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Dirac algebroids in Lagrangian and Hamiltonian mechanics☆

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ABSTRACT

We present a unified approach to constrained implicit Lagrangian and Hamiltonian systems based on the introduced concept of *Dirac algebroid*. The latter is a certain almost Dirac structure associated with the Courant algebroid $\mathsf{T}E^* \oplus_M \mathsf{T}^*E^*$ on the dual E^* to a vector bundle $\tau : E \to M$. If this almost Dirac structure is integrable (Dirac), we speak about a Dirac–Lie algebroid. The bundle E plays the role of the bundle of kinematic configurations (quasi-velocities), while the bundle E^* – the role of the phase space. This setting is totally intrinsic and does not distinguish between regular and singular Lagrangians. The constraints are part of the framework, so the general approach does not change when nonholonomic constraints are imposed, and produces the (implicit) Euler–Lagrange and Hamilton equations in an elegant geometric way. The scheme includes all important cases of Lagrangian and Hamiltonian systems, no matter if they are with or without constraints, autonomous or non-autonomous etc., as well as their reductions; in particular, constrained systems on Lie algebroids. we prove also some basic facts about the geometry of Dirac and Dirac–Lie algebroids.

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1. Introduction

The concept of *Dirac structure*, proposed by Dorfman [1] in the Hamiltonian framework of integrable evolution equations and defined in [2] as a subbundle of the Whitney sum $\mathsf{T}N \oplus_N \mathsf{T}^*N$ of the tangent and the cotangent bundle (the *extended tangent* or the *Pontryagin bundle*) satisfying certain conditions, was thought-out as a common generalization of Poisson and presymplectic structures. It was designed also to deal with constrained systems, including constraints induced by degenerate Lagrangians, as was investigated by Dirac [3], which is the reason for the name.

The need of extending the geometrical tools of the Lagrangian formalism from tangent bundles to Lie algebroids was caused by the fact that reductions usually move us out of the environment of the tangent bundles [4] (think on the reduction to $\mathfrak{so}(3, \mathbb{R})$ for the rigid body). It is similar to the better-known situation of passing from the symplectic to the Poisson structures by a reduction in the Hamiltonian formalism.

Note that the use of Lie algebroids and Lie groupoids for describing some systems of Analytical Mechanics was proposed by Libermann [5] and Weinstein [6], and then developed by many authors, for instance [7–10], making use of Lie algebroids in various aspects of Analytical Mechanics and Classical Field Theory.

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Since a Lie algebroid structure on a vector bundle $\tau : E \to M$ can be viewed as a linear Poisson structure Π on the dual bundle $\pi : E^* \to M$, a properly defined 'linear' Dirac structure should be viewed as a generalization of the concept of Lie algebroid. Linear structures of different kinds on a vector bundle can be viewed, in turn, as associated with certain *double vector bundles*. The double vector bundles, introduced in [11,12] (see also [13,14]) as manifolds with two 'compatible' vector bundle structures, have been successfully applied in [15,16] to geometric formalisms of Analytical Mechanics, including nonholonomic constraints [17,18]. To be more precise, note first that canonical examples of double vector bundles are: the tangent bundle $\mathsf{T}E$, and the cotangent bundle T^*E of the vector bundle E. The double vector bundles

$$\begin{tikzcd} \mathsf{T}^*E^* \arrow[r, "\mathsf{T}^*\pi"] \arrow[d, "\pi_{E^*}"'] & E \arrow[d, "\tau"] \\ E^* \arrow[r, "\pi"] & M \end{tikzcd}, \qquad \begin{tikzcd} \mathsf{T}^*E \arrow[r, "\mathsf{T}^*\tau"] \arrow[d, "\tau_{E^*}"'] & E^* \arrow[d, "\pi"] \\ E \arrow[r, "\tau"] & M \end{tikzcd}$$

are canonically isomorphic (cf. [13,19]). In particular, all arrows correspond to vector bundle structures and all pairs of vertical and horizontal arrows are vector bundle morphisms. Double vector bundles have been recently characterized [14] in a simple way as two vector bundle structures whose Euler vector fields commute.

In [16,15], a Lie algebroid (and its generalizations) on E has been viewed as a double vector bundle morphism

$$\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^* \tag{1.1}$$

covering the identity on E^*. This is because the linearity of a bivector field (e.g. a Poisson tensor) Π_ε on the dual bundle E^* can be geometrically expressed as respecting the double vector bundle structures by the induced vector bundle morphism

$$\widetilde{\Pi}_\varepsilon : \mathsf{T}^*E^* \to \mathsf{T}E^*. \tag{1.2}$$

We obtain ε as the composition of the canonical isomorphism of double vector bundles $\mathcal{R}_\tau : \mathsf{T}^*E \to \mathsf{T}^*E^*$ with $\widetilde{\Pi}_\varepsilon$, $\varepsilon = \widetilde{\Pi}_\varepsilon \circ \mathcal{R}_\tau$.

An application of this approach to Analytical Mechanics, in which $\tau : E \to M$ plays the role of kinematic configurations, is based on some ideas of Tulczyjew and Urbański [20–22].

Note that we can represent the morphism (1.2) of double vector bundles by its graph D_ε in the Whitney sum bundle

$$\mathcal{T}E^* = \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^*. \tag{1.3}$$

The *Pontryagin bundle* $\mathcal{T}E^*$ is canonically a double vector bundle: over E^* and over $\mathsf{T}M \oplus_M E$, and the fact that ε is a morphism means that D_ε is a double vector subbundle. Moreover, since D_ε is the graph of a Poisson tensor (in the case when E is a standard Lie algebroid), the subbundle D_ε is a Dirac structure on E^*. This immediately leads to a generalization of the concept of Lie algebroid: we replace the graph D_ε with any Dirac structure D on E^* which is linear, i.e., which is a double vector subbundle of $\mathcal{T}E^*$. We will call such an object a *Dirac–Lie algebroid*.

As was observed already in [23], the construction of phase dynamics associated with a given Lagrangian does not use the fact that the bivector field Π_ε is Poisson (which, on the other hand, induces nice properties of the dynamics), so we will use also almost Dirac structures, imposing no integrability assumptions. Thus, a *Dirac algebroid* on E will be a linear almost Dirac structure on E^*. We introduce also affine analogs of Dirac and Dirac–Lie algebroids.

The main applications we propose go back again to Analytical Mechanics. To some extent, our concepts are similar to that of [24,25], where (almost) Dirac structures have been used in the description of 'implicit' Lagrangian systems. However, we find our approach much more general (we work with arbitrary vector bundles) and much simpler. This is because we obtain 'implicit Lagrangian systems' (in fact both: implicit phase dynamics and implicit Euler–Lagrange equations), as well as implicit Hamilton equations, just composing relations, instead of working with the somehow artificial concept of *partial vector fields*. This generality allows us to cover a large variety of Lagrangian and Hamiltonian systems, including reduced systems, nonholonomic or vakonomic constraints, and time-dependent systems, with no regularity assumptions on the Lagrangian or Hamiltonian.

The paper is organized as follows. In Section 2 we recall basic facts concerning the double vector bundle approach to Lie algebroids and their generalizations. Dirac algebroids, Dirac–Lie algebroids, and their affine counterparts are introduced in Section 3, together with the main examples. In Section 4 we investigate closer the structure of Dirac algebroids, finding a short exact sequence of Lie algebroids associated with a Dirac–Lie algebroid and providing a local form of Dirac algebroids. Section 5 is devoted to inducing new Dirac algebroids by means of nonholonomic constraints. In Section 6 we present the general schemes, based on Dirac algebroids, for Lagrangian and Hamiltonian formalisms. We end up with a number of examples in Section 7 and concluding remarks in Section 8.

2. Lie algebroids as double vector bundle morphisms

We start with recalling basic facts and introducing some notation.

Let M be a smooth manifold and let (x^a), $a = 1, \ldots, n$, be a coordinate system in M. We denote with $\tau_M : \mathsf{T}M \to M$ the tangent vector bundle and by $\pi_M : \mathsf{T}^*M \to M$ the cotangent vector bundle. We have the induced (adapted) coordinate systems, $(x^a, \dot{x}^b)$ in $\mathsf{T}M$ and (x^a, p_b) in T^*M. More generally, let $\tau: E \to M$ be a vector bundle and let $\pi: E^* \to M$ be the dual bundle. Let $(e_1, \ldots, e_m)$ be a basis of local sections of $\tau: E \to M$ and let $(e^1_*, \ldots, e^m_*)$ be the dual basis of local sections of $\pi: E^* \to M$. We have the induced coordinate systems: (x^a, y^i), $y^i = \iota(e^i_*)$, in E, and (x^a, ξ_i), $\xi_i = \iota(e_i)$, in E^*, where the linear functions $\iota(e)$ are given by the canonical pairing $\iota(e)(v_x) = \langle e(x), v_x\rangle$. In this way we get local coordinates

$$(x^a, y^i, \dot{x}^b, \dot{y}^j) \quad \text{in } \mathsf{T}E, \qquad (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \quad \text{in } \mathsf{T}E^*,$$
$$(x^a, y^i, p_b, \pi_j) \quad \text{in } \mathsf{T}^*E, \qquad (x^a, \xi_i, p_b, \varphi^j) \quad \text{in } \mathsf{T}^*E^*.$$

The cotangent bundles T^*E and T^*E^* are examples of so-called *double vector bundles*. They are fibred over E and E^* and canonically isomorphic, with the isomorphism $\mathcal{R}_\tau: \mathsf{T}^*E \longrightarrow \mathsf{T}^*E^*$, being simultaneously an anti-symplectomorphism (cf. [13,16]). In local coordinates, $\mathcal{R}_\tau$ is given by

$$\mathcal{R}_\tau(x^a, y^i, p_b, \pi_j) = (x^a, \pi_i, -p_b, y^j). \tag{2.1}$$

This means that we can identify coordinates π_j with ξ_j, coordinates φ^j with y^j, and use the coordinates (x^a, y^i, p_b, ξ_j) in T^*E and the coordinates (x^a, ξ_i, p_b, y^j) in T^*E^* in full agreement with (2.1). According to [14], the double vector bundle structure is completely characterized by a pair of commuting Euler vector fields defining the two vector bundle structures (or by the pair of the corresponding families of homotheties). In local coordinates the Euler vector fields on T^*E^* read

$$\nabla^E_{\mathsf{T}^*E} = p_b\partial_{p_b} + \xi_i\partial_{\xi_i}, \qquad \nabla^{E^*}_{\mathsf{T}^*E} = p_b\partial_{p_b} + y^i\partial_{y^i}. \tag{2.2}$$

Double vector (and vector-affine) bundles will play an important role in our concepts and we refer to [26,14,27,13,19] for the general theory.

It is well known that Lie algebroid structures on a vector bundle E correspond to linear Poisson tensors on E^*. A 2-contravariant tensor Π on E^* is called *linear* if the corresponding mapping $\widetilde{\Pi}: \mathsf{T}^*E^* \to \mathsf{T}E^*$ induced by the contraction, $\widetilde{\Pi}(\nu) = i_\nu \Pi$, is a morphism of double vector bundles. One can equivalently say that the corresponding bracket of functions is closed on (fiber-wise) linear functions. The commutative diagram

$$\begin{tikzcd}
\mathsf{T}^*E^* \arrow[r, "\widetilde{\Pi}"] & \mathsf{T}E^* \\
\mathsf{T}^*E \arrow[u, "\mathcal{R}_\tau"] \arrow[ur, "\varepsilon"'] &
\end{tikzcd},$$

describes a one-to-one correspondence between linear 2-contravariant tensors Π_ε on E^* and morphisms ε (covering the identity on E^*) of the following double vector bundles (cf. [13,16]):

$$\begin{tikzcd}
& \mathsf{T}^*E \arrow[rrr, "\varepsilon"] \arrow[dr, "\pi_E"] \arrow[ddl, "\mathsf{T}^*\tau"'] & & & \mathsf{T}E^* \arrow[dr, "\mathsf{T}\pi"] \arrow[dl, "\tau_{E^*}"'] & \\
& & E \arrow[rrr, "\rho"] \arrow[ddl, "\tau"'] & E^* & & \mathsf{T}M \arrow[ddl, "\tau_M"'] \\
E^* \arrow[rrr, "id"] \arrow[dr, "\pi"] & & & E^* \arrow[dr, "\pi"] & & \\
& M \arrow[rrr, "id"] & & & M &
\end{tikzcd} \tag{2.3}$$

In local coordinates, every such ε is of the form

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x)y^k, c^k_{ij}(x)y^i\xi_k + \sigma^a_j(x)p_a) \tag{2.4}$$

(summation convention assumed) and it corresponds to the linear tensor

$$\Pi_\varepsilon = c^k_{ij}(x)\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^a_j(x)\partial_{x^a} \otimes \partial_{\xi_j}.$$

The morphism (2.3) of double vector bundles covering the identity on E^* has been called an *algebroid* in [16]. We will consider only *skew algebroids*, i.e., algebroids ε for which the tensor Π_ε is skew-symmetric, i.e., is a bivector field. If Π_ε is a Poisson tensor, we deal with a *Lie algebroid*. The relation to the canonical definition of Lie algebroid is given by the following theorem (cf. [15,16]).

Theorem 2.1. *A skew algebroid structure (E, ε) can be equivalently defined as a skew-symmetric bilinear bracket $[\cdot,\cdot]_\varepsilon$ on the space* $\mathrm{Sec}(E)$ *of sections of $\tau: E \to M$, together with a vector bundle morphism $\rho: E \to \mathsf{T}M$ (called the anchor), such that*

$$[X, fY]_\varepsilon = \rho(X)(f)Y + f[X, Y]_\varepsilon$$

for $f \in \mathcal{C}^\infty(M)$, $X, Y \in \mathrm{Sec}(E)$. The bracket and the anchor are related to the bracket $\{\varphi, \psi\}_{\Pi_\varepsilon} = \langle \Pi_\varepsilon, \mathrm{d}\varphi \otimes \mathrm{d}\psi \rangle$ in the algebra of functions on E^*, associated with the bivector field Π_ε, by the formulae

$$\iota([X, Y]_\varepsilon) = \{\iota(X), \iota(Y)\}_{\Pi_\varepsilon},$$
$$\pi^*(\rho(X)(f)) = \{\iota(X), \pi^* f\}_{\Pi_\varepsilon},$$

where $\iota(X)$ is the linear function on E^* associated with the section X of E.

3. Dirac algebroids and affine Dirac algebroids

Let N be a smooth manifold. There is a natural symmetric pairing $(\cdot|\cdot)$ on the vector bundle $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^*N$ (called sometimes the *Pontryagin bundle*) given by

$$(X_1 + \alpha_1 | X_2 + \alpha_2) = \frac{1}{2}(\alpha_1(X_2) + \alpha_2(X_1)),$$

for all sections $X_i + \alpha_i$, $i = 1, 2$, of $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^*N$. Furthermore, the space $\mathrm{Sec}(\mathcal{T}N)$ of smooth sections of $\mathcal{T}N$ is endowed with the Courant–Dorfman bracket,

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - i_{X_2}\mathrm{d}\alpha_1, \tag{3.1}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields, $\mathcal{L}_X$ is the Lie derivative along the vector field X, and i_X is the contraction (inner product) with X. An *almost Dirac structure (or bundle)* on the smooth manifold N is a subbundle D of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot|\cdot)$. If additionally the space of sections of D is closed under the Courant–Dorfman bracket, we speak about a *Dirac structure (or bundle)* [2,1].

Standard examples of almost Dirac structures are the graphs

$$\mathrm{graph}(\Pi) = \{X_p + \alpha_p \in \mathcal{T}_pN : p \in N,\ X_p = \widetilde{\Pi}(\alpha_p)\},$$
$$\mathrm{graph}(\omega) = \{X_p + \alpha_p \in \mathcal{T}_pN : p \in N,\ \alpha_p = \widetilde{\omega}(X_p)\},$$

of bivector fields Π or 2-forms ω viewed as vector bundle morphisms,

$$\widetilde{\Pi} : \mathsf{T}^*N \to \mathsf{T}N, \qquad \widetilde{\Pi}(\alpha_p) = i_{\alpha_p}\Pi(p),$$
$$\widetilde{\omega} : \mathsf{T}N \to \mathsf{T}^*N, \qquad \widetilde{\omega}(X_p) = -i_{X_p}\omega(p).$$

These graphs are actually Dirac structures if and only if Π is a Poisson tensor and ω is a closed 2-form, respectively.

Remark 3.1. A vector subbundle of a vector bundle over N is often understood as a vector bundle over the whole base manifold N. It is however clear by many reasons (see e.g. [14, Theorem 2.3]) that we must consider also vector subbundles supported on submanifolds of N. Throughout this paper the term *vector subbundle* always means a subbundle of the original vector bundle supported on a submanifold $N_0 \subset N$. In this sense, our definitions of almost Dirac and Dirac structure are slightly more general than those usually available in the literature. By 'being closed' with respect to the bracket we clearly mean that the bracket of any two sections of $\mathcal{T}N$, extending sections of D, does not depend over N_0 on the extensions chosen and gives a section extending a section of D. This uniquely defines a bracket on sections of D which is known to be a Lie algebroid bracket.

Since the projection $\mathrm{pr}_{\mathsf{T}N} : \mathcal{T}N \to \mathsf{T}N$ is the left anchor for the Courant–Dorfman bracket, i.e.,

$$[\![X_1 + \alpha_1, f(X_2 + \alpha_2)]\!] = f[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] + X_1(f)(X_2 + \alpha_2), \tag{3.2}$$

it is a straightforward observation that the bracket of extensions of sections of a subbundle D, supported on a submanifold N_0 of N, does not depend on the extensions if and only if

$$\mathrm{pr}_{\mathsf{T}N}(D) \subset \mathsf{T}N_0. \tag{3.3}$$

Indeed, if f is 0 on N_0, by (3.2) $X_1(f)$ must be 0 on N_0 for any section $X_1 + \alpha_1$ which belongs to D along N_0. The condition (3.3) we will call the *first integrability condition* for the Dirac–Lie algebroid. Under this condition the Courant–Dorfman bracket restricts to

$$[\![\cdot, \cdot]\!]_D : \mathrm{Sec}(D) \times \mathrm{Sec}(D) \to \mathrm{Sec}(\mathcal{T}N). \tag{3.4}$$

Then, the *second integrability condition* says that $[\![\cdot, \cdot]\!]_D$ takes values in $\mathrm{Sec}(D)$:

$$[\![\cdot, \cdot]\!]_D : \mathrm{Sec}(D) \times \mathrm{Sec}(D) \to \mathrm{Sec}(D) \subset \mathrm{Sec}((\mathcal{T}N)_{|N_0}), \tag{3.5}$$

which, according to (3.2) and (3.3), is sufficient to be checked on a generating set of sections of D:

$$[\![\sigma_k, \sigma_l]\!]_D \in \mathrm{Sec}(D) \quad \text{for } \{\sigma_i\} \subset \mathrm{Sec}(D) \text{ generating } D. \tag{3.6}$$

By definition, an almost Dirac structure is a Dirac structure if and only if it satisfies both the integrability conditions, (3.3) and (3.6).

Remark 3.2. Suppose that an almost Dirac structure D satisfies the first integrability condition, i.e., the Courant–Dorfman bracket $[\![\cdot, \cdot]\!]_D$ of sections of D supported on N_0 is well defined. If we have chosen a subbundle K of $\mathcal{T}N$ complementary to

D over N_0, we can define the bracket

$$[\![\cdot,\cdot]\!]_D^K : \mathrm{Sec}(D)\times\mathrm{Sec}(D)\to\mathrm{Sec}(D) \tag{3.7}$$

by projecting the value of $[\![\cdot,\cdot]\!]_D$ onto $\mathrm{Sec}(D)$ along K. Of course, if D is a Dirac structure, $[\![\cdot,\cdot]\!]_D^K$ does not depend on the choice of K and is just the Lie algebroid bracket on sections of D.

In Geometric Mechanics there is often a need to use affine bundles and affine versions of algebroids [28–34] (*affgebroids* in the terminology introduced in [29,30]). We will use the following concept.

Definition 3.1. Let A be an affine subbundle of a Lie algebroid $E\to M$ with the bracket $[\cdot,\cdot]$ and the anchor $\rho : E\to \mathsf{T}M$, supported on a submanifold $S\subset M$. Let $V=\mathsf{v}(A)$ be its model vector bundle viewed as a vector subbundle of E. We call A an *affine Lie subalgebroid in E*, if the brackets of sections of A lie in $\mathrm{Sec}(\mathsf{v}(A))$, i.e., $\rho(A)\subset \mathsf{T}S$ (thus the bracket of sections of A is well defined over S) and $[\sigma,\sigma']\in\mathrm{Sec}(V)$ for all $\sigma,\sigma'\in\mathrm{Sec}(A)$.

For a more extensive treatment of brackets on affine bundles we refer to [29,30] (see also [28,33,34]).

To consider also affine versions of (almost) Dirac structures, we propose the following (compare [29,30]).

Definition 3.2. An *affine almost Dirac structure* on a manifold N is an affine subbundle D of $\mathcal{T}N$, supported on a submanifold N_0 of N, whose model vector bundle $\mathsf{v}(D)\subset\mathcal{T}N$ (canonically represented by a subbundle of $\mathcal{T}N$) is an almost Dirac structure on N. An affine almost Dirac structure is called an *affine Dirac structure*, if the Courant–Dorfman bracket of sections of D makes sense (like the analogous concept for Dirac–Lie algebroids) and takes values in the set of sections of $\mathsf{v}(D)$, i.e., (3.3) is satisfied, so that (3.4) is well defined and

$$[\![\cdot,\cdot]\!]_D : \mathrm{Sec}(D)\times\mathrm{Sec}(D)\to\mathrm{Sec}(\mathsf{v}(D))\subset\mathrm{Sec}(\mathcal{T}N). \tag{3.8}$$

The following is straightforward.

Proposition 3.1. *If D is an affine Dirac structure, then $\mathsf{v}(D)$ is a Dirac structure.*

Now let F be a vector bundle over a manifold M. Since both, $\mathsf{T}F$ and T^*F, are canonically double vector bundles, their Whitney sum carries a structure of a canonical double vector bundle as well. From the general theory we easily derive the following (cf. [13,14]).

Theorem 3.1. *If F is a vector bundle over M, its Pontryagin bundle $\mathcal{T}F=\mathsf{T}F\oplus_F\mathsf{T}^*F$, canonically isomorphic to $\mathsf{T}F\oplus_F\mathsf{T}^*F^*$, is also canonically a double vector bundle structure with two compatible vector bundle structures: $\tau_1:\mathcal{T}F\to F$ and $\tau_2:\mathcal{T}F\to\mathsf{T}M\oplus_M F^*$.*

The core bundle of $\mathcal{T}F$, i.e., a vector bundle over M being the intersection of the kernels of the both projections, is in this case canonically isomorphic to $\mathsf{T}^*M\oplus_M F$. Moreover, the fibration

$$(\tau_1,\tau_2):\mathcal{T}F\to F\oplus_M\mathsf{T}M\oplus_M F^*$$

is an affine bundle modeled on the pull-back core bundle, i.e., the core bundle $\mathsf{T}^*M\oplus_M F$ over M pulled-back to $F\oplus_M\mathsf{T}M\oplus_M F^*$ via the canonical projection $F\oplus_M\mathsf{T}M\oplus_M F^*\to M$.

Definition 3.3. We call a submanifold D of a double vector bundle its *double vector subbundle*, if D is a subbundle for each of the two vector bundle structures.

Following the ideas of [14], one can easily prove that this means that the two Euler vector fields defining the double vector bundle structure are tangent to D. One can also equivalently say that D is invariant with respect to both commuting families of homotheties defined by the two vector bundle structures (cf. [14]).

Proposition 3.2. *Let D be a double vector subbundle of a double vector bundle*

$$\begin{tikzcd} K \arrow[r, "\tau_2"] \arrow[d, "\tau_1"] & K_2 \arrow[d, "\tau_1'"] \\ K_1 \arrow[r, "\tau_2'"] & M \end{tikzcd} \tag{3.9}$$

Then, D inherits a double vector bundle structure with projections onto vector bundles $S_i=\tau_i(D),\ i=1,2$, where S_i is a vector subbundle of K_i.

Proof. It is easy to see that the homothety h^1_t, being the multiplication of vectors of the bundle $\tau_1:K\to K_1$ by $t\in\mathbb{R}$, coincides on K_2 with the homothety of the vector bundle $K_2\to M$. The submanifold D, being h^1_t-invariant, has the base $S_2\subset K_2$ which is h^1_t-invariant, thus is a vector subbundle of $K_2\to M$ [14, Theorem 2.3]. □

Definition 3.4. A *Dirac algebroid* (resp., *Dirac–Lie algebroid*) *structure* on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathsf{T}M \oplus_M E$.

Remark 3.3. The above definition gives an analog of linearity of a Poisson or a presymplectic structure.

We will consider also *affine Dirac algebroids* (*Dirac affgebroids* in short).

Definition 3.5. An *affine Dirac algebroid* on a vector bundle E is an affine subbundle D of $\mathcal{T}E^*$ whose model vector bundle $\mathsf{v}(D) \subset \mathcal{T}E^*$ (represented by vertical vectors tangent to the fibers of D) is a Dirac algebroid. An affine Dirac algebroid is called an *affine Dirac–Lie algebroid*, if D is an affine Dirac structure, i.e., if the Courant–Dorfman bracket of sections of D is a section of $\mathsf{v}(D)$.

According to Proposition 3.1, if D is an affine Dirac–Lie algebroid, then $\mathsf{v}(D)$ is a Dirac–Lie algebroid.

Remark 3.4. We can consider other affine types of Dirac structures as well, defined on affine or special affine bundles, by considering vector-affine bundles of different types (see e.g. [27]), but we skip these considerations here in order not to multiply technical difficulties.

In view of Proposition 3.2, a Dirac algebroid $D \subset \mathcal{T}E^*$ projects onto two vector subbundles: $\mathsf{Ph}_D = \tau_1(D) \subset E^*$ and $\mathsf{Vel}_D = \tau_2(D) \subset \mathsf{T}M \oplus_M E$, both based on a submanifold M_D of M, giving rise to a single projection,

$$\tau^D = (\tau_1^D, \tau_2^D) : D \to \mathsf{Ph}_D \oplus_{M_D} \mathsf{Vel}_D \subset E^* \oplus_M (\mathsf{T}M \oplus_M E), \tag{3.10}$$

which, according to Theorem 3.1, is an affine bundle modeled on the core $\mathcal{C}_D$ of D pulled-back to $\mathsf{Ph}_D \oplus_{M_D} \mathsf{Vel}_D$, i.e., on $(\mathsf{Ph}_D \oplus_{M_D} \mathsf{Vel}_D) \times_{M_D} \mathcal{C}_D$. Note that the core $\mathcal{C}_D$ is a subbundle (supported on M_D) of $\mathsf{T}^*M \oplus_M E^*$—the core of the double vector bundle $\mathcal{T}E$.

We will call the first component in $\mathsf{Ph}_D \oplus_{M_D} \mathsf{Vel}_D$ the *phase bundle* and the second—the *anchor relation* (or the *velocity bundle*) of the Dirac algebroid D. The anchor relation is just a linear relation between vectors of E ('quasi-velocities') and vectors tangent to M ('actual velocities') and gives rise to the *anchor map*

$$\rho_D : \mathsf{T}M \oplus_M E \supset \mathsf{Vel}_D \to \mathsf{T}M_D \tag{3.11}$$

being the projection onto the first summand.

To express linearity of an almost Dirac (or Dirac) subbundle of $\mathcal{T}E^*$ in a more explicit way, consider adapted coordinates $(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k)$ on $\mathcal{T}E^*$. The two commuting Euler vector fields are:

$$\nabla_1 = p_a\partial_{p_b} + \dot{\xi}_j\partial_{\dot{\xi}_j} + y^i\partial_{y^i} + \dot{x}^b\partial_{\dot{x}^b},$$

corresponding to the vector bundle structure over E^* with coordinates (x, ξ), and

$$\nabla_2 = p_a\partial_{p_b} + \xi_i\partial_{\xi_i} + \dot{\xi}_j\partial_{\dot{\xi}_j},$$

corresponding to vector bundle structure over $\mathsf{T}M \oplus_M E$ with coordinates $(x, \dot{x}, y)$. The corresponding homotheties read

$$h_t^1(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) = (x^a, \xi_i, t\dot{x}^b, t\dot{\xi}_j, tp_c, ty^k), \tag{3.12}$$

$$h_s^2(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) = (x^a, s\xi_i, \dot{x}^b, s\dot{\xi}_j, sp_c, y^k), \tag{3.13}$$

and a linear almost Dirac subbundle in $\mathcal{T}E^*$ (Dirac algebroid) should be invariant with respect to both sets of homotheties. Note that the canonical symmetric pairing is represented by the quadratic function $Q(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) = p_a\dot{x}^a + y^i\dot{\xi}_i$ which vanishes on Dirac algebroids.

Example 3.1. The graph of any linear bivector field

$$\Pi = \frac{1}{2}c_{ij}^k(x)\xi_k\partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x)\partial_{\xi_i} \wedge \partial_{x^b},$$

where $c_{ij}^k = -c_{ji}^k$, is a Dirac algebroid:

$$\mathrm{graph}(\Pi) = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : \dot{x}^b = \rho_k^b(x)y^k,\ \dot{\xi}_j = c_{ij}^k(x)y^i\xi_k - \rho_j^a(x)p_a\}.$$

It is clear that Q vanishes on $\mathrm{graph}(\Pi)$. This graph is a double vector subbundle, since the constraint functions

$$\dot{x}^b - \rho_k^b(x)y^k, \qquad \dot{\xi}_j - c_{ij}^k(x)y^i\xi_k + \rho_j^a(x)p_a \tag{3.14}$$

are homogeneous with respect to the Euler vector fields ∇_1, ∇_2. The phase bundle here is E^* and the anchor relation is actually the graph of the vector bundle morphism $\rho : E \to \mathsf{T}M$ (the anchor map) given in local coordinates by $\rho(x^a, y^i) = (x^a, \rho_i^b(x)y^i)$. This means that skew-algebroids are particular examples of Dirac algebroids. We will call the Dirac algebroids of this form, associated with a bivector field Π, *Π-graph Dirac algebroids* on E and denote it by D_Π. The Dirac algebroid D_Π is a Dirac–Lie algebroid if and only if Π is a Poisson tensor, i.e., if and only if we deal with a Lie algebroid.

Example 3.2. The graph of any linear 2-form

$$\omega = \frac{1}{2}c^k_{ab}(x)\xi_k \mathrm{d}x^a \wedge \mathrm{d}x^b + \rho^i_b(x)\mathrm{d}\xi_i \wedge \mathrm{d}x^b,$$

where $c^k_{ab} = -c^k_{ba}$, is a Dirac algebroid:

$$\mathrm{graph}(\omega) = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : y^i = \rho^i_a(x)\dot{x}^a,\ p_a = c^k_{ab}(x)\xi_k\dot{x}^b - \rho^i_a(x)\dot{\xi}_i\}.$$

It is clear that Q vanishes on $\mathrm{graph}(\omega)$. This graph is a double vector subbundle, since the constraint functions

$$y^i - \rho^i_a(x)\dot{x}^a, \qquad p_a - c^k_{ab}(x)\xi_k\dot{x}^b + \rho^i_a(x)\dot{\xi}_i \tag{3.15}$$

are homogeneous with respect to the Euler vector fields ∇_1, ∇_2. The phase bundle here is E^* and the anchor relation is in fact the graph of the vector bundle morphism $\rho : \mathsf{T}M \to E$ given in local coordinates by $\rho(x^a, \dot{x}^b) = (x^a, \rho^i_b(x)\dot{x}^b)$. We will call the Dirac algebroids of this form, associated with a 2-form ω, *ω-graph Dirac algebroids* and denote it by D_ω. The Dirac algebroid D_ω is a Dirac–Lie algebroid (*presymplectic Dirac–Lie algebroid*) if and only if ω is closed.

Example 3.3. The *canonical Dirac–Lie algebroid* $D_M = D_{\Pi_M} = D_{\omega_M}$, corresponding to the canonical Lie algebroid $E = \mathsf{T}M$, belongs to the both above types. It is associated with the canonical symplectic form ω_M on $E^* = \mathsf{T}^*M$ and, simultaneously, to the canonical Poisson tensor $\Pi_M = \omega_M^{-1}$ on T^*M. In our local coordinates, the equations defining D_M are

$$\dot{x^a} = y^a, \qquad \dot{\xi_b} = -p_b.$$

Example 3.4. Suppose we have a Dirac (Dirac–Lie) algebroid D on $E \to M$. Let us consider the extension $E_0 = E \times \mathbb{R}$ as a vector bundle over $M_0 = M \times \mathbb{R}$ in the obvious way. Then, $E_0^* = E^* \times \mathbb{R}$, $\mathsf{T}E_0^* = \mathsf{T}E^* \times \mathsf{T}\mathbb{R}$, and $\mathsf{T}^*E_0^* = \mathsf{T}^*E^* \times \mathsf{T}^*\mathbb{R}$. The subbundle $D_0 = D \times A_0$ in $\mathcal{T}E_0^* = \mathcal{T}E^* \times \mathcal{T}\mathbb{R}$, where A_0 is the affine subbundle in $\mathcal{T}\mathbb{R}$ defined by the constraint $\dot{x}_0 = 1$ in the natural coordinates $(x_0, \dot{x_0}, p_0)$ on $\mathcal{T}\mathbb{R}$, is an affine Dirac (Dirac–Lie) algebroid on E_0.

4. The structure of a Dirac algebroid

Let us start this paragraph recalling that any section $\sigma : N \to F$ of a vector bundle $F \to N$ (actually, of any fibration) is uniquely determined by its image $\sigma(N)$—a submanifold of F. We will denote this submanifold by $[\sigma]$.

Definition 4.1. Let K be a double vector bundle (3.9). We say that a section $\widetilde{\sigma} : K_1 \to K$ *projects* on the section $\sigma : M \to K_2$, if τ_2 projects $[\widetilde{\sigma}]$ onto $[\sigma]$. We will write $\widetilde{\sigma}^{\tau_2} = \sigma$ and call such $\widetilde{\sigma}$ *projectable*.

We say that a section $\widetilde{\sigma} : K_1 \to K$ is *suitable*, if $[\widetilde{\sigma}]$ is a vector subbundle of the vector bundle $\tau_2 : K \to K_2$.

It is easy to see the following

Theorem 4.1. *Any suitable section $\widetilde{\sigma}$ is projectable and $[\widetilde{\sigma}^{\tau_2}]$ is the image under τ_2 of the zero-section 0_{K_1} of τ_1. Moreover, the set of suitable sections,* $\mathrm{Suit}(K)$*, is canonically a $C^\infty(M)$-module and the module morphism $[\tau_2] : \widetilde{\sigma} \mapsto \widetilde{\sigma}^{\tau_2}$ is an epimorphism onto* $\mathrm{Sec}(K_2)$.

Suitable sections which project on the zero-section of the bundle K_2 we will call 0-*suitable*. So the set $\mathrm{Suit}_0(K)$ of 0-suitable sections is the kernel of the map $[\tau_2] : \mathrm{Suit}(K) \to \mathrm{Sec}(K_2)$. A standard argument shows that that the $C^\infty(M)$-modules $\mathrm{Suit}(K)$ and $\mathrm{Suit}_0(K)$ are the modules of sections of certain vector bundles over M, $\mathsf{Suit}(K)$ and $\mathsf{Suit}_0(K)$, respectively, but we will not go into details here.

All this can be applied to the situation of the Pontryagin bundle over the vector bundle E^*,

$$\begin{tikzcd} \mathcal{T}E^* \arrow[r, "\tau_2"] \arrow[d, "\tau_1"'] & \mathsf{T}M \oplus_M E \arrow[d, "\tau_M \times \tau"] \\ E^* \arrow[r, "\pi"] & M \end{tikzcd} \tag{4.1}$$

and easily explained in our standard local coordinates $(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k)$. The image of a section $\widetilde{\sigma}$ of τ_1 consists of points

$$(x^a, \xi_i, \dot{x}^b(x, \xi), \dot{\xi}_j(x, \xi), p_c(x, \xi), y^k(x, \xi)) \in \mathcal{T}E^*.$$

This section is projectable if and only if the coefficients $\dot{x}^b$ and y^k depend on x only,

$$\dot{x}^b = \dot{x}^b(x), \qquad y^k = y^k(x),$$

thus $\widetilde{\sigma}$ projects onto the section

$$\sigma(x) = (x, \dot{x}^b(x), y^k(x))$$

of $\mathsf{T}M \oplus_M E$. Since being a vector subbundle means exactly being a submanifold invariant with respect to homotheties [14], $\widetilde{\sigma}$ is suitable if the submanifold

$$[\widetilde{\sigma}] = \{(x, \xi_i, \dot{x}^b(x,\xi), \dot{\xi}_j(x,\xi), p_c(x,\xi), y^k(x,\xi)) \in \mathcal{T}E^*\}$$

is invariant with respect to homotheties (3.13), i.e.,

$$\dot{x}^b(x, s\xi) = \dot{x}^b(x,\xi), \qquad y^k(x, s\xi) = y^k(x,\xi),$$
$$\dot{\xi}_j(x, s\xi) = s\dot{\xi}_j(x,\xi), \qquad p_c(x, s\xi) = sp_c(x,\xi).$$

As smooth homogeneous functions are linear, we get finally that $\dot{x}^b$ and y^k do not depend on ξ ($\widetilde{\sigma}$ is projectable) and that $\dot{\xi}_j$ and p_c linearly depend on ξ,

$$\dot{\xi}_j(x,\xi) = \dot{\xi}^i_j(x)\xi_i, \qquad p_c(x,\xi) = p^i_c(x)\xi_i. \tag{4.2}$$

Recall that the section $\widetilde{\sigma}$ is $X + \alpha$, where the vector field on E^* reads

$$X = \dot{x}^b(x,\xi)\partial_{\dot{x}^b} + \dot{\xi}_j(x,\xi)\partial_{\dot{\xi}_j}$$

and the 1-form α is

$$\alpha = p_c(x,\xi)\mathrm{d}x^c + y^k(x,\xi)\mathrm{d}\xi_k.$$

Since linearity is measured by homogeneity with respect to the Euler vector field in the bundle, this implies immediately the following.

Theorem 4.2. *Let ∇ be the Euler vector field in the vector bundle E^*. A section $X + \alpha$ of $\tau_1 : \mathcal{T}E^* \to E^*$ is suitable if and only if $\mathcal{L}_\nabla X = 0$ and $\mathcal{L}_\nabla \alpha = \alpha$.*

Such vector fields and 1-forms are sometimes called, with some abuse of terminology, *linear*. Hence, $X + \alpha$ is suitable if and only if X and α are *linear*. This allows one to identify the bundle $\mathsf{Suit}(\mathcal{T}E^*)$ with $\mathsf{Der}(E) \oplus_M (\mathsf{Der}(E)^* \otimes_M E)$ with $\mathsf{Der}(E)$ being the bundle of *quasi-derivations* (or *derivative endomorphisms* or *quasi-derivations*) in E (see [35]). We will not go into details here.

A fundamental observation is now the following.

Theorem 4.3. *If $\widetilde{\sigma}_i$, $i = 1, 2$, are suitable sections of $\mathcal{T}E^*$, then $[\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!]$ and $\mathrm{d}\,(\widetilde{\sigma}_1|\widetilde{\sigma}_2)$ are suitable. Moreover, if $\widetilde{\sigma}_2$ is additionally 0-suitable, then $[\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!]$ and*

$$[\![\widetilde{\sigma}_2, \widetilde{\sigma}_1]\!] - 2\mathrm{d}(\widetilde{\sigma}_1|\widetilde{\sigma}_2)$$

are 0-suitable.

In particular, suitable sections of $\mathcal{T}E^*$ are closed with respect to the Courant–Dorfman bracket and 0-suitable sections form a left-ideal inside.

Proof. If X_i and α_i are linear, $i = 1, 2$, then of course $[X_1, X_2]$ and $\mathcal{L}_{X_1}\alpha_2 - i_{X_2}\alpha_1$, as well as $\mathrm{d}\left(i_{X_1}\alpha_2 + i_{X_2}\alpha_1\right)$, are linear. To find the projection $[\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!]^{\tau_2}$ in coordinates, let us write

$$\widetilde{\sigma}_1(x,\xi) = \dot{x}^b(x)\partial_{x^b} + f^j_i(x)\xi_j\partial_{\xi_i} + y^i(x)\mathrm{d}\xi_i + g^j_a(x)\xi_j\mathrm{d}x,$$
$$\widetilde{\sigma}_2(x,\xi) = \dot{\bar{x}}^b(x)\partial_{x^b} + \bar{f}^j_i(x)\xi_j\partial_{\xi_i} + \bar{y}^i(x)\mathrm{d}\xi_i + \bar{g}^j_a(x)\xi_j\mathrm{d}x.$$

Then, direct calculations of the Courant–Dorfman bracket show that $[\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!]^{\tau_2}$ is represented by the tensor

$$\left(\dot{x}^c\frac{\partial\dot{\bar{x}}^b}{\partial x^c} - \dot{\bar{x}}^c\frac{\partial\dot{x}^b}{\partial x^c}\right)(x)\partial_{x^b} + \left(\dot{x}^b\frac{\partial\bar{y}^i}{\partial x^b} - \dot{\bar{x}}^b\frac{\partial y^i}{\partial x^b} + \bar{y}^j f^i_j + \dot{\bar{x}}^b g^i_b\right)(x)\mathrm{d}\xi_i. \tag{4.3}$$

If $\dot{\bar{x}}$ and $\bar{y}$ are 0, we get 0. If $\dot{x}$ and y are 0, we get

$$(\bar{y}^j f^i_j + \dot{\bar{x}}^b g^i_b)(x)\mathrm{d}\xi_i = 2(\mathrm{d}(\widetilde{\sigma}_1|\widetilde{\sigma}_2))^{\tau_2}. \quad \square$$

It is clear that having a double vector subbundle D, e.g. Dirac algebroid, we can consider suitable sections of D in the same manner. As the scalar products $(\widetilde{\sigma}_1|\widetilde{\sigma}_2)$ vanish for sections of a Dirac algebroid, out of Theorem 4.3 we can easily derive the following. Let us fix a Dirac algebroid with an anchor relation Vel_D inducing an anchor map $\rho_D : \mathsf{Vel}_D \to \mathsf{T}M_D$.

Theorem 4.4. *If D is a Dirac algebroid satisfying the first-integrability condition, then the Courant–Dorfman bracket induces on the module of suitable sections of D a skew-symmetric bracket*

$$[\![\cdot,\cdot]\!]_D : \mathrm{Suit}(D) \times \mathrm{Suit}(D) \to \mathrm{Suit}((\mathcal{T}E^*)_{|\mathsf{Ph}_D})$$

such that

$$[\![\widetilde{\sigma}_1, f\widetilde{\sigma}_2]\!]_D = f[\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!]_D + \rho_D(\widetilde{\sigma}_1^{\tau_2})(f)\widetilde{\sigma}_2$$

for all $f \in C^\infty(M_D)$. Moreover, if one of the sections is 0-suitable, the resulted bracket is 0-suitable.

In the case when D is a Dirac–Lie algebroid, the Courant–Dorfman bracket is a Lie algebra bracket on $\mathsf{Suit}(\mathcal{T}E^*)$ for which $\mathsf{Suit}_0(\mathcal{T}E^*)$ is a Lie ideal and turns the bundles $\mathsf{Suit}(D)$ and $\mathsf{Suit}_0(D)$ into Lie algebroids. Moreover, in this situation, as $\mathsf{Suit}(D)/\mathsf{Suit}_0(D) \simeq \mathsf{Sec}(\mathsf{Vel}_D)$, we get a Lie algebroid bracket on the anchor bundle Vel_D that gives rise to a canonical short exact sequence of Lie algebroids associated with the Dirac–Lie algebroid D.

$$0 \longrightarrow \mathsf{Suit}_0(D) \longrightarrow \mathsf{Suit}(D) \longrightarrow \mathsf{Vel}_D \longrightarrow 0.$$

In the case of a Lie algebroid E associated with a linear Poisson structure Π on E^* the Lie bracket of sections of E can be recognized inside the Lie algebroid on sections of D_Π as the bracket of sections $\widetilde{\Pi}(\alpha) + \alpha$, associated with 'linear 1-forms' α, in coordinates $\alpha = y^i(x)\mathrm{d}\xi_i$. The above theorem provides a generalization of this fact and, for each Dirac–Lie algebroid, describes the induced Lie algebroid structure on its velocity bundle.

The next theorem characterizes the core bundle of a Dirac algebroid in terms of its anchor relation.

Theorem 4.5. *The core bundle $\mathcal{C}_D \subset \mathsf{T}^*M \oplus_M E^*$ of a Dirac algebroid $D \subset \mathcal{T}E^*$ is the annihilator subbundle $\mathsf{Vel}_D^0 \subset \mathsf{T}^*M \oplus_M E^*$ of the anchor relation $\mathsf{Vel}_D \subset \mathsf{T}M \oplus_M E$.*

Proof. To an element $d \in D$ that projects onto $(\tau_1, \tau_2)(d) = (\mu_x, v_x)$ we can add any element u_x of the x-fiber of the core not changing the projections, so, due to isotropy, $\langle v_x, u_x\rangle = 0$ for all $v_x \in (\mathsf{Vel}_D)_x$ and $\mathcal{C}_D \subset \mathsf{Vel}_D^0$. The equality follows from the conditions on the rank. In coordinates, d is represented by

$$d = \dot{x}^a\partial_{x^a} + f_i\partial_{\xi_i} + y^i\mathrm{d}\xi_i + g_a\mathrm{d}x^a$$

and

$$u_x = \dot{\xi}_i\partial_{\xi_i} + p_a\mathrm{d}x^a.$$

Since $(d|d) = 0$ and $(d + u_x|d) = 0$, we have $\langle\tau_2(d), u_x\rangle = 0$, i.e.

$$\dot{x}^a p_a + y^i\dot{\xi}_i = 0. \quad \square$$

In order to describe the local form of a Dirac algebroid D, note first that, since an arbitrary Dirac algebroid $D \subset \mathcal{T}E^*$ is the restriction to the phase bundle $\mathsf{Ph}_D \subset E^*$ of a Dirac algebroid supported on the whole bundle E^*, we can assume at the beginning for simplicity that $\mathsf{Ph}_D = E^*$. As the Pontryagin bundle $\mathcal{T}E^*$ is, as the bundle over the projection

$$(\tau_1, \tau_2) : \mathcal{T}E^* \to (\mathsf{T}M \oplus_M E) \oplus_M E^*$$

an affine bundle modeled on the pull-back bundle of the core bundle $\mathsf{T}^*M \oplus_M E^*$ (Theorem 3.1), we can write

$$\mathcal{T}E^* \simeq (E^* \oplus_M \mathsf{T}M \oplus_M E) \times_M (\mathsf{T}^*M \oplus_M E^*). \tag{4.4}$$

Note that the product $\times_M$ in the above expression is not canonical, but it can be used to express the fact that we can add elements of $\mathsf{T}^*_xM \oplus E^*_x$ to elements of $E^*_x \oplus \mathsf{T}_xM \oplus E_x$ and to serve for introducing local coordinates. Instead of the coordinates we have already used, it will be more convenient to introduce affine coordinates $(x^a, \eta^i, \widehat{\eta}^j)$ in $\mathsf{T}_xM \oplus E_x$ and dual affine coordinates $(x^a, \zeta_i, \widehat{\zeta}_j)$ in $\mathsf{T}^*_xM \oplus E^*_x$, so that $(\eta^i, \widehat{\eta}^j)$ represent linear coordinates in fibers of the anchor relation Vel_D and its (non-canonical) complementary subbundle V, $\mathsf{T}_xM \oplus E_x = \mathsf{Vel}_D \oplus_M V$, respectively, and the coordinates $(\zeta_i, \widehat{\zeta}_j)$ are linear coordinates in the annihilators $\mathsf{T}^*_xM \oplus E^*_x = V^0 \oplus_M \mathsf{Vel}_D^0$, respectively. Note that V^0 represents the dual bundle Vel_D^*.

The points of D then satisfy $\widehat{\eta}^j = 0$. Since we can add elements of the core $\mathcal{C}_D = \mathsf{Vel}_D^0$, coordinates $\widehat{\zeta}_j$ are arbitrary. Therefore, there are sections $\widetilde{\sigma}_i$ of D associated with the canonical local basis σ_i of sections of Vel_D, $\eta^{i'}(\sigma_i(x)) = \delta_i^{i'}$, which read

$$\sigma_i(x^a, \xi_j) = (x^a, \xi_j, \eta^{i'} = \delta_i^{i'}, 0, \zeta_{i'} = c^j_{ii'}(x)\xi_j, 0).$$

Due to isotropy, we have skew-symmetry $c^j_{ii'}(x) = -c^j_{i'i}(x)$. Now, we can add linear constraint Ph_D in E^* by introducing affine coordinates, say $(x, \widehat{x}, \xi, \widehat{\xi})$, such that Ph_D is expressed by $\widehat{x} = 0$, $\widehat{\xi} = 0$. In this way we get the following

Theorem 4.6 (*Local form of a Dirac Algebroid*)**.** *In the introduced local affine coordinates the Dirac algebroid D consists of points $(x, \widehat{x}, \xi, \widehat{\xi}, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$ for which*

$$\widehat{x} = 0, \qquad \widehat{\xi} = 0, \qquad \widehat{\eta} = 0, \qquad \zeta_k = c^j_{ik}(x)\eta^i\xi_j. \tag{4.5}$$

Moreover, $c^j_{ik}(x) = -c^j_{ki}(x)$.

Let us note that the above constraints can be viewed as a common generalization of (3.14) and (3.15). The functions c^j_{ik} play the role of structure functions and $\widehat{\eta}=0$ defines the anchor relation. We can write $(\eta,\widehat{\eta})$ as linear functions of variables $(\dot{x},y)$ and $(\zeta,\widehat{\zeta})$ as linear functions of $(p,\dot{\xi})$ (with coefficients being functions of x) to derive constraints

$$\widehat{\eta}(x,\dot{x},y)=0,\qquad \zeta_i(x,p,\dot{\xi})+c^j_{ik}(x)\eta^k(x,\dot{x},y)\xi_j=0. \tag{4.6}$$

Example 4.1. For the Π-graph Dirac algebroid as described in Example 3.1 we have

$$\eta=y,\qquad \widehat{\eta}^b=\dot{x}^b-\rho^b_k(x)y^k,\qquad \zeta_j=\dot{\xi}_j+\rho^a_j(x)p_a,\qquad \widehat{\zeta}_b=p_b$$

and the Eq. (4.6) read

$$\dot{x}^b-\rho^b_k(x)y^k=0,\qquad \dot{\xi}_j+\rho^a_j(x)p_a+c^k_{ji}(x)y^i\xi_k=0,$$

exactly as in (3.14).

5. Induced Dirac algebroids

In this section we will show how appropriate linear (or affine) 'nonholonomic constraints' in the velocity bundle Vel_D give rise to new (induced) Dirac algebroids. These constructions may be viewed as a generalization of the similar construction for the canonical Lie algebroid $E=\mathsf{T}M$ in [24].

Consider a Dirac algebroid $D\subset\mathcal{T}E^*$ and let V be a vector subbundle of the velocity bundle $\mathsf{Vel}_D\subset\mathsf{T}M\oplus_M E$ supported on $S\subset M_D\subset M$. Let $\widetilde{V}=(\tau^D_2)^{-1}(V)$ be the restriction of the vector bundle $\tau^D_2:D\to\mathsf{Vel}_D$ to the submanifold V in the base, and let $V^0\subset\mathsf{T}^*M\oplus E^*$ be the annihilator of V. Of course, V^0 is supported on S as well and $V^0\supset\mathsf{Vel}^0_D=\mathcal{C}_D$. Since $\mathsf{T}^*M\oplus E^*$ is the core of $\mathcal{T}E^*$, we may add vectors from V^0 to vectors of the vector bundle $\tau_1:\mathcal{T}E^*\to E^*$ not changing any of two projections. In this sense, $D^V=\widetilde{V}+V^0$ is again a double vector subbundle of $\mathcal{T}E^*$ which is no longer D, but still projects on V *via* τ_2, and on Ph_D *via* τ_1.

Theorem 5.1. *The double vector subbundle D^V in $\mathcal{T}E^*$ is a Dirac algebroid on E.*

Proof. The subbundle D^V is isotropic by definition, since $\widetilde{V}$ is isotropic as a subbundle of D, and V^0 is isotropic and orthogonal to $\widetilde{V}$. The rank of this bundle is maximal, since first we lose rank by $\dim(\mathsf{Vel}_D/V)$ and then we gain $\dim(V^0/\mathsf{Vel}^0_D)=\dim(\mathsf{Vel}_D/V)$. □

Definition 5.1. We will call the Dirac algebroid D^V the Dirac algebroid *induced from D by the subbundle $V\subset\mathsf{Vel}_D$*.

Quite similarly, we can induce affine Dirac algebroids using an affine subbundle A of Vel_D based on a submanifold $S\subset M$. Let $V=\mathsf{v}(A)$ be its model vector bundle viewed as a vector subbundle of Vel_D. Let us put $\widetilde{A}=(\tau^D_2)^{-1}(A)$, and let $V^0\subset\mathsf{T}^*M\oplus E^*$ be the annihilator of V. The vector subbundle $\widetilde{A}$ of $\tau^D_2:D\to\mathsf{Vel}_D$ is simultaneously an affine subbundle of $\tau^D_1:D\to\mathsf{Ph}_D$, thus *vector-affine* subbundle. Similarly as above, $D^A=\widetilde{A}+V^0$ is again a vector-affine subbundle of $\mathcal{T}E^*$ which still projects on A *via* τ_2, and on Ph_D *via* τ_1. Analogously to Theorem 5.1 one can prove the following.

Theorem 5.2. *The vector-affine subbundle D^A in $\mathcal{T}E^*$ is an affine Dirac algebroid on E.*

Definition 5.2. We will call the affine Dirac algebroid D^A the affine Dirac algebroid *induced from D by the affine subbundle $A\subset\mathsf{Vel}_D$*.

Example 5.1. Consider a Dirac algebroid D_Π on a vector bundle $\tau:E\to M$, associated with a linear bivector field Π on E^*. Since the anchor relation Vel_{D_Π} is in this case the graph of the anchor map $\rho:E\to\mathsf{T}M$, subbundles V of Vel_{D_Π} may be identified with subbundles V_0 of E, $V=\{\rho(v)+v;\, v\in V_0\}$.

It is convenient to see all this in local coordinates $(x^a,\xi_i,\dot{x}^b,\dot{\xi}_j,p_c,y^k)$ in $\mathcal{T}E^*$. We may choose local coordinates in $(x^a)=(x^\alpha,x^A)$ in M, so that S is given locally by $x^A=0$. Let us also use linear coordinates (y^i) in the fibers of E, so that $y=(y^i)=(y^\iota,y^I)$ and the subbundle V_0 is defined by the constraint $y^I=0$. On $\mathcal{T}E^*$ we have then local coordinates $(x^a,\dot{x}^b,\dot{\xi}_l,p_c,y^\iota,y^I)$ where we have also decompositions $(\xi_k)=(\xi_\kappa,\xi_K)$ and $(\dot{\xi}_l)=(\dot{\xi}_\lambda,\dot{\xi}_L)$ associated with the decomposition $(y^i)=(y^\iota,y^I)$. The double subbundle $\widetilde{V}$ is defined by the constraints (cf. Example 3.1)

$$\widetilde{V}=\{(x^a,\xi_i,\dot{x}^b,\dot{\xi}_j,p_c,y^k):x^A=0,\ y^I=0,\dot{x}^b=\rho^b_\iota(x)y^\iota,\dot{\xi}_k=c^j_{\iota k}(x)y^\iota\xi_j-\rho^a_k(x)p_a\}.$$

The points $(x^a,p_b,\dot{\xi}_i)$ of $\mathsf{T}^*M\oplus_M E^*$ belong to V^0 if and only if $x^A=0$ and $p_b\rho^b_\iota(x)y^\iota+\dot{\xi}_\iota y^\iota=0$ for all (y^ι), thus $\dot{\xi}_\iota=-\rho^b_\iota(x)p_b$ and $\dot{\xi}_I$ are arbitrary. As the first condition agrees with the original constraints, we get the final constraints defining D^V_Π:

$$x^A=0,\qquad \dot{x}^b=\rho^b_\iota(x)y^\iota,\qquad \dot{\xi}_\kappa=c^j_{\iota\kappa}(x)y^\iota\xi_j-\rho^a_\kappa(x)p_a,\qquad y^I=0, \tag{5.1}$$

as adding V^0 makes $\dot{\xi}_K$ arbitrary.

Let us assume now that Π is a Poisson tensor, i.e., D_Π is a Dirac–Lie algebroid. The first integrability condition for D^V_Π is now $\mathrm{pr}_{\mathsf{T}E^*} \subset \mathsf{T}E^*_{|S}$, i.e., $\dot{x}^B = \rho^B_\iota(x^\alpha, 0)y^\iota = 0$ for all y^ι, thus

$$\rho^B_\iota(x^\alpha, 0) = 0. \tag{5.2}$$

To check the second integrability condition, let us note first that D^V_Π is locally generated by the sections ∂_{ξ_I} and $R_{\xi_\iota} = \widetilde{\Pi}(\mathrm{d}\xi_\iota) + \mathrm{d}\xi_\iota$. Since, by the assumption that Π is Poisson,

$$[\![R_{\xi_\iota}, R_{\xi_{\iota'}}]\!] = R_{\Pi(\mathrm{d}\xi_\iota, \mathrm{d}\xi_{\iota'})},$$

Ith components of $\mathrm{d}(\Pi(\mathrm{d}\xi_\iota, \mathrm{d}\xi'_\iota))$ must vanish along $E^*_{|S}$, i.e.,

$$c^I_{\iota\iota'}(x^\alpha, 0) = 0. \tag{5.3}$$

The vector fields ∂_{ξ_I} commute, so it remains to check whether $[\![\partial_{\xi_I}, R_{\xi_\iota}]\!]$ are sections of D^V_Π over $E^*_{|S}$. But

$$[\![\partial_{\xi_I}, R_{\xi_\iota}]\!] = [\partial_{\xi_I}, \widetilde{\Pi}(\mathrm{d}\xi_\iota)] \tag{5.4}$$

and, according to (5.2),

$$\widetilde{\Pi}(\mathrm{d}\xi_\iota)(x^\alpha, 0, \xi_i) = c^{\iota''}_{\iota\iota'}(x^\alpha, 0)\xi_{\iota''}\partial_{\xi'_\iota} + f^{I'}(x^\alpha, 0, \xi_i)\partial_{\xi_{I'}}$$

for some functions $f^{I'}$. Hence

$$[\![\partial_{\xi_I}, R_{\xi_\iota}]\!](x^\alpha, 0, \xi_i) = \partial_{\xi_I}(f^{I'})(x^\alpha, 0, \xi_i)\partial_{\xi_{I'}},$$

so that the expression in (5.4) is spanned over $E^*_{|S}$ by ∂_{ξ_I}. Thus we get that D^V_Π is a Dirac–Lie algebroid if and only if (5.2) and (5.3) are satisfied. This, in turn, means that V_0 is a Lie subalgebroid in the Lie algebroid on E associated with Π, so V is a Lie subalgebroid of the Lie algebroid Vel_D—the graph of the anchor map.

Theorem 5.3. *If D_Π is a Π-graph Dirac–Lie algebroid, then D^V_Π is a Dirac–Lie algebroid if and only if V is a Lie subalgebroid of* Vel_D.

Example 5.2. A particular case of the above example is the canonical Dirac–Lie algebroid D_M. In this case we recover the induced Dirac structure considered in [24], i.e., the set

$$D_V = \{(X+\alpha) \in \mathsf{T}\mathsf{T}^*M \oplus_{\mathsf{T}^*M} \mathsf{T}^*\mathsf{T}^*M;\ X \in (\mathsf{T}\pi_M)^{-1}(V_0),\ \forall\, W \in (\mathsf{T}\pi_M)^{-1}(V_0) : \langle \alpha, X\rangle = \omega_M(X, W)\},$$

where ω_M is the canonical symplectic form on T^*M. Let us show that this indeed is the case.

According to our definition the canonical Dirac–Lie algebroid D_M on the cotangent bundle is given by the canonical Poisson structure Π_M or the canonical symplectic structure ω_M on T^*M, i.e.,

$$D_M = \mathrm{graph}(\Pi_M) = \mathrm{graph}(\omega_M).$$

The velocity bundle $\mathsf{Vel}_{D_M} \subset \mathsf{T}M \oplus_M \mathsf{T}M$ is in this case the graph of the identity map on $\mathsf{T}M$, the phase bundle Ph_{D_M} is the whole cotangent bundle T^*M and the core $\mathcal{C}_{D_M} \subset \mathsf{T}^*M \oplus_M \mathsf{T}^*M$ is the graph of the minus identity map on T^*M.

Any subbundle V of the velocity bundle is given by a subbundle V_0 of the tangent bundle $\mathsf{T}M$ and is of the form $V = \{v + v;\ v \in V_0\}$. Then we get

$$\widetilde{V} = (\tau_2^{D_M})^{-1}(V) = \{X + \tilde{\omega}_M(X) \in \mathcal{T}\mathsf{T}^*M : \mathsf{T}\pi_M(X) \in V_0\}.$$

The annihilator V^0 consists of all pairs of covectors (φ, ψ) at the same point in M such that $\varphi + \psi \in (V_0)^0$. Since the induced Dirac structure is $D^V_M = \tilde{V} + V^0$, we have that

$$D^V_M = \{(X+\varphi) + (\tilde{\omega}_M(X) + \psi);\ \mathsf{T}\pi_M(X) \in V_0, \varphi + \psi \in (V_0)^0\}.$$

The "+" sign in brackets in the above formula stands for adding an element of a core to an element of the double vector bundle. To compare D^V_M with the Dirac structure considered in [24] let us observe, that the projection of D^V_M on $\mathsf{T}\mathsf{T}^*M$ gives the whole $(\mathsf{T}\pi_M)^{-1}(V_0)$. Adding elements of a core of a double vector bundle does not change projections, therefore adding φ to X produces another element Y of $(\mathsf{T}\pi_M)^{-1}(V_0)$. Since $\tilde{\omega}_M$ is a double vector bundle isomorphism, it respects the structure of the double vector bundle. In particular, it maps the core of $\mathsf{T}\mathsf{T}^*M$ to the core of $\mathsf{T}^*\mathsf{T}^*M$. Both cores are isomorphic to T^*M, and $\tilde{\omega}_M$ restricted to the core is the identity map. We have

$$\tilde{\omega}_M(Y) + \psi = \tilde{\omega}_M(X+\varphi) + \psi = \tilde{\omega}_M(X) + (\varphi + \psi),$$

so

$$D^V_M = \{X + (\tilde{\omega}_M(X) + \psi);\ \mathsf{T}\pi_M(X) \in V_0,\ \psi \in (V_0)^0\}.$$

Evaluating $\tilde{\omega}_M(X)+\psi$ on any $W\in(\mathsf{T}\pi_M)^{-1}(V_0)$, we get that

$$\langle\tilde{\omega}_M(X)+\psi,W\rangle=\langle\tilde{\omega}_M(X),W\rangle+\langle\psi,\mathsf{T}\pi_M(W)\rangle=\langle\tilde{\omega}_M(X),W\rangle=\omega_M(X,W),$$

thus $D_M^V\subset D_V$. For dimensional reasons the inclusion is in fact equality.

6. Lagrangian and Hamiltonian formalisms based on Dirac algebroids

6.1. Implicit differential equations

Let us start with an explanation what we will understand as implicit dynamics on a manifold N.

Definition 6.1. An *ordinary first-order (implicit) differential equation (implicit dynamics)* on a manifold N will be understood as a subset $\mathcal{D}$ of the tangent bundle $\mathsf{T}N$. We say that a smooth curve $\gamma:\mathbb{R}\to N$ (or a smooth path $\gamma:[t_0,t_1]\to N$) *satisfies the equation* $\mathcal{D}$ (or *is a solution of* $\mathcal{D}$), if its tangent prolongation $\dot{\gamma}:\mathbb{R}\to\mathsf{T}N$ (resp., $\dot{\gamma}:[t_0,t_1]\to\mathsf{T}N$) takes values in $\mathcal{D}$. We call a curve (or a path) $\widetilde{\gamma}$ in $\mathsf{T}N$ *admissible*, if it is the tangent prolongation of its projection $\widetilde{\gamma}_N$ on N.

According to the above definition, solutions of an implicit dynamics $\mathcal{D}$ on a manifold N are projections $\widetilde{\gamma}_N$ of admissible curves $\widetilde{\gamma}$ lying in D. Note, however, that different implicit differential equations may have the same set of solutions. First of all, if $\mathcal{D}$ is supported on a subset N_0, $\tau_N(\mathcal{D})=N_0$, only vectors from $\mathcal{D}\cap\mathsf{T}N_0$ do matter, if solutions are concerned. Hence, $\mathcal{D}'=\mathcal{D}\cap\mathsf{T}N_0$ has the same solution as $\mathcal{D}$, and $\mathcal{D}\subset\mathsf{T}N_0$ is the *first integrability condition*. Of course, replacing $\mathcal{D}$ with $\mathcal{D}'$ may turn out to be an infinite procedure, but we will not discuss the integrability problems in this paper.

All this can be generalized to ordinary implicit differential equations of arbitrary order. In this case we consider $\mathcal{D}$ as a subset of higher jet bundles, the n-th tangent bundle T^nN in case of an equation of order n, and consider γ as a solution when its n-th jet prolongation takes values in $\mathcal{D}$. If we call the n-th jet prolongations *admissible* in T^nN, then solutions of $\mathcal{D}$ are exactly projections $\widetilde{\gamma}_N$ to N of admissible curves (or paths) $\widetilde{\gamma}$ in T^nN lying in $\mathcal{D}$.

Remark 6.1. The implicit differential equations described above are called by some authors *differential relations*. Let us explain that we use the most general definition, not requiring from $\mathcal{D}$ any differentiability properties, since in real life the dynamics $\mathcal{D}$ we encounter are often not submanifolds. This generality is also very convenient, as it allows us to skip technical difficulties in the corresponding Lagrangian and Hamiltonian formalisms. Of course, what is an encumbrance in defining implicit dynamics can be very useful in solving the equations, but in our opinion, solving could be considered case by case, while geometric formalisms of generating dynamics should be as general as possible. Note also that for any subset N_0 of a manifold N the tangent prolongations $\mathsf{T}N_0$, T^2N_0, etc., make precisely sense as subsets of $\mathsf{T}N$, T^2N, etc. They are simply understood as families of the corresponding jets of appropriately smooth curves in N which take values in N_0.

Admissibility of a path in $\mathsf{T}N$ has a natural generalization for paths γ in an algebroid E. This concept plays a fundamental role in the 'integration' of Lie algebroids to Lie groupoids [36] and appears as natural consequence of the algebroid version of the Euler–Lagrange equations [23,17]. We propose the following extension of this concept to Dirac algebroids, which reduces to the standard definition for Π-graph Dirac algebroids and Lie algebroids.

Note first that given a smooth curve or path γ with values in E we have a unique 'tangent prolongation' of γ to a curve (or path) $\widehat{\boldsymbol{\gamma}}:\mathbb{R}\to\mathsf{T}M\oplus_M E$ (resp., $\widehat{\boldsymbol{\gamma}}:[t_0,t_1]\to\mathsf{T}M\oplus_M E$), defined in an obvious way by

$$\widehat{\boldsymbol{\gamma}}(t)=\dot{\gamma}_M(t)\oplus\gamma(t),\tag{6.1}$$

where γ_M is the projection of γ to M, $\gamma_M=\tau\circ\gamma$.

Definition 6.2. Let D be a Dirac algebroid on $\tau:E\to M$ and let $\mathsf{Vel}_D\subset\mathsf{T}M\oplus_M E$ be its anchor relation. We say that a curve $\gamma:\mathbb{R}\to E$ (or a path $\gamma:[t_0,t_1]\to E$) is *D-admissible*, if its tangent prolongation $\widehat{\boldsymbol{\gamma}}$ takes values in Vel_D,

$$\forall\, t\in\mathbb{R}\,[\widehat{\boldsymbol{\gamma}}(t)=\dot{\gamma}_M(t)+\gamma(t)\in\mathsf{Vel}_D\subset\mathsf{T}M\oplus_M E].$$

Remark 6.2. It is easy to see that in the case of a Π-graph Dirac algebroid, when Vel_D is the graph of the anchor map $\rho:E\to\mathsf{T}M$, a curve γ in E is D-admissible if and only if $\rho(\gamma(t))=\dot{\gamma}_M(t)$ that coincides with the concept of admissibility for Lie algebroids. In particular, for the canonical Lie algebroid $E=\mathsf{T}M$ and the corresponding canonical Dirac algebroid D_M, a curve γ in $\mathsf{T}M$ is D_M-admissible if and only if it is admissible, i.e., it is the tangent prolongation of its projection γ_M on M, $\gamma(t)=\dot{\gamma}_M(t)$.

6.2. Phase dynamics, Hamilton, and Euler–Lagrange equations

Our experience in working with (constrained) systems on skew-algebroids [23,17] suggests the following approach. Let us fix a Dirac algebroid D on a vector bundle E,

$$D\subset\mathsf{T}^*E^*\oplus_{E^*}\mathsf{T}E^*\simeq\mathsf{T}^*E\oplus_{E^*}\mathsf{T}E^*.$$

In generalized Lagrangian and Hamiltonian formalisms we will view D as a differential relation

$$\varepsilon_D : \mathsf{T}^*E \longrightarrow\!\!\!\!\triangleright\ \mathsf{T}E^*$$

or

$$\beta_D : \mathsf{T}^*E^* \longrightarrow\!\!\!\!\triangleright\ \mathsf{T}E^*,$$

respectively. We use the symbol '$\longrightarrow\!\!\!\!\triangleright$' to stress that we deal with relations having domains in T^*E or T^*E^* (not necessarily the whole T^*E or T^*E^*) and with ranges being subsets of $\mathsf{T}E^*$. Note that $\varepsilon_D = \beta_D \circ \mathcal{R}_\tau$ and β_D is a relation over the identity on the support of D in E^*—the phase bundle Ph_D. The bundle E plays the role of the bundle of generalized velocities (quasi-velocities), and its dual, E^*, the role of the phase space.

A Lagrangian function $L : E \to \mathbb{R}$ and a Hamiltonian $H : E^* \to \mathbb{R}$ give rise to maps associated with their derivatives, $\mathrm{d}L : E \to \mathsf{T}^*E$ and $\mathrm{d}H : E^* \to \mathsf{T}^*E^*$, respectively. The Lagrangian produces the *phase dynamics* $\varepsilon_D[\mathrm{d}L]$ as the image of E under the composition of relations $\Lambda^L_D = \varepsilon_D \circ \mathrm{d}L$:

$$\varepsilon_D[\mathrm{d}L] = \Lambda^L_D(E) \subset \mathsf{T}E^*. \tag{6.2}$$

The relation Λ^L_D we call the *Tulczyjew differential of* L. Similarly, when using the composition of relations $\Phi^H_D = \beta_D \circ \mathrm{d}H$, that projects onto the relation $\chi^H_D = \tau_{E^*} \circ \Phi^H_D$ being the identity on a subset of E^*, the *Hamiltonian dynamics* generated by the Hamiltonian H is defined by

$$\beta_D[\mathrm{d}H] = \Phi^H_D(E^*) \subset \mathsf{T}E^*. \tag{6.3}$$

The phase dynamics $\varepsilon_D[\mathrm{d}L]$ associated with the Lagrangian L *has a Hamiltonian description*, if there is a Hamiltonian H with the same dynamics, $\varepsilon_D[\mathrm{d}L] = \beta_D[\mathrm{d}H]$.

Of course, the actual phase spaces associated with L and H are projections of the phase dynamics on E^*, $\mathsf{Ph}^L_D = \tau_{E^*}(\varepsilon_D[\mathrm{d}L])$ and $\mathsf{Ph}^H_D = \tau_{E^*}(\beta_D[\mathrm{d}H])$.

Since, as easily seen, the projection of the relation $\Lambda^L_D = \varepsilon_D \circ \mathrm{d}L$ to $E \oplus_M E^*$ is actually a function,

$$\lambda^L_D = \tau_{E^*} \circ \Lambda^L_D = \mathsf{T}^*\tau \circ \mathrm{d}L_{|\mathsf{Vel}^L_D},$$

called the *Legendre map* associated with the Lagrangian L. The domain of the Legendre map will be denoted Vel^L_D and called the *Euler–Lagrange domain*. It is easy to see that Ph^L_D is the image of the Legendre map

$$\lambda^L_D = \tau_{E^*} \circ \Lambda^L_D : E \supset \mathsf{Vel}^L_D \to \mathsf{Ph}^L_D \subset E^*,$$

and the Legendre map is the restriction of the vertical derivative $\mathrm{d}^{\mathsf{v}}L : E \to E^*$ to

$$\mathsf{Vel}^L_D = \{v \in \mathrm{pr}_E(\mathsf{Vel}_D) : \mathrm{d}^{\mathsf{v}}L(v) \in \mathsf{Ph}_D\}. \tag{6.4}$$

In local coordinates,

$$\mathrm{d}^{\mathsf{v}}L(x, y) = \left(x, \frac{\partial L}{\partial y^i}(x, y)\right).$$

If D is a Π-graph Dirac algebroid, then $\mathsf{Vel}^L_D = E$.

The diagram for the corresponding *Tulczyjew triple* containing: T^*E (the Lagrangian side), the canonically isomorphic (via $\mathcal{R}_\tau$) double vector bundle T^*E^* (the Hamiltonian side), and $\mathsf{T}E^*$ (the phase dynamics side) is the following (here, the arrows denote relations):

$$\begin{tikzcd}
\mathsf{T}^*E \arrow[rr, "\varepsilon_D"] \arrow[rrrr, bend left, "\mathcal{R}_\tau"] \arrow[d, shift left, "\pi_E"] & & \mathsf{T}E^* \arrow[d, "\tau_{E^*}"] & & \mathsf{T}^*E^* \arrow[ll, "\beta_D"'] \arrow[d, shift left, "\pi_{E^*}"] \\
E \arrow[u, shift left, "\mathrm{d}L"] \arrow[rr, "\lambda^L_D"'] \arrow[urr, dotted, "\Lambda^L_D"] & & E^* & & E^* \arrow[ll, "\chi^H_D"] \arrow[u, shift left, "\mathrm{d}H"] \arrow[ull, dotted, "\Phi^H_D"']
\end{tikzcd} \quad . \tag{6.5}$$

The Euler–Lagrange equation associated with L will be viewed as an implicit dynamics on E. It will make sense for curves in E taking values in the Euler–Lagrange domain $\mathsf{Vel}^L_D \subset E$.

Definition 6.3. We say that a curve $\gamma : \mathbb{R} \to \mathsf{Vel}^L_D$ *satisfies* (or *is a solution of*) *the Euler–Lagrange equation*, if γ is Λ^L_D-related to an admissible curve $\widetilde{\gamma}$ in $\mathsf{T}E^*$ (i.e., $\widetilde{\gamma}$ is the tangent prolongation of its projection $\widetilde{\gamma}_{E^*}$ onto E^*). In particular, γ is λ^L_D-related to the curve $\widetilde{\gamma}_{E^*}$ which satisfies the phase equation.

To describe the Euler–Lagrange equation explicitly, consider the tangent prolongation of the relation Λ_D^L,

$$\mathsf{T}\Lambda_D^L = \mathsf{T}\varepsilon_D \circ \mathsf{T}\mathrm{d}L : \mathsf{T}E \longrightarrow\!\!\triangleright\ \mathsf{T}\mathsf{T}E^*.$$

In $\mathsf{T}\mathsf{T}E^*$ we can distinguish *holonomic vectors*, i.e., vectors $X_v \in \mathsf{T}_v\mathsf{T}E^*$ such that v equals the tangent projection of X_v onto $\mathsf{T}E^*$, i.e., $v = \mathsf{T}\tau_{E^*}(X_v)$. The set of holonomic vectors can be seen as the second tangent bundle T^2E^*. We define the *(implicit) Euler–Lagrange dynamics* as the subset of $\mathsf{T}E$ defined by the inverse image

$$\mathcal{E}_D^L = (\mathsf{T}\Lambda_D^L)^{-1}(\mathsf{T}^2E^*) \subset \mathsf{T}E.$$

Theorem 6.1. *If a curve $\gamma : \mathbb{R} \to E$ satisfies the Euler–Lagrange equation, then its tangent prolongation takes values in $\mathcal{E}_D^L$. In particular, γ is D-admissible.*

Proof. Let $\widetilde{\gamma}$ be an admissible curve, $\widetilde{\gamma} = \dot{\widetilde{\gamma}}_{E^*}$, contained in $\varepsilon_D[\mathrm{d}L]$ and Λ_D^L-related to γ. Then, its tangent prolongation $\dot{\widetilde{\gamma}}$ is $\mathsf{T}\Lambda_D^L$-related to the tangent prolongation $\dot{\gamma}$ of γ. But $\dot{\widetilde{\gamma}}$ is the 2-tangent prolongation of $\widetilde{\gamma}_{E^*}$, thus lies in T^2E^*. □

Note that the converse is 'almost true'. Indeed, if $\dot{\gamma}$ lies in $\mathsf{T}(\Lambda_D^L)^{-1}(\mathsf{T}^2E^*)$, we only need to know that we can pick up a curve in T^2E^* being $\mathsf{T}\Lambda_D^L$-related to $\dot{\gamma}$. This can be assured, for instance, by some smooth transversality assumptions. As we do not want to consider these questions here, let us only mention that the converse of Theorem 6.2 is always true in the case when Λ_D^L is a map, for instance for Π-graph Dirac algebroids.

Remark 6.3. Let us observe that in our setting the Euler–Lagrange equation is a first-order equation on E, in full agreement with the fact that the Hamilton equation is first-order as well. In the standard setting, the Euler–Lagrange equation is viewed as second-order, but for curves in the base M. This can be explained as follows. The solutions of the Euler–Lagrange equations are always D-admissible. In the case of the canonical algebroid $E = \mathsf{T}M$ the admissible curves in $\mathsf{T}M$ are exactly the tangent prolongations of curves in the base M, thus we may view the corresponding Euler–Lagrange equations as first-order equations on tangent prolongations, so second-order equations for curves on the base.

6.3. Hyperregular Lagrangians

Let us assume that we have a *hyperregular* Lagrangian $L : E \to \mathbb{R}$, i.e., such a Lagrangian that its vertical derivative $\mathfrak{L} = \mathrm{d}^{\mathsf{v}}L : E \to E^*$ is a diffeomorphism. For instance, L can be *of mechanical type*, being the sum of a 'kinetic energy' (associated with a 'metric' on the vector bundle E) and a potential (a basic function). It is well known [23] that in this case the Hamiltonian $H : E^* \to \mathbb{R}$ defined by

$$H = (\nabla_E(L) - L) \circ \mathfrak{L}^{-1}, \tag{6.6}$$

where ∇_E is the Euler vector field on the vector bundle E, defines the same Lagrangian submanifold in T^*E^* as L in T^*E, when we identify canonically both bundles:

$$\mathrm{d}H(E^*) = \mathcal{R}_\tau(\mathrm{d}L(E)).$$

In local coordinates, $\xi_i = \frac{\partial L}{\partial y_i}(x, y)$ and

$$H(x, \xi) = \xi_i \cdot y^i(x, \xi) - L(x, y(x, \xi)).$$

It is then easy to see that the Legendre map λ_D^L is a diffeomorphism of Vel_D^L on Ph_D^L, and that the phase dynamics associated with L and H coincide.

Theorem 6.2. *If L is a hyperregular Lagrangian, then, for any Dirac algebroid D, the phase dynamics $\varepsilon_D[\mathrm{d}L]$ coincides with the phase dynamics $\beta_D[\mathrm{d}H]$ for the Hamiltonian H defined by (6.6). In this sense, for hyperregular Lagrangians, the Lagrangian and Hamiltonian formalisms are equivalent.*

6.4. Constraints

Nonholonomic linear (or affine) constraints in our Dirac algebroid setting are understood as represented by vector (affine) subbundles V of the velocity bundle Vel_D. This could look strange at the first sight, but it becomes quite natural, if we recall that the solution of the Euler–Lagrange equations are admissible curves γ in the bundle E of quasi-velocities. Since there is a canonical tangent prolongation $\widehat{\boldsymbol{\gamma}}$ of γ, with $\widehat{\boldsymbol{\gamma}}$ lying in Vel_D, the constraint V gives us equations for γ with $\widehat{\boldsymbol{\gamma}}$ in V. The general principle is the following.

Definition 6.4 (*Nonholonomic Constraints*)**.** The phase dynamics and the Euler–Lagrange equations for a constrained Lagrangian system on a Dirac algebroid D over a vector bundle $\tau : E \to M$, and associated with the Lagrangian $L : E \to \mathbb{R}$ and the linear (affine) constraint bundle $V \subset \mathsf{Vel}_D$, is the dynamics associated with the same Lagrangian but on the induced Dirac algebroid D^V over E.

Another type of constraint we can consider in our setting are *vakonomic constraints* represented by a submanifold (not necessary an affine subbundle) C of E (cf. [37]). Let us recall that with any submanifold C of E and any function $L : C \to \mathbb{R}$ we can associate a Lagrangian submanifold $[\mathrm{d}L_C]$ of T^*E defined by

$$[\mathrm{d}L_C] = \{\eta \in \mathsf{T}^*_yE : y \in C \text{ and } \forall v \in \mathsf{T}_yC \quad \langle \eta, v\rangle = \langle \mathrm{d}L(y), v\rangle\}. \tag{6.7}$$

We can view $[\mathrm{d}L_C]$ as a relation $[\mathrm{d}L_C] : E \longrightarrow\!\!\!\triangleright\ \mathsf{T}^*E$. Now we can define the constrained phase dynamics and the Euler–Lagrange equations completely analogously to unconstrained ones, but replacing the relation $\mathrm{d}L(E)$ with $[\mathrm{d}L_C]$.

Definition 6.5 (*Vakonomic Constraints*)**.** The phase dynamics for a constrained Lagrangian system on a Dirac algebroid D over a vector bundle $\tau : E \to M$, associated with the Lagrangian $L : E \to \mathbb{R}$ and a vakonomic constraints represented by a submanifold C of E, is the dynamics represented by the subset $\varepsilon_D([\mathrm{d}L_C])$ of $\mathsf{T}E^*$. We say that a curve $\gamma : \mathbb{R} \to [\mathrm{d}L_C]$ *satisfies the vakonomically constrained Euler–Lagrange equation*, if γ is ε_D-related to an admissible curve in $\mathsf{T}E^*$.

Remark 6.4. Note first that, by definition, the phase dynamics for vakonomic constraints depends on the restriction of the Lagrangian L to C only. Second, we recover the old dynamics in the unconstrained case, as $\mathrm{d}L(E) = [\mathrm{d}L_E]$. This can look strange at first sight that we define a solution of the Euler–Lagrange equations as curves in $[\mathrm{d}L_C]$ and not in $\mathsf{Vel}_D \subset E$, but when the constraints are absent there is no real difference between Vel_D and $[\mathrm{d}L_{\mathsf{Vel}_D}]$, since the projection π_E establishes a diffeomorphism. In the presence of a constraint we no longer have this diffeomorphism. Of course, we could say that a curve in Vel_D satisfies the constrained Euler–Lagrange equation, if it is a projection of an appropriate curve in $[\mathrm{d}L_C]$, but our approach seems to be more natural. It could happen that one curve is the projection of different curves in $[\mathrm{d}L_C]$ that is a geometric interpretation of the presence of ‘Lagrange multipliers’.

7. Examples

Example 7.1 (*Mechanics on a General Dirac Algebroid*)**.** The very general scheme of the phase or the Euler–Lagrange dynamics on a Dirac algebroid $D \subset \mathcal{T}E^*$ can be described in local coordinates as follows. Let us choose the standard adapted coordinates (slightly reordered) $(x, \xi, \dot{x}, y, p, \dot{\xi})$ in $\mathcal{T}E^*$. Starting with a Lagrangian $L : E \to \mathbb{R}$ we can define the associated subset $[\mathrm{d}L]$ in $\mathcal{T}E^*$ as consisting of points with coordinates for which $\xi = \frac{\partial L}{\partial y}(x, y)$ and $p = -\frac{\partial L}{\partial x}(x, y)$. Next, we intersect $[\mathrm{d}L]$ with D getting the (implicit) Euler–Lagrange equations defined by the following relations (in coordinates of (4.6)):

$$\left(x, \frac{\partial L}{\partial y}(x, y)\right) \in \mathsf{Ph}_D, \quad \widehat{\eta}(x, \dot{x}, y) = 0, \tag{7.1}$$

$$\zeta_i\left(x, -\frac{\partial L}{\partial x}(x, y), \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y}(x, y)\right)\right) + c^j_{ik}(x)\eta^k(x, \dot{x}, y)\frac{\partial L}{\partial y^j}(x, y) = 0. \tag{7.2}$$

Similarly, starting with a Hamiltonian $H : E^* \to \mathbb{R}$ and defining the subset $[\mathrm{d}H]$ by putting the constraints $y = \frac{\partial H}{\partial \xi}(x, \xi)$ and $p = \frac{\partial H}{\partial x}(x, \xi)$, we get after intersecting with D the following (implicit) phase dynamics

$$(x, \xi) \in \mathsf{Ph}_D, \quad \widehat{\eta}\left(x, \dot{x}, \frac{\partial H}{\partial \xi}(x, \xi)\right) = 0, \tag{7.3}$$

$$\zeta_i\left(x, \frac{\partial H}{\partial x}(x, \xi), \dot{\xi}\right) + c^j_{ik}(x)\eta^k\left(x, \dot{x}, \frac{\partial H}{\partial \xi}(x, \xi)\right)\xi_j = 0. \tag{7.4}$$

For the canonical Dirac algebroid D_M we have in adapted coordinates $\widehat{\eta}^a = \dot{x}^a - y^a$, $\zeta_a = \dot{\xi}_a + p_a$, and $c^k_{ij} = 0$, so we get the standard Euler–Lagrange

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = y^a, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^a}\right)(x, y) = \frac{\partial L}{\partial x^a}(x, y)$$

and Hamilton

$$\frac{\mathrm{d}\xi_a}{\mathrm{d}t} = -\frac{\partial H}{\partial x^a}(x, \xi), \qquad \frac{\mathrm{d}x^b}{\mathrm{d}t} = \frac{\partial H}{\partial \xi_b}(x, \xi)$$

equations. Changing the symbols y, ξ for velocities and momenta into the standard ones, $\dot{x}$, p, we end up with the traditional Euler–Lagrange and Hamilton equations.

Example 7.2 (*Pontryagin Maximum Principle for General Dirac Algebroids*)**.** Starting with a general Dirac algebroid as above, let us impose a vakonomic constraint $C \subset E$ parametrized by $f : M \times U \to C$, with U being a manifold of ‘control parameters’. In local coordinates (x, y) in E and u in U, the parametrization yields $y = f(x, u)$. Note that, classically, for $E = \mathsf{T}M$ and $y = \dot{x}$, the constraint C represents the differential equation $\dot{x} = f(x, u)$.

A Lagrangian $L : C \to \mathbb{R}$ may be seen now as a function $L : M \times U \to \mathbb{R}$, and $[\mathrm{d}L_C]$ consists of points $(x, y, p, \xi) \in \mathsf{T}^*E$ (we skip the indices) such that

$$y = f(x, u), \qquad p = \frac{\partial L}{\partial x} - \xi\frac{\partial f}{\partial x}, \qquad \xi\frac{\partial f}{\partial u} = \frac{\partial L}{\partial u}. \tag{7.5}$$

The above identities define a subset $[\mathrm{d}L_C]$ in $\mathcal{T}E^*$ which, similarly as above, leads to implicit Euler–Lagrange equations

$$(x, \xi) \in \mathsf{Ph}_D, \quad \widehat{\eta}(x, \dot{x}, f(x, u)) = 0, \tag{7.6}$$

$$\zeta_i\left(x, \xi\frac{\partial f}{\partial x} - \frac{\partial L}{\partial x}(x, y), \dot{\xi}\right) + c^j_{ik}(x)\eta^k(x, \dot{x}, f(x, u))\xi = 0, \tag{7.7}$$

constrained additionally by

$$\xi\frac{\partial f}{\partial u} - \frac{\partial L}{\partial u} = 0. \tag{7.8}$$

Let us note that Eqs. (7.6) and (7.7) are the same as the Hamilton Eqs. (7.3) and (7.4) with the Hamiltonian

$$H(x, u, \xi) = \xi \cdot f(x, u) - L(x, u) \tag{7.9}$$

depending on the parameter u. Moreover, the Eq. (7.8) reads $\frac{\partial H}{\partial u}(x, u, \xi) = 0$ that is an infinitesimal form of the Pontryagin Maximum Principle (PMP): our solutions choose control parameters which are critical for the Hamiltonian. The whole picture is an obvious generalization of (PMP), this time for Dirac algebroids, of course in its smooth and infinitesimal version.

Example 7.3 (*Mechanics on Skew Algebroids*)**.** Consider the Dirac algebroid D_Π associated with a linear bivector field Π on E^*, as described in Example 3.1. Since in this case D_Π is the graph of the map $\widetilde{\Pi}$, the relation ε_{D_Π} is a map. Hence, $\Lambda^L_{D_\Pi} = \varepsilon_{D_\Pi} \circ \mathrm{d}L$ is also a map $\Lambda^L_{D_\Pi} : E \to \mathsf{T}E^*$ which in local coordinates reads

$$\Lambda^L_{D_\Pi}(x^a, y^i) = \left(x^a, \frac{\partial L}{\partial y^i}(x, y), \rho^b_k(x)y^k, c^k_{ij}(x)y^i\frac{\partial L}{\partial y^k}(x, y) + \rho^a_j(x)\frac{\partial L}{\partial x^a}(x, y)\right). \tag{7.10}$$

The Legendre relation $\lambda^L_{D_\Pi}$ is also a map which reads

$$\lambda^L_{D_\Pi}(x^a, y^i) = \left(x^a, \frac{\partial L}{\partial y^i}(x, y)\right). \tag{7.11}$$

Let $\gamma(t) = (x(t), y(t))$ be a smooth curve in E. Since $\widetilde{\gamma} = \Lambda^L_{D_\Pi} \circ \gamma$ is the only curve in $\mathsf{T}E^*$ which is $\Lambda^L_{D_\Pi}$-related to γ, the latter satisfies the Euler–Lagrange equation if and only if $\widetilde{\gamma}$ is admissible, i.e., $\widetilde{\gamma} = \dot{\underline{\widetilde{\gamma}}}$. In local coordinates,

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(x) = \rho^a_k(x)y^k, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right)(x, y) = c^k_{ij}(x)y^i\frac{\partial L}{\partial y^k}(x, y) + \rho^a_j(x)\frac{\partial L}{\partial x^a}(x, y), \tag{7.12}$$

in full agreement with the Euler–Lagrange equation for Lie (and general skew) algebroids as described in [23,17,8,9,6]. Note that we do not assume any regularity of the Lagrangian.

As for the Hamilton equations, let us note that also in this case the relation β_{D_Π} is a map, $\beta_{D_\Pi} = \widetilde{\Pi}$,

$$\Pi(x^a, \xi_j, p_b, y^i) = (x^a, \xi_j, \rho^b_k(x)y^k, c^k_{ij}(x)y^j\xi_k - \rho^a_j(x)p_a). \tag{7.13}$$

The corresponding phase dynamics are explicit and associated with the Hamiltonian vector field

$$\mathcal{X}_H(x, \xi) = \left(c^k_{ij}(x)\xi_k\frac{\partial H}{\partial \xi_i}(x, \xi) - \rho^a_j(x)\frac{\partial H}{\partial x^a}(x, \xi)\right)\partial_{\xi_j} + \rho^b_i(x)\frac{\partial H}{\partial \xi_i}(x, \xi)\partial_{x^b}, \tag{7.14}$$

i.e.,

$$\dot{\xi}_j = \left(c^k_{ij}(x)\xi_k\frac{\partial H}{\partial \xi_i}(x, \xi) - \rho^a_j(x)\frac{\partial H}{\partial x^a}(x, \xi)\right) \qquad \dot{x}^b = \rho^b_i(x)\frac{\partial H}{\partial \xi_i}(x, \xi).$$

In the particular case of the canonical Lie algebroid $E = \mathsf{T}M$, we can take for coordinates y in the fiber the coordinates $\dot{x}^a$ induced from the base. As now $c^a_{bc} = 0$ (coordinate vector fields commute) and $\rho^a_b = \delta^a_b$ (the anchor map is the identity), we get the traditional Euler–Lagrange equations

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \dot{x}^a, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}\right)(x, \dot{x}) = \frac{\partial L}{\partial x^a}(x, \dot{x}),$$

as a particular case. Also the Hamilton equations become completely traditional in coordinates ξ replaced by the corresponding momenta:

$$\dot{p}_a = -\frac{\partial H}{\partial x^a}(x, p), \qquad \dot{x}^b = \frac{\partial H}{\partial p_b}(x, p).$$

Example 7.4 (*Mechanics on Presymplectic Manifolds*)**.** Consider the Dirac algebroid D_ω associated with a linear 2-form ω on E^*, as described in Example 3.2. Since in this case D_ω is the graph of the map $\widetilde{\omega} : \mathsf{T}E^* \to \mathsf{T}^*E^* \simeq \mathsf{T}^*E$, the implicit phase dynamics associated with a Lagrangian and a Hamiltonian are inverse images of the images of dL and dH, respectively. In coordinates,

$$\beta_D[\mathrm{d}H] = \left\{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) : \rho^i_a(x)\dot{x}^a = \frac{\partial H}{\partial \xi_i}(x, \xi),\ c^k_{ab}(x)\xi_k\dot{x}^b - \rho^i_a(x)\dot{\xi}_i = \frac{\partial H}{\partial x^a}(x, \xi)\right\}$$

and

$$\varepsilon_D[\mathrm{d}L] = \left\{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) : \exists y \left[\xi_i = \frac{\partial L}{\partial y^i}(x, y), \rho^i_a(x)\dot{x}^a = y^i,\ c^k_{ab}(x)\xi_k\dot{x}^b - \rho^i_a(x)\dot{\xi}_i = \frac{\partial L}{\partial x^a}(x, y)\right]\right\}.$$

The implicit Euler–Lagrange equations (Euler–Lagrange relations) take the form

$$\rho^i_a(x)\frac{\mathrm{d}x^a}{\mathrm{d}t}(x) = y^i, \qquad \rho^i_a(x)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}\right)(x, y) = c^k_{ab}(x)\frac{\mathrm{d}x^b}{\mathrm{d}t}(x)\frac{\partial L}{\partial y^k}(x, y) - \frac{\partial L}{\partial x^a}(x, y). \tag{7.15}$$

Of course, for the canonical symplectic structure $\omega_M = \mathrm{d}p_a \wedge \mathrm{d}x^a$ on $E^* = \mathsf{T}^*M$ we get the classical dynamics as above. But also in the case of a regular presymplectic form of rank r,

$$\omega = \sum_{a\le r} \mathrm{d}p_a \wedge \mathrm{d}x^a,$$

we get the equations for the presymplectic reduction by the characteristic distribution to the reduced symplectic form: the coordinates x^a and $\dot{x}^a$ with $a > r$ are simply forgotten,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}\right)(x, \dot{x}) = -\frac{\partial L}{\partial x^a}(x, \dot{x}), \quad a \le r.$$

Example 7.5 (*Non-Autonomous Systems*)**.** Consider the affine Dirac algebroid D_0 on $E_0 = E \times \mathbb{R}$ described in Example 3.4, for the Π-graph Dirac algebroid $D = D_\Pi$ on E, as in Example 7.3. In coordinates,

$$D_0 = \{(x^0, x^a, \xi_i, \dot{x}^0, \dot{x}^b, \dot{\xi}_j, p_0, p_c, y^k) : \dot{x}^0 = 1,\ \dot{x}^b = \rho^b_k(x)y^k,\ \dot{\xi}_j = c^k_{ij}(x)y^i\xi_k - \rho^a_j(x)p_a\}.$$

For a Lagrangian $L : E \times \mathbb{R} \to \mathbb{R}$ we get the Tulczyjew differential $\Lambda^L_{D_0} : E_0 \longrightarrow\!\triangleright\ \mathsf{T}E^*_0$ of L which is the map which in coordinates reads

$$\Lambda^L_{D_0}(x^0, x^a, y^i) = (x^0, x^a, \xi_i, \dot{x}^0, \dot{x}^b, \dot{\xi}_j)$$

such that

$$\xi_i = \frac{\partial L}{\partial y^i}(x^0, x^a, y^i), \qquad \dot{x}^0 = 1, \qquad \dot{x}^b = \rho^b_k(x)y^k, \qquad \dot{\xi}_j = c^k_{lj}(x)y^l\frac{\partial L}{\partial y^k}(x^0, x^a, y^i) + \rho^b_j(x)\frac{\partial L}{\partial x^b}(x^0, x^a, y^i).$$

Identifying x^0 with the time parameter t, we get the corresponding Euler–Lagrange equations in the form

$$\frac{\mathrm{d}x^b}{\mathrm{d}t} = \rho^b_k(x)y^k, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}(t, x^a, y^i)\right) = c^k_{lj}(x)y^l\frac{\partial L}{\partial y^k}(t, x^a, y^i) + \rho^b_j(x)\frac{\partial L}{\partial x^b}(t, x^a, y^i).$$

This is exactly the Euler–Lagrange equation on a skew algebroid for time-dependent Lagrangians. Such equations have been obtained also as the Euler–Lagrange equations for affgebroids [30,31,33,34,28]. For the canonical Lie algebroid $E = \mathsf{T}M$, we get

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \dot{x}^a, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}(t, x, \dot{x})\right) = \frac{\partial L}{\partial x^a}(t, x, \dot{x}).$$

Example 7.6 (*Nonholonomic Constraints*)**.** Consider once more the Dirac algebroid D_Π associated with a linear bivector field Π on E^*, as described in Example 3.1. Consider also a *nonholonomic constraint* defined by a vector subbundle V of E supported on a submanifold $S \subset M$. Using coordinates $(x^a) = (x^\alpha, x^A)$ in M, so that S is given locally by $x^A = 0$, and linear coordinates (y^i) in the fibers of E, so that $y = (y^i) = (y^\iota, y^I)$ and the subbundle V is defined by the constraint $y^I = 0$, on $\mathcal{T}E^*$ we have then local coordinates $(x^a, \dot{x}^b, \dot{\xi}_I, p_c, y^\iota, y^I)$, with decompositions $(\xi_k) = (\xi_\kappa, \xi_K)$ and $(\dot{\xi}_I) = (\dot{\xi}_\lambda, \dot{\xi}_L)$ associated with the decomposition $(y^i) = (y^\iota, y^I)$. The Dirac algebroid induced from D_Π by the nonholonomic constraint V in local coordinates reads

$$D^V_\Pi = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : x^A = 0,\ \dot{x}^b = \rho^b_\iota(x) y^\iota,\ \dot{\xi}_\kappa = c^j_{\iota\kappa}(x) y^\iota \xi_j - \rho^a_\kappa(x) p_a,\ y^I = 0\}.$$

The Tulczyjew differential $\Lambda^L_{D^V_\Pi}$ associated with a Lagrangian $L : E \to \mathbb{R}$ is defined on V and associates with every $(x^\alpha, 0, y^\iota, 0) \in V$ the set

$$\Lambda^L_{D^V_\Pi}(x^\alpha, 0, y^\iota, 0) = \Big\{(x^\alpha, 0, \xi_i, \dot{x}^b, \dot{\xi}_j) \in \mathsf{T}E^* : \dot{x}^b = \rho^b_\iota(x) y^\iota,\ \xi_i = \frac{\partial L}{\partial y^i}(x^\alpha, 0, y^\iota, 0),$$
$$\dot{\xi}_\kappa = c^j_{\iota\kappa}(x) y^\iota \frac{\partial L}{\partial y^j}(x^\alpha, 0, y^\iota, 0) + \rho^a_\kappa(x) \frac{\partial L}{\partial x^a}(x^\alpha, 0, y^\iota, 0)\Big\}. \tag{7.16}$$

Note that the coordinates $\dot{\xi}_A$ of points from $\Lambda^L_{D^V_\Pi}(x^\alpha, 0, y^\iota, 0)$ are arbitrary. Curves $\Lambda^L_{D^V_\Pi}$-related to a curve $\gamma(t) = (x^\alpha(t), 0, y^\iota(t), 0)$ in V have thus arbitrary coordinates $\dot{\xi}_I$, but the remaining coordinates, if the curve is admissible, satisfy the *nonholonomically constrained Euler–Lagrange equations* (cf. [17,18,38]):

$$x^A = 0, \qquad y^I = 0, \qquad \frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho^a_\iota(x^\alpha, 0) y^\iota, \tag{7.17}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^\kappa}(x^\alpha, 0, y^\iota, 0)\right) = c^j_{\iota\kappa}(x^\alpha, 0) y^\iota \frac{\partial L}{\partial y^j}(x^\alpha, 0, y^\iota, 0) + \rho^a_\kappa(x^\alpha, 0) \frac{\partial L}{\partial x^a}(x^\alpha, 0, y^\iota, 0).$$

Note that a minimal integrability requirement is the first integrability condition for D^V_Π, saying that $\rho^A_\iota(x^\alpha, 0) = 0$.

The constraint V is *generalized holonomic* if, independently on the Lagrangian, the above equations depend on the restriction of L to V only. Hence, $c^I_{\iota\kappa}(x^\alpha, 0) = 0$ and $\rho^A_\kappa(x^\alpha, 0) = 0$, so that V is generalized holonomic if and only if V is a subalgebroid of E. In the classical situation of a canonical Lie algebroid $\mathsf{T}M$, the constraint V is generalized holonomic if and only if V is involutive, for instance $V = \mathsf{T}M_0$ for a submanifold M_0 in M. This is the traditional understanding of being holonomic.

Example 7.7 (*Affine Constraints*)**.** We can perform a similar procedure with an affine nonholonomic constraint instead of the linear one. Let us distinguish one variable y^0 from $y^I = (y^0, y^{\bar{I}})$ such that the affine constraint $A \subset E$ is defined by $x^A = 0,\ y^0 = 1,\ y^{\bar{I}} = 0$. The model vector bundle $V = \mathsf{v}(A)$ is as above and, as easily checked, the constrained Euler–Lagrange equations are

$$x^A = 0, \qquad y^0 = 1, \qquad y^I = 0, \qquad \frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho^a_0(x^\alpha, 0) + \rho^a_\iota(x^\alpha, 0) y^\iota,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^\kappa}(x^\alpha, 0, y^\iota, 0)\right) = (c^j_{0\kappa}(x) + c^j_{\iota\kappa}(x^\alpha, 0) y^\iota) \frac{\partial L}{\partial y^j}(x^\alpha, 0, y^\iota, 0) + \rho^a_\kappa(x^\alpha, 0) \frac{\partial L}{\partial x^a}(x^\alpha, 0, y^\iota, 0), \tag{7.18}$$

exactly as in [17].

Example 7.8 (*Rolling Disc*)**.** To show how our method of Dirac algebroid works for an explicit constrained system, let us reconsider the case of a vertical rolling disc on a plane studied in [24]. The position configuration space is the Lie group $N = \mathbb{R}^2 \times S^1 \times S^1$ with coordinates $(x^1, x^2, \theta, \varphi)$. The Lagrangian on $\mathsf{T}N$ in the adapted coordinates $(x^1, x^2, \theta, \varphi, \dot{x}^1, \dot{x}^2, \dot{\varphi}, \dot{\theta})$ reads

$$L(x^1, x^2, \varphi, \theta, \dot{x}^1, \dot{x}^2, \dot{\varphi}, \dot{\theta}) = \frac{1}{2} m((\dot{x}^1)^2 + (\dot{x}^2)^2) + \frac{1}{2} J_1 \dot{\varphi}^2 + \frac{1}{2} J_2 \dot{\theta}^2. \tag{7.19}$$

The kinematic constraint due to the rolling contact without slipping on the plane is

$$\dot{x}^1 = R\dot{\theta}\cos\varphi, \qquad \dot{x}^2 = R\dot{\theta}\sin\varphi.$$

Since the Lagrangian and the constraints are invariant with respect to translation with respect to x^1, x^2, θ, we have an obvious Lie algebroid reduction to $E = \mathsf{T}N/(\mathbb{R}^2 \times S^1) = \mathsf{T}\mathbb{R} \times \mathbb{R}^3$ which is a vector bundle of rank 4 over S^1 with coordinates

$$(\varphi, \dot{\varphi}, \dot{x}^1, \dot{x}^2, \dot{\theta}),$$

associated with the basis of (global) sections $(f_1 = \partial_\varphi, f_2, f_3, f_4)$, where f_2, f_3, f_4 come from the reductions of $\partial_\theta, \partial_{x^1}, \partial_{x^2}$, respectively. The anchor $\rho : E \to \mathsf{T}S^1$ is the projection onto $\mathsf{T}S^1$, and all the basic sections commute. We will also denote the reduced Lagrangian L, as it takes values exactly like in (7.19).

The constraint subbundle V of E is spanned by the sections $e_1 = f_1$ and $e_2 = f_2 + R\cos\varphi \cdot f_3 + R\sin\varphi \cdot f_4$, so we can use the basis $e_1 = f_1, e_2, e_3 = f_3, e_4 = f_4$, and the corresponding coordinates (φ, y) on E. The basis (e_1, e_2, e_3, e_4) induces the coordinate system (φ,ξ) in E^* and adapted coordinates $(\varphi,\xi,\dot\varphi,\dot\xi)$ in $\mathsf{T}E^*$ and (φ,ξ,p,y) in T^*E^*. The constraint is now described by the equations $y^3=y^4=0$, but we get non-trivial commutation relations

$$[e_1,e_2]=R\cos\varphi\cdot e_4-R\sin\varphi\cdot e_3.$$

In other words, the corresponding Poisson tensor Π on E^* in the adapted coordinates reads

$$\Pi=R(\cos\varphi\cdot\xi_4-\sin\varphi\cdot\xi_3)\partial_{\xi_1}\wedge\partial_{\xi_2}+\partial_{\xi_1}\wedge\partial_\varphi\,,$$

and the Dirac structure induced by the constraints is

$$\begin{aligned}D^V_\Pi=\{&(\varphi,\xi,\dot\varphi,\dot\xi,p,y):y^3=y^4=0,\dot\varphi=y^1,\\&\dot\xi_1=Ry^2\xi_3\sin\varphi-Ry^2\xi_4\cos\varphi-p,\dot\xi_2=-Ry^1\xi_3\sin\varphi+Ry^1\xi_4\cos\varphi\}.\end{aligned}\tag{7.20}$$

Hence, the nonholonomically constrained Euler–Lagrange Eq. (7.17) takes the form

$$\begin{aligned}&y^3=y^4=0,\qquad\frac{\mathrm{d}\varphi}{\mathrm{d}t}=y^1,\\&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^1}(\varphi,y^1,y^2,0,0)\right)=R\sin\varphi\cdot y^2\frac{\partial L}{\partial y^3}(\varphi,y^1,y^2,0,0)-R\cos\varphi\cdot y^2\frac{\partial L}{\partial y^4}(\varphi,y^1,y^2,0,0)\\&\qquad\qquad+\frac{\partial L}{\partial\varphi}(\varphi,y^1,y^2,0,0),\\&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^2}(\varphi,y^1,y^2,0,0)\right)=-R\sin\varphi\cdot y^1\frac{\partial L}{\partial y^3}(\varphi,y^1,y^2,0,0)+R\cos\varphi\cdot y^1\frac{\partial L}{\partial y^4}(\varphi,y^1,y^2,0,0).\end{aligned}\tag{7.21}$$

Since

$$\dot x^1=y^3+Ry^2\cos\varphi,\qquad\dot x^2=y^4+Ry^2\sin\varphi,\qquad\dot\varphi=y^1,\qquad\dot\theta=y^2,$$

the Lagrangian in coordinates (φ,y) reads

$$L(\varphi,y^1,y^2,y^3,y^4)=\frac12m((y^3)^2+(y^4)^2)+\frac12J_1(y^1)^2+\frac12(mR^2+J_2)(y^2)^2+mRy^2(y^3\cos\varphi+y^4\sin\varphi)$$

and, as shown in the straightforward calculations, the Euler–Lagrange Eq. (7.21) takes the form

$$y^3=y^4=0,\qquad\frac{\mathrm{d}\varphi}{\mathrm{d}t}=y^1,\qquad(mR^2+J_2)\frac{\mathrm{d}y^2}{\mathrm{d}t}=0,\qquad J_1\frac{\mathrm{d}y^1}{\mathrm{d}t}=0.\tag{7.22}$$

Going back to the original coordinates, we finally get

$$\dot x^1=R\dot\theta\cos\varphi,\qquad\dot x^2=R\dot\theta\sin\varphi,\qquad\ddot\varphi=0,\qquad\ddot\theta=0,\tag{7.23}$$

with obvious explicit solutions.

If the phase dynamics is concerned, in view of (7.16), we get that $\varepsilon_{D^V_\Pi}[\mathrm{d}L]=\Lambda^L_{D^V_\Pi}(V)$ is parametrized by (ϕ,y^1,y^2) as follows:

$$\begin{aligned}\varepsilon_{D^V_\Pi}[\mathrm{d}L]=\{&(\varphi,\xi_1,\xi_2,\xi_3,\xi_4,\dot\varphi,\dot\xi_1,\dot\xi_2,\dot\xi_3,\dot\xi_4):\xi_1=J_1y^1,\ \xi_2=(mR^2+J_2)y^2,\\&\xi_3=mRy^2\cos\varphi,\ \xi_4=mRy^2\sin\varphi,\ \dot\varphi=y^1,\ \dot\xi_1=0,\ \dot\xi_2=0\}.\end{aligned}$$

Here $\dot\xi_3$ and $\dot\xi_4$ are arbitrary, but the integrability condition allows us to describe them as well. Let us note that the phase space $\mathrm{Ph}^L_D\subset E^*$ is defined by the equations

$$\xi_3=\mu\cos\varphi\cdot\xi_2,\qquad\xi_4=\mu\sin\varphi\cdot\xi_2,\tag{7.24}$$

where $\mu=\frac{mR}{mR^2+J_2}$. Hence, the first integrability condition gives

$$\dot\xi_3=-\mu\xi_2\sin\varphi\cdot\dot\varphi=-\frac{\mu}{J_1}\xi_1\xi_2\sin\varphi,\qquad\dot\xi_4=\mu\xi_2\cos\varphi\cdot\dot\varphi=\frac{\mu}{J_1}\xi_1\xi_2\cos\varphi.$$

The dynamics are Hamiltonian, since the Hamiltonian

$$H(\varphi,\xi)=\frac{1}{2J_1}(\xi_1)^2+\frac{1}{2J_2}(\xi_2-R\xi_3\cos\varphi-R\xi_4\sin\varphi)^2+\frac{1}{2m}((\xi_3)^2+(\xi_4)^2)$$

defined on E^* induces the dynamics $\beta_{D_\Pi^v}[\mathrm{d}H] = \varepsilon_{D_\Pi^v}[\mathrm{d}L]$. The equality can be checked by straightforward calculations. Let us only note that, since

$$y^3 = \frac{\partial H}{\partial \xi_3} = -\frac{R}{J_2}\xi_2 \sin\varphi + \left(\frac{R^2}{J_2}\cos^2\varphi + \frac{1}{m}\right)\xi_3 + \frac{R^2}{J_2}\xi_4 \sin\varphi\cos\varphi,$$
$$y^4 = \frac{\partial H}{\partial \xi_4} = -\frac{R}{J_2}\xi_2 \sin\varphi + \frac{R^2}{J_2}\xi_3 \cos\varphi\sin\varphi + \left(\frac{R^2}{J_2}\sin^2\varphi + \frac{1}{m}\right)\xi_4,$$

imposing the conditions $y^3 = 0$, $y^4 = 0$ of the Dirac structure, we recover the Hamiltonian constraints (7.24).

8. Concluding remarks

We have introduced the concepts of Dirac and Dirac–Lie algebroid as a natural common generalization of a skew (resp., Lie) algebroid and a linear presymplectic structure. Aside from its interesting geometrical structure, Dirac algebroids, as well as their affine counterparts–affine Dirac algebroids, provide a powerful geometrical tool for description of mechanical systems by means of generalized Lagrangian and Hamiltonian formalisms.

The kinematic configurations (quasi-velocities) form in this framework a subset of a vector bundle $\tau : E \to M$ and are related to the actual velocities from $\mathsf{T}M$ by the so-called anchor relation, while the phase space is a subset of the dual bundle, E^*. The phase dynamics induced by a Lagrangian or a Hamiltonian is an implicit dynamics in the phase space described by a subset of the tangent bundle $\mathsf{T}E^*$, and the associated Euler–Lagrange equations are defined by an implicit dynamics in E.

We proposed a well-described procedure of inducing a new Dirac algebroid out of a given one by imposing certain linear constraints in the anchor relation (velocity bundle), that on the Lagrangian formalism level corresponds to imposing nonholonomic constraints. Since imposing constraints we end up in a Dirac algebroid again, our approach does not really distinguish between constrained and unconstrained systems, as well as between regular and singular Lagrangians. Since the use of algebroids already includes reductions to the picture, our approach covers all main examples of mechanical systems: regular or singular, constrained or not, autonomous or non-autonomous etc.

The Dirac algebroid, especially the Dirac–Lie algebroids, possess a rich and intriguing geometrical structure whose investigations have been started in the present paper. We are strongly convinced that these objects, as well as their possible generalization, will allow us to find a proper intrinsic framework also for field theories and other areas of mathematical physics.

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