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Local obstructions to projective surfaces admitting skew-symmetric Ricci tensor

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A B S T R A C T

The equation determining whether a projective structure admits a connection in its given projective class that has skew-symmetric Ricci tensor is an overdetermined system of semi-linear partial differential equations which we call the projective Einstein–Weyl (pEW) equation. In 2-dimensions, we give local obstructions for projective surfaces to admit such a connection in its projective class. The obstructions are the resultants of polynomial equations that have to be satisfied for there to admit any pEW solution.

1. Introduction

A projective structure on a smooth oriented manifold is an equivalence class of torsion-free affine connections that have the same unparameterised curves as geodesics. For projective manifolds $(M, [\nabla])$, a natural geometric problem is to find an affine connection in the projective class with Ricci tensor identically zero. This can be seen as a projective analogue of Einstein's equation and can be reformulated as solving an overdetermined system of linear partial differential equations, known as the projective Ricci-flat equation in [1]. A generalisation of the Ricci-flat condition is to find an affine connection in the projective class with the symmetric part of the Ricci tensor identically zero, i.e. the Ricci tensor is skew-symmetric. The overdetermined system of partial differential equations associated to this condition, which we call the projective Einstein–Weyl (pEW) equation, now becomes semi-linear. In this paper we derive local obstructions to the existence of solutions to the pEW equation on 2-dimensional projective manifolds. The obstructions are the resultants of polynomials with coefficients given by invariants of the projective structure. Computing these obstructions and checking that they do not vanish tell us that the projective surface does not admit any skew-symmetric Ricci tensor. In dimension 2, affine structures with skew-symmetric Ricci-tensor are of interest, as investigated in [2,3]. We first set up the closed system for the pEW equation in 2-dimensions, then derive the algebraic constraints that give rise to the obstructions. We conclude with two examples for which the obstructions do not vanish. Abstract index notation as explained in [4] is used throughout the paper to describe tensors on the manifold.

2. Projective differential geometry and the pEW equation

In this section we review the projective differential geometry needed for the results. More details can be found in [5]. Let $(M^n, [\nabla])$ be a smooth oriented n-dimensional manifold with a smooth projective structure $[\nabla]$ with $n \geq 2$. Two

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torsion-free affine connections ∇ and $\widehat{\nabla}$ are projectively equivalent (and in $[\nabla]$) if and only if they have the same geodesics up to reparameterisation. Equivalently,

$$\widehat{\nabla}_a\omega_b = \nabla_a\omega_b - \Upsilon_a\omega_b - \Upsilon_b\omega_a \tag{1}$$

for some 1-form Υ_a. The curvature of any affine connection $\nabla \in [\nabla]$ decomposes as follows:

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c\mathrm{P}_{bd} - \delta_b{}^c\mathrm{P}_{ad} - 2\mathrm{P}_{[ab]}\delta_d{}^c \tag{2}$$

where $R_{ab} = R_{ca}{}^c{}_b$ is the Ricci curvature and $\mathrm{P}_{ab} = \frac{n}{n^2-1}R_{ab} + \frac{1}{n^2-1}R_{ba}$ is the projective rho tensor. The Ricci tensor (and the projective rho tensor P_{ab}) is not necessarily symmetric. The totally trace-free part of $R_{ab}{}^c{}_d$ denoted by $W_{ab}{}^c{}_d$ is called the projective Weyl tensor and is a projective invariant, that is to say it is a universal polynomial in the jets of the projective structure that is invariantly defined as a weighted tensor (the notion of projective weights is discussed later on). A projectively invariant quantity does not change under a projective change of connection. In dimension 2, the Weyl tensor $W_{ab}{}^c{}_d$ vanishes by symmetry considerations. The skew-symmetric part of the projective rho tensor $\mathrm{P}_{[ab]}$ is called the Faraday 2-form F_{ab}. It is always closed as a consequence of the Bianchi identity. The condition of a projective structure admitting skew-symmetric Ricci tensor is equivalent to finding a torsion-free affine connection $D \in [\nabla]$ such that the symmetric part of its projective rho tensor can be made to vanish, i.e. $\mathrm{P}_{(ab)} = 0$. By definition, this condition depends only on the projective class and is therefore projectively invariant. In this case $\mathrm{P}_{ab} = \mathrm{P}_{[ab]} = F_{ab}$ and formula (2) reduces to

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^cF_{bd} - \delta_b{}^cF_{ad} - 2F_{ab}\delta_d{}^c.$$

Choosing any ∇ from $[\nabla]$, we have

$$D_a\omega_b = \nabla_a\omega_b + \alpha_a\omega_b + \alpha_b\omega_a \tag{3}$$

for some 1-form α_a by the projective transformation formula (1). We can use the background connection ∇_a and its associated curvature to write down conditions on α_a to admit skew-symmetric Ricci tensor. This gives the pEW equation. We shall work with a preferred background connection ∇_a called special that will be explained in the next section.

2.1. Projective densities and special connections

The projective transformation formula (1) extends to connections acting on n-forms by the Leibniz rule; in particular, any volume form $\omega_{bc\ldots d} \in \Gamma(\Lambda^n)$ is an n-form and we have

$$\widehat{\nabla}_a\omega_{bc\ldots d} = \nabla_a\omega_{bc\ldots d} - (n+1)\Upsilon_a\omega_{bc\ldots d}. \tag{4}$$

In the case that M^n is oriented, we define the density line bundle $\mathcal{E}(w)$ to be the $-\frac{w}{n+1}$-th root of Λ^n (see [5] for more details). Sections of $\mathcal{E}(w)$ are projective densities of weight w. Given σ a section of $\mathcal{E}(w)$, we have under a projective change of connection that

$$\widehat{\nabla}_a\sigma = \nabla_a\sigma + w\Upsilon_a\sigma.$$

By definition, $\mathcal{E}(-(n+1)) = \Lambda^n = \mathcal{E}_{[bc\ldots d]}$ and the line bundle $\mathcal{E}_{[bc\ldots d]}(n+1)$ is trivial. Let $\epsilon_{bc\ldots d}$ be the tautological section of this bundle, which satisfies $\nabla_a\epsilon_{bc\ldots d} = 0$. The inverse of the tautological form $\epsilon^{bc\ldots d}$ identifies the bundle of n-forms Λ^n with the density line bundle $\mathcal{E}(-(n+1))$. For projective structures there is always a connection in $[\nabla]$ that we can choose to preserve any given volume form $\omega_{bc\ldots d}$ locally (this connection is necessarily flat on the bundle Λ^n). To see this, given any connection $\nabla \in [\nabla]$, we have $\nabla_a\omega_{bc\ldots d} = \mu_a\omega_{bc\ldots d}$ for some 1-form μ_a, since the bundle of volume forms is 1-dimensional. Now take the projective transformation to be given by $\Upsilon_a = \frac{1}{n+1}\mu_a$, and by (4) we have $\widehat{\nabla}_a\omega_{bc\ldots d} = 0$, where $\widehat{\nabla}$ is projectively related to the ∇ we started with. Then relabel $\widehat{\nabla}$ as ∇ to get a parallel volume form. For a chosen volume form $\omega_{bc\ldots d}$ the associated connection will be called special. A special connection has the property that its Ricci tensor is symmetric. Given any projective structure, we can always locally restrict to this class of special connections and the projective changes will be restricted to locally exact 1-forms Υ_a. Note however that the connection D_a we are looking for with skew-symmetric Ricci tensor does not lie in the special class of connections. A choice of volume form $\omega_{bc\ldots d}$ is often referred to as a choice of scale. Any other volume form consistent with the orientation is of the form $\Omega^{n+1}\omega_{bc\ldots d}$ for some positive smooth function Ω and then by (4) the transformation rule (1) holds with $\Upsilon_a = \nabla_a \log \Omega$.

2.2. The pEW equation in 2 dimensions

In the 2-dimensional setting, we shall take the tautological section (weighted volume form) ϵ_{ab} of weight 3 to satisfy $\epsilon^{ac}\epsilon_{bc} = \delta_b{}^a$ so that $\epsilon^{ab}\epsilon_{ab} = 2! = 2$. We shall restrict to the class of special connections so that the projective rho tensor $\mathrm{P}_{ab} = \mathrm{P}_{(ab)}$ is now symmetric. The condition of a projective surface admitting a torsion-free affine connection with skew-symmetric Ricci tensor in the projective class is equivalent to solving for the pEW equation

$$\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} = 0 \tag{5}$$

for ∇_a the special connection in the projective class and α_a the 1-form given in (3). Under projective transformations, α_a changes by a gauge $\widehat{\alpha}_a = \alpha_a + \Upsilon_a$ where now Υ_a is locally exact. The pEW equation is an overdetermined system of semilinear PDEs, and specialises to the projective Ricci-flat (or projective to Einstein) equation when $\nabla_{[a}\alpha_{b]} = 0$ (i.e. α_a is locally exact). We note that in dimension 2, the projective Cotton–York tensor defined by $Y_{abc} := \nabla_a\mathrm{P}_{bc} - \nabla_b\mathrm{P}_{ac}$ is projectively invariant. Using the inverse weighted volume form to dualise, we can write $Y_{abc} = \frac{1}{2}\epsilon_{ab}Y_c$, where $Y_a = \epsilon^{bc}Y_{bca}$ is projectively invariant of weight -3. The vanishing of Y_a characterises projectively flat surfaces. Using the weighted volume form to raise and lower indices, we have $Y^a = \epsilon^{ab}Y_b$, $Y_b = Y^a\epsilon_{ab}$. Let

$$\phi = 2\nabla_a Y^a.$$

The scalar quantity ϕ has projective weight -6 and transforms projectively as $\widehat{\phi} = \phi - 6\Upsilon_a Y^a$. Introduce the vector $W^a = Y^b\nabla_b Y^a - \frac{2\phi}{3}Y^a$. This is projectively invariant of weight -12. To see this, we have

$$\begin{aligned}\widehat{W}^b = \widehat{Y^a\nabla_a Y^b} - \frac{2}{3}\widehat{\phi}\widehat{Y}^b &= Y^a\nabla_a Y^b - 4(\Upsilon_a Y^a)Y^b - \frac{2}{3}(\phi - 6\Upsilon_a Y^a)Y^b \\ &= Y^a\nabla_a Y^b - \frac{2}{3}\phi Y^b \\ &= W^b.\end{aligned}$$

The density $\rho := Y_a W^a$ is projectively invariant of weight -15.

2.3. Deriving the closed system for pEW on projective surfaces

A common procedure to treat equations such as (5) is through prolongation [6]. This involves expressing first derivatives of the dependent variables in terms of the variables themselves. Introduce $F_{ab} = \nabla_{[a}\alpha_{b]}$ as the extra dependent variable. Using the weighted volume form to dualise, we can write $F_{ab} = \frac{1}{2}\epsilon_{ab}F$ where $F = \epsilon^{ab}F_{ab}$ is a projective scalar density of projective weight -3. We can rewrite (5) as

$$\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} = \frac{1}{2}\epsilon_{ab}F. \tag{6}$$

Differentiating (6) and using (6) to eliminate derivatives of α_a gives

$$\frac{1}{2}\epsilon_{ab}\nabla_c F = \nabla_c\nabla_a\alpha_b + \frac{1}{2}\epsilon_{ca}F\alpha_b - \mathrm{P}_{ca}\alpha_b + \frac{1}{2}\epsilon_{cb}F\alpha_a - \alpha_a\mathrm{P}_{cb} - 2\alpha_a\alpha_b\alpha_c + \nabla_c\mathrm{P}_{ab}.$$

Skewing with the weighted volume form ϵ^{ac}, we find that

$$\nabla_a F = -3F\alpha_a - Y_a \tag{7}$$

is a consequence of the original equation. Eqs. (6) and (7) form the first order closed system for the pEW equation, and from this we can derive algebraic constraints for (5) to hold by further differentiating the system.

3. Statement of results

In the flat case when $Y_a = 0$, we necessarily have $F = 0$ by further differentiating (7) and skewing (see Eq. (11)). The 1-form α_a is therefore exact, and Eq. (5) specialises to the projective Ricci-flat equation. We shall now restrict our attention to non-flat projective surfaces, that is one with Y_a non-zero. This ensures that $F \neq 0$. It turns out in deriving the constraint equations for (5) to hold we have to distinguish between the cases where ρ vanishes or not. Let $\mathrm{Res}(P(t), Q(t))$ be the resultant of polynomials $P(t)$ and $Q(t)$ in the single variable t. $\mathrm{Res}(P(t), Q(t)) = 0$ is a necessary and sufficient condition for $P(t)$ and $Q(t)$ to share a common root. We have the following.

Theorem 3.1. *Let $(M^2, [\nabla])$ be a projective surface with $\rho \neq 0$. Suppose M^2 admits a solution to (5). Then there exist polynomials $P_1(t), P_2(t), P_3(t)$ in the single variable t with coefficients given by invariants of the projective structure such that when $t = F$,*

$$P_1(F) = P_2(F) = P_3(F) = 0$$

must hold.

The polynomial constraints $P_1(F) = P_2(F) = P_3(F) = 0$ are explicitly computed in Section 4. As a corollary, we obtain local obstructions for there to be solutions for (5).

Corollary 3.2. *Let $(M^2, [\nabla])$ be a projective surface with $\rho \neq 0$. Suppose M^2 admits a solution to (5). Then*

$$Res(P_1(t), P_2(t)) = Res(P_2(t), P_3(t)) = Res(P_1(t), P_3(t)) = 0$$

must hold.

The case where $\rho = 0$ is discussed in Section 5. We have the following.

Theorem 3.3. *Let $(M^2, [\nabla])$ be a projective structure with $\rho = 0$. Suppose M^2 admits a solution to (5). Then*

$$15m^2 + 3fmk - hk^2 = 0, \tag{8}$$

$$kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m) = 0 \tag{9}$$

must hold, where f, h, k, m are further quantities obtained from the projective structure on M^2 to be defined later in Section 5.

4. Proof of Theorem 3.1

We shall now derive the polynomial equations $P_1(F) = 0, P_2(F) = 0, P_3(F) = 0$ that arise for (5) to hold. Differentiating (7) and using the closed system for the pEW equation gives

$$\nabla_a \nabla_b F = 9F\alpha_a\alpha_b + 3Y_a\alpha_b - \frac{3}{2}\epsilon_{ab}F^2 + 3\alpha_a\alpha_b F + 3\mathrm{P}_{ab}F - \nabla_a Y_b \tag{10}$$

and skewing with the weighted volume form ϵ^{ab} gives

$$3\alpha_a Y^a + 3F^2 + \nabla_a Y^a = 0 \tag{11}$$

as the first constraint of the system and we can rewrite Eq. (11) as

$$\alpha_a Y^a = -F^2 - \frac{\phi}{6}. \tag{12}$$

Differentiating (12) and using (6), (7) and (12), we find that

$$\alpha_a W^a = -5F^4 + \frac{5}{36}\phi^2 + \mathrm{P}_{ab}Y^aY^b - \frac{Y^a\nabla_a\phi}{6}.$$

Let

$$\ell = \frac{5\phi^2}{12} + 3\mathrm{P}_{ac}Y^aY^c - \frac{Y^a\nabla_a\phi}{2}.$$

We thus have

$$\alpha_a W^a = -5F^4 + \frac{\ell}{3} \tag{13}$$

as our second constraint. We can now solve for α_a assuming that ρ is non-zero. It is given by

$$\alpha_a = \frac{1}{3\rho}\left(\frac{\phi}{2} + 3F^2\right)W_a - \frac{1}{3\rho}\left(15F^4 - \ell\right)Y_a. \tag{14}$$

Substituting (14) back into Eq. (5) yields further constraints on F. They are polynomial equations $P_1(F) = 0$, $P_2(F) = 0$, $P_3(F) = 0$ that arise for (5) to hold. The first polynomial constraint $P_1(F) = 0$ comes from computing $F = \nabla_a\alpha^a$ using (14). It is given by

$$\begin{aligned}
P_1(F) &= -90F^6 + 15\left(\frac{Y^a\nabla_a\rho}{\rho} - \frac{5\phi}{2}\right)F^4 - \left(\frac{3W^a\nabla_a\rho}{\rho} + 6\ell - 3\nabla_a W^a\right)F^2 \\
&\quad - 9\rho F + \left(\frac{W^a\nabla_a\phi}{2} + \frac{\phi}{2}\nabla_a W^a + Y^a\nabla_a\ell + \frac{\phi\ell}{2} - \frac{\phi W^a\nabla_a\rho}{2\rho} - \ell\frac{Y^a\nabla_a\rho}{\rho}\right) \\
&= 0.
\end{aligned}$$

It can be verified that the coefficients appearing in the polynomial $P_1(t)$ are all projectively invariant. For example, under projective rescalings, the coefficient of the degree 4 term in $P_1(t)$ transforms as follows:

$$\begin{aligned}
15\left(\widehat{\frac{Y^a\nabla_a\rho}{\rho}} - \frac{5\phi}{2}\right) = 15\left(\frac{Y^a\widehat{\nabla}_a\rho}{\rho} - \frac{5\widehat{\phi}}{2}\right) &= 15\left(\frac{Y^a\nabla_a\rho - 15\Upsilon_aY^a\rho}{\rho} - \frac{5(\phi - 6\Upsilon_aY^a)}{2}\right) \\
&= 15\left(\frac{Y^a\nabla_a\rho}{\rho} - 15\Upsilon_aY^a - \frac{5\phi}{2} + 15\Upsilon_aY^a\right) \\
&= 15\left(\frac{Y^a\nabla_a\rho}{\rho} - \frac{5\phi}{2}\right).
\end{aligned}$$

It is therefore projectively invariant. The second and third polynomial constraints come from substituting (14) back into (5) and contracting with W^aW^b and W^aY^b respectively. Another possible contraction with Y^aY^b yields an equation that is identically zero. Evaluating $W^aW^b\nabla_a\alpha_b + \alpha_a\alpha_bW^aW^b + \mathrm{P}_{ab}W^aW^b = 0$ for α_a in (14) gives

$$\begin{aligned}
P_2(F) &= -275F^8 + \left(\frac{-5W^eY^d\nabla_eW_d}{\rho} + \frac{50\ell}{3}\right)F^4 + 20\rho F^3 + \left(\frac{W^eW^a\nabla_eW_a}{\rho}\right)F^2 \\
&\quad + \frac{\phi W^eW^a\nabla_eW_a}{6\rho} + \mathrm{P}_{ea}W^eW^a + \frac{\ell^2}{9} + \frac{\ell W^eY^d\nabla_eW_d}{3\rho} + \frac{W^e\nabla_e\ell}{3} \\
&= 0,
\end{aligned}$$

while evaluating $W^aY^b\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b W^aY^b + \mathrm{P}_{ab}W^aY^b = 0$ gives

$$\begin{aligned} P_3(F) &= -40F^6 + \left(\frac{-5Y^a(W^e\nabla_eY_a + Y^e\nabla_aW_e)}{2\rho} - \frac{25\phi}{6}\right)F^4 \\ &\quad + \left(\frac{2\ell}{3} + \frac{W^e(W^d\nabla_eY_d + Y^d\nabla_dW_e)}{2\rho}\right)F^2 + \rho F - \frac{W^e\nabla_e\phi}{12} + \frac{\phi W^bW^e\nabla_eY_b}{12\rho} \\ &\quad + \frac{\ell W^eY^a\nabla_eY_a}{6\rho} + \mathrm{P}_{ae}W^eY^a + \frac{Y^e\nabla_e\ell}{6} + \frac{\phi W^eY^a\nabla_aW_e}{12\rho} - \frac{\ell\phi}{18} + \frac{\ell Y^bY^a\nabla_aW_b}{6\rho} \\ &= 0. \end{aligned}$$

Replacing F with the indeterminate t, we obtain polynomials $P_1(t)$, $P_2(t)$, $P_3(t)$ with coefficients given by projectively invariant densities. This proves Theorem 3.1. We shall now explain a more concise way of extracting the obstructions.

4.1. Concise way of extracting obstructions

We can eliminate the single odd degree term so that even degree terms remain in $P_1(t), P_2(t), P_3(t)$. Namely, define

$$\begin{aligned} &Q_1(t^2) = -20t^2P_3(t) + P_2(t), \qquad Q_2(t^2) = -9P_3(t) - P_1(t), \\ &Q_3(t^2) = -\frac{20}{9}t^2P_1(t) - P_2(t). \end{aligned}$$

Then the three polynomials $Q_1(t^2)$, $Q_2(t^2)$, $Q_3(t^2)$ will be quartic, cubic and quartic polynomials of t^2 respectively since only even powers of t remain. This allows the obstructions to be extracted easily since now the resultant of any of these 2 polynomials will at most be the determinant of a 8 by 8 matrix. Let

$$\begin{aligned} P_1(t) &= -90t^6 + a_1t^4 + a_2t^2 - 9\rho t + a_3, \\ P_2(t) &= -275t^8 + b_1t^4 + 20\rho t^3 + b_2t^2 + b_3, \\ P_3(t) &= -40t^6 + c_1t^4 + c_2t^2 + \rho t + c_3, \end{aligned}$$

where

$$\begin{aligned} a_1 &= 15\left(\frac{Y^a\nabla_a\rho}{\rho} - \frac{5\phi}{2}\right) \\ a_2 &= -\left(\frac{3W^a\nabla_a\rho}{\rho} + 6\ell - 3\nabla_aW^a\right) \\ a_3 &= \left(\frac{W^a\nabla_a\phi}{2} + \frac{\phi}{2}\nabla_aW^a + Y^a\nabla_a\ell + \frac{\phi\ell}{2}\right) - \frac{\phi W^a\nabla_a\rho}{2\rho} - \ell\frac{Y^a\nabla_a\rho}{\rho} \\ b_1 &= \frac{-5W^eY^d\nabla_eW_d}{\rho} + \frac{50\ell}{3} \\ b_2 &= \frac{W^eW^a\nabla_eW_a}{\rho} \\ b_3 &= \frac{\phi W^eW^a\nabla_eW_a}{6\rho} + \mathrm{P}_{ea}W^eW^a + \frac{\ell^2}{9} + \frac{\ell W^eY^d\nabla_eW_d}{3\rho} + \frac{W^e\nabla_e\ell}{3} \\ c_1 &= \frac{-5Y^a(W^e\nabla_eY_a + Y^e\nabla_aW_e)}{2\rho} - \frac{25\phi}{6} \\ c_2 &= \frac{2\ell}{3} + \frac{W^e(W^d\nabla_eY_d + Y^d\nabla_dW_e)}{2\rho} \\ c_3 &= -\frac{W^e\nabla_e\phi}{12} + \frac{\phi W^bW^e\nabla_eY_b}{12\rho} + \frac{\ell W^eY^a\nabla_eY_a}{6\rho} + \mathrm{P}_{ae}W^eY^a + \frac{Y^e\nabla_e\ell}{6} + \frac{\phi W^eY^a\nabla_aW_e}{12\rho} - \frac{\ell\phi}{18} + \frac{\ell Y^bY^a\nabla_aW_b}{6\rho}, \end{aligned}$$

then a computation gives

$$\begin{aligned} Q_1(X) &= 525X^4 - 20c_1X^3 + (b_1 - 20c_2)X^2 + (b_2 - 20c_3)X + b_3, \\ Q_2(X) &= 450X^3 - (9c_1 + a_1)X^2 - (9c_2 + a_2)X - (9c_3 + a_3), \\ Q_3(X) &= 475X^4 - \frac{20}{9}a_1X^3 - \left(\frac{20}{9}a_2 + b_1\right)X^2 - \left(\frac{20}{9}a_3 + b_2\right)X - b_3, \end{aligned}$$

where $X = t^2$. The local obstructions are therefore

$$\begin{aligned} Q_{12} &= \mathrm{Res}(Q_1(X), Q_2(X)) \\ &= \begin{vmatrix} 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 & 0 & 0 \\ 0 & 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 & 0 \\ 0 & 0 & 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 \\ 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) & 0 & 0 & 0 \\ 0 & 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) & 0 & 0 \\ 0 & 0 & 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) & 0 \\ 0 & 0 & 0 & 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) \end{vmatrix}, \end{aligned}$$

$$\begin{aligned} Q_{23} &= \mathrm{Res}(Q_2(X), Q_3(X)) \\ &= \begin{vmatrix} 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) & 0 & 0 & 0 \\ 0 & 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) & 0 & 0 \\ 0 & 0 & 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) & 0 \\ 0 & 0 & 0 & 450 & -(9c_1+a_1) & -(9c_2+a_2) & -(9c_3+a_3) \\ 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 & 0 & 0 \\ 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 & 0 \\ 0 & 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 \end{vmatrix}, \end{aligned}$$

and

$$\begin{aligned} Q_{13} &= \mathrm{Res}(Q_1(X), Q_3(X)) \\ &= \begin{vmatrix} 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 & 0 & 0 & 0 \\ 0 & 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 & 0 & 0 \\ 0 & 0 & 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 & 0 \\ 0 & 0 & 0 & 525 & -20c_1 & (b_1-20c_2) & (b_2-20c_3) & b_3 \\ 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 & 0 & 0 & 0 \\ 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 & 0 & 0 \\ 0 & 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 & 0 \\ 0 & 0 & 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2+b_1\right) & -\left(\frac{20}{9}a_3+b_2\right) & -b_3 \end{vmatrix}, \end{aligned}$$

and the knowledge of the values of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ at a point will let us determine the values of the obstruction at that point.

5. Proof of Theorem 3.3

In the case where $\rho = 0$, the vectors W^a and Y^a are linearly dependent everywhere on M^2 and we can express $W^a = fY^a$ for some density f of weight -6. Under the conditions that both $Y^a \neq 0$ and $\rho = 0$ hold on the projective structure, the quantity f is projectively invariant but not in the classical sense since it is rational in the jets of the projective structure. Constraint (13) becomes

$$f\alpha_a Y^a = -5F^4 + \frac{\ell}{3},$$

upon which multiplying (12) by f and eliminating $\alpha_a Y^a$ gives the quartic equation

$$15F^4 - 3fF^2 - \left(\ell + \frac{f\phi}{2}\right) = 0. \tag{15}$$

Let $h = \ell + \frac{f\phi}{2}$. Differentiating (15) gives

$$60F^3(\nabla_a F) - 6fF(\nabla_a F) - 3(\nabla_a f)F^2 - \nabla_a h = 0,$$

which we contract with Y^a and use (7) and (12) to get

$$180F^6 + (30\phi - 18f)F^4 - 3(\phi f + Y^a\nabla_a f)F^2 - Y^a\nabla_a h = 0. \tag{16}$$

Multiplying (15) by $12F^2$ gives

$$180F^6 - 36fF^4 - 12hF^2 = 0,$$

which can be used to eliminate the term of degree 6 in (16) to obtain

$$(30\phi + 18f)F^4 - 3(\phi f + Y^a\nabla_a f - 4h)F^2 - Y^a\nabla_a h = 0. \tag{17}$$

Using (15) once more, we can further eliminate the term of degree 4 in (17) to obtain the quadratic equation

$$\left(3\phi f - 3Y^a\nabla_a f + 12h + \frac{18}{5}f^2\right)F^2 + \frac{6hf}{5} - Y^a\nabla_a h + 2\phi h = 0. \tag{18}$$

Let us call

$$k = 3\phi f - 3Y^a\nabla_a f + 12h + \frac{18}{5}f^2,$$

$$m = \frac{6hf}{5} - Y^a\nabla_a h + 2\phi h,$$

so that (18) becomes

$$kF^2 + m = 0. \tag{19}$$

Substituting $F^2 = -\frac{m}{k}$ given by (19) into (15) gives

$$15\left(\frac{m}{k}\right)^2 + 3f\left(\frac{m}{k}\right) - h = 0,$$

and we clear the denominator k^2 to obtain the vanishing of the first obstruction (8) in Theorem 3.3. Differentiating (19) gives

$$(\nabla_a k)F^2 + 2Fk\nabla_a F + \nabla_a m = 0,$$

which contracted into Y^a and again using (7) and (12) gives

$$6kF^4 + (k\phi + Y^a\nabla_a k)F^2 + Y^a\nabla_a m = 0.$$

Again substituting (19) and clearing denominator k yields

$$6m^2 - m(k\phi + Y^a\nabla_a k) + kY^a\nabla_a m = kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m) = 0,$$

which is the desired second obstruction (9) in Theorem 3.3.

6. Examples

In this section we give 2 different projective structures on $\mathbb{R}^2$ that yield non-vanishing obstructions, one with the projective invariant $\rho \neq 0$ and the other with $\rho = 0$.

6.1. *Example with $\rho \neq 0$ and non-vanishing obstruction*

This projective structure on $\mathbb{R}^2$ has the connection coefficients given by

$$\Pi^1_{22} = xy, \qquad \Pi^2_{11} = -y, \qquad \Pi^1_{11} = \Pi^1_{21} = \Pi^2_{12} = \Pi^2_{22} = 0.$$

We compute the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ at an arbitrary point $p \in \mathbb{R}^2$ where $\rho(p) \neq 0$. Taking p to be given in local coordinates by $(x, y) = (1, 1)$, we find $\rho(p) = 328$, and the polynomials at the point p are given by

$$P_1(t) = -90t^6 + \frac{185760}{328}t^4 - \frac{528608}{328}t^2 - 2952t - \frac{134912}{328},$$

$$P_2(t) = -275t^8 + \frac{13774080}{2952}t^4 + 6560t^3 - \frac{601856}{8856}t^2 + \frac{523957248}{26568},$$

$$P_3(t) = -40t^6 + \frac{30960}{328}t^4 + \frac{125360}{984}t^2 + 328t + \frac{31603200}{26568}.$$

Using the software MAPLE, we find that

$$Q_{12}(p) = -\frac{14578904595741615923392000000}{1681},$$

$$Q_{13}(p) = -\frac{188610437798501965389961756672000000000}{452190681},$$

$$Q_{23}(p) = \frac{14578904595741615923392000000}{1681}.$$

At the point p and hence near it the quantities $\rho(p)$, $Q_{12}(p)$, $Q_{13}(p)$ and $Q_{23}(p)$ are non-zero, and so there are no solutions to pEW near p. Moreover, since $Q_{12}(p)$, $Q_{13}(p)$ and $Q_{23}(p)$ are polynomials in p we can conclude that there is no solution to pEW anywhere on $\mathbb{R}^2$, as the set where these polynomials are non-zero is Zariski open and hence dense in $\mathbb{R}^2$.

6.2. *Example with $\rho = 0$ and non-vanishing obstruction*

This projective structure on $\mathbb{R}^2$ has the connection coefficients given by

$$\Pi_{11}^1 = -\frac{x^2}{6}, \qquad \Pi_{22}^1 = -\frac{x^2}{2}, \qquad \Pi_{21}^2 = \frac{x^2}{6}, \qquad \Pi_{11}^2 = \Pi_{21}^1 = \Pi_{22}^2 = 0.$$

A computation shows that $\rho = 0$. With the help of MATLAB, we find that the obstructions are

$$\begin{aligned} 15m^2 + 3fmk - hk^2 = {} & \frac{208}{5}x^{30} + \frac{4248}{5}x^{27} - 768x^{24} - \frac{349776}{5}x^{21} + 61008x^{18} + 1468824x^{15} \\ & - 1651200x^{12} - 1164000x^9 + 1543200x^6 - 212000x^3 + 24000 \end{aligned}$$

and

$$\begin{aligned} k(Y^a\nabla_a m) - m(Y^a\nabla_a k + k\phi - 6m) = {} & \frac{32}{75}x^{30} + \frac{144}{25}x^{27} - 208x^{24} - \frac{117408}{25}x^{21} + \frac{311104}{5}x^{18} + 136208x^{15} \\ & + 71536x^{12} - 996160x^9 + 978560x^6 - \frac{824000}{3}x^3 + 27200. \end{aligned}$$

We conclude that this projective structure does not admit any solution to pEW locally.

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