagger \cdot \Gamma_0 \cdot \phi) \end{aligned} \quad (73)$$

where on the right $\psi^\dagger := {}^t\psi^*$ is the Hermitian conjugate, Notation 18, and where “ \cdot ” denotes matrix multiplication over \mathbb{K} .⁴ Furthermore, this is bilinear, skew-symmetric and $\text{Spin}(\dim_{\mathbb{R}} \mathbb{K} + 2, 1)$ -invariant.

3.3. The A_1 -type singularity

Here we spell out basics of the linear representations of $\text{Sp}(1)_{L/R} \simeq \text{SU}(2)_{L/R}$ by left/right quaternion multiplication on the quaternion space \mathbb{H} . The resulting orbifold quotient $\mathbb{H} // \mathbb{Z}_{n+1}$ for $\mathbb{Z}_{n+1} \subset \text{SU}(2)_L$ is the A_n -type singularity (e.g. [120]).

Since some prefactors in the following crucially matter, we begin by making fully explicit:

The exceptional isomorphism $\text{Spin}(4) \cong \text{SU}(2)_L \times \text{SU}(2)_R$. As recalled in (45), the 6-dimensional real Lie algebra $\mathfrak{so}(4)$ has a distinguished basis $\{R_{a_1 a_2}\}_{1 \leq a_1 < a_2 \leq 4}$ with commutation relations

$$[R_{ij}, R_{kl}] = \delta_{jk} R_{il} + \delta_{il} R_{jk} - \delta_{jl} R_{ik} - \delta_{ik} R_{jl}. \quad (74)$$

Define elements $J_{L/R}^i \in \mathfrak{so}(4)$ by

$$J_L^{i-1} := -\frac{1}{2} R^{1i} - \frac{1}{4} \epsilon^{1ijk} R_{jk}, \quad J_R^{i-1} := \frac{1}{2} R^{1i} - \frac{1}{4} \epsilon^{1ijk} R_{jk}, \quad i, j, k \in \{2, 3, 4\}. \quad (75)$$

More explicitly:

$$\begin{aligned} J_L^1 &= -\frac{1}{2} R_{12} - \frac{1}{2} R_{34}, & J_L^2 &= -\frac{1}{2} R_{13} - \frac{1}{2} R_{42}, & J_L^3 &= -\frac{1}{2} R_{14} - \frac{1}{2} R_{23}, \\ J_R^1 &= \frac{1}{2} R_{12} - \frac{1}{2} R_{34}, & J_R^2 &= \frac{1}{2} R_{13} - \frac{1}{2} R_{42}, & J_R^3 &= \frac{1}{2} R_{14} - \frac{1}{2} R_{23}. \end{aligned} \quad (76)$$

It is immediate that $\{J_L^i, J_R^j\}_{i,j}$ is a linear basis of $\mathfrak{so}(4)$. Moreover, with (74), one finds the relations

$$[J_L^i, J_L^j] = \epsilon^{ijk} J_L^k, \quad [J_R^i, J_R^j] = \epsilon^{ijk} J_R^k, \quad [J_L^i, J_R^j] = 0.$$

Therefore, the subspaces

$$\mathfrak{su}(2)_L := \langle J_L^1, J_L^2, J_L^3 \rangle \subset \mathfrak{so}(4) \quad \text{and} \quad \mathfrak{su}(2)_R := \langle J_R^1, J_R^2, J_R^3 \rangle \subset \mathfrak{so}(4) \quad (77)$$

are two mutually commuting Lie subalgebras of $\mathfrak{so}(4)$, both canonically isomorphic to $\mathfrak{su}(2)$ (40), whose joint embedding is a Lie algebra isomorphism. This consequently induces an isomorphism between the corresponding simply connected compact Lie groups:

$$\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \xrightarrow{\cong} \mathfrak{so}(4), \quad \text{SU}(2)_L \times \text{SU}(2)_R \xrightarrow{\cong} \text{Spin}(4). \quad (78)$$

Remark 28 (Clifford representation of $\mathfrak{su}(2)_{L/R}$). The images $\tau_{L/R}^i$ of the elements $J_{L/R}^i \in \mathfrak{so}(4)$ (75) under the embedding $\mathfrak{so}(4) \hookrightarrow \text{Cl}_{\mathbb{R}}(4)$ (46) are given by

$$\begin{aligned} \tau_L^1 &= -\frac{1}{4} \Gamma_1 \Gamma_2 - \frac{1}{4} \Gamma_3 \Gamma_4, & \tau_L^2 &= -\frac{1}{4} \Gamma_1 \Gamma_3 - \frac{1}{4} \Gamma_4 \Gamma_2, & \tau_L^3 &= -\frac{1}{4} \Gamma_1 \Gamma_4 - \frac{1}{4} \Gamma_2 \Gamma_3, \\ \tau_R^1 &= \frac{1}{4} \Gamma_1 \Gamma_2 - \frac{1}{4} \Gamma_3 \Gamma_4, & \tau_R^2 &= \frac{1}{4} \Gamma_1 \Gamma_3 - \frac{1}{4} \Gamma_4 \Gamma_2, & \tau_R^3 &= \frac{1}{4} \Gamma_1 \Gamma_4 - \frac{1}{4} \Gamma_2 \Gamma_3. \end{aligned} \quad (79)$$

Lemma 29 (\mathbb{Z}_2 inside $\text{SU}(2)$). The 1-parameter subgroup $\{\exp(2\pi t J_L^1)\}_{t \in \mathbb{R}/\mathbb{Z}} \subseteq \text{SU}(2)_L$ (78) generated by the infinitesimal rotation J_L^1 (75) is a copy of $\text{U}(1)$. Inside it we have a copy of $\mathbb{Z}_2 \cong \{1, -1\}$ given by $\{1, \exp(2\pi J_L^1)\}$ (49). The image (79) of the generator of this copy of \mathbb{Z}_2 in the Clifford algebra (46) is

$$\exp(2\pi \tau_L^1) = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4. \quad (80)$$

Proof. Observing that $(\Gamma_1 \Gamma_2)^2 = (\Gamma_3 \Gamma_4)^2 = -1$ and that $\Gamma_1 \Gamma_2$ commutes with $\Gamma_3 \Gamma_4$ we have, with (79):

$$\exp(2\pi \tau_L^1) = \exp\left(-\frac{\pi}{2} \Gamma_1 \Gamma_2 - \frac{\pi}{2} \Gamma_3 \Gamma_4\right) = \exp\left(-\frac{\pi}{2} \Gamma_1 \Gamma_2\right) \exp\left(-\frac{\pi}{2} \Gamma_3 \Gamma_4\right) = (-\Gamma_1 \Gamma_2)(-\Gamma_3 \Gamma_4) = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4,$$

where in the third step we used Euler's formula (48) in the Clifford algebra. \square

⁴ The triple matrix product in (73) is associative even over \mathbb{O} , since the components of $\Gamma_0 = \begin{bmatrix} 0 & -1_{2 \times 2} \\ 1_{2 \times 2} & 0 \end{bmatrix}$ are real.

Example 30 (Defining representation of the symplectic group). The defining action of $\mathrm{Sp}(1) := \{q \in \mathbb{H} | qq^* = 1\}$ on the space \mathbb{H} of quaternions is a real 4-dimensional irreducible representation, to be denoted

$$\mathbf{4} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1)). \quad (81)$$

A priori there are two distinct such representations, given by left and quaternion multiplication, respectively

$$\begin{array}{ccc} \mathrm{Sp}(1) \times \mathbb{H} & \longrightarrow & \mathbb{H} \\ (q, v) & \longmapsto & q \cdot v \end{array} \quad , \quad \begin{array}{ccc} \mathrm{Sp}(1) \times \mathbb{H} & \longrightarrow & \mathbb{H} \\ (q, v) & \longmapsto & v \cdot q^* \end{array} \quad (82)$$

However, these are clearly isomorphic, via conjugation on all quaternions $\mathbb{H} \xrightarrow[\simeq_{\mathbb{R}}]{(-)^*} \mathbb{H}$.

Example 31 ($\mathrm{Spin}(4)$ and two copies of $\mathrm{Sp}(1)$). Under the exceptional isomorphism (78)

$$\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \xrightarrow{\simeq} \mathrm{Spin}(4) \quad (83)$$

the vector representation $\mathbf{4}$ of $\mathrm{Spin}(4)$ from (81) is given by combined left and conjugate right multiplication (82)

$$\begin{array}{ccc} \mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R \times \mathbb{H} & \longrightarrow & \mathbb{H} \\ (q_A, q_V, v) & \longmapsto & q_A \cdot v \cdot \bar{q}_V \end{array} \quad (84)$$

All three of these actions are irreducible in themselves, so that under restriction (33) along any of the two inclusions (78) $\mathrm{Sp}(1)_{L/R} \xrightarrow{\iota_{L/R}} \mathrm{Spin}(4)$ there is “no branching” in that $(\iota_{A/V})^* \mathbf{4} = \mathbf{4}$.

Example 32 (Reduction to $\mathrm{Spin}(3)$). Under the exceptional isomorphism (78) and the further exceptional isomorphism $\mathrm{Sp}(1) \xrightarrow{\simeq} \mathrm{Spin}(3)$ the canonical inclusion $\mathrm{Spin}(3) \xrightarrow{\iota} \mathrm{Spin}(4)$ is identified with the diagonal map on $\mathrm{Sp}(1)$:

$$\begin{array}{ccc} \mathrm{Spin}(3) & \xhookrightarrow{\iota} & \mathrm{Spin}(4) \\ \mathrm{Sp}(1) \upharpoonright^{\downarrow} & \xrightarrow{\mathrm{diag}} & \mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R \upharpoonright^{\downarrow} \\ \mathbf{1} \oplus \mathbf{3} & \xleftarrow{\iota^*} & \mathbf{4} \end{array}$$

and hence by restriction of the 4-dimensional vector representation in its quaternion form (84) it follows that the resulting $\mathbf{3} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1))$ is given by the diagonal of the actions (82)

$$\begin{array}{ccc} \mathrm{Sp}(1) \times \mathbb{H}_{\mathrm{im}} & \longrightarrow & \mathbb{H}_{\mathrm{im}} \\ (q, v) & \longmapsto & q \cdot v \cdot \bar{q} \end{array} \quad (85)$$

Here $\mathbb{H}_{\mathrm{im}} \subset \mathbb{H}$ is the real 3-dimensional space of imaginary quaternions.

The Lie algebra \mathfrak{sp}_1 via quaternions. As $\mathrm{Sp}(1)$ is the unit sphere of the skew-field \mathbb{H} of quaternions, the identification $\mathbb{H} \cong \mathbb{R}^4$ given by the standard \mathbb{R} -basis $\{e_0, e_1, e_2, e_3\}$ of \mathbb{H} with

$$e_0 = 1, \quad e_i e_i = -1 = -e_0 \quad \text{for } i \in \{1, 2, 3\}, \quad e_{\sigma(1)} e_{\sigma(3)} = \mathrm{sgn}(\sigma) e_{\sigma(3)} \quad \text{for } \sigma \in \mathrm{Sym}(3), \quad (86)$$

identifies the Lie algebra \mathfrak{sp}_1 of $\mathrm{Sp}(1)$ with the tangent space at S^3 in \mathbb{R}^4 at the point $(1, 0, 0, 0)$, and so it is identified with the space \mathbb{H}_{im} of imaginary quaternions. The Lie algebra structure on \mathfrak{sp}_1 is easily obtained by noticing that the 1-parameter subgroups generated by the basis elements $\{e_1, e_2, e_3\}$ of \mathfrak{sp}_1 are given by $e_i(t) = \cos(t) + e_i \sin(t)$. From this we get

$$[e_1, e_2] = \frac{1}{2} \frac{d^2}{ds dt} \Big|_{(s,t)=(0,0)} e_1(t) e_2(s) e_1(t)^{-1} e_2(s)^{-1} = e_3,$$

and similarly

$$[e_2, e_3] = e_1 \quad \text{and} \quad [e_3, e_1] = e_2.$$

Therefore the basis vectors $\{e_i\}$ are the standard Lie algebra basis for $\mathfrak{so}_3 \cong \mathfrak{sp}_1$. Finally, by differentiating the action, one sees that the Lie algebra representations corresponding to the representations $\mathbf{4}_{l/r} \cong \mathbb{H}$ from (82), and $\mathbf{3} \cong \mathbb{H}_{\text{im}}$ from (85) are given, respectively, by

$$\begin{aligned} \mathfrak{sp}_1 \otimes \mathbf{4} &\longrightarrow \mathbf{4}, & \mathfrak{sp}_1 \otimes \mathbf{4} &\longrightarrow \mathbf{4}, & \mathfrak{sp}_1 \otimes \mathbf{3} &\longrightarrow \mathbf{3}. \\ e_i \otimes v &\longmapsto e_i v & e_i \otimes v &\longmapsto v e_i & e_i \otimes v &\longmapsto [e_i, v] \end{aligned} \quad (87)$$

Here the multiplications and the commutators on the right are taken in the associative algebra \mathbb{H} of quaternions. As $\text{Sp}(1)$ is a compact and simply connected Lie group, its Lie algebra \mathfrak{sp}_1 knows everything about its representation theory. An example of application of this principle are the proofs of the following lemmas.

Lemma 33 (Decomposition of irreps of $\mathbf{4} \wedge \mathbf{4}$). *The second exterior power $\mathbf{4} \wedge \mathbf{4}$ (see (32)) of the defining real 4-dimensional irrep $\mathbf{4} \in \text{Rep}_{\mathbb{R}}(\text{Sp}(1))$ (see (81)) is the direct sum of the real 3-dimensional irrep $\mathbf{3} \in \text{Rep}_{\mathbb{R}}(\text{Sp}(1))$ (see (85)) with the 3-dimensional trivial rep:*

$$\mathbf{4} \wedge \mathbf{4} \simeq 3 \cdot \mathbf{1} \oplus 1 \cdot \mathbf{3} \in \text{Rep}_{\mathbb{R}}(\text{Sp}(1)). \quad (88)$$

Proof. By (87), the \mathfrak{sp}_1 -action on $\mathbf{4}$ is given on the canonical linear basis (86)

$$\mathbf{4} \simeq \langle e_0, e_1, e_2, e_3 \rangle_{\mathbb{R}} \quad (89)$$

by $e_i \otimes e_j \mapsto e_i e_j$, with $i \in \{1, 2, 3\}$ and $j \in \{0, 1, 2, 3\}$. Consider then the following linear basis of $\mathbf{4} \wedge \mathbf{4}$:

$$\mathbf{4} \wedge \mathbf{4} \simeq_{\mathbb{R}} \left\langle \begin{aligned} a_1^L &:= e_0 \wedge e_1 + e_2 \wedge e_3, & a_1^R &:= e_0 \wedge e_1 - e_2 \wedge e_3, \\ a_2^L &:= e_0 \wedge e_2 + e_3 \wedge e_1, & a_2^R &:= e_0 \wedge e_2 - e_3 \wedge e_1, \\ a_3^L &:= e_0 \wedge e_3 + e_1 \wedge e_2, & a_3^R &:= e_0 \wedge e_3 - e_1 \wedge e_2 \end{aligned} \right\rangle. \quad (90)$$

The induced $\mathfrak{sp}(1)$ Lie algebra action is given by

$$e_i \cdot (e_j \wedge e_k) = (e_i e_j) \wedge e_k + e_j \wedge (e_i e_k).$$

From this we find for e_1 :

$$\begin{aligned} e_1 \cdot a_1^{L/R} &= e_1 \cdot (e_0 \wedge e_1 \pm e_2 \wedge e_3) \\ &= \underbrace{(e_1 \wedge e_1)}_{=0} + \underbrace{(e_0 \wedge (-e_0))}_{=0} \pm \underbrace{(e_3 \wedge e_3)}_{=0} + \underbrace{(e_2 \wedge (-e_2))}_{=0} = 0 \\ e_1 \cdot a_2^{L/R} &= e_1 \cdot (e_0 \wedge e_2 \pm e_3 \wedge e_1) \\ &= \underbrace{(e_1 \wedge e_2)}_{=e_1 \wedge e_2} + \underbrace{(e_0 \wedge e_3)}_{=e_0 \wedge e_3} \pm \underbrace{((-e_2) \wedge e_1)}_{=e_1 \wedge e_2} + \underbrace{(e_3 \wedge (-e_0))}_{=e_0 \wedge e_3} = \begin{cases} 2a_2^L \\ 0 \end{cases} \\ e_1 \cdot a_3^{L/R} &= e_1 \cdot (e_0 \wedge e_3 \pm e_1 \wedge e_2) \\ &= \underbrace{(e_1 \wedge e_3)}_{=-e_3 \wedge e_1} + \underbrace{(e_0 \wedge (-e_2))}_{=-e_0 \wedge e_2} \pm \underbrace{((-e_0) \wedge e_2)}_{=-e_0 \wedge e_2} + \underbrace{(e_1 \wedge e_3)}_{=-e_3 \wedge e_1} = \begin{cases} -2a_2^L \\ 0 \end{cases} \end{aligned}$$

Since everything here is invariant under cyclic permutation of the three non-zero indices, it follows generally that

$$\left(\frac{1}{2}e_i\right) \cdot a_j^L = \sum_k \epsilon_{ijk} a_k^L, \quad \left(\frac{1}{2}e_i\right) \cdot a_j^R = 0 \quad \text{for all } i, j \in \{1, 2, 3\}.$$

This identifies $\{a_1^L, a_2^L, a_3^L\}$ and $\{a_i^R\}$ as $\mathbf{3}$ and $\mathbf{1}$, respectively, as representations of \mathfrak{sp}_1 and hence as representations of $\text{Sp}(1)$:

$$\langle a_1^L, a_2^L, a_3^L \rangle \simeq \mathbf{3}, \quad \langle a_i^R \rangle \simeq \mathbf{1} \in \text{Rep}_{\mathbb{R}}(\text{Sp}(1)). \quad \square \quad (91)$$

Lemma 34 (Left-right exchange of $\mathbf{4} \wedge \mathbf{4}$). *Under the exceptional isomorphism $\text{Sp}(1)_R \times \text{Sp}(1)_L \xrightarrow{\cong} \text{Spin}(4)$ (from (83)), the second exterior power $\wedge^2 \mathbf{4}$ of the vector representation $\mathbf{4}$ of $\text{Spin}(4)$ splits as the direct sum of the outer tensor products (from (35)) of the $\mathbf{3}$ (from (85)) of one of the $\text{Sp}(1)$ factors with the $\mathbf{1}$ of the other factor:*

$$\wedge^2 \mathbf{4} \simeq \mathbf{3} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{3} \in \text{Rep}_{\mathbb{R}}(\text{Sp}(1)_L \times \text{Sp}(1)_R).$$

Proof. We have to show that we have an \mathbb{R} -vector space splitting

$$\wedge^2 \mathbf{4} = V \oplus W$$

with both V and W representations of $\mathrm{Sp}(1)_{A/V}$ via the restrictions along the two inclusions $\mathrm{Sp}(1)_{L/R} \xrightarrow{L/R} \mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R$ and with $V \cong \mathbf{3}$ and $W \cong \mathbf{3} \cdot \mathbf{1}$ in $\mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1)_L)$, and $V \cong \mathbf{3} \cdot \mathbf{1}$ and $W \cong \mathbf{3}$ in $\mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1)_R)$. Set, in the notation of Lemma 33,

$$V := \langle a_1^L, a_2^L, a_3^L \rangle; \quad W := \langle a_1^R, a_2^R, a_3^R \rangle.$$

Then the identification

$$V \oplus W = \mathbf{3} \oplus \mathbf{3} \cdot \mathbf{1} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1)_L)$$

is precisely the content of Lemma 33, and the identification

$$V \oplus W = \mathbf{3} \cdot \mathbf{1} \oplus \mathbf{3} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1)_R)$$

is proved analogously, by considering the \mathfrak{sp}_1 -action on \mathbb{H} induced by the $\mathrm{Sp}(1)$ -action on the right (see equation (87)). \square

Lemma 35 (The third wedge power). *The fourth exterior power $\wedge^4 \mathbf{4}$ of $\mathbf{4} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1))$ is the trivial representation $\mathbf{1}$, and the third exterior power $\wedge^3 \mathbf{4}$ of $\mathbf{4} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1))$ is isomorphic to $\mathbf{4}$ itself:*

$$\wedge^3 \mathbf{4} \simeq \mathbf{4} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}(1)).$$

Proof. For any $i \in \{0, 1, 2, 3\}$ let $b_i \in \wedge^3 \mathbf{4}$ be the element

$$b_i := (-1)^i e_0 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_3,$$

where the factor e_i is omitted. Then $\{b_0, b_1, b_2, b_3\}$ is an \mathbb{R} -basis of $\wedge^3 \mathbf{4}$ and we have an isomorphism of \mathbb{R} -vector spaces $\wedge^3 \mathbf{4} \rightarrow \mathbf{4}$ given by $\phi: b_i \mapsto e_i$. Direct inspection shows that ϕ is actually an isomorphism of \mathfrak{sp}_1 representations, and so of $\mathrm{Sp}(1)$ representations. This follows by direct inspection. For instance, for the Lie action of e_1 we find:

$$\begin{aligned} \phi(e_1 \cdot b_0) &= \phi(e_1 \cdot (e_1 \wedge e_2 \wedge e_3)) \\ &= \phi\left(\underbrace{(-e_0) \wedge e_2 \wedge e_3}_{=b_1} + \underbrace{e_1 \wedge e_3 \wedge e_3}_{=0} + \underbrace{e_1 \wedge e_2 \wedge (-e_2)}_{=0}\right) = \phi(b_1) = e_1 = e_1 \cdot e_0 = e_1 \cdot \phi(b_0), \\ \phi(e_1 \cdot b_1) &= \phi(e_1 \cdot (-e_0 \wedge e_2 \wedge e_3)) \\ &= \phi\left(\underbrace{-e_1 \wedge e_2 \wedge e_3}_{b_0} - \underbrace{e_0 \wedge e_3 \wedge e_3}_{=0} - \underbrace{e_0 \wedge e_2 \wedge (-e_2)}_{=0}\right) = \phi(-b_0) = -e_0 = e_1 \cdot e_1 = e_1 \cdot \phi(b_1), \end{aligned}$$

and similarly for the other cases. \square

In summary, we have proven the first statement of Theorem 7:

Proposition 36 (The $\mathbf{3}$ inside $\mathbf{4} \wedge \mathbf{3}$). *The $\mathrm{SU}(2)_L$ -fixed locus inside the $\wedge^2 \mathbf{4}$ of $\mathrm{Spin}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is the $\mathbf{3}$ of $\mathrm{SU}(2)_R$, while the $\mathrm{SU}(2)_L$ -fixed locus in $\wedge^3 \mathbf{4}$ (and in $\mathbf{4}$) is trivial.*

Proof. Using the fact that fixed loci are the direct summands of the corresponding trivial representations (37), this follows from Lemma 34 and Lemma 35. \square

3.4. 11d spinors at the A_1 -singularity

We now combine §3.2 with §3.3 to discuss the representation theory of 11d spinors restricted to an A_1 -singularity.

Definition 37 (Identifying \mathbb{Z}_2 subgroup). With respect to the inclusion $\mathrm{Spin}(5, 1) \times \mathrm{Spin}(4) \subset \mathrm{Spin}(9, 1)$ given by (62), consider now the corresponding inclusion of the subgroup from Lemma 29:

$$\mathbb{Z}_2^A := \{1, \exp(2\pi J_L^1)\} \subset \mathrm{SU}(2)_L \subset \mathrm{Spin}(4) \subset \mathrm{Spin}(10, 1). \quad (92)$$

The following lemma is essentially the content of [73, Lemma 4.13]:

Lemma 38 (The fermionic \mathbb{Z}_2^A -fixed locus). The fixed locus (36) of \mathbb{Z}_2^A (92) in the real $\text{Spin}(9, 1)$ representations **16** and $\overline{\mathbf{16}}$ (65) is, as a residual $\text{Spin}(5, 1)$ representation, the **8** and $\overline{\mathbf{8}}$ from (65) respectively, given under Lemma 20 by the inclusion $\mathbb{H}^2 \subset \mathbb{O}^2$ (59)

$$\mathbf{16}^{\mathbb{Z}_2^A} = \mathbf{8}, \quad \overline{\mathbf{16}}^{\mathbb{Z}_2^A} = \overline{\mathbf{8}} \quad \in \text{Rep}_{\mathbb{R}}(\text{Spin}(5, 1)). \quad (93)$$

Proof. By Proposition 20 we need to prove that, under the identifications of the $\text{SO}(9, 1)$ representations **16** and $\overline{\mathbf{16}}$ with \mathbb{O}^2 , the \mathbb{Z}_2^A fixed locus in **16** and $\overline{\mathbf{16}}$ is identified with the real subspace \mathbb{H}^2 of \mathbb{O}^2 (59). According to Lemma 29 and to Proposition 20, we are equivalently asking for the fixed locus of the consecutive left action of the octonionic Pauli matrices $\sigma_6, \sigma_7, \sigma_8, \sigma_9$ from Lemma 19 on the space \mathbb{O}^2 :

$$\exp(2\pi \tau_L^1) \psi = \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9 \psi = \sigma_6(\sigma_7(\sigma_8(\sigma_9 \psi))) = e_4(e_5(e_6(e_7 \psi))). \quad (94)$$

Now writing $\psi = \eta_1 + \eta_2 \ell$ with $\eta_1, \eta_2 \in \mathbb{H}^2$, Lemma 17 gives

$$\exp(2\pi \tau_L^1)(\eta_1 + \eta_2 \ell) = \eta_1 - \eta_2 \ell.$$

Therefore $\exp(2\pi \tau^L) \psi = \psi$ is equivalent to $\psi = \eta_1$. \square

Lemma 39 (Clifford action of $\mathfrak{su}(2)_R$ is by quaternion right action). Under the identification $\mathbf{16}^{\mathbb{Z}_2^A} \cong \mathbf{8} \cong \mathbb{H}^2$ from Lemma 38, the action of $\tau_R^i \in \mathfrak{su}(2)_R$ (79) is given by right multiplication with half the quaternion unit e_i :

$$\tau_L^i \psi = \psi \quad \Rightarrow \quad \tau_R^i \psi = \frac{1}{2} \psi e_i \quad \in \mathbb{H}^2. \quad (95)$$

Proof. Using (80), (20) and (56) we compute as follows:

$$\begin{aligned} \tau_R^1 \psi &= \left(\frac{1}{4} \Gamma_6 \Gamma_7 - \frac{1}{4} \Gamma_8 \Gamma_9 \right) \psi = \frac{1}{4} \Gamma_6 \Gamma_7 \psi - \frac{1}{4} \Gamma_8 \Gamma_9 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9 = \frac{1}{4} \Gamma_6 \Gamma_7 + \frac{1}{4} \Gamma_6 \Gamma_7 \\ &= \frac{1}{2} \sigma_6(\sigma_7 \psi) = \frac{1}{2} e_4(e_5 \psi) = \frac{1}{2} \ell((i\ell)\psi) = \frac{1}{2} \ell((i\psi^*)\ell) = \frac{1}{2} \psi i = \frac{1}{2} \psi e_1. \end{aligned}$$

A directly analogous computation shows the statement in the other cases. \square

Lemma 40 (The **8** of $\text{Spin}(5, 1)$ as representation of $\text{Spin}(3, 1) \times \text{SU}(2)_R$). When regarded as a $\text{Spin}(3, 1) \times \text{SU}(2)_R$ representation, along (3), the **8** and $\overline{\mathbf{8}}$ of (93) are isomorphic, as real representations, to the outer tensor product (35) of the left and right complex 2-dimensional Weyl representation $\mathbf{2}_{\mathbb{C}}, \overline{\mathbf{2}}_{\mathbb{C}}$ of $\text{Spin}(3, 1)$ (66) with the fundamental complex representation $\mathbf{2}_{\mathbb{C}}$ of $\text{SU}(2)$ (38):

$$\mathbf{8} = \mathbf{2}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}}, \quad \overline{\mathbf{8}} = \overline{\mathbf{2}}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}} \quad \in \text{Rep}_{\mathbb{R}}(\text{Spin}(3, 1) \times \text{SU}(2)_R). \quad (96)$$

Proof. From Remark 21, and in view of the chain of inclusions (59), one manifestly sees that the restricted representations without the $\text{SU}(2)_R$ -action considered are

$$\begin{array}{ccc} \text{Spin}(5, 1) & \xleftarrow{i} & \text{Spin}(3, 1) \\ \mathbf{8} & \xrightarrow{\quad} & \mathbf{2}_{\mathbb{C}} \oplus \mathbf{2}_{\mathbb{C}} \\ \downarrow \wr & & \downarrow \wr \\ \mathbb{H}^2 & & \mathbb{C}^2 \oplus \mathbb{C}^2 j \end{array} \quad \begin{array}{ccc} \text{Spin}(5, 1) & \xleftarrow{i} & \text{Spin}(3, 1) \\ \overline{\mathbf{8}} & \xrightarrow{\quad} & \overline{\mathbf{2}}_{\mathbb{C}} \oplus \overline{\mathbf{2}}_{\mathbb{C}} \\ \downarrow \wr & & \downarrow \wr \\ \mathbb{H}^2 & & \mathbb{C}^2 \oplus \mathbb{C}^2 j \end{array} \quad (97)$$

So it remains to show that, with respect to the $\text{SU}(2)_R$ -action we have $\mathbf{2}_{\mathbb{C}} \cdot \mathbf{2} \simeq_{\mathbb{R}} \mathbf{2}_{\mathbb{C}} \boxtimes \mathbf{2}_{\mathbb{C}}$. By Lemma 39, the action of $\text{SU}(2)_R$ on spinors $\mathbb{C}^2 \oplus \mathbb{C}^2 \ni [\psi^+ \ \psi^-] \longleftrightarrow \psi^+ + \psi^- j \in \mathbb{C}^2 \oplus \mathbb{C}^2 j$ is by right multiplication by the unit quaternions, under the isomorphism $\text{SU}(2) \simeq \text{Sp}(1)$. Therefore, using the quaternion algebra, we check explicitly that this right multiplication gives the fundamental complex representation $\mathbf{2}_{\mathbb{C}} \in \text{Rep}_{\mathbb{C}}(\text{SU}(2))$ via the Pauli matrix representation (39):

$$\begin{aligned} 2\tau_R^1 [\psi \ \phi] &= [\psi \ \phi] e_1 & 2\tau_R^2 [\psi \ \phi] &= [\psi \ \phi] e_2 & 2\tau_R^3 [\psi \ \phi] &= [\psi \ \phi] e_3 \\ &= (\psi + \phi j) i & &= (\psi + \phi j) j & &= (\psi + \phi j) k \\ &= \psi i - (\phi i) j & &= \psi j - \phi & &= \underbrace{\psi k}_{=ij} + \underbrace{\phi jk}_{=i} \\ &= [\psi i \ -\phi i] & &= [-\phi \ \psi] & &= [\phi i \ \psi i] \\ &= [\psi \ \phi] \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, & &= [\psi \ \phi] \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & &= [\psi \ \phi] \cdot \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \quad \square \end{aligned} \quad (98)$$

Lemma 41 (Residual $U(1)_V$ -action on spinors). When regarded as a $\text{Spin}(3, 1) \times U(1)_V$ representation, along (3), the $\mathbf{8}$ and $\bar{\mathbf{8}}$ of (93) are isomorphic, as real representations, to the outer tensor product (35) of the left and right complex 2-dimensional Weyl representation $\mathbf{2}_{\mathbb{C}}, \bar{\mathbf{2}}_{\mathbb{C}}$ of $\text{Spin}(3, 1)$ (see (66)) with the complex irrep $\mathbf{1}_{\mathbb{C}}^-$ of $U(1)$ (see (41)).

Proof. We explicitly compute the action of R_{45} , via its Clifford representation $\frac{1}{2}\Gamma_4\Gamma_5$ (46) on a spinor $\mathbb{C}^2 \oplus \mathbb{C}^2 \ni [\psi \ \phi] \longleftrightarrow \psi + \phi j \in \mathbb{C}^2 \oplus \mathbb{C}^2 j$ using (64):

$$\begin{aligned} 2\frac{1}{2}\Gamma_4\Gamma_5[\psi \ \phi] &= \sigma_4(\sigma_5(\psi + \phi j)) = -e_2(e_3(\psi + \phi j)) \\ &= -jk(\psi + \phi j) = -i(\psi + \phi j) = (-\psi i) + (-\phi i)j \\ &= [\psi \ \phi] \cdot \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}, \end{aligned} \quad (99)$$

where we also used that quaternionic multiplication is associative and then that complex multiplication, furthermore, is commutative. \square

In summary, we have proven the second statement of Theorem 7:

Proposition 42 (The reduced Spin representation). The fixed locus (36) under the action of $\mathbb{Z}_2^A \subset \text{SU}(2)_L$ (92) in the $\text{Spin}(10, 1)$ representation $\mathbf{32}$ (from (71)), regarded with its residual action of $\text{Spin}(3, 1) \times U(1) \times \text{SU}(2)_R$ is:

$$\begin{aligned} \text{Spin}(10, 1) &\longleftarrow \text{Spin}(3, 1) \times U(1)_V \times \text{SU}(2)_R. \\ \mathbf{32} &\longrightarrow \mathbf{32}^{\mathbb{Z}_2^A} = (\mathbf{2}_{\mathbb{Z}} \oplus \bar{\mathbf{2}}_{\mathbb{Z}}) \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^- \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}} \end{aligned} \quad (100)$$

Proof. This follows immediately from combining Lemma 40 and Lemma 41. \square

For the record, and as this is somewhat subtle, we highlight the following equivalent and non-equivalent versions of the statement in (100):

Lemma 43 (Real isomorphism classes of chiral charged 4d spinor reps). Consider the real representations of $\text{Spin}(3, 1) \times U(1)_V \times \text{SU}(2)_R$.

(i) There are isomorphisms of real representations as follows:

$$\mathbf{2}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^{\mp} \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}} \simeq_{\mathbb{R}} \bar{\mathbf{2}}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^{\pm} \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}}. \quad (101)$$

(ii) There are, however, no isomorphisms, not even as real representations, between $\mathbf{2}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^- \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}}$ and $\mathbf{2}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^+ \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}}$ nor between $\bar{\mathbf{2}}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^- \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}}$ and $\bar{\mathbf{2}}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}}^+ \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}}$.

Proof. Notice that under the canonical embedding of $\text{SU}(2)$ as the spatial $\text{Spin}(3)$ -group inside the Lorentzian $\text{Spin}(3, 1)$, the two complex Weyl spinor representation (66) both restrict to the fundamental representation of $\text{SU}(2)$ from (38):

$$\begin{aligned} \text{Spin}(3, 1) &\longleftarrow \text{Spin}(3) \simeq \text{SU}(2). \\ \mathbf{2}_{\mathbb{C}} &\longrightarrow \mathbf{2}_{\mathbb{C}} \\ \bar{\mathbf{2}}_{\mathbb{C}} &\longrightarrow \mathbf{2}_{\mathbb{C}} \end{aligned}$$

Now consider the identification of complex representations

$$\mathbf{2}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}} \simeq_{\mathbb{C}} \text{Mat}(2 \times 2, \mathbb{C}),$$

where on the right the $\text{Spin}(3, 1)$ -action is by left matrix multiplication with \mathbb{C} -Pauli matrices (65) $\frac{1}{2}\sigma_{a_1}\sigma_{a_2}$ for $a_i \in \{0, 1, 2, 3\}$, while the $\text{SU}(2)_R$ -action is by right matrix multiplication with transposed Pauli matrices ${}^t(\frac{1}{2}\sigma_{a_1}\sigma_{a_2})$ for $a_i \in \{1, 2, 3\}$. Therefore, by the proof of Lemma 22, we have an isomorphism of real representations given by

$$\begin{aligned}
\mathbf{2}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}} &\longrightarrow \overline{\mathbf{2}}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{1}_{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}} . \\
\mathrm{Mat}(2 \times 2, \mathbb{C}) &\longrightarrow \mathrm{Mat}(2 \times 2, \mathbb{C}) \\
A &\longmapsto \epsilon \cdot A^* \cdot {}^t \epsilon
\end{aligned} \tag{102}$$

Since the real-linear map in the last row is complex anti-linear, this same real-linear map gives the claimed isomorphisms (101). Moreover, since $\mathbf{2}_{\mathbb{C}}$ is not isomorphic to $\overline{\mathbf{2}}_{\mathbb{C}}$ as a complex representation, no real isomorphism as in the first line of (102) can be complex-linear, hence all such isomorphisms must be complex anti-linear. This gives the second claim. \square

In closing this discussion of octonionic spinor algebra, we notice the following identifications, which will be useful in §4 below:

Notation 44 (4d chirality operator). If we introduce for the 4d sector of the octonionic Dirac matrices (70) the following standard notation for complex Dirac matrices

$$\begin{aligned}
\gamma_{\mu} &:= \Gamma_{\mu} \quad \mu \in \{0, 1, 2, 3\} \\
\gamma_5 &:= i\gamma_0\gamma_1\gamma_2\gamma_3
\end{aligned} \tag{103}$$

then the chirality operator γ_5 is

$$\begin{aligned}
\gamma_5 &= \Gamma_{5'} \\
i\gamma_5 &= \Gamma_4\Gamma_5\Gamma_{5'} .
\end{aligned}$$

Lemma 45 (Branching of spinor pairing). The pairing in Proposition 27 applied to 6d spinors and expressed in terms of their branching, from Proposition 40, into iso-doublets of 4d spinors

$$\begin{aligned}
\begin{bmatrix} \psi^+ & \psi^- \end{bmatrix} &\in (\mathbf{2}_{\mathbb{C}} \oplus \overline{\mathbf{2}}_{\mathbb{C}}) \boxtimes_{\mathbb{C}} \mathbf{2}_{\mathbb{C}} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(3, 1) \times \mathrm{SU}(2)_R) \\
\parallel \\
\psi = \psi^+ + \psi^- j &\in \mathbf{8} \oplus \overline{\mathbf{8}} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(5, 1))
\end{aligned}$$

equals the sum over iso-doublet components of the pairing in Proposition 27 applied to 4d spinors:

$$\begin{aligned}
\langle [\psi^+ \ \psi^-], [\phi^+ \ \phi^-] \rangle &:= \langle \psi, \phi \rangle \\
&= \langle \psi^+, \phi^+ \rangle + \langle \psi^-, \phi^- \rangle .
\end{aligned}$$

Proof. Noticing that

$$(c \in \mathbb{C} \hookrightarrow \mathbb{O}) \Rightarrow \begin{cases} (cj)^* = -cj \\ -jcd^*j = c^*d \end{cases} \Rightarrow \begin{cases} \mathrm{Re}(cj) = 0 \\ \mathrm{Re}(-jcd^*j) = \mathrm{Re}(cd^*) \end{cases}$$

we compute as follows:

$$\begin{aligned}
\langle \psi, \phi \rangle &= \langle \psi^+ + \phi^- j, \psi^+ + \phi^- j \rangle \\
&= \mathrm{Re}\left((\psi^+ + \psi^- j)^{\dagger} \cdot \Gamma_0 \cdot (\phi^+ + \phi^- j)\right) \\
&= \mathrm{Re}\left((\psi^+)^{\dagger} - j((\psi^-)^{\dagger}) \cdot \Gamma_0 \cdot (\phi^+ + \phi^- j)\right) \\
&= \mathrm{Re}((\psi^+)^{\dagger} \cdot \Gamma_0 \cdot \phi^+) + \mathrm{Re}((\psi^-)^{\dagger} \cdot \Gamma_0 \cdot \phi^-) \\
&= \langle \psi^+, \phi^+ \rangle + \langle \psi^-, \phi^- \rangle . \quad \square
\end{aligned}$$

4. Analysis of the WZW-type terms

With the full field content (4) in hand, via the proof in §3, here we spell out two noteworthy contributions to the super-exceptional PS-Lagrangian (20) that both originate from the super-exceptional WZW term (30), and we comment on the relation of both to quantum hadrodynamics. A more comprehensive discussion of the emergent $\mathrm{SU}(2)$ -flavor gauge theory encoded by the Lagrangian (20) is beyond the scope of this article, but see §5 for some outlook.

We need the following:

Notation 46 (*Radial holographic Kaluza-Klein mode expansion*). In the following, we consider a choice of sequence of smooth functions of the coordinate x^4 on the M5-brane worldvolume (21),

$$h_{(n)} : x^4 \mapsto h_{(n)}(x^4) \in \mathbb{R}, \quad n \in \mathbb{N} \quad (104)$$

to be regarded as KK-modes for flat space holography. These functions (104) may, for instance, be any of the following:

- **(Flat hard-wall holography)**: Trigonometric functions of x^4 defined on a closed interval and vanishing at the boundary points, as considered in [130, 5.1].
- **(Atiyah-Manton holography)**: Hermite functions defined on all $x^4 \in \mathbb{R}$ as considered in [134, 3][135, 3].
- **(Circle-compactification of M5)**: Periodic functions of x^4 for some given radius of periodicity, as for the KK-modes in the model considered in [77].

With such a choice understood, let V be a function or differential form on the M5-worldvolume (21) that is pulled back from $\mathbb{R}^{3,1}$. Then we have

$$V(\{x^\mu\}, x^4) = V^{(n)}(\{x^\mu\})h_{(n)}(x^4) \in \Omega^\bullet(\mathbb{R}^{3,1}) \hookrightarrow \Omega^\bullet(\mathbb{R}^5, 1), \quad (105)$$

where a sum over $n \in \mathbb{N}$ left notationally implicit, for the corresponding mode expansion (assumed to exist), with the $V^{(n)}$ being functions or differential forms just on $\mathbb{R}^{3,1}$. The details of the super-exceptional M5-brane model will depend on the choice of this mode expansion (105), its assumed completeness relations and boundary conditions, etc. However, for the present purpose of identifying just the general form of a few main terms in the super-exceptional Lagrangian (20), we only need to assume that any expansion as in (105) has been chosen.

The point of this, in the following, is that acting upon the expansion (105) by the de Rham differential in 5d decomposes the exceptional vielbein field values $d(V \dots)$

$$\begin{aligned} d_5 V &= d_5(V^{(n)}h_{(n)}) \\ &= h_{(n)}d_4 V^{(n)} + h'_{(n)}V^{(n)} \wedge dx^4 \end{aligned}$$

with $d_4 V^{(n)}$ being the de Rham differential form on $\mathbb{R}^{3,1}$, and $h'_{(n)}$ denoting the derivative of the mode function $h_{(n)}$ (with respect to its only variable x^4). As a result, every summand in the super-exceptional Lagrangian (23) is a wedge product of fields with precisely one of the factor fields not differentiated itself, but instead contributing with the differential $h'_{(n)}dx^4$ of its mode function.

Remark 47 (*Role of KK-mode expansion*). We highlight the following:

- (i) A mode expansion as in (105) also governs the field content in the Witten-Sakai-Sugimoto model [115, (3.20)][116, (2.10)], reviewed in [134][24][135].
- (ii) There it is crucial, for the details of the meson mass spectrum (highlighted in [135, (53)]), that the geometry is curved (specifically: asymptotically AdS). Here we are concerned with flat geometry (or at least: parallelizable, see Remark 53 below), since the whole complexity of the super-exceptional M5-brane model in §2 emerges [50] from the super M-brane cocycle in rational Cohomotopy theory on flat $D = 11$ super Minkowski spacetime [47].
- (iii) Heuristically, this matches the idea that our model should pertain to a *single* heterotic M5-brane ($N = 1$, deep M-theory regime), in contrast to a stack of a large number of coincident branes backreacting on their ambient spacetime geometry ($N \gg 1$, supergravity regime) as considered in the Witten-Sakai-Sugimoto model. Alternatively the flat super-exceptional spacetime ought to be regarded as the local tangent frame of a super-exceptional curved Cartan geometry [73, p. 7s], with only the lowest modes relevant locally.
- (iv) In any case, mode expansions (105) in flat space holography have been considered in [130, 5.1][134, 3][135, 3] and suffice for the following purpose of analyzing the form of the interaction terms that appear.

4.1. Vector meson coupling to Skyrme baryon current

We show here that the contribution to the super-exceptional WZW term (30) by those bosonic fields that transforms as **3** of $SU(2)_R$ in (4) has the form which, in quantum hadrodynamics, is characteristic of the coupling of the neutral vector mesons to the baryon current, with the baryons appearing as Skyrme solitons in the pion field (see [109]).

Remark 48 (*Linear basis for $\mathbf{4} \wedge \mathbf{4}$*). By (90) and Proposition 36 we have on the \mathbb{Z}_2^A -fixed locus the relations

$$\left. \begin{aligned} e^{I6} &= \epsilon^{IJK} e_{JK} \\ e_{a_1 a_2 a_3}^{I6} &= \epsilon^{IJK} e_{a_1 a_2 a_3 JK} \end{aligned} \right\} \in \text{CE}(\mathbb{R}_{\text{ex}_s}^{10,1|32})^{\mathbb{Z}_2^A}. \quad (106)$$

Definition 49 (*The π field*). For σ a super-exceptional field configuration (21), write

$$d\pi^I := \sigma^*(e^{I6}) = \sigma^*\left(\frac{1}{2}(e^{I6} + \epsilon^{IJK}e_{JK})\right) \quad (107)$$

for its component corresponding to the M2-brane wrapping modes on the vanishing 2-cycle in the A_1 -singularity. On the right in (107) we have highlighted the equivalent expressions, using (106).

Proposition 50 (*Emergence of isospin WZW term*). Consider a super-exceptional sigma-model field (21) with π -field component according to Definition 49. Then the value of the second factor in the super-exceptional WZW-term $\mathbf{L}_{\text{ex}_s}^{\text{WZW}}$ (30), for indices ranging in $a_i \in \{6, 7, 8, 9\}$, is:

$$\sigma^*(e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1}) = 4\epsilon_{IJK} d\pi^I \wedge d\pi^J \wedge d\pi^K. \quad (108)$$

Proof. We directly compute as follows:

$$\begin{aligned} & \sigma^*(e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1}) \\ &= \sigma^*(e_{IJ} \wedge e^{JK} \wedge e_K^I + 3e_{6I} \wedge e^{IJ} \wedge e_J^6) \\ &= \epsilon_{IJJ'} \epsilon^{JKK'} \epsilon_K^I d\pi^{J'} \wedge d\pi_{K'} \wedge d\pi^{I'} - 3d\pi^I \wedge \epsilon_{IJJ'} d\pi^{J'} \wedge d\pi^{I'} \\ &= \epsilon_{J'}^{K'} d\pi^{J'} \wedge d\pi_{K'} \wedge d\pi^{I'} + 3\epsilon_{IJJ'} d\pi^I \wedge d\pi^{J'} \wedge d\pi^{I'} \\ &= 4\epsilon_{IJK} d\pi^I \wedge d\pi^J \wedge d\pi^K, \end{aligned}$$

where in the second step we identified the π -field factors according to (107). \square

Definition 51 (*The A-field*). Let A be a 1-form on $\mathbb{R}^{4,1}$ with mode expansion

$$A(\{x^\mu\}, x^4) = A^{(n)}(\{x^\mu\})h_{(n)}(x^4)$$

as in (105), and consider a super-exceptional σ -model field (21) with

$$\sigma^*(e_{\mu 5'}) = d(A_\mu) = d(A_\mu^{(n)} h_{(n)}). \quad (109)$$

Example 52 (*Photon-pion coupling term*). Consider a super-exceptional sigma-model field (21) with π -field component according to Definition 49 and with A-field component according to Definition 51. Then the super-exceptional WZW-term (30) has, by Proposition 50, the contribution:

$$\sigma^*(e_{a_5'} \wedge e^a \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1}) = \underbrace{4A^{(0)} \wedge \epsilon_{IJK} d\pi^I \wedge d\pi^J \wedge d\pi^K}_{\gamma\pi\pi\pi\text{-coupling}} \wedge h'_{(0)} dx^4 \wedge dx^{5'} + \mathcal{O}(h_{(n>0)}) \quad (110)$$

If we interpret $A^{(0)}$ as the photon field (in accordance with the super-exceptional Perry-Schwarz mechanism (28)) and π^I with the pion field, then this expression (110) is the characteristic form of the $\gamma\pi\pi\pi$ -coupling term [12, (2.2)][137, (22)][22, (4)].

Remark 53 (*Skyrme baryon current*). We may consider also a curved but parallelizable geometry, where the super-exceptional vielbein components e^{I6} become a basis of three left-invariant 1-forms on the group manifold $\text{SU}(2)_R \simeq S^3$. Then the field identification (107) becomes

$$\sigma^*(e^{I6}) = e^{-\vec{\pi}} de^{\vec{\pi}} = (e^{\vec{\pi}})^* \theta \quad (111)$$

for

$$e^{\vec{\pi}} : \mathbb{R}^{3,1} \longrightarrow \text{SU}(2)_R \quad (112)$$

an $\text{SU}(2)_R$ -valued function and $\theta \in \Omega_{\text{L}}^1(\text{SU}(2), \mathfrak{su}(2))$ the Maurer-Cartan form on the group manifold, characterized by the condition $d\theta + [\theta \wedge \theta] = 0$. In this situation, the proof of Proposition 50 gives

$$\begin{aligned} \frac{1}{4} \sigma^*(e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1}) &= \overbrace{\text{Tr}(e^{-\vec{\pi}} de^{\vec{\pi}} \wedge e^{-\vec{\pi}} de^{\vec{\pi}} \wedge e^{-\vec{\pi}} de^{\vec{\pi}})}^{\text{Skyrme baryon current}} \\ &= (e^{\vec{\pi}})^* \text{Tr}(\theta \wedge \theta \wedge \theta), \end{aligned} \quad (113)$$

where $\text{Tr}(\theta \wedge \theta \wedge \theta)$ is proportional to the canonical volume form on the group manifold $\text{SU}(2)_R \simeq S^3$. If we still interpret π as the pion field, as in Example 52, then this is the characteristic form of the baryon current [59, (6)][137, (29)][138, (2)][5, (11)] in the Skyrme model of quantum hadrodynamics (review in [109]).

Example 54 (*Photon-Skyrmion coupling*). Accordingly, in the global case of Remark 53 the coupling in Example 52 becomes

$$\sigma^*(e_{a5'} \wedge e^a \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1}) = \underbrace{4A^{(0)} \wedge \text{Tr}(e^{-\tilde{\pi}} de^{\tilde{\pi}} \wedge e^{-\tilde{\pi}} de^{\tilde{\pi}} \wedge e^{-\tilde{\pi}} de^{\tilde{\pi}})}_{\gamma\text{-Skyrmion coupling}} \wedge h'_{(0)} dx^4 \wedge dx^{5'} + \mathcal{O}(h_{(n>0)}) \quad (114)$$

This is the form of the photon-Skyrmion coupling [137, (19)][82, (13)].

Definition 55 (*Dual graviton*). We shall say that a super-exceptional sigma-model field (3) satisfies *higher self-duality* if it takes the same value on the two copies of $\mathbf{7} \boxtimes \mathbf{1}$ in (4). By the super-exceptional embedding condition (22) this means, in particular, that

$$\sigma^*(e^{\mu}_{6789}) = \sigma^*(e^{\mu}) = dx^{\mu}. \quad (115)$$

Definition 56 (*The ω -field*). Let ω be a 1-form on $\mathbb{R}^{4,1}$ with mode expansion

$$\omega(\{x^{\mu}\}, x^4) = \omega^{(n)}(\{x^{\mu}\})h_{(n)}(x^4)$$

as in (105), and consider a super-exceptional σ -model field (21) with

$$\sigma^*(e_{\mu_1 \mu_2 \mu_3}^{45}) = d((\star_4 \omega)_{\mu_1 \mu_2 \mu_3}) = d((\star_4 \omega^{(n)})_{\mu_1 \mu_2 \mu_3} h_{(n)}). \quad (116)$$

Example 57 (*The ω -meson couplings to pions and Skyrmions*). Consider a super-exceptional sigma-model field (21) satisfying higher self-duality in the sense of Definition 55 and with π -field component as in Definition 49 and ω -field component as in Definition 56. Then the super-exceptional WZW-term (30) has the contribution

$$\begin{aligned} & \sigma^*(\varepsilon^{\mu_1 \mu_2 \mu_3 455'} \varepsilon^{\mu_4 6789} e_{\mu_1 \mu_2 \mu_3 45} \wedge e_{\mu_4 6789} \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \wedge e^{5'}) \\ &= \varepsilon^{\mu_1 \mu_2 \mu_3 45 \mu_4} (\star_4 \omega^{(0)})_{\mu_1 \mu_2 \mu_3} \wedge dx_{\mu_4} \wedge \varepsilon_{IJK} d\pi^I \wedge d\pi^J \wedge d\pi^K \wedge 4h'_0 dx^4 \wedge dx^{5'} + \mathcal{O}(h_{(n>0)}) \\ &= \underbrace{\omega^{(0)} \wedge \varepsilon_{IJK} d\pi^I \wedge d\pi^J \wedge d\pi^K}_{\omega\pi\pi\pi\text{-coupling}} \wedge h'_0 dx^4 \wedge dx^{5'} + \mathcal{O}(h_{(n>0)}). \end{aligned}$$

If we interpret π as the pion field, following Example 52, and in addition interpret $\omega^{(0)}$ as the ω -meson field, then this is the characteristic form of the $\omega\pi\pi\pi$ -coupling term [113, (2)][56, (1)]. More generally, in the global situation of Remark 53 this contribution becomes

$$\begin{aligned} & \sigma^*(\varepsilon^{\mu_1 \mu_2 \mu_3 455'} \varepsilon^{\mu_4 6789} e_{\mu_1 \mu_2 \mu_3 45} \wedge e_{\mu_4 6789} \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \wedge e^{5'}) \\ &= \underbrace{\omega^{(0)} \wedge \text{Tr}(e^{-\tilde{\pi}} de^{\tilde{\pi}} \wedge e^{-\tilde{\pi}} de^{\tilde{\pi}} \wedge e^{-\tilde{\pi}} de^{\tilde{\pi}})}_{\omega\text{-Skyrmion coupling}} \wedge 4h'_0 dx^4 \wedge dx^{5'} + \mathcal{O}(h_{(n>0)}) \end{aligned} \quad (117)$$

This is the characteristic form of ω -meson coupling to the Skyrme baryon current (113) due to [4, (2)], see also [79, (12)][69, (2.1)][62, (2.1)].

Remark 58 (*Chiral partner of the ω -meson*). The difference in interpretation between the photon-pion coupling in Example 52 and the omega-pion coupling in Example 57 is that, by the classification result of (4), the vector field in the latter case is part of an $\mathfrak{u}(2) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ -multiplet of vector fields whose $\mathfrak{su}(2)_R$ -partner comes from the super-exceptional vielbein component $e_{\mu_1 \mu_2 \mu_3}^{6l}$.

Therefore, we are led to interpreting this field component as in the following definition and subsequent example.

Definition 59 (*The ρ -field*). Let ρ be a $\mathfrak{su}(2)$ -valued 1-form on $\mathbb{R}^{4,1}$ with mode expansion

$$\rho_l(\{x^{\mu}\}, x^4) = \rho_l^{(n)}(\{x^{\mu}\})h_{(n)}(x^4)$$

as in (105), and consider a super-exceptional σ -model field (21) with

$$\sigma^*(e_{\mu_1 \mu_2 \mu_3}^{6l}) = d((\star_4 \rho)^l_{\mu_1 \mu_2 \mu_3}) = d((\star_4 (\rho^l)^{(n)})_{\mu_1 \mu_2 \mu_3} h_{(n)}). \quad (118)$$

Example 60 (*The ω/ρ -meson coupling to iso-doublet fermions*). Consider a super-exceptional sigma-model field (21) with ω -field component according to Definition 49 and with ρ -field component according to Definition 59. Then the fermionic coupling terms of these fields appearing in (9) are of the form

$$e_{a_1 \dots a_5} \Gamma^{a_1 \dots a_5} = \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} i \gamma_5 \gamma_{\mu_4} \left(e_{\mu_1 \mu_2 \mu_3}^{45} (-i) + e_{\mu_1 \mu_2 \mu_3}^{6I} \tau_I \right) + \dots$$

$$\quad \quad \quad \downarrow \sigma^*$$

$$i \gamma_5 \underbrace{\left(-\omega^\mu i + \rho_I^\mu \tau^I \right)}_{\omega/\rho - \text{fermion coupling}} \gamma_\mu + \dots \quad (119)$$

where we identified the 4d spacetime Dirac matrices γ_μ , γ_5 by (103) and the isospin Pauli matrices τ^I according to (79). This combination (119) of couplings to isodoublet fermions (100) is characteristic of that of the ω - and ρ -mesons in quantum hadrodynamics ([129, (3.12)], see also [81, (24)]).

Finally we show one example of couplings involving fermions, to illustrate how the left-invariance of the super-exceptional vielbein, via the last line of (10), makes the fermionic pairing terms in the super-exceptional Lagrangian (20) expand out to the usual fermionic currents:

Definition 61 (*The N -field*). Let N be a smooth function on $\mathbb{R}^{4,1}$ with values in Dirac spinor iso-doublets, and with mode expansion

$$N(\{x^\mu\}, x^4) = N^{(n)}(\{x^\mu\}) h_{(n)}(x^4)$$

as in (105), and consider a super-exceptional σ -model field (21) with

$$\sigma^*(\psi) = d(N) = d(N^{(n)} h_{(n)}), \quad (120)$$

where the first equality uses the identification of Theorem 7.

Example 62 (*fermion-pion coupling*). Consider a super-exceptional sigma-model field (21) with π -field component according to Definition 49 and with N -field component according to (61). Then the super-exceptional WZW-term (30) has, by Proposition 50 and using the last line of (10), the contribution:

$$\sigma^* \left(\langle \eta \wedge \Gamma_{5'} \cdot \psi \rangle \wedge e_{a_1 a_2} \wedge e^{a_2 a_3} \wedge e_{a_3}^{a_1} \wedge e^{5'} \right) \quad a_i \in \{6, 7, 8, 9\}$$

$$= (s+1) \underbrace{\langle N^{(0)}, \gamma_\mu \gamma_5 N^{(0)} \rangle}_{\text{fermion-pion coupling}} dx^\mu \wedge \varepsilon_{IJK} d\pi^I \wedge d\pi^J \wedge d\pi^K h'_{(0)} dx^4 \wedge dx^{5'} + \dots \quad (121)$$

Noticing that the spinor pairing over the brace is the sum over iso-doublet components of the 4d spinor pairing (by Lemma 45) this is the form of spinor-pion coupling seen in [102, Table 3, item 30][54, Table 1, item 8] – except for the appearance of the chirality operator γ_5 , via (44).

Notice that in passing from the super-exceptional MK6-locus to the actual $\frac{1}{2}$ M5-locus (Definition (13)) the right-handed spinors get projected out anyway, so that the value of the chirality operator becomes the identity.

4.2. Non-linear electromagnetic coupling

We show here that the contribution to the super-exceptional WZW term (30) by the purely electromagnetic terms (24) and (27) has the form of the first DBI-correction of non-linear electromagnetism, and we discuss how this is compatible with the appearance of hadrodynamics as above.

Proposition 63 (*First Born-Infeld correction from super-exceptional WZW term*). Consider a super-exceptional sigma-model field (21) with electromagnetic field component according to (24) and (27). Then the contribution of the super-exceptional WZW term (30) for indices ranging as $\alpha_i \in \{0, 1, 2, 3, 5'\}$ is:

$$\sigma^* \left(e_{\alpha 5'} \wedge e^\alpha \wedge e_{\alpha_1 \alpha_2} \wedge e^{\alpha_2 \alpha_3} \wedge e_{\alpha_3}^{\alpha_1} \wedge e^{5'} \right) = \frac{3}{2} \left(\frac{F \wedge F}{\text{dvol}_4} \right)^2 \text{dvol}_{6'} \quad (122)$$

Proof. We calculate as follows:

$$\begin{aligned}
& \sigma^*(e_{\mu 5'} \wedge e^{\mu} \wedge e_{\alpha_1 \alpha_2} \wedge e^{\alpha_2 \alpha_3} \wedge e_{\alpha_3}^{\alpha_1} \wedge e^{5'}) \\
&= \frac{3}{2} F_{\mu_1 \mu_2} dx^{\mu_1} \wedge dx^{\mu_2} \wedge F_{\nu_1 \mu_3} dx^{\mu_3} \wedge F_{\nu_2 \mu_4} dx^{\mu_4} \wedge \varepsilon^{\nu_1 \nu_2 \nu_3 \nu_4} F_{\nu_3 \nu_4} \wedge dx^4 \wedge dx^{5'} \\
&= \frac{3}{2} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} F_{\mu_1 \mu_2} F_{\nu_1 \mu_3} F_{\nu_2 \mu_4} \varepsilon^{\nu_1 \nu_2 \nu_3 \nu_4} F_{\nu_3 \nu_4} \wedge \text{dvol}_6 \\
&= \frac{3}{2} \left(\frac{F \wedge F}{\text{dvol}_4} \right)^2 \text{dvol}_{6'} .
\end{aligned} \tag{123}$$

Here in the first step we used that $\sigma^*(e_{\mu 5'} \wedge e^{\mu}) = d(A_{\mu}) \wedge dx^{\mu} = F$ from (24) has no factor of dx^4 , while $\sigma^*(e_{\mu_1 \mu_2}) = \frac{1}{2}(\star 4)_{\mu_1 \mu_2} dx^4$ from (27) does have this factor, so that in the triple wedge product the latter term contributes only linearly, and as such once in each of the three factors. The last step is the following computation:

$$\begin{aligned}
& F_{\mu_1 \mu_2} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} F_{\mu_3 \nu_1} F_{\mu_4 \nu_2} \varepsilon^{\nu_1 \nu_2 \nu_3 \nu_4} F_{\nu_3 \nu_4} \\
&= 4 E_i \varepsilon^{ijk} \varepsilon_{jj'j''} B^{j''} \varepsilon_{kk'k''} B^{k''} \varepsilon^{i'j'k'} E_{i'} + 4 B^l \varepsilon_{lij} \varepsilon^{ijk} E_{i'} B^{l'} \varepsilon_{l'k'j'} \varepsilon^{i'j'k'} E_{k'} + 4 B^i E_{i'} E_i B^{i'} \\
&= 4 E_i B^i \underbrace{\varepsilon_{kk'k''} B^{k''} \varepsilon^{i'kk'} E_{i'}}_{2 E_{i'} B^{i'}} - E_i B^k \underbrace{\varepsilon_{kk'k''} B^{k''} \varepsilon^{i'ik'} E_{i'}}_{=0} - 4 B^i E_{i'} E_i B^{i'} \\
&= 4 (E \cdot B)^2 = \left(\frac{F \wedge F}{\text{dvol}_4} \right)^2 .
\end{aligned} \tag{124}$$

Here in the first step the three summands shown are the contributions from two, one or no occurrences, respectively, of the index “0” in the two outer factors of F . \square

We close by highlighting how such non-linear corrections to electromagnetism are to be expected in a theory of hadrodynamics.

Born-Infeld correction. The term (122) is of the form of the first correction in Born-Infeld electromagnetism [25, p. 437] (as reviewed in [123, (22)][99, 9.4])

$$L^{\text{BI}} := \sqrt{-\det(\eta + 2\pi \ell_s^2 F)} \text{dvol}_4 . \tag{125}$$

$$\begin{aligned}
\text{with } \det((\eta_{\mu\nu}) + (F_{\mu\nu})) &= -1 - \underbrace{\frac{1}{2}(F \wedge \star_4 F)/\text{dvol}_4}_{\text{ordinary Maxwell term}} + \underbrace{\left(\frac{1}{2}(F \wedge F)/\text{dvol}_4\right)^2}_{\text{DBI correction term}} \\
&= -1 + E \cdot E - B \cdot B + (B \cdot E)^2 .
\end{aligned} \tag{126}$$

which string perturbation theory suggests appears in the low-energy effective action on D-branes [52][1] [88] (reviewed in [123][136][127][99, 9.4]). Notice that DBI-action (125) encodes a critical value of the electric field strength

$$E_{\text{BI}} := T \sqrt{\frac{T^2 + B^2}{T^2 + B_{\parallel}^2}} \tag{127}$$

(as in [65, (2.6)]) since

$$-\det((\eta_{\mu\nu}) + \frac{1}{T}(F_{\mu\nu})) \geq 0 \quad \Leftrightarrow \quad E \leq T \sqrt{\frac{T^2 + B^2}{T^2 + B_{\parallel}^2}} \tag{128}$$

where $B_{\parallel} := \frac{1}{\sqrt{E \cdot E}} B \cdot E$ for the component of the magnetic field parallel to the electric field.

We may observe that the same critical field strength is implied by electromagnetic vacuum polarization if the electromagnetic field is assumed to fill a leptonic/hadronic vacuum:

The Schwinger effect. Consider a constant electromagnetic field (\vec{E}, \vec{B}) on 4d Minkowski spacetime, away from the measure-0 subset of configurations with $\vec{E} \cdot \vec{B} = 0$, hence assuming that $\vec{E} \cdot \vec{B} \neq 0$. Then there exists a Lorentz transformation $(\vec{E}, \vec{B}) \mapsto (\vec{E}', \vec{B}')$ such that $\vec{E}' \parallel \vec{B}'$. The *Schwinger effect* of vacuum polarization, in this case, predicts that in a theory with electrically charged particles of charge e and mass m (reviewed for electron/positron pair creation in [38, (1.28)], and for quark/anti-quark pair creation in [66, (2)]) an electric field strength around the *Schwinger limit* scale

$$E'_{\text{crit}} := \frac{m^2 c^3}{e \hbar} \quad (129)$$

(reviewed in [38, (1.3)][92, (40)]) causes a sizable density of particle/anti-particle pairs to be created out of the vacuum,⁵ causing non-linear corrections to the electromagnetic dynamics or even decay of the vacuum, and hence in any case a breakdown of ordinary electromagnetism.

Now observe that a Lorentz-invariant expression for the field strength E' above, as a function of the electromagnetic field (\vec{E}, \vec{B}) in any frame, is given by the following expression (see also [38, (1.6)]):

$$E'(\vec{E}, \vec{B}) = \sqrt{\left(\frac{1}{2}(\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B})\right)^2 + (\vec{B} \cdot \vec{E})^2 + \frac{1}{2}(\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B})}.$$

This follows immediately from the fact that this expression is a Lorentz invariant (being a function of the basic Lorentz invariants $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}$ and $\vec{E} \cdot \vec{B}$, e.g. [44]) and that it evidently reduces to the absolute value $E' = \sqrt{\vec{E}' \cdot \vec{E}'}$ in any Lorentz frame where $\vec{E}' \parallel \vec{B}'$. But then basic algebra reveals ([65, (2.6)]) that

$$E'(\vec{E}, \vec{B}) = E'_{\text{crit}} \quad \Leftrightarrow \quad E = E'_{\text{crit}} \sqrt{\frac{E'^2_{\text{crit}} + B^2}{E'^2_{\text{crit}} + B^2_{\parallel}}}.$$

This is exactly the formula for the critical field strength (127) in Born-Infeld -theory, if we identify the string tension with the Schwinger limit field strength $T = E'_{\text{crit}}$, as befits the picture that it is the flux tube strings connecting a quark/anti-quark which counteracts their separation.

5. Conclusion

Irrespective of the physics interpretation of the full field content in the big table (4), found to emerge in the super-exceptional M5-brane model [50], we have identified elusive field theoretic structure in the previously somewhat mysterious expression (17) for the super-exceptional 3-flux form (the “hidden superalgebra of 11d supergravity” [34][18][8]) including:

- (i) a Skyrme current-type term (Lemma 50) and
- (ii) the first DBI-correction term (Proposition 63).

Specifically the DBI-term (108) is the expected first-order interaction correction to the free Perry-Schwarz Lagrangian (28) in the full interacting M5-brane model [106, (63)] as seen explicitly after double dimensional reduction to the D4-brane model [6, 6][7, 6].

In the same manner one may now identify further interaction terms in the full super-exceptional PS-Lagrangian (20). For instance, following Example 62 there are evident vector-, axial- and isospin-current terms coupling the two fermion iso-doublets to the various vector fields. This is to be discussed elsewhere.

Thereby one possible interpretation of the field content in §3 certainly suggests itself: We seem to be seeing the emergence of a variant/sector of SU(2)-flavor chiral hadrodynamics (e.g. [129][124][23]), on the single M5-brane at an A_1 -singularity, in a way that is different from but akin to (see also Remark 47) the hadrodynamics seen on flavor D-branes [80] in holographic QCD models such as the Witten-Sakai-Sugimoto (WSS) model [108][109].⁶

This seems remarkable, since the only input for the super-exceptional M5-brane model §2 is the pair of M-brane supercocycles (15) and (19), which jointly form a single cocycle in super rational Cohomotopy theory [118, 2.5][45][73, 3.2] (review in [47, (57)]). Elsewhere we have shown that taking seriously this cohomotopical nature of the M-brane charges (“Hypothesis H”) implies a fair bit of topological/homotopical structure expected in M-theory in general [48][120] and for the M5-brane specifically [49][120] (reviewed in [125]). In particular, it implies [51][110] the emergence of a topological sector of an SU(2)_R-gauge field on the heterotic M5-brane, similar to that considered in [114], whose origin matches that of the SU(2)_R-valued local differential form data found here (see Remark 53). All taken together, we seem to be seeing a coherent non-abelian M5-brane model specifically for single heterotic M5-branes with SU(2)-flavor gauge fields on their worldvolume.

This raises the question of the extent to which the SU(2)-flavor gauge theory emerging in the super-exceptional cohomotopical M5-brane model might be related to actual real-world hadrodynamics, possibly as a version in the deep M-theoretic regime (on $N = 1$ brane!) of the D-brane models for holographic QCD that have to work in the (unrealistic) $N \gg 1$ supergravity regime.

⁵ Even without the detailed formula for the Schwinger mechanism decay rate, in any of its variants, one may understand the critical electric field strength (129) as that whose work done by its Lorentz force eE'_{crit} over the Compton wavelength $\lambda_m := \hbar/mc$ equals the rest mass mc^2 of the given particles: $E'_{\text{crit}} = \frac{mc^2}{e\lambda_m}$.

⁶ Recently it has been pointed out [77] that the core mechanism of the hadron sector in the WSS model, namely KK-reduction of 5d Yang-Mills theory, also appears on the M5-brane.

- (i) On the one hand, the super-exceptional PS-Lagrangian (20) contains a wealth of interaction terms, all of which being Spin-, Isospin- and Hypercharge-invariant (due to Theorem 7, by Spin(10, 1)-invariance of the unreduced model), hence identifiable among the list [102, 5.2][54, Tab. 1][57] of possible interactions in the effective field theory of hadrons. Interestingly, the super-exceptional M5-brane model organizes these hadron interaction terms in a supersymmetric (9) KK-tower (105). Notice here that while both
- (a) supersymmetry and
 - (b) KK-modes
- remain notoriously hypothetical in the unconfined color-charged sector of the standard model, both are in fact observed, to a reasonable degree of accuracy, in the confined flavor-sector of nature. This is, respectively, the phenomenon of
- (a) *hadron supersymmetry* [95][96][30] [31] (reviewed in [89][36][26][27]) and
 - (b) the success of the *dimensional deconstruction* of vector meson models [130][115, 5.2][115, 3.3] in holographic QCD (see [133, Fig. 15.2]).
- (ii) On the other hand, various standard Lagrangian terms considered in the literature on chiral perturbation theory and quantum hadrodynamics are clearly missing in the super-exceptional PS-Lagrangian (20); not the least the kinetic terms for the mesons. But since the one kinetic term that does appear, the one for the photon (29), crucially appears from the self-duality constraint (27) on the ordinary 3-form flux, this might just indicate that the super-exceptional version of the 3-flux self-duality remains to be understood and to be implemented.⁷

It is worth recalling that none of the existing models of hadrodynamics, be it chiral perturbation theory, quark-bag models, vector meson dominance, Walecka-type QHD models etc., have been derived from fundamental principles – notably not from QCD. This is the open “Confinement Problem” [61], one of the “Millennium Problems” [33][78], the “Holy Grail” of nuclear physics [68][63, 13.1.9]. Instead they are all models in effective field theory using clues from experiment. Therefore, even a partial hint for a possible emergence of confined hadrodynamics from first (cohomotopical) principles of M5-brane theory seems profound and interesting.

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⁷ Note that in the presence of H_3 one can consider a noncommutative worldvolume for the M5-brane – see [93].

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