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Relative gerbes

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Abstract

This paper introduces the notion of "relative gerbes" for smooth maps of manifolds, and discusses their differential geometry. The equivalence classes of relative gerbes are further classified by the relative integral cohomology in degree 3.

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1. Introduction

Giraud [9] first introduced the concept of gerbes in the early 1970s to study non-Abelian second cohomology. Later, Brylinski [4] defined gerbes as sheaves of groupoids with certain axioms, and discussed their differential geometry. He proved that the group of equivalence classes of gerbes gives a geometric realization of integral 3 cohomology classes on manifolds. Through a more elementary approach, Chatterjee and Hitchin [5,16] introduced

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gerbes in terms of transition line bundles for a given cover of the manifold. From this point of view, a gerbe is a one-degree-up generalization of a line bundle, where the line bundle is presented by transition maps. A notable example of a gerbe arises as the obstruction for the existence of a lift of a principal G-bundle to a central extension of the Lie group. Another example is the associated grebe of an oriented codimension 3 submanifold of an oriented manifold. The third example is what is called "basic gerbe", which corresponds to the generator of the degree 3 integral cohomology of a compact, simple and simply connected Lie group. The basic gerbe over G is closely related to the basic central extension of the loop group, and it was constructed, from this point of view, by Brylinski [4]. Later, Gawedski–Reis [8], for G= SU(n), and Meinrenken [21], in the general case, gave a finite-dimensional construction along with an explicit description of the grebe connection.

This paper introduces the notion of *relative gerbes* for smooth maps of manifolds, and discusses their differential geometry. The equivalence classes of relative gerbes are classified by the relative integral cohomology in degree 3.

The organization of this paper is as follows. In Section 2, the relative (co)homology of a smooth map between two manifolds is discussed. When the map is inclusion, the singular relative (co)homology of the map coincides with the singular relative (co)homology of the pair. Also, for a continuous map of topological spaces, the relative (co)homology of the map is isomorphic to the (co)homology of the mapping cone. In Section 3, following the Chatterjee–Hitchin perspective on gerbes, the notion of *relative gerbe* is defined for a smooth map $\Phi \in C^\infty(M, N)$ between two manifolds M and N as a gerbe over the target space together with a quasi-line bundle for the pull-back gerbe. It is also proven that the group of equivalence classes of relative gerbes can be characterized by the integral degree 3 relative cohomology of the same map.

Another objective of this paper is to develop the differential geometry of relative gerbes. More specifically, in Section 4, the concepts of relative connection, relative connection curvature, relative Cheeger–Simons differential character, and relative holonomy are introduced. As well, it is proven that a given closed relative three-form arises as a curvature of some relative grebe with connection if and only if the relative three-form is integral. Further, it is shown that a relative gerbe with connection for a smooth map $\Phi : M \to N$ generates a *relative* line bundle with connection for the corresponding map of loop paces, $L\Phi : LM \to LN$.

2. Relative homology/cohomology

2.1. Algebraic mapping cone for chain complexes

Definition 2.1. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a chain map between chain complexes over R where R is a commutative ring. The algebraic mapping cone of f [7] is defined as a chain complex $\mathrm{Cone}_\bullet(f)$ where

$$\mathrm{Cone}_n(f) = X_{n-1} \oplus Y_n$$

with the differential

$$\partial(\theta, \eta) = (\partial\theta, f(\theta) - \partial\eta).$$

Since $\partial^2 = 0$, we can consider the homology of this chain complex. Define relative homology of $f_\bullet$ as

$$H_n(f) := H_n(\mathrm{Cone}_\bullet(f)).$$

The short exact sequence of chain complexes

$$0 \to Y_n \xrightarrow{j} \mathrm{Cone}_n(f) \xrightarrow{k} X_{n-1} \to 0,$$

where $j(\beta) = (0, \beta)$ and $k(\alpha, \beta) = \alpha$ gives a long exact sequence in homology

$$\cdots \to H_n(Y) \xrightarrow{j} H_n(f) \xrightarrow{k} H_{n-1}(X) \xrightarrow{\delta} H_{n-1}(Y) \to \cdots, \tag{2.1}$$

where δ is the connecting homomorphism.

Lemma 2.2. *The connecting homomorphism δ is given by $\delta[\gamma] = [f(\gamma)]$ for $\gamma \in X_{n-1}$.*

Proof. For $\gamma \in X_{n-1}$, we have $k(\gamma, 0) = \gamma$. The short exact sequence of chain complexes gives an element $\gamma' \in Y_{n-1}$ such that $j(\gamma') = \partial(\gamma, 0) = (\partial\gamma, f(\gamma))$. δ is defined by $\delta[\gamma] = [\gamma']$. But, by definition of j, $j(\gamma') = (0, \gamma')$. Therefore $f(\gamma) = \gamma'$. This shows $\delta[\gamma] = [f(\gamma)]$. □

Definition 2.3. We call a chain map $f_\bullet : X_\bullet \to Y_\bullet$ a quasi-isomorphism if it induces isomorphism in cohomology, i.e., $H_\bullet(X) \xrightarrow{\cong} H_\bullet(Y)$.

Corollary 2.4. *$f_\bullet : X_\bullet \to Y_\bullet$ is a quasi-isomorphism if and only if $H_\bullet(f) = 0$.*

Proof. f is a quasi-isomorphism, if and only if the connecting homomorphism in the long exact sequence (2.1) is an isomorphism. □

Definition 2.5. A homotopy operator between two chain complexes $f, g : X_\bullet \to Y_\bullet$ is a linear map $h : X_\bullet \to Y_{\bullet+1}$ such that

$$h\partial + \partial h = f - g \quad (\star)$$

In that case, f and g are called chain homotopic and we denote it by $f \simeq g$.

Two chain maps $f : X_\bullet \to Y_\bullet$ and $g : Y_\bullet \to X_\bullet$ are called homotopy inverse if $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ are both homotopic to the identity. If $f : X_\bullet \to Y_\bullet$ admits a homotopy inverse, it is called a homotopy equivalence. In particular, every homotopy equivalence is a quasi-isomorphism.

Proposition 2.6. *Any homotopy between chain maps $f, g : X_\bullet \to Y_\bullet$ induces an isomorphism of chain complexes* $\mathrm{Cone}(f)_\bullet$ *and* $\mathrm{Cone}(g)_\bullet$.

Proof. Given a homotopy operator h satisfying $(\star)$, define a map $F : \mathrm{Cone}_\bullet(f) \to \mathrm{Cone}_\bullet(g)$ by

$$F(\alpha, \beta) = (\alpha, -h(\alpha) + \beta).$$

Since

$$\partial F(\alpha, \beta) = (\partial\alpha, g(\alpha) + \partial h(\alpha) + \partial\beta) = (\partial\alpha, f(\alpha) - h\partial(\alpha) + \partial\beta) = F\partial(\alpha, \beta),$$

F is a chain map and its inverse map is $F^{-1}(\alpha, \beta) = (\alpha, h(\alpha) + \beta)$. $\square$

Lemma 2.7. *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_\bullet & \longrightarrow & Y_\bullet & \longrightarrow & Z_\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{X}_\bullet & \longrightarrow & \widetilde{Y}_\bullet & \longrightarrow & \widetilde{Z}_\bullet & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain maps with exact rows. If two of vertical maps are quasi-isomorphisms, then so is the third.

Proof. The statement follows from the five-lemma applied to the corresponding diagram in homology, $\square$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_\bullet(X) & \longrightarrow & H_\bullet(Y) & \longrightarrow & H_\bullet(Z) & \longrightarrow & H_{\bullet-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_\bullet(\widetilde{X}) & \longrightarrow & H_\bullet(\widetilde{Y}) & \longrightarrow & H_\bullet(\widetilde{Z}) & \longrightarrow & H_{\bullet-1}(\widetilde{X}) & \longrightarrow & \cdots
\end{array}$$

Proposition 2.8. *Suppose that we have the following commutative diagram of chain maps,*

$$\begin{array}{ccc}
X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\
{\scriptstyle \Phi_\bullet}\downarrow & & {\scriptstyle \Psi_\bullet}\downarrow \\
\widetilde{X}_\bullet & \xrightarrow{\tilde{f}_\bullet} & \widetilde{Y}_\bullet
\end{array}$$

such that Φ and Ψ are quasi-isomorphisms. Then the induced map

$$F : \mathrm{Cone}_\bullet(f) \to \mathrm{Cone}_\bullet(\tilde{f}), \qquad (\alpha, \beta) \mapsto (\Phi(\alpha), \Psi(\beta))$$

is a quasi-isomorphism.

Proof. The map F is a chain map since,

$$\begin{aligned}
\partial F(\alpha, \beta) &= \partial(\Phi(\alpha), \Psi(\beta)) = (\partial\Phi(\alpha), \tilde{f}(\Phi(\alpha)) - \partial\Psi(\beta)) = (\Phi(\partial\alpha), \Psi(f(\alpha) - \partial\beta)) \\
&= F(\partial\alpha, f(\alpha) - \partial\beta) = F\partial(\alpha, \beta).
\end{aligned}$$

The chain map F fits into a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_\bullet & \longrightarrow & \mathrm{Cone}_\bullet(f) & \longrightarrow & X_{\bullet-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{Y}_\bullet & \longrightarrow & \mathrm{Cone}_\bullet(\widetilde{f}) & \longrightarrow & \widetilde{X}_{\bullet-1} & \longrightarrow & 0 \end{array}$$

Since Φ and Ψ are quasi-isomorphisms, so is F by Lemma 2.7. □

Proposition 2.9. *For any chain map $f_\bullet : X_\bullet \to Y_\bullet$, there is a long exact sequence*

$$\cdots \to H_{n-1}(\ker f) \xrightarrow{j} H_n(f) \xrightarrow{k} H_n(\mathrm{coker} f) \xrightarrow{\delta} H_{n-2}(\ker f) \to H_{n-1}(f) \to \cdots,$$

where j, k and the connecting homomorphism δ are defined by

$$j[\theta] = [(\theta, 0)], \qquad k[(\theta, \eta)] = [\eta \bmod f(X)],$$

$$\delta[(\eta \bmod f(X))] = [\partial\theta] \in H_{n-2}(\ker f).$$

Here, $\eta \in Y_n$ and $\partial\eta = f(\theta)$ for some $\theta \in X_{n-1}$. In particular, if f is an injection then $H_n(f) = H_n(\mathrm{coker} f)$, and if it is onto then $H_n(f) = H_{n-1}(\ker f)$.

Proof. Let $\tilde{f}_\bullet : X_\bullet \to \mathrm{im}(f_\bullet) \subseteq Y_\bullet$ be the chain map $f_\bullet$, viewed as a map into the subcomplex $f_\bullet(X_\bullet) \subseteq Y_\bullet$. We have the following short exact sequence

$$0 \to \mathrm{Cone}_n(\tilde{f}) \xrightarrow{i} \mathrm{Cone}_n(f) \xrightarrow{k} \mathrm{coker}(f_n) \to 0,$$

where k is as above and i is the inclusion map. Therefore, there exists a long exact sequence

$$\cdots \to H_n(\tilde{f}) \xrightarrow{i} H_n(f) \xrightarrow{k} H_n(\mathrm{coker} f) \to H_{n-1}(\tilde{f}) \to \cdots. \tag{2.2}$$

Let $\tilde{f}'_\bullet : X_\bullet/\ker f_\bullet \to \mathrm{im}(f_\bullet)$ be the map induced by f. Notice that since $\tilde{f}'$ is an isomorphism, therefore $H_\bullet(\tilde{f}') = 0$. By using the long exact sequence corresponding to the short exact sequence

$$0 \to \ker f_{\bullet-1} \xrightarrow{\tilde{j}} \mathrm{Cone}_\bullet(\tilde{f}) \xrightarrow{\pi} \mathrm{Cone}_\bullet(\tilde{f}') \to 0,$$

where $\tilde{j}(\theta) = (\theta, 0)$, and $\pi(\theta, \eta) = (\theta \bmod \ker f, \eta)$, we see that $\tilde{j}$ is a quasi-isomorphism. Since $j = i \circ \tilde{j}$, we obtain the long exact sequence

$$\cdots \to H_{n-1}(\ker f) \xrightarrow{j} H_n(f) \xrightarrow{k} H_n(\mathrm{coker} f) \to H_{n-2}(\ker f) \to \cdots.$$

To find connecting homomorphism, assume $[\eta \bmod f(X)] \in H_n(\mathrm{coker} f)$ for $\eta \in Y_n$. Then $\partial\eta \in f(X)$, i.e., $\partial\eta = f(\theta)$ for some $\theta \in X_{n-1}$. Since

$$f(\partial\theta) = \partial f(\theta) = \partial\partial\eta = 0$$

then $\partial\theta \in \ker(f)$. Also $k(\theta,\eta) = \eta \bmod f(X)$ and $j(\theta) = i \circ \tilde{j}(\partial\theta) = i(\partial\theta, 0) = (\partial\theta, 0) = \partial(\theta,\eta)$. Thus, we have

$$\delta[(\eta \bmod f(X))] = [\partial\theta] \in H_{n-2}(\ker f). \quad \square$$

2.2. Algebraic mapping cone for co-chain complexes

If $f^\bullet : X^\bullet \to Y^\bullet$ is a co-chain map between co-chain complexes, the algebraic mapping cone of f is defined as a co-chain complex $\mathrm{Cone}^\bullet(f)$ where

$$\mathrm{Cone}^n(f) = Y^{n-1} \oplus X^n$$

with the differential

$$\mathrm{d}(\alpha,\beta) = (f(\beta) - \mathrm{d}\alpha, \mathrm{d}\beta).$$

Since $\mathrm{d}^2 = 0$, we can consider the cohomology of this co-chain complex. Define relative cohomology of $f^\bullet$ as

$$H^n(f) := H^n(\mathrm{Cone}^\bullet(f)).$$

Remark 2.10. Any cochain complex $(X^\bullet, d)$ may be viewed as a chain complex $(\tilde{X}_\bullet, \partial)$, where $\tilde{X}_n = X^{-n}$ and $\partial_n = d^{-n} (n \in \mathbb{Z})$. This correspondence takes cochain maps $f^\bullet : X^\bullet \to Y^\bullet$ into chain maps $\tilde{f}_\bullet : \tilde{X}_\bullet \to \tilde{Y}_\bullet$, where $\tilde{f}_n = f^{-n}$, and identifies $\mathrm{Cone}(\tilde{f})$ and $\widetilde{\mathrm{Cone}(f)}$ up to a degree shift:

$$\widetilde{\mathrm{Cone}(f)}_n = \mathrm{Cone}(f)^{-n} = Y^{-n-1} \oplus X^{-n},$$
$$\mathrm{Cone}(\tilde{f})_n = \tilde{X}_{n-1} \oplus \tilde{Y}_n = X^{-n+1} \oplus Y^{-n}.$$

Thus, $\widetilde{\mathrm{Cone}(f)}_n \cong \mathrm{Cone}(\tilde{f})_{n+1}$.

Using this correspondence, the results for the mapping cone of chain maps are directly carried over to cochain maps.

2.3. Kronecker pairing

For a chain complex $X_\bullet$, the dual co-chain complex $(X')^\bullet$ is defined by $(X')^n = \mathrm{Hom}(X_n, R)$ with the dual differential.

Proposition 2.11. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map between chain complexes, and $(f')^\bullet : (Y')^\bullet \to (X')^\bullet$ be its dual cochain map. Then the bilinear pairing*

$$\mathrm{Cone}^n(f') \times \mathrm{Cone}_n(f) \to R$$

given by the formula

$$\langle(\alpha,\beta),(\theta,\eta)\rangle = \langle\alpha,\theta\rangle - \langle\beta,\eta\rangle$$

for $(\alpha,\beta)\in \mathrm{Cone}^n(f')$ *and* $(\theta,\eta)\in \mathrm{Cone}_n(f)$ *induces a pairing in cohomology/homology*

$$H^n(f')\times H_n(f)\to R.$$

Proof. It is enough to show that a cocycle paired with a boundary is zero and a coboundary paired with a cycle is zero. Let $(\alpha,\beta)=\partial(\alpha',\beta')$ and $\partial(\theta,\eta)=0$. Therefore, by definition

$$\alpha = f'\beta' - \mathrm{d}\alpha', \qquad \beta=\mathrm{d}\beta',$$

and

$$\partial\eta = f(\theta), \qquad \partial\theta = 0.$$

$$\begin{aligned}\langle(\alpha,\beta),(\theta,\eta)\rangle &= \langle\alpha,\theta\rangle - \langle\beta,\eta\rangle = \langle f'\beta',\theta\rangle - \langle \mathrm{d}\alpha',\theta\rangle - \langle \mathrm{d}\beta',\eta\rangle\\ &= \langle f'\beta',\theta\rangle - \langle\alpha',\partial\theta\rangle - \langle\beta',\partial\eta\rangle = \langle f'\beta',\theta\rangle - \langle\beta', f(\theta)\rangle = 0. \end{aligned} \tag{2.3}$$

Similarly we can prove that a co-boundary paired with a cycle is zero. □

Lemma 2.12. *If* $f_\bullet : X_\bullet \to Y_\bullet$ *is a chain map, and* $(f')^\bullet : (Y')^\bullet \to (X')^\bullet$ *be its dual cochain map, then* $\mathrm{Cone}^\bullet(f') = (\mathrm{Cone}_\bullet(f))'$.

Proof. Notice that $\mathrm{Cone}^n(f') = (\mathrm{Cone}_n(f))' = (X^{n-1})'\oplus(Y^n)'$. It follows from definitions that

$$\langle \mathrm{d}(\alpha,\beta),(\theta,\eta)\rangle = \langle(\alpha,\beta),\partial(\theta,\eta)\rangle.$$

Therefore differential of $\mathrm{Cone}^n(f')$ is dual of differential of $\mathrm{Cone}_n(f)$. □

2.4. Singular, de Rham, Čech theory

In this section, two manifolds M and N and a map $\Phi\in C^\infty(M,N)$ are fixed.

Singular relative homology: Consider the push-forward map $\Phi_* : S_q(M,R)\to S_q(N,R)$, where R is a commutative ring and $S_q(M,R)$, $S_q(N,R)$ are the singular chain complexes of M and N, respectively. Singular relative homology is the homology of the chain complex $\mathrm{Cone}_\bullet(\Phi_*)$, and is denoted $H_\bullet(\Phi,R)$.

Singular relative cohomology: Consider the pull-back map $\Phi^* : S^q(N,R)\to S^q(M,R)$, where R is a commutative ring, and $S^q(M,R)$ and $S^q(N,R)$ are the singular co-chain complex of M and N, respectively. Singular relative cohomology is the cohomology of the co-chain complex $\mathrm{Cone}^\bullet(\Phi^*)$, and is denoted $H^\bullet(\Phi,R)$.

de Rham relative cohomology: For $\Phi\in C^\infty(M,N)$, consider the pull back-map

$$\Phi^* : \Omega^q(N)\to\Omega^q(M)$$

between differential co-chain complexes. In this paper, the cohomology of $\mathrm{Cone}^\bullet(\Phi^*)$ is denoted as $H^\bullet_{\mathrm{dR}}(\Phi)$ and called it "de Rham relative cohomology."

Čech relative cohomology: Let A be a R-module, and $\mathcal{U} = \{U_\alpha\}$ be a good cover of a manifold M, i.e., all the finite intersections are contractible. For any collection of indices $\alpha_0, \cdots, \alpha_p$ such that $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \emptyset$, let

$$U_{\alpha_0\cdots\alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}.$$

A Čech-p-cochain $f \in \check{C}^p(\mathcal{U}, A)$ is a function

$$f = \coprod_{\alpha_0\cdots\alpha_p} f_{\alpha_0\cdots\alpha_p} : \coprod_{\alpha_0\cdots\alpha_p} U_{\alpha_0\cdots\alpha_p} \to A,$$

where $f_{\alpha_0\cdots\alpha_p}$ is locally constant and anti-symmetric in indices. The differential is defined by

$$(\mathrm{d}f)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}},$$

where the "hat" sign means that the index has been omitted. Since $d \circ d = 0$, one can define Čech cohomology groups with coefficients in A as

$$\check{H}^p(M, A) := H^p(\check{C}(\mathcal{U}, A)).$$

Let $\mathcal{U} = \{U_i\}_{i\in I}$, $\mathcal{V} = \{V_j\}_{j\in J}$ be good covers of M and N, respectively, such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$. Let $\check{C}^\bullet(M, A)$ and $\check{C}^\bullet(N, A)$ be the Čech complexes for given covers, where A is an R-module. Using the pull-back map $\Phi^* : \check{C}^\bullet(N, A) \to \check{C}^\bullet(M, A)$, the relative Čech cohomology is defined as the cohomology of $\mathrm{Cone}^\bullet(\Phi^*)$. Denote this cohomology by $\check{H}^\bullet(\Phi, A)$.

Suppose that $\underline{A}$ is one of the sheaves [11,4] $\underline{\mathbb{Z}}, \underline{\mathbb{R}}, \underline{\mathrm{U}(1)}, \underline{\Omega^q}$. Denote the space of k-cochains of the sheaf $\underline{A}$ on M and N, respectively, as $C^k(M, \underline{A})$ and $C^k(N, \underline{A})$. Here, the differential is defined as above. Again, we have an induced map

$$\Phi^* : C^k(N, \underline{A}) \to C^k(M, \underline{A}).$$

Denote the cohomology of $\mathrm{Cone}^\bullet(\Phi^*)$ as $H^*(\Phi, \underline{A})$.

Theorem 2.13. *There is a canonical isomorphism* $H^n_{\mathrm{dR}}(\Phi) \cong H^n(\Phi, \mathbb{R})$.

Proof. Let $S^\bullet_{\mathrm{sm}}(M, \mathbb{R})$ and $S^\bullet_{\mathrm{sm}}(N, \mathbb{R})$ be the smooth singular cochain complex of M and N, respectively [1]. Consider the following diagram:

$$\begin{array}{ccc} \Omega^n(N) & \xrightarrow{\Phi^*} & \Omega^n(M) \\ {\scriptstyle g^n}\downarrow & & {\scriptstyle f^n}\downarrow \\ S^n_{sm}(N,\mathbb{R}) & \xrightarrow{\Phi^*} & S^n_{sm}(M,\mathbb{R}) \end{array}$$

where f^n is defined by $f^n(\omega) : \sigma \mapsto \int_{\Delta_n} \sigma^*\omega$, for $\omega \in \Omega^n(M)$ and $\sigma \in S_n^{\rm sm}(M)$ is a smooth singular n-simplex. $g^\bullet$ is defined in a similar fashion. From these definitions, it is clear that the diagram commutes. $f^\bullet$ and $g^\bullet$ are quasi-isomorphisms by de Rham Theorem [1]. Define $k^\bullet : \Omega^\bullet(\Phi, \mathbb{R}) \to S^\bullet_{\rm sm}(\Phi, \mathbb{R})$ by $k^\bullet(\alpha, \beta) = (f^{\bullet -1}(\alpha), g^\bullet(\beta))$. One can use Proposition 2.8 and deduce that $k^\bullet$ is a quasi-isomorphism. There is a co-chain map

$$l^\bullet : S^\bullet(M, \mathbb{R}) \to S^\bullet_{\rm sm}(M, \mathbb{R})$$

given by the dual of the inclusion map in chain level. In [24, p. 196] it is shown that $l^\bullet$ is a quasi-isomorphism. Therefore, by using Proposition 2.8 again,

$$H^n(\Phi, \mathbb{R}) \cong H^n_{\rm sm}(\Phi, \mathbb{R}).$$

Together, one can have $H^\bullet(\Phi, \mathbb{R}) \cong H^\bullet_{\rm dR}(\Phi, \mathbb{R})$. □

Theorem 2.14. *For $\Phi \in C^\infty(M, N)$, there is an isomorphism $H^q_{\rm dR}(\Phi) \cong \check{H}^q(\Phi, \mathbb{R})$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ be good covers of M and N together with a map $r : I \to J$, such that $\Phi(U_i) \subseteq V_{r(i)}$. Define the double complex $E^{p,q}(M) = \check{C}^p(M, \Omega^q)$, where $\check{C}^p(\mathcal{U}, \Omega^q)$ is the set of q-forms $\omega_{\alpha_0\cdots\alpha_p} \in \Omega^q(U_{\alpha_0\cdots\alpha_p})$ anti-symmetric in indices with the differential 'd' defined as before. Let $E^n(M) = \oplus_{p+q=n} E^{p,q}(M)$ be the associated total complex. The map $\Phi : M \to N$ induces chain maps $\Phi^* : E^n(N) \to E^n(M)$. Let us denote the corresponding algebraic mapping cone as $E^n(\Phi)$. The inclusion $\check{C}^n(M, \mathcal{U}) \to E^n(M)$ is a quasi-isomorphism [1, p. 97]. There exists a similar quasi-isomorphism for N, and since inclusion maps commute with pull-back of Φ, one gets a quasi-isomorphism $\check{C}^n(\Phi) \to E^n(\Phi)$. Thus, the following isomorphism is obtained.

$$\check{H}^n(\Phi, \mathbb{R}) \cong H^n(E(\Phi)). \tag{2.4}$$

The map $\Omega^n(M) \to E^{0,n}(M) \subset E^n(M)$, given by restrictions of forms $\alpha \mapsto \alpha|_{U_i}$, is a quasi-isomorphism [1, p. 96]. Again, these maps commute with pull back, and hence define a quasi-isomorphism $\Omega^n(\Phi) \to E^n(\Phi)$ that means

$$H^n_{\rm dR}(\Phi) \cong H^n(E(\Phi)). \tag{2.5}$$

By combining Eqs. (2.4) and (2.5), one obtains $\check{H}^\bullet(\Phi, \mathbb{R}) \cong H^\bullet_{\rm dR}(\Phi)$. □

Remark 2.15. A modification of this argument, working instead with the double complex $\check{C}^p(M, S^q)$ given by collection of $S^q(U_{\alpha_0\cdots\alpha_p})$, gives isomorphism between Čech relative cohomology and singular relative cohomology with integer coefficients, hence

$$\check{H}^q(\Phi, \mathbb{Z}) \cong H^q_S(\Phi, \mathbb{Z}).$$

2.5. Topological definition of relative homology

Let $\Phi : M \to N$ be an inclusion map, then the push-forward map $\Phi_* : S_\bullet(M, R) \to S_\bullet(N, R)$ is injection. Proposition 2.9 shows that $H_\bullet(\Phi) \cong H_\bullet(S(N)/S(M)) =$

$H_\bullet(N, M; R)$. $H_\bullet(N, M; R)$ is known as relative homology. Obviously, this is a special case of what the author defined as a relative singular homology of an arbitrary map $\Phi : M \to N$.

Given a continuous map $f : X \to Y$ of topological spaces, define mapping cylinder

$$\mathrm{Cyl}_f = \frac{(X \times I) \sqcup Y}{(x, 1) \sim f(x)},$$

and mapping cone [14]

$$\mathrm{Cone}_f = \frac{\mathrm{Cyl}_f}{X \times \{0\}}.$$

Let $\mathrm{Cone}(X) := X \times I / X \times \{0\}$. There are natural maps

$$i : Y \hookrightarrow \mathrm{Cone}_f, \quad j : \mathrm{Cone}(X) \to \mathrm{Cone}_f.$$

Note that j is an inclusion only if f is an inclusion. There is a canonical map,

$$h : S_{n-1}(X) \to S_n(\mathrm{Cone}(X))$$

with the property $h \circ \partial + \partial \circ h = k$, where h is defined by replacing a singular n-simplex with its cone, and $k : X \hookrightarrow \mathrm{Cone}(X)$ is the inclusion map. Define the map

$$l_n : \mathrm{Cone}_n(f_*) \to S_n(\mathrm{Cone}_f), \quad (x, y) \mapsto j_*(h(x)) - i_*(y).$$

Theorem 2.16. *$l_\bullet$ is a chain map and a quasi-isomorphism. Thus,*

$$H_n(f) \cong H_n(\mathrm{Cone}_f).$$

Proof. Recall that $\partial(x, y) = (\partial x, f_*(x) - \partial y)$. Since

$$l(\partial(0, y)) + \partial l(0, y) = l((0, -\partial y)) - \partial i_* y = i_*(\partial y) - \partial i_* y = 0, \tag{2.6}$$

and

$$\begin{aligned} l(\partial(x, 0)) + \partial l(x, 0) &= l((\partial x, f(x))) + \partial j_* h(x) = j_* h(\partial x) - i_* f(x) + \partial j_* h(x) \\ &= j_* k_*(x) - i_* f(x) = 0 \end{aligned} \tag{2.7}$$

therefore $\partial l + l\partial = 0$. Consider diagram

$$\begin{tikzcd} 0 \arrow[r] & S_n(\mathrm{Cyl}_f) \arrow[r] & S_n(\mathrm{Cone}_f) \arrow[r] & S_n(\mathrm{Cone}_f, \mathrm{Cyl}_f) \arrow[r] & 0 \\ 0 \arrow[r] & S_n(Y) \arrow[r] \arrow[u] & \mathrm{Cone}_n(f_*) \arrow[r] \arrow[u, "l"] & S_{n-1}(X) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where the first row corresponds to the pair $(\mathrm{Cone}_f, \mathrm{Cyl}_f)$ and the right vertical arrow comes from

$$S_{n-1}(X) \to S_n(\mathrm{Cone}(X), X) \underset{\text{exision}}{\cong} S_n(\mathrm{Cone}_f, \mathrm{Cyl}_f).$$

The diagram commutes, and the rows are exact. Since the right and left vertical maps are quasi-isomorphisms, hence so is the middle map. □

2.6. *An integrality criterion*

If A and B are R-modules, then any homomorphism $\kappa : A \to B$ induces homomorphisms $\kappa : H^n(\Phi, A) \to H^n(\Phi, B)$ and $\kappa : H_n(\Phi, A) \to H_n(\Phi, B)$. In particular, the injection $\iota : \mathbb{Z} \to \mathbb{R}$ induces a homomorphism

$$\iota : H^n(\Phi, \mathbb{Z}) \to H^n(\Phi, \mathbb{R}).$$

A class $[\gamma] \in H^n(\Phi, \mathbb{R})$ is called integral in case $[\gamma]$ lies in the image of the map ι.

Proposition 2.17. *A class* $[(\alpha, \beta)] \in H^n(\Phi, \mathbb{R})$ *is integral if and only if* $\int_\theta \alpha - \int_\eta \beta \in \mathbb{Z}$ *for all cycles* $(\theta, \eta) \in \mathrm{Cone}_n(\Phi, \mathbb{Z})$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H^n(\Phi,\mathbb{R}) & \xrightarrow{\cong} & Hom(H_n(\Phi,\mathbb{R}),\mathbb{R}) & \longrightarrow & 0 \\
 & & \uparrow & & \iota\uparrow & & \tilde{\iota}\uparrow & & \\
0 & \longrightarrow & Ext(H_n(\Phi,\mathbb{Z})) & \longrightarrow & H^n(\Phi,\mathbb{Z}) & \xrightarrow{\tau} & Hom(H_n(\Phi,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where $H^n(\Phi, \mathbb{R}) \to \mathrm{Hom}(H_n(\Phi, \mathbb{R}), \mathbb{R})$ and τ are pairing given by integral. The map $\tilde{\iota}$ is inclusion map, considering the fact that

$$\mathrm{Hom}(H_n(\Phi, \mathbb{R}), \mathbb{R}) = \mathrm{Hom}(H_n(\Phi, \mathbb{Z}), \mathbb{R}).$$

Thus, $[(\alpha, \beta)] \in H^n(\Phi, \mathbb{R})$ is integral if $\int_\theta \alpha - \int_\eta \beta \in \mathbb{Z}$ for all cycles $(\theta, \eta) \in \mathrm{Cone}_n(\Phi, \mathbb{Z})$. □

2.7. *Bohr–Sommerfeld condition*

Let (N, ω) be a symplectic manifold. Recall that an immersion $\Phi : M \to N$ is isotropic if $\Phi^*\omega = 0$. It is called Lagrangian if furthermore $\dim M = \frac{1}{2} \dim N$. Suppose that $H_1(N, \mathbb{Z}) = 0$ and ω is integral. A Lagrangian immersion $\Phi : M \to N$ is said to satisfy the Bohr–Sommerfeld condition [12,18] if for all one-cycles $\gamma \in S_1(M)$

$$\frac{1}{2\pi} \int_D \omega \in \mathbb{Z}, \quad \text{where } \partial D = \Phi(\gamma).$$

Note that since ω is integral, the above condition does not depend on the choice of D. Also, if $\omega = \mathrm{d}\theta$ is exact (for example for the cotangent bundles), the condition means that

$$\frac{1}{2\pi}\int_{\gamma} \Phi^{*}\theta \in \mathbb{Z} \quad \text{for all one-cycles } \gamma.$$

In terms of relative cohomology, the above condition means that $(0, \omega) \in \Omega^2(\Phi)$ defines an integral class in $H^2_{\mathrm{dR}}(\Phi)$. The interesting feature of this situation is that the forms on M, N are fixed, and it defines a condition on the map Φ.

Example 2.18. Let $N = \mathbb{R}^2$, $M = S^1$, $\omega = \mathrm{d}x \wedge \mathrm{d}y$, $\Phi =$ inclusion map. Then, the immersion $\Phi : S^1 \hookrightarrow \mathbb{R}^2$ satisfies the Bohr–Sommerfeld condition.

3. Geometric interpretation of integral relative cohomology groups

Let $\Phi \in C^\infty(M, N)$, where M and N are manifolds. Let $U = \{U_i\}_{i\in I}$, $V = \{V_j\}_{j\in J}$ be good covers of M and N, respectively, such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$.

Proposition 3.1. *$H^q(\Phi, \mathbb{Z}) \cong H^{q-1}(\Phi, \underline{U(1)})$ for $q \geq 1$.*

Proof. Consider the following long exact sequence,

$$\cdots \to H^{q-1}(M, \underline{\mathbb{R}}) \to H^q(\Phi, \underline{\mathbb{R}}) \to H^q(N, \underline{\mathbb{R}}) \to H^q(M, \underline{\mathbb{R}}) \to \cdots.$$

Since $H^\bullet(M, \underline{\mathbb{R}}) = 0$ and $H^\bullet(N, \underline{\mathbb{R}}) = 0$, one can see that $H^q(\Phi, \underline{\mathbb{R}}) = 0$ for $q > 0$. By using the long exact sequence associated to exponential sequence

$$0 \to \mathbb{Z} \to \underline{\mathbb{R}} \overset{\exp}{\to} \underline{U(1)} \to 0 \quad (\star)$$

one can deduce that $H^q(\Phi, \mathbb{Z}) \cong H^{q-1}(\Phi, \underline{U(1)})$ for $q \geq 1$. □

3.1. *Geometric interpretation of $H^1(\Phi, \mathbb{Z})$*

Let X be a manifold. Function $f \in C^\infty(X, U(1))$ has global logarithm if there exists a function $k \in C^\infty(X, \mathbb{R})$ such that $f = \exp((2\pi\sqrt{-1})k)$.

Definition 3.2. The two maps $f, g : X \to U(1)$ are equivalent if f/g has a global logarithm.

The short exact sequence of sheaves $(\star)$ gives an exact sequence of Abelian groups

$$0 \to H^0(X, \mathbb{Z}) \to C^\infty(X, \mathbb{R}) \overset{\exp}{\to} C^\infty(X, U(1)) \to H^1(X, \mathbb{Z}) \to 0.$$

This shows that there is a one-to-one correspondence between equivalence classes and elements of $H^1(X,\mathbb{Z})$. One should look for a geometric realization of $H^1(\Phi,\mathbb{Z})$ for a smooth map $\Phi : M \to N$. Let

$$\mathcal{L} := \{(k, f)|\Phi^* f = \exp((2\pi\sqrt{-1})k)\} \subset C^\infty(M,\mathbb{R}) \times C^\infty(N, U(1)).$$

$\mathcal{L}$ has a natural group structure. There is a natural group homomorphism,

$$\tau : C^\infty(N,\mathbb{R}) \to \mathcal{L},$$

where τ is defined for $l \in C^\infty(N,\mathbb{R})$ by

$$\tau(l) = (\Phi^* l, \exp((2\pi\sqrt{-1})l)).$$

Definition 3.3. $(k, f), (k', f') \in \mathcal{L}$ are equivalent if $f/f' = \exp((2\pi\sqrt{-1})h)$ for some function $h \in C^\infty(N,\mathbb{R})$ such that

$$\Phi^* h = k - k'.$$

The set of equivalence classes is a group $\mathcal{L}/\tau(C^\infty(N,\underline{\mathbb{R}}))$.

Theorem 3.4. *There exists an exact sequence of groups*

$$C^\infty(N,\mathbb{R}) \xrightarrow{\tau} \mathcal{L} \to H^1(\Phi,\mathbb{Z}) \to 0.$$

Thus, $H^1(\Phi,\mathbb{Z})$ *parameterizes equivalence classes of pairs* (k, f).

Proof. The first step is to construct a group homomorphism

$$\chi : \mathcal{L} \to H^1(\Phi,\mathbb{Z}).$$

Given (k, f), let $l_j \in C^\infty(V_j,\mathbb{R})$ be local logarithms for $f|_{V_j}$, that is $f|_{V_j} = \exp((2\pi\sqrt{-1})l_j)$. On overlaps, $a_{jj'} := l_{j'} - l_j : V_{jj'} \to \mathbb{Z}$ defines a Čech cocycle in $\check{C}^1(N,\mathbb{Z})$. Let

$$b_i := \Phi^* l_{r(i)} - k|_{U_i} : U_i \to \mathbb{Z}.$$

Since $b'_i - b_i = \Phi^* a_{r(i)r(i')}$, so that (b, a) defines a Čech cocycle in $\check{C}^1(\Phi,\mathbb{Z})$. Given another choice of local logarithms $\tilde{l}_j$, the Čech cocycle changes to

$$\tilde{b}_i = b_i + \Phi^* c_{r(i)}, \quad \tilde{a}_{jj'} = a_{jj'} + c_{j'} - c_j$$

where $c_j = \tilde{l}_j - l_j : V_j \to \mathbb{Z}$. Thus, $(\tilde{b}, \tilde{a}) = (b, a) + \mathrm{d}(0, c)$, and $\chi(k, f) := [(b, a)] \in H^1(\Phi,\mathbb{Z})$ is well-defined. Similarly, if $(b, a) = \mathrm{d}(0, c)$ then the new local logarithms

$\tilde{l}_j = l_j - c_j$ satisfy $\tilde{a}_{jj'} = 0$, which means that $\tilde{l}_j$ patches to a global logarithm $\tilde{l}$. $b_i = \Phi^* c_{r(i)}$ implies that $k|_{U_i} = \Phi^* \tilde{l}_{r(i)}$, which means $k = \Phi^* \tilde{l}$. This shows that the kernel of χ consists of (k, f) such that there exists $l \in C^\infty(N, \mathbb{R})$ with $f = \exp((2\pi\sqrt{-1})l)$ and $k = \Phi^* l$, i.e., $\ker(\chi) = \mathrm{im}(\tau)$.

Finally, it is shown below that χ is surjective. Suppose that $(b, a) \in \check{C}^1(\Phi, \mathbb{Z})$ is a cocycle. Then

$$a_{j'j''} - a_{jj''} + a_{jj'} = 0, \tag{3.1}$$

$$\Phi^* a_{r(i)r(i')} = b_{i'} - b_i. \tag{3.2}$$

Choose a portion of unity $\sum_{j \in J} h_j = 1$ subordinate to the open cover $V = \{V_j\}_{j \in J}$. Define $f_j \in C^\infty(V_j, U(1))$ by

$$f_j = \exp\left(2\pi\sqrt{-1} \sum_{p \in J} a_{jp} h_p\right).$$

By applying (3.1) on $V_j \cap V_{j'}$ one has

$$f_j f_{j'}^{-1} = \exp\left(2\pi\sqrt{-1} \sum_{p \in J} a_{jp} h_p\right) \exp\left(-2\pi\sqrt{-1} \sum_{p \in J} a_{j'p} h_p\right)$$
$$= \exp\left(2\pi\sqrt{-1} \sum_{p \in J} a_{jj'} h_p\right) = 1.$$

Hence f_i defines a map $f \in C^\infty(N, U(1))$ such that $f_{|V_j} = f_j$. Define $k_i \in C^\infty(U_i, \mathbb{R})$ by

$$k_i = \sum_{p \in J} (\Phi^* a_{r(i)p} + b_i) \Phi^* h_p. \tag{3.3}$$

Since $b_i \in \mathbb{Z}$, $\exp((2\pi\sqrt{-1})k_i) = \Phi^* f|_{U_i}$. One can check that on overlaps $U_i \cap U_{i'}$, $k_i - k_{i'} = 0$, so that $\{k_i\}$ defines a global function $k \in C^\infty(M, \mathbb{R})$ with $\Phi^* f = \exp((2\pi\sqrt{-1})k)$. Indeed, by applying (3.1) and (3.2) on $U_i \cap U_{i'}$ one can obtain

$$\sum_{p \in J} (\Phi^* a_{r(i)p} + \Phi^* a_{pr(i')} + b_i - b_{i'}) \Phi^* h_p = \sum_{p \in J} (\Phi^* a_{r(i)r(i')} + b_i - b_{i'}) \Phi^* h_p = 0.$$

By construction $\chi(k, f) = [(b, a)]$, which shows χ is surjective. $\square$

Remark 3.5. Any $(k, f) \in \mathcal{L}$ defines a $U(1)$-valued function on the mapping cone, $\mathrm{Cone}_\phi = N \cup_\Phi \mathrm{Cone}(M)$, given by f on N and by $\exp((2\pi\sqrt{-1})tk)$ on $\mathrm{Cone}(M)$. Here, $t \in I$ is the cone parameter. Hence, one obtains a map

$$\mathcal{L} \to H^1(\mathrm{Cone}_\Phi, \mathbb{Z}) \cong H^1(\Phi, \mathbb{Z}).$$

This gives an alternative way of proving Theorem 3.4.

3.2. Geometric interpretation of $H^2(\Phi, \mathbb{Z})$

Denote the group of Hermitian line bundles over M with $\mathrm{Pic}(M)$ and the subgroup of Hermitian line bundles over M which admits a unitary section with $\mathrm{Pic}_0(M)$. Recall that there is an exact sequence of Abelian groups

$$0 \to \mathrm{Pic}_0(M) \hookrightarrow \mathrm{Pic}(M) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \to 0$$

defined as follows. For the line bundle L over M with transition maps $c_{ii'} \in C^\infty(U_{ii'}, U(1))$ over good cover $\{U_i\}_{i\in I}$ for M, $\delta(L)$ is the cohomology class of the two-cocycle $a_{ii'i''} : U_{ii'i''} \to \mathbb{Z}$ given as

$$a_{ii'i''} := \left(\frac{1}{2\pi\sqrt{-1}}(\log c_{i'i''} - \log c_{ii''} + \log c_{ii'})\right) \in \mathbb{Z}.$$

Thus, one can say two Hermitian line bundles L_1 and L_2 over M are equivalent if and only if $L_1 L_2^{-1}$ admits a unitary section. The exact sequence shows that $H^2(M, \mathbb{Z})$ parameterizes the equivalence classes of line bundles [19]. The class $\delta(L) := c_1(L)$ is called the first Chern class of L. Similarly, for a smooth map $\Phi : M \to N$ one should look for a geometric realization of $H^2(\Phi, \mathbb{Z})$.

Definition 3.6. Suppose that $\Phi \in C^\infty(M, N)$ and L_1, L_2 are two Hermitian line bundles over N, and σ_1, σ_2 are unitary sections of $\Phi^* L_1, \Phi^* L_2$. Then, (L_1, σ_1) is equivalent to (L_2, σ_2) if $L_1 L_2^{-1}$ admits a unitary section τ, and there is a map $f \in C^\infty(M, \mathbb{R})$ such that $(\Phi^*\tau)/\sigma_1\sigma_2^{-1} = \exp((2\pi\sqrt{-1})f)$.

This defines an equivalence relation among (σ, L), where L is a Hermitian line bundle over N and σ is a unitary section of $\Phi^* L$.

Definition 3.7. A relative line bundle for $\Phi \in C^\infty(M, N)$ is a pair (σ, L), where L is a Hermitian line bundle over N and σ is an unitary section for $\Phi^* L$. Define the group of relative line bundles

$$\mathrm{Pic}(\Phi) = \{(\sigma, L) | L \in \mathrm{Pic}(N), \sigma \text{ a unitary section of } \Phi^* L\},$$

and a subgroup of it

$$\mathrm{Pic}_0(\Phi) = \{(\sigma, L) \in \mathrm{Pic}(\Phi) | \exists \text{ a unitary section } \tau \text{ of } L \text{ and } k \in C^\infty(M, \mathbb{R}) \text{ with } \Phi^*\tau/\sigma = \exp((2\pi\sqrt{-1})k)\}.$$

Example 3.8. Let (N, ω) be a compact symplectic manifold of dimension $2n$, and let $L \to N$ be a line bundle with connection ∇ whose curvature is ω, i.e., L is a pre-quantum line bundle with connection. A Lagrangian submanifold M satisfies the Bohr–Sommerfeld condition if there exists a global non-vanishing covariant constant(=flat) section σ_M of

$\Phi^* L$, where $\Phi : M \to N$ is inclusion map (1.6 [18]). For any Lagrangian submanifold M, $(\sigma_M, L) \in \mathrm{Pic}(\Phi)$.

Theorem 3.9. *There is a short exact sequence of Abelian groups*

$$0 \to \mathrm{Pic}_0(\Phi) \to \mathrm{Pic}(\Phi) \to H^2(\Phi, \mathbb{Z}) \to 0.$$

Thus, $H^2(\Phi, \mathbb{Z})$ parameterizes the set of equivalence classes of pairs (σ, L).

Proof. One can identify $H^2(\Phi, \mathbb{Z})$ with $H^1(\Phi, \underline{U(1)})$ by Proposition 3.1. Let $(\sigma, L) \in \mathrm{Pic}(\Phi)$. Let $\{V_j\}_{j\in J}$ be a good cover of N and $\{U_i\}_{i\in I}$ be a good cover of M such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$. Choose unitary sections σ_j of $L|_{V_j}$. The corresponding transition functions for L are

$$g_{jj'} \in C^\infty(V_{jj'}, U(1)), \quad j, j' \in J,\ g_{jj'}\sigma_j = \sigma_{j'} \quad \text{on} \quad V_{jj'}$$

Define $f_i = \Phi^*(\sigma_{r(i)})/\sigma$ on U_i. Then

$$f_i f_{i'}^{-1} = (\Phi^*(\sigma_{r(i)})/\sigma) \cdot (\Phi^*(\sigma_{r(i')})/\sigma)^{-1} = \Phi^* g_{r(i)r(i')}. \tag{3.4}$$

Since

$$(\delta g)_{r(i)r(i')r(i'')} = 1,$$

then $(f_i, g_{r(i)r(i')})$ is a cocycle in $\check{C}^1(\Phi, \mathbb{Z})$. If one changes local sections σ_j, $j \in J$, then $(f_i, g_{r(i)r(i')})$ will shift by a co-boundary. Define

$$\chi : \mathrm{Pic}(\Phi) \longrightarrow H^1(\Phi, \underline{U(1)}), \qquad (\sigma, L) \mapsto [(f, g)].$$

To find the kernel of χ, suppose that $(f, g) = \delta(t, c)$. Thus, $g = \delta c$ and $f = \phi^*(c)\exp(2\pi i h)^{-1}$, where h is the global logarithm of t. Define local section $\tau_j := \sigma_j/c_j$ on V_j. Since on $V_{jj'}$

$$\sigma_j/c_j = \sigma_{j'}/c_{j'},$$

then we obtain a global section τ. On the other hand,

$$\Phi^*\sigma_{r(i)}/\sigma = f_i = \phi^* c_{r(i)} \exp((2\pi\sqrt{-1})h)^{-1}.$$

Therefore, $\phi^*\tau/\sigma = \exp((2\pi\sqrt{-1})h)^{-1}$. This exactly shows that the kernel of χ is $\mathrm{Pic}_0(\Phi)$.

Next, it will be shown that χ is onto. Let $(f_i, g_{jj'}) \in C^1(\Phi, \underline{U(1)})$ be a cocycle. Pick a line bundle L over N with $g_{jj'}$ corresponding to local sections σ_j. $\Phi^*\sigma_{r(i)}/f_i$ defines local sections for $\Phi^* L$ over U_i. On $U_i \cap U_{i'}$

$$\Phi^*\sigma_{r(i)}/f_i = \Phi^*\sigma_{r(i')}/f_{i'},$$

which defines a global section σ for $\Phi^* L$. By construction, $\chi(\sigma, L) = [(f, g)]$. This shows that χ is onto. $\square$

Remark 3.10. A relative line bundle (L, σ) for the map $\Phi : M \to N$ defines a line bundle over the mapping cone, $\mathrm{Cone}_\Phi = N \cup_\Phi \mathrm{Cone}(M)$. This line bundle is given by L on $N \subset \mathrm{Cone}_\Phi$ and by the trivial line bundle on $\mathrm{Cone}(M)$. The section σ is used to glue these two bundles. Hence, one obtains a map

$$\mathrm{Pic}(\Phi) \to H^2(\mathrm{Cone}_\Phi, \mathbb{Z}) \cong H^2(\Phi, \mathbb{Z}).$$

3.3. *Gerbes*

The main references for this section are [16,5,15].

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover for a manifold M. It will be convenient to introduce the following notations. Suppose that there is a collection of line bundles $L_{i^{(0)},\dots,i^{(n)}}$ on $U_{i^{(0)},\dots,i^{(n)}}$. Consider the inclusion maps,

$$\delta_k : U_{i^{(0)},\dots,i^{(n+1)}} \to U_{i^{(0)},\dots,i^{(k)},\dots,i^{(n+1)}} \quad (k = 0, \cdots, n+1),$$

and define Hermitian line bundles $(\delta L)_{i^{(0)},\dots,i^{(n+1)}}$ over $U_{i^{(0)},\dots,i^{(n+1)}}$ by

$$\delta L := \bigotimes_{n+1}^{k=0} (\delta_k^* L)^{(-1)^k}.$$

Notice that $\delta(\delta L)$ is canonically trivial. If one has a unitary section $\lambda_{i^{(0)},\dots,i^{(n)}}$ of $L_{i^{(0)},\dots,i^{(n)}}$ for each $U_{i^{(0)},\dots,i^{(n)}} \neq \emptyset$, then one can define $\delta\lambda$ in a similar fashion. Note that $\delta(\delta\lambda) = 1$ as a section of trivial line bundle.

Definition 3.11. A gerbe on a manifold M on an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M is defined by Hermitian line bundles $L_{ii'}$ on each $U_{ii'}$ such that $L_{ii'} \cong L_{i'i}^{-1}$, and a unitary section $\theta_{ii'i''}$ of δL on $U_{ii'i''}$ such that $\delta\theta = 1$ on $U_{ii'i''i'''}$. Denote this data as $\mathcal{G} = (\mathcal{U}, L, \theta)$.

Denote the set of all gerbes on M on the open cover $\mathcal{U} = \{U_i\}_{i\in I}$ as $\mathrm{Ger}(M, \mathcal{U})$. Recall that an open cover $\mathcal{V} = \{V_j\}_{j\in J}$ is a refinement of open cover $\mathcal{U} = \{U_i\}_{i\in I}$ if there is a map $r : J \to I$ with $V_j \subset U_i$. In this case, one gets a map

$$\mathrm{Ger}(M, \mathcal{U}) \hookrightarrow \mathrm{Ger}(M, \mathcal{V}).$$

Define the group of gerbes on M as

$$\mathrm{Ger}(M) = \lim_{\to} \mathrm{Ger}(M, \mathcal{U}).$$

Define the product of two gerbes $\mathcal{G}$ and $\mathcal{G}'$ to be the gerbe $\mathcal{G} \otimes \mathcal{G}'$ consisting of an open cover of M, $\mathcal{V} = \{V_i\}_{i\in I}$ (common refinement of open covers of $\mathcal{G}$ and $\mathcal{G}'$), line bundles $L_{ii'} \otimes L'_{ii'}$ on $V_{ii'}$ and unitary sections $\theta_{ii'i''} \otimes \theta'_{ii'i''}$ of $\delta(L \otimes L')$ on $V_{ii'i''}$.

$\mathcal{G}^{-1}$, the dual of a gerbe $\mathcal{G}$, is defined by dual bundles $L^{-1}_{ii'}$ on $U_{ii'}$ and sections θ^{-1} of $\delta(L^{-1})$ over $U_{ii'i''}$. Therefore, one get a group structure on $\mathrm{Ger}(M)$. If $\Phi : M \to N$ be a smooth map between two manifolds and $\mathcal{G}$ be a gerbe on N with open cover $\mathcal{V} = \{V_j\}_{j\in J}$, the pull-back grebe $\Phi^*\mathcal{G}$ is simply defined on $\mathcal{U} = \{U_i\}_{i\in I}$, where $\Phi(U_i) \subset V_{r(i)}$ for a map $r : I \to J$, line bundles $\Phi^* L_{r(i)r(i')}$ on $U_{ii'}$, and unitary sections θ of $\delta(\Phi^* L)$ on $U_{ii'i''}$.

Definition 3.12 (Quasi-line bundle). A quasi-line bundle for the gerbe $\mathcal{G}$ on a manifold M on the open cover $\mathcal{U} = \{U_i\}_{i\in I}$ is defined as:

(1) a Hermitian line bundle E_i over each U_i.
(2) Unitary sections $\psi_{ii'}$ of

$$(\delta E^{-1})_{ii'} \otimes L_{ii'}$$

such that $\delta\psi = \theta$.

Denote this quasi-line bundle as $\mathcal{L} = (E, \psi)$.

Proposition 3.13. *Any two quasi-line bundles over a given gerbe differ by a line bundle.*

Proof. Consider two quasi-line bundles $\mathcal{L} = (E, \psi)$ and $\tilde{\mathcal{L}} = (\tilde{E}, \tilde{\psi})$ for the gerbe $\mathcal{G} = (\mathcal{U}, L, \theta)$. $\psi_{ii'} \otimes \tilde{\psi}^{-1}_{ii'}$ is a unitary section for

$$\begin{aligned} E_{i'} \otimes E_i^{-1} \otimes L^{-1}_{ii'} \otimes \tilde{E}^{-1}_{i'} \otimes \tilde{E}_i \otimes L_{ii'} &\cong E_{i'} \otimes E_i^{-1} \otimes \tilde{E}^{-1}_{i'} \otimes \tilde{E}_i \\ &\cong E_{i'} \otimes \tilde{E}^{-1}_{i'} \otimes E_i^{-1} \otimes \tilde{E}_i. \end{aligned}$$

Therefore, $E \otimes \tilde{E}^{-1}$ defines a line bundle over M. $\square$

Denote the group of all gerbes on M related to the open cover $\mathcal{U} = \{U_i\}_{i\in I}$ that admits a quasi-line bundle as $\mathrm{Ger}_0(M, \mathcal{U})$. Define

$$\mathrm{Ger}_0(M) = \varinjlim \mathrm{Ger}_0(M, \mathcal{U}).$$

Proposition 3.14. *There exists a short exact sequence of groups*

$$0 \to \mathrm{Ger}_0(M) \hookrightarrow \mathrm{Ger}(M) \xrightarrow{\chi} H^3(M, \mathbb{Z}) \to 0.$$

Proof. Identify $H^3(M, \mathbb{Z})$ with $H^2(M, \underline{U(1)})$. Consider the gerbe $\mathcal{G}$ on M. Refine the cover such that any $L_{ii'}$ admits unitary sections $\sigma_{ii'}$. Define

$$t := (\delta\sigma)\theta^{-1}.$$

Thus, $\delta t = 1$, which means t is a cocycle. Define

$$\chi(\mathcal{G}) := [t].$$

Different sections shift the cocycle by $\delta \check{\mathrm{C}}^1(M, \underline{U(1)})$, which shows that χ is well-defined. Also, $\chi(\mathcal{G} \otimes \mathcal{G}') = [tt'] = \chi(\mathcal{G})\chi(\mathcal{G}')$, which proves that χ is a group homomorphism. Next, it will be shown that the kernel of χ is $\mathrm{Ger}_0(M)$. For $\mathcal{G} \in \mathrm{Ger}_0(M)$, choose a quasi-line bundle $\mathcal{L} = (E, \psi)$. Thus, $t = \delta(\sigma\psi^{-1})$. Hence, $\chi(\mathcal{G}) = [t] = 1$. Conversely, if $[t] = 1$, then

$$t = \delta t',$$

and by defining the new sections $\sigma' = t'\sigma$ one infers that $\delta\sigma' = t\delta\sigma = \theta$, which shows that $\mathcal{G}$ admits a quasi-line bundle.

Finally, it will be shown that χ is onto. If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of M and $t_{ii'i''}$ is a cocycle $\check{\mathrm{C}}^2(M, \underline{U(1)})$, then define a gerbe $\mathcal{G}$ on M by trivial line bundle $L_{ii'}$ on $U_{ii'}$ and unitary sections $\sigma_{ii'}$ on $U_{ii'}$. Define $\theta = t\delta\sigma$. Since $\delta t = 1$, then $\delta\theta = 1$. By construction, $\chi(\mathcal{G}) = [t]$. $\square$

Definition 3.15. Let $\mathcal{G} \in \mathrm{Ger}(M)$. $\chi(\mathcal{G}) \in H^2(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z})$ is called Dixmier–Douady class of the gerbe $\mathcal{G}$, which is denoted as $\mathrm{DD}\mathcal{G}$.

A gerbe admits a quasi-line bundle if and only if its Dixmier–Douady class is zero by Proposition 3.14.

Example 3.16. Let G be a Lie group, and $1 \to U(1) \to \hat{G} \to^{\kappa} G \to 1$ be a central extension. Suppose that $\pi : P \to M$ is a principal G-bundle. A lift of $\pi : P \to M$ is a principal $\hat{G}$-bundle $\hat{\pi} : \hat{P} \to M$ together with a map $q : \hat{P} \to P$ such that $\hat{\pi} = \pi \circ q$ and the following diagram commutes:

$$\begin{array}{ccc} \widehat{G} \times \widehat{P} & \longrightarrow & \widehat{P} \\ {\scriptstyle (\kappa,q)}\downarrow & & {\scriptstyle q}\downarrow \\ G \times P & \longrightarrow & P \end{array}$$

In the above diagram the horizontal maps are respective group actions. Suppose that $\{U_i\}_{i\in I}$ is an open cover of M such that $P|_{U_i} := P_i$ has a lift $\hat{P}_i$. Define $\hat{G}$-equivariant Hermitian line bundles as

$$E_i = \hat{P}_i \times_{U(1)} \mathbb{C} \to P|_{U_i}.$$

Since $U(1)$ acts by weight 1 on E_i, it acts by weight 0 on $E_i \otimes E_{i'}^{-1} := E_{ii'}$ on $U_{ii'}$. Therefore, G acts on $E_{ii'}$, and $E_{ii'}/G$ is a well-defined Hermitian line bundle, namely $L_{ii'}$. By construction, δL is trivial on $U_{ii'i''}$, therefore one can pick trivial section θ that obviously satisfies the relation $\delta\theta = 1$. This shows the obstruction to lifting P to $\hat{P}$ defines a grebe $\mathcal{G}$.

If $E_i \to U_i$ defines a quasi-line bundle $\mathcal{L}$ for $\mathcal{G}$, then the line bundles $\widetilde{E_i} := E_i \otimes \pi^* L_i^{-1}$ patch together to a global $\hat{G}$-equivariant line bundle $\hat{E} \to P$, and the unit circle bundle defines a global lift $\hat{P} \to P$. Conversely, if P admits a global lift $\hat{P}$ and $\widetilde{P}_i =: \hat{P}|_{U_i}$, then $L_{ii'}$ is trivial, which shows that the resulting gerbe is a trivial one.

Example 3.17. Let $N \subset M$ be an oriented codimension 3 submanifold of an n-oriented manifold M. The tubular neighborhood U_0 of N has the form $P \times_{\mathrm{SO}(3)} \mathbb{R}^3$, where $P \to N$ is the frame bundle. Let $U_1 = M - N$. Then, $U_0 \cap U_1 \cong P \times_{\mathrm{SO}(3)} (\mathbb{R}^3 - 0)$. Over $(\mathbb{R}^3 - 0) \cong S^2 \times (0, \infty)$, one has degree 2 line bundle E that is $SO(3)$ equivariant. Thus,

$$L_{01} := P \times_{\mathrm{SO}(3)} E$$

is a line bundle over $U_0 \cap U_1$, which defines the only transition line bundle. Since there is no triple intersection, this data defines a gerbe over M.

3.4. *Geometric interpretation of $H^3(\Phi, \mathbb{Z})$*

Definition 3.18. A relative gerbe for $\Phi \in C^\infty(M, N)$ is a pair $(\mathcal{L}, \mathcal{G})$, where $\mathcal{G}$ is a gerbe over N and $\mathcal{L}$ is a quasi-line bundle for $\Phi^*\mathcal{G}$.

Notation: Let $\Phi \in C^\infty(M, N)$. Then

$$\mathrm{Ger}(\Phi) = \{(\mathcal{L}, \mathcal{G}) | (\mathcal{L}, \mathcal{G}) \text{ is a relative gerbe for } \Phi \in C^\infty(M, N)\},$$

$$\mathrm{Ger}_0(\Phi) = \{(\mathcal{L}, \mathcal{G}) \in \mathrm{Ger}(\Phi) | \mathcal{G} \text{ admits a quasi-line bundle } \mathcal{L}' \text{ sth the line bundle } \mathcal{L} \otimes \Phi^*\mathcal{L}'^{-1} \text{ admits a unitary section}\}.$$

Example 3.19. Consider a smooth map $\Phi : M \to N$ with $\dim M \leq 2$. Let $\mathcal{G}$ be a grebe on N. Since $\Phi^*\mathcal{G}$ admits a quasi-line bundle say $\mathcal{L}$, $(\mathcal{L}, \mathcal{G})$ is a relative gerbe.

Theorem 3.20. *There exists a short exact sequence of Abelian groups*

$$0 \to \mathrm{Ger}_0(\Phi) \hookrightarrow \mathrm{Ger}(\Phi) \xrightarrow{\kappa} H^3(\Phi, \mathbb{Z}) \longrightarrow 0.$$

Proof. One can identify $H^3(\Phi, \mathbb{Z}) \cong H^2(\Phi, \underline{U(1)})$. Let $\{V_j\}_{j \in J}$ be a good cover of N and $\{U_i\}_{i \in I}$ be a good cover of M such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$. Let $(\mathcal{L}, \mathcal{G}) \in \mathrm{Ger}(\Phi)$. Refine the gerbe $\mathcal{G} = (\mathcal{U}, L, \theta)$ sufficiently such that all $L_{jj'}$ admit unitary sections $\sigma_{jj'}$. Then, define $t_{jj'j''} \in \check{\mathrm{C}}^2(N, U(1))$ by

$$t := (\delta\sigma)(\theta)^{-1}.$$

Since $\delta\theta = 1$ and $\delta(\delta\sigma) = 1$, then $\delta t = 1$. Let $\mathcal{L} = (E, \psi)$ be a quasi-line bundle for $\Phi^*\mathcal{G}$ with unitary sections $\psi_{ii'}$ for line bundles $((\delta E)_{ii'})^{-1} \otimes \Phi^* L_{r(i)r(i')}$. Define $s_{ii'} \in \check{C}^1(M, U(1))$ by

$$s_{ii'} := (\psi_{ii'})^{-1}((\delta\lambda)_{ii'}^{-1} \otimes \Phi^*\sigma_{r(i)r(i')}),$$

where λ_i is a unitary section for E_i. Now

$$(\Phi^* t^{-1})\delta(\Phi^*\sigma) = \Phi^*\theta = \delta\psi = (\delta s)^{-1}(\delta\delta\lambda^{-1} \otimes \delta\Phi^*\sigma) = (\delta s)^{-1}\delta\Phi^*\sigma. \tag{3.5}$$

This proves that $\delta s = \Phi^* t$. Define the map

$$\kappa : \mathrm{Ger}(\Phi) \to H^2(\Phi, \underline{U(1)}), \qquad \kappa(\mathcal{L}, \mathcal{G}) = [(s, t)].$$

It is straightforward to check that this map is well-defined, i.e., it is independent of the choice of $\sigma_{jj'}$ and λ_i. Conversely, given $[(s, t)] \in H^2(\Phi, \underline{U(1)})$, one can pick $\mathcal{G}$ such that $\theta = t^{-1}(\delta\sigma)$ and define

$$\psi_{ii'} = s_{ii'}^{-1}((\delta\lambda^{-1})_{ii'} \otimes \Phi^*\sigma_{r(i)r(i')}).$$

Since $\delta s = \Phi^* t$, then $\mathcal{L} = (E, \psi)$ defines a quasi-line bundle for $\Phi^*\mathcal{G}$. The construction shows $\kappa(\mathcal{L}, \mathcal{G}) = [(s, t)]$. Therefore κ is onto.

It is now shown that $\ker(\kappa) = \mathrm{Ger}_0(\Phi)$. Assume $\kappa(\mathcal{L}, \mathcal{G}) = [(s, t)]$ is a trivial class. Therefore, there exists $(\rho, \tau) \in \check{C}^1(\Phi, \underline{U(1)})$ such that $(s, t) = \delta(\rho, \tau) = (\Phi^*\tau(\delta\rho)^{-1}, \delta\tau)$. $t = \delta\tau$ shows that $\mathcal{G}$ admits a quasi-line bundle $\mathcal{L}'$. Thus, $\mathcal{L} \otimes \Phi^*\mathcal{L}'^{-1}$ defines a line bundle over M. The first Chern class of this line bundle is given by the cocycle $s(\Phi^*\tau)^{-1}$. The condition $s = (\Phi^*\tau)\delta\rho^{-1}$ shows that this cocycle is exact, i.e., the line bundle $\mathcal{L} \otimes \Phi^*\mathcal{L}'^{-1}$ admits a unitary section. Thus, $\ker(\kappa) \subseteq \mathrm{Ger}_0(\Phi)$. Conversely, if $(\mathcal{L}, \mathcal{G}) \in \mathrm{Ger}_0(\Phi)$ then the above argument, read in reverse, shows that (s, t) is exact. Hence, $\mathrm{Ger}_0(\Phi) \subseteq \ker(\kappa)$. □

Remark 3.21. A relative (topological) gerbe $(\mathcal{L}, \mathcal{G}) \in \mathrm{Ger}(\Phi)$ defines a (topological) gerbe over the mapping cone by "gluing" the trivial gerbe over $\mathrm{Cone}(M)$ with the gerbe $\mathcal{G}$ over $N \subset \mathrm{Cone}_\Phi$. Here, the line bundles E_i that define $\mathcal{L}$ play the role of transition line bundles. For gluing the gerbes see [22].

Example 3.22. Let $1 \to U(1) \to \hat{G} \to G \to 1$ be a central extension of a Lie group G. Suppose $\Phi \in C^\infty(M, N)$ and $Q \to N$ is a principal G-bundle. If $P = \Phi^* Q \to M$ admit a lift $\hat{P}$, then one obtains an element of $H^3(\Phi, \mathbb{Z})$.

Example 3.23. Suppose that G is a compact Lie group. Recall that the universal bundle $EG \to BG$ is a (topological) principal G-bundle with the property that any principal G-bundle $P \to B$ is obtained as the pull-back by some classifying map $\Phi : B \to BG$. While the classifying bundle is infinite-dimensional, it can be written as a limit of finite-dimensional bundles $E_n G \to B_n G$. For instance, if $G = U(k)$, one can take $E_n G$ the Stiefel manifold of unitary k-frames over the Grassmanian $Gr_{\mathbb{C}}(k, n)$. Furthermore, if B is given, any G-bundle $P \to B$ is given by a classifying map $\Phi : B \to B_n G$ for some fixed, sufficiently large n depending only on $\dim B$ [17].

It can be shown that $H^3(BG, \mathbb{Z})$ classifies central extension $1 \to U(1) \to \hat{G} \to G \to 1$ [3]. For n sufficiently large, $H^3(B_n G, \mathbb{Z}) = H^3(BG, \mathbb{Z})$. Hence, $H^3(\Phi, \mathbb{Z})$ classifies pairs $(\hat{G}, \hat{P})$, where $\hat{G}$ is a central extension of G by $U(1)$ and $\hat{P}$ is a lift of $\Phi^* EG$ to $\hat{G}$.

4. Differential geometry of relative gerbes

4.1. Connections on line bundles

Let L be a Hermitian line bundle with Hermitian connection ∇ over a manifold M. In terms of local unitary sections σ_i of $L|_{U_i}$ and the corresponding transition maps

$$g_{ii'} : U_{ii'} \to U(1),$$

connection one-forms A_i on U_i are defined by $\nabla\sigma_i = (2\pi\sqrt{-1})A_i\sigma_i$. On $U_{ii'}$,

$$(2\pi\sqrt{-1})(A_{i'} - A_i) = g_{ii'}^{-1}\,\mathrm{d}g_{ii'}.$$

Hence, the differentials $\mathrm{d}A_i$ agree on overlaps. The curvature two-form F is defined by $F|_{U_i} =: \mathrm{d}A_i$. The cohomology class of F is independent of the chosen connection. The cohomology class of F is the image of the Chern class $c_1(L) \in H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. A given closed two-form $F \in \Omega^2(M, \mathbb{R})$ arises as a curvature of some line bundle with connection if and only if F is integral [4].

The line bundle with connection (L, ∇) is called flat if $F = 0$. In this case, define the holonomy of (L, ∇) as follows. Assume that the open cover $\{U_i\}_{i\in I}$ is a good cover of M. Therefore, $A_i = \mathrm{d}f_i$ on U_i, where $f_i : U_i \to \mathbb{R}$ is a smooth map on U_i. Then,

$$\mathrm{d}(2\pi\sqrt{-1}(f_{i'} - f_i) - \log g_{ii'}) = 0.$$

Thus,

$$c_{ii'} := (2\pi\sqrt{-1}(f_{i'} - f_i) - \log g_{ii'})$$

are constants. Since $\log g$ is only defined modulo $2\pi\sqrt{-1}\,\mathbb{Z}$, so there exists a collection of constants $\tilde{c}_{ii'} := c_{ii'} \bmod \mathbb{Z}$. Different choices of f_i, shift this cocyle with a coboundary. The one-cocycle $\tilde{c}_{ii'}$ represents a Čech class in $\check{H}^1(M, U(1))$, which is called the *holonomy* of the flat line bundle L with connection ∇.

Let $L \to M$ be a line bundle with connection ∇, and $\gamma : S^1 \to M$ a smooth curve. The holonomy of ∇ around γ is defined as the holonomy of the line bundle γ^*L with flat connection $\gamma^*\nabla$.

4.2. Connections on gerbes

Definition 4.1. Let $\mathcal{G} = (\mathcal{U}, L, \theta)$ be a gerbe on a manifold M. A gerbe connection on $\mathcal{G}$ consists of connections $\nabla_{ii'}$ on line bundles $L_{ii'}$ such that $(\delta\nabla)_{ii'i''}\theta_{ii'i''} := (\nabla_{i'i''} \otimes \nabla_{ii''}^{-1} \otimes \nabla_{ii'})\theta_{ii'i''} = 0$, together with two-forms $\varpi_i \in \Omega^2(U_i)$ such that on $U_{ii'}$,

$$(\delta\varpi)_{ii'} = F_{ii'} = \text{the curvature of } \nabla_{ii'}.$$

This connection gerbe is denoted as a pair (∇, ϖ).

Since $F_{ii'}$ is a closed two-form, the de Rham differential $\kappa|_{U_i} := \mathrm{d}\varpi_i$ defines a global three-form κ, which is called the *curvature of the gerbe connection*. $[\kappa] \in H^3(M, \mathbb{R})$ is the image of the Dixmier–Douady class of the gerbe under the induced map by inclusion

$$\iota : H^3(M, \mathbb{Z}) \to H^3(M, \mathbb{R}).$$

A given closed three-form $\kappa \in \Omega^2(M, \mathbb{R})$ arises as a curvature of some gerbe with connection if and only if $i\kappa$ is integral [16].

Example 4.2. Suppose that $\pi : P \to B$ is a principal G-bundle, and

$$1 \to U(1) \to \hat{G} \to G \to 1$$

a central extension. In Example 3.16, a gerbe $\mathcal{G}$ is described whose Dixmier–Douady class is the obstruction to the existence of a lift $\hat{\pi} : \hat{P} \to B$. Following Brylinski [4] (also see [10]) one can define a connection on this gerbe. Two ingredients are required:

(i) a principal connection $\theta \in \Omega^1(P, \mathfrak{g})$,
(ii) a splitting $\tau : P \times_G \hat{\mathfrak{g}} \to B \times \mathbb{R}$ of the sequence of vector bundles

$$0 \to B \times \mathbb{R} \to P \times_G \hat{\mathfrak{g}} \to P \times_G \mathfrak{g} \to 0$$

associated to the sequence of Lie algebras $0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$. For a given lift $\hat{\pi} : \hat{P} \to B$, with corresponding projection $q : \hat{P} \to P$, one say that a principal connection $\hat{\theta} \in \Omega^1(\hat{P}, \hat{\mathfrak{g}})$ lifts θ if its image under $\Omega^1(\hat{P}, \hat{\mathfrak{g}}) \to \Omega^1(\hat{P}, \mathfrak{g})$ coincides with $q^*\theta$. Given such a lift with curvature

$$F^\theta = \mathrm{d}\hat{\theta} + \frac{1}{2}[\hat{\theta}, \hat{\theta}] \in \Omega^2(\hat{P}, \hat{\mathfrak{g}})_{\mathrm{basic}} = \Omega^2(B, P \times_G \mathfrak{g}),$$

let $K^\theta := \tau(F^\theta) \in \Omega^2(B, \mathbb{R})$ be its "scalar part." Any pair of lifts of (P, θ) differs by a line bundle with connection (L, ∇^L) on B. Twisting a given lift $(\hat{P}, \hat{\theta})$ by such a line bundle, the scalar part changes by the curvature of the line bundle [4]

$$K^\theta + \frac{1}{2\pi\sqrt{-1}} \operatorname{curv}(\nabla^L). \tag{4.1}$$

In particular, the exact three-form $\mathrm{d}K^\theta \in \Omega^3(B)$ only depends on the choice of splitting and the connection θ. (It does not depend on choice of lift.) In general, a global lift $\hat{P}$ of P does not exist. However, let us choose local lifts $(\hat{P}_i, \hat{\theta}_i)$ of $(P|_{U_i}, \theta)$. Denote the scalar part of F^{θ_i} with $\varpi_i \in \Omega^2(U_i)$, and let $L_{ii'} \to U_{ii'}$ be the line bundle with connection $\nabla^{L_{ii'}}$ defined by two lifts $(\hat{P}_i|_{U_{ii'}}, \hat{\theta}_i)$ and $(\hat{P}_{i'}|_{U_{ii'}}, \hat{\theta}_{i'})$. By Eq. (4.1),

$$(\delta\varpi)_{ii'} = \frac{1}{2\pi\sqrt{-1}} \operatorname{curv}(\nabla^{L_{ii'}}).$$

On the other hand, the connection $\delta\nabla^L$ on $(\delta L)_{ii'i''} = L_{ii'}L_{ii''}^{-1}L_{ii'}$ is just the trivial connection on the trivial line bundle. Hence, a gerbe connection is defined.

A quasi-line bundle (E, ψ) with connection ∇^E for this gerbe with connection gives rise to a global lift $(\hat{P}, \hat{\theta})$ of (P, θ), where $\hat{P}|_{U_i}$ is obtained by twisting $\hat{P}_i$ by the line bundle with connection (E_i, ∇^{E_i}). The error two-form is the scalar part of $F^{\hat{\theta}}$ (see Definition 4.3).

Definition 4.3. Let $\mathcal{G}$ be a gerbe with connection with a quasi-line bundle $\mathcal{L} = (E, \psi)$. A connection on a quasi-line bundle consists of connections ∇_i^E on line bundles E_i with curvature F_i^E such that

$$(\delta\nabla^E)_{ii'} := \nabla_{i'}^E \otimes (\nabla_i^E)^{-1} \cong \nabla_{ii'}.$$

Also, the two-curvatures obey $(\delta F^E)_{ii'} = F_{ii'}$. Denote this quasi-line bundle with connection by $(\mathcal{L}, \nabla^E)$. Locally defined two-forms $\omega|_{U_i} = \varpi_i - F_i^E$ patch together to define a global two-form ω, which is called the error two-form [15].

Remark 4.4. The difference between two quasi-line bundles with connections is a line bundle with connection, with the curvature equal to the difference of the error two-forms.

Let $\mathcal{G} = (\mathcal{U}, L, \theta)$ be a gerbe with connection on M. Again, assume that $\mathcal{U}$ is a good cover. Let $t \in \check{C}^2(M, \underline{U(1)})$ be a representative for the Dixmier–Douady class of $\mathcal{G}$. Then, one can have a collection of one-forms $A_{ii'} \in \Omega^1(U_{ii'})$ and two-forms $\varpi_i \in \Omega^2(U_i)$ such that

$$\kappa|_{U_i} = \mathrm{d}\varpi_i, \qquad \delta\varpi = \mathrm{d}A, \qquad (2\pi\sqrt{-1})\delta A = t^{-1}\,\mathrm{d}t.$$

If $\kappa = 0$, the gerbe is called flat. In this case by using Poincaré lemma, $\varpi_i = \mathrm{d}\mu_i$ on U_i and on $U_{ii'}$,

$$(\delta\varpi)_{ii'} = \mathrm{d}\delta(\mu)_{ii'} = \mathrm{d}A_{ii'}.$$

Thus, again by Poincaré lemma

$$A_{ii'} - (\delta\mu)_{ii'} = \mathrm{d}h_{ii'}.$$

By using $(2\pi\sqrt{-1})\delta A = t^{-1}\,\mathrm{d}t$,

$$\mathrm{d}((2\pi\sqrt{-1})\delta h - \log t) = 0.$$

Therefore, there exists a collection of constants $c_{ii'i''} \in \check{C}^2(M, \mathbb{R})$. Since log is defined modulo $2\pi\sqrt{-1}\,\mathbb{Z}$, we define

$$\tilde{c}_{ii'i''} := c_{ii'i''} \bmod \mathbb{Z}.$$

The two-cocycle $\tilde{c}_{ii'i''}$ represents a Čech class in $\check{H}^2(M, U(1))$, which is called the *holonomy of the flat gerbe with connection*. Let $\sigma : \Sigma \to M$ be a smooth map, where Σ is a closed surface. The holonomy of $\mathcal{G}$ around Σ is defined as the holonomy of the pullback gerbe $\sigma^*\mathcal{G}$ with the flat connection $\sigma^*(\nabla, \varpi)$ [16,20].

4.3. Connections on relative gerbes

Let $\Phi \in C^\infty(M, N)$ and $U = \{U_i\}_{i\in I}$, $V = \{V_j\}_{j\in J}$ are good covers of M and N, respectively, such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$.

Definition 4.5 (A). relative connection on a relative gerbe $(\mathcal{L}, \mathcal{G})$ consists of gerbe connection (∇, ϖ) on $\mathcal{G}$ and a connection ∇^E on the quasi-line bundle $\mathcal{L} = (E, \psi)$ for the $\Phi^*\mathcal{G}$.

Consider a relative connection on a relative gerbe $(\mathcal{L}, \mathcal{G})$. Define the two-form τ on M by

$$\tau|_{U_i} := \Phi^*\varpi_{r(i)} - F_i^E.$$

Thus, $(\tau, \kappa) \in \Omega^3(\Phi)$ is a relative closed three-form which is called here the *curvature of the relative connection*.

Theorem 4.6. *A given closed relative three-form $(\tau, \kappa) \in \Omega^3(\Phi)$ arises as a curvature of some relative gerbe with connection if and only if (τ, κ) is integral.*

Proof. Let $(\tau, \kappa) \in \Omega^3(\Phi)$ be an integral relative three-form. By Proposition 2.17,

$$\int_\alpha \kappa - \int_\beta \tau \in \mathbb{Z}, \tag{4.2}$$

where $\alpha \subset N$ is a smooth three-chain and $\Phi(\beta) = \partial\alpha$, i.e., $(\beta, \alpha) \in \mathrm{Cone}_3(\Phi, \mathbb{Z})$ is a cycle. If α is a cycle then $(0, \alpha) \in \mathrm{Cone}_3(\Phi, \mathbb{Z})$ is a cycle. In this case, Eq. (4.2) shows that for all cycles $\alpha \in S_3(N, \mathbb{Z})$,

$$\int_\alpha \kappa \in \mathbb{Z}.$$

Therefore, one can pick a gerbe $\mathcal{G} = (\mathcal{V}, L, \theta)$ with connection (∇, ϖ) over N with curvature three-form κ. Denote $\tau_i := \tau|_{U_i}$. Define $F_i^E \in \Omega^2(U_i)$ by

$$F_i^E = \Phi^*\varpi_{r(i)} - \tau_i.$$

Let $(\alpha_i, \beta_i) \in \mathrm{Cone}_3(\Phi|_{U_i}, \mathbb{Z})$ be a cycle. Then,

$$\int_{\beta_i} F_i^E = \int_{\beta_i} (\Phi^*\varpi_{r(i)} - \tau_i) = \int_{\Phi(\beta_i)} \varpi - \int_{\beta_i} \tau = \int_{\alpha_i} \mathrm{d}\varpi - \int_{\beta_i} \tau = \int_{\alpha_i} \kappa - \int_{\beta_i} \tau \in \mathbb{Z}.$$

Therefore, one can find a line bundle E_i with connection over U_i whose curvature is equal to F_i^E. Over $U_{ii'}$, the curvature of two line bundles $\Phi^*L_{ii'}$ and $E_{i'} \otimes E_i^{-1}$ agrees. Assume that the open cover $\mathcal{U} = \{U_i\}_{i\in I}$ is a good cover of M. Thus, there is a unitary section $\psi_{ii'}$ for the line bundle $E_i \otimes E_{i'}^{-1} \otimes \Phi^* L_{ii'}$ such that $\delta\psi = \Phi^*\theta$. Therefore, one obtains a quasi-line bundle $\mathcal{L} = (E, \psi)$ with connection for $\Phi^*\mathcal{G}$. By construction the curvature of the relative gerbe $(\mathcal{L}, \mathcal{G})$ is (τ, κ). Conversely, for a given relative gerbe with connection $(\mathcal{L}, \mathcal{G})$ one can have $\int_{\beta_i} F_i^E \in \mathbb{Z}$, where $\beta_i \subset U_i$ is a two-cycle which gives (4.2). □

Suppose that $\mathcal{G}$ is a gerbe with a flat connection (∇, ϖ) on N and $\mathcal{L}$ a quasi-line bundle with connection for $\Phi^*\mathcal{G}$. Since $\kappa = 0$, as explained in the previous section, there exist two-cocycles $\tilde{c}_{ii'i''}$ that represent a cohomology class in $\check{H}^2(M, U(1))$. Since $\Phi^*\mathcal{G}$ is trivializable, there is a collection of maps $f_{ii'}$ on $U_{ii'}$ such that $\delta f = \Phi^* t$, where $j = r(i)$ and $j' = r(i')$. Define $k_{ii'} \in \mathbb{R}$ as

$$k_{ii'} =: (2\pi\sqrt{-1})\Phi^* h_{ii'} - \log f_{ii'},$$

and

$$\tilde{k}_{ii'} := k_{ii'} \bmod \mathbb{Z}.$$

Thus,

$$\Phi^*\tilde{c} = \delta\tilde{k}.$$

Define the *relative holonomy* of the pair $(\mathcal{G}, \mathcal{L})$ by the relative class $[(\tilde{k}, \tilde{c})] \in H^2(\Phi, U(1))$.

Definition 4.7. Let the following diagram be commutative:

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[d, "\psi"] & \Sigma \arrow[d, "\tilde{\psi}"] \\ M \arrow[r, "\Phi"] & N \end{tikzcd}$$

where Σ is a closed surface, i is inclusion map and all other maps are smooth. Suppose that $\mathcal{G}$ is a gerbe with connection on N, and $\Phi^*\mathcal{G}$ admits a quasi-line bundle $\mathcal{L}$ with connection. Clearly, $\tilde{\psi}^*\mathcal{G}$ is a flat gerbe and since $i^*\tilde{\psi}^*\mathcal{G} = \psi^*\Phi^*\mathcal{G}$ then $i^*\tilde{\psi}^*\mathcal{G}$ admits a quasi line bundle with connection that is equal to $\psi^*\mathcal{L}$. Define the holonomy of the relative gerbe around the commutative diagram as holonomy of the pair $(\psi^*\mathcal{L}, \tilde{\psi}^*\mathcal{G})$.

4.4. Cheeger–Simons differential characters

In this section, a relative version of Cheeger–Simons differential characters is developed [23,2,13,25]. Denote the smooth singular chain complex on a manifold M as $S_\bullet^{\mathrm{sm}}(M)$. Let $Z_\bullet^{\mathrm{sm}}(M) \subseteq S_\bullet^{\mathrm{sm}}(M)$ be the sub-complex of smooth cycles. Recall that a differential character of degree k on a manifold M is a homomorphism

$$j : Z_{k-1}^{\mathrm{sm}}(M) \to U(1),$$

such that there is a closed form $\alpha \in \Omega^k(M)$ with

$$j(\partial x) = \exp\left(2\pi\sqrt{-1}\int_x \alpha\right)$$

for any $x \in S_k^{\mathrm{sm}}(M)$ [6].

A connection on a line bundle defines a differential character of degree 2, where j is the holonomy map. Similarly, a connection on a gerbe defines a differential character of degree 3. Specifically, any smooth k-chain $x \in S_k^{\mathrm{sm}}(M)$ is realized as a piecewise smooth map

$$\varphi_x : K_x \to M,$$

where K_x is a k-dimensional simplicial complex [14]. Then, by definition

$$\int_{K_x} \alpha = \int_x \alpha, \quad \alpha \in \Omega^k(M).$$

Suppose that $y = \Sigma \epsilon_i \sigma_i \in Z_2^{\mathrm{sm}}(M)$, where $\epsilon_i = \pm 1$. Assume that $\mathcal{G}$ is a gerbe with connection over M. Since $H^3(K_y, \mathbb{Z}) = 0$, $\varphi_y^* \mathcal{G}$ admits a piecewise smooth quasi-line bundle $\mathcal{L}$ with connection. That is, a quasi-line bundle $\mathcal{L}_i$ for all $\varphi^* \mathcal{G} \mid_{\Delta_{\sigma_i}^k}$, such that all $\mathcal{L}_i$ agree on the matching boundary faces. Let $\omega \in \Omega^2(K_y)$ be the error two-form and define

$$j(y) := \exp\left(2\pi\sqrt{-1}\int_{K_y} \omega\right).$$

Any two quasi-line bundles differ by a line bundle, and hence different choices for $\mathcal{L}_i$, change ω by an integral two-form. Therefore, j is well-defined. Assume that $y = \partial x$. Since the components of K_x with empty boundary do not contribute, one can assume that each component of K_x has non-empty boundary. Since $H^3(K_x, \mathbb{Z}) = 0$, choose a quasi-line bundle with connection for $\varphi_x^* \mathcal{G}$ with error two-form ω. Let k be the curvature of $\mathcal{G}$. Since $\varphi_x^* k = \mathrm{d}\omega$, by stokes' theorem

$$\int_{K_x} k = \int_{K_x} \mathrm{d}\omega = \int_{\partial K_x} \omega = \int_{K_y} \omega.$$

This shows that j is a differential character of degree 3.

Definition 4.8. Let $\Phi \in C^\infty(M, N)$ be a smooth map between manifolds. A relative differential character of degree k for the map Φ is a homomorphism

$$j : Z_{k-1}^{\mathrm{sm}}(\Phi) \to U(1),$$

such that there is a closed relative form $(\beta, \alpha) \in \Omega^k(\Phi)$ with

$$j(\partial(y, x)) = \exp\left(2\pi\sqrt{-1}\left(\int_y \beta - \int_x \alpha\right)\right)$$

for any $(y, x) \in S_k^{\mathrm{sm}}(\Phi)$.

Theorem 4.9. *A relative connection on a relative gerbe defines a relative differential character of degree 3.*

Proof. Let $\Phi \in C^{\infty}(M, N)$ be a smooth map between manifolds, and consider a relative grebe $(\mathcal{L}, \mathcal{G})$ with connection. Let $(y, x) \in S_1^{\mathrm{sm}}(\Phi)$ be a smooth relative singular cycle, i.e.,

$$\partial y = 0,$$

and

$$\Phi_*(y) = \partial x.$$

Suppose that K_y and K_x are the corresponding simplicial complex, and

$$\Phi : K_y \to K_x$$

is the induced map. Given a relative connection, choose a quasi-line bundle $\mathcal{L}'$ for $\varphi_x^* \mathcal{G}$, and a unitary section σ of the line bundle $H := \varphi_y^* \mathcal{L} \otimes (\Phi^* \mathcal{L}')^{-1}$. Let $\tilde{\omega} \in \Omega^2(N)$ be the error two-form for $\mathcal{L}'$, and $A \in \Omega^1(M)$ be the connection one-form for H with respect to σ. Define a map j by

$$j(y, x) := \exp\left(2\pi\sqrt{-1}\left(\int_{K_x} \tilde{\omega} - \int_{K_y} A\right)\right).$$

Choose another quasi-line bundle for $\varphi_x^* \mathcal{G}$. Then, the difference of error two-forms is an integral two-form. Changing the section σ will shift connection one-form A to $A + A'$, where A' is an integral one-form. Thus,

$$j : Z_2^{\mathrm{sm}}(\Phi) \to U(1)$$

is well-defined. Let k be the curvature three-form for $\mathcal{G}$, and ω be the error two-form for $\mathcal{L}$. Then $(\omega, k) \in \Omega^3(\Phi)$, and

$$\begin{aligned} j(\partial(y, x)) = j(\partial y, \Phi_*(y) - \partial x) &= \exp\left(2\pi\sqrt{-1}\left(\int_{K_{(\Phi_*(y)-\partial x)}} \tilde{\omega} - \int_{K_{\partial y}} A\right)\right) \\ &= \exp\left(2\pi\sqrt{-1}\left(\int_{K_{\Phi_*(y)}} \tilde{\omega} - \int_{K_{\partial x}} \tilde{\omega} - \int_{K_{\partial y}} A\right)\right) \\ &= \exp\left(2\pi\sqrt{-1}\left(\int_{K_y} (\Phi^* \tilde{\omega} - \mathrm{d}A) - \int_{K_x} \mathrm{d}\tilde{\omega}\right)\right) \\ &= \exp\left(2\pi\sqrt{-1}\left(\int_{K_y} \omega - \int_{K_x} k\right)\right). \end{aligned}$$

Thus, j is a relative differential character in degree 3. □

4.5. Transgression

For a manifold M, denote its loop space as LM. In this section, a line bundle with connection over LM is first constructed by transgressing a gerbe with connection over M. A map $\Phi \in C^{\infty}(M, N)$ induces a map $L\Phi \in C^{\infty}(LM, LN)$. Next, it is proven that a relative gerbe with connection on Φ produces a relative line bundle with connection on $L\Phi$ by transgression.

Proposition 4.10 (Parallel transportation). *Suppose that $\mathcal{G}$ is a gerbe with connection on $M \times [0, 1]$ and $\mathcal{G}_0 = \mathcal{G}|_{(M\times\{0\})}$. There is a natural quasi-line bundle with connection for the grebe $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}$, where π is the projection map*

$$\pi : M \times [0, 1] \to M \times \{0\}.$$

Proof. It is obvious that one can obtain a quasi-line bundle with connection for the gerbe $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}$. Specify a quasi-line bundle $\mathcal{L}_{\mathcal{G}}$ for the grebe $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}$ by the following requirements:

1. The pull-back $\iota^*\mathcal{L}_{\mathcal{G}}$ is trivial, while ι is inclusion map

$$\iota : M \times \{0\} \hookrightarrow M \times [0, 1].$$

2. Let $\eta \in \Omega^3(M \times [0, 1])$ be the curvature three-form for $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}$. Note that $\iota^*\eta = 0$. Let $\chi \in \Omega^2(M \times [0, 1])$ be the canonical primitive of η given by transgression. Then, choose a connection on $\mathcal{L}_{\mathcal{G}}$ such that its error two-form is χ. Any two such quasi-line bundles differ by a flat line bundle over $M \times [0, 1]$. This line bundle is a trivial line bundle over $M \times \{0\}$. □

Theorem 4.11. *A grebe $\mathcal{G}$ with connection on $M \times S^1$, induces a line bundle $E_{\mathcal{G}}$ with connection on M. Also, a quasi-line bundle with connection for $\mathcal{G}$ induces a unitary section of $E_{\mathcal{G}}$.*

Proof. $M \times S^1 = M \times [0, 1]/\sim$, where the equivalence relation is defined by $(m, 0) \sim (m, 1)$ for $m \in M$. Therefore, $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}|_{M\times\{1\}/\sim}$ is a trivial gerbe, and $\mathcal{L}_{\mathcal{G}}|_{M\times\{1\}/\sim}$ is a quasi-line bundle with connection for this trivial gerbe, i.e., a line bundle with connection $E_{\mathcal{G}}$ for M. If one change $\mathcal{L}_{\mathcal{G}}$ to another natural quasi-line bundle with connection, the difference between two quasi-line bundles over $M \times S^1$ is a trivial line bundle. Thus, the assignment $\mathcal{G} \to E_{\mathcal{G}}$ is well-defined.

Suppose that the gerbe $\mathcal{G}$ admits a quasi-line bundle $\mathcal{L}$. Then, $(\pi^*\mathcal{L}_0) \otimes (\mathcal{L}^{-1})$ and $\mathcal{L}_{\mathcal{G}}$ are two quasi-line bundles for the gerbe $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}$, where $\mathcal{L}_0 = \mathcal{L}|_{M\times\{0\}}$. Thus, $\pi^*\mathcal{L}_0 \otimes \mathcal{L}^{-1} \otimes (\mathcal{L}_{\mathcal{G}})^{-1}$ defines a line bundle over $M \times S^1 = M \times [0, 1]/\sim$. This line bundle over M defines a map $s : M \to U(1)$. $(\pi^*\mathcal{L}_0) \otimes (\mathcal{L}^{-1})|_{M\times\{0\}/\sim}$ is the trivial line bundle E. Since $E_{\mathcal{G}} \otimes E^{-1} = s$, $E_{\mathcal{G}}$ admits a unitary section. □

Remark 4.12. Let $\mathcal{G}$ be a gerbe with connection on M. Consider the evaluation map

$$e : LM \times S^1 \to M.$$

Thus, $e^*\mathcal{G}$ induces a line bundle with connection on LM.

Theorem 4.13. *For a given map $\Phi \in C^\infty(M, N)$, a relative gerbe with connection $\mathcal{G}_\Phi$ induces a relative line bundle with connection $E_{L\Phi}$.*

Proof. The relative grebe $\mathcal{G}_\Phi$ is a gerbe $\mathcal{G}$ on N together with a quasi-line bundle with connection $\mathcal{L}$ for the pull-back gerbe $\Phi^*\mathcal{G}$. The grebe $\mathcal{G}$ induces a line bundle with connection $E_\mathcal{G}$. Further, the quasi-line bundle with connection $\mathcal{L}$ for $\Phi^*\mathcal{G}$ induces a unitary section s for the line bundle with connection $(L\Phi)^* E_\mathcal{G}$ by Theorem 4.11. Thus, the pair $(s, E_\mathcal{G})$ defines a relative line bundle with connection $E_{L\Phi}$. □

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