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Enumeration of curves with one singular point

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ABSTRACT

In this paper we obtain an explicit formula for the number of curves in $\mathbb{P}^2$, of degree d, passing through $(d(d+3)/2-k)$ generic points and having a singularity $\mathfrak{X}$, where $\mathfrak{X}$ is of type $A_{k\le 7}$, $D_{k\le 7}$ or $E_{k\le 7}$. Our method comprises of expressing the enumerative problem as the Euler class of an appropriate bundle and using a purely topological method to compute the degenerate contribution to the Euler class. These numbers have also been computed by M. Kazarian using the existence of universal formulas for Thom polynomials.

1. Introduction

Enumerative geometry is a branch of mathematics concerned with the following question:

How many geometric objects are there which satisfy prescribed constraints?

A well known class of enumerative problems is that of singular curves in $\mathbb{P}^2$ (complex projective plane) passing through the appropriate number of points. This question has been studied by algebraic geometers for a long time. However, in this paper we use topological methods to tackle this problem. Although all our spaces and vector bundles will be algebraic, we will often be considering C^∞-sections of these bundles as opposed to *holomorphic* sections.

If p is a non-zero vector in $\mathbb{C}^3$, we will denote its equivalence class in $\mathbb{P}^2$ by $\tilde{p}$.[1]

Definition 1.1. Let f be a homogeneous degree d polynomial in three variables and $\tilde{p} \in \mathbb{P}^2$. A point $\tilde{p} \in f^{-1}(0)$ is of singularity type A_k, D_k, E_6, E_7, or E_8 if there exists a locally analytic coordinate system $(x, y) : (\mathcal{U}, \tilde{p}) \to (\mathbb{C}^2, 0)$ such that $f^{-1}(0) \cap \mathcal{U}$ is given by

$$A_k : y^2 + x^{k+1} = 0 \quad k \ge 0, \qquad D_k : y^2x + x^{k-1} = 0 \quad k \ge 4,$$
$$E_6 : y^3 + x^4 = 0, \qquad E_7 : y^3 + yx^3 = 0, \qquad E_8 : y^3 + x^5 = 0.$$

In more common terminology, $\tilde{p}$ is a *smooth* point of $f^{-1}(0)$ if it is a singularity of type A_0; a *simple node* if its singularity type is A_1; a *cusp* if its type is A_2; a *tacnode* if its type is A_3; an *ordinary triple point* if its type is D_4.

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[1] In this paper we will use the symbol $\tilde{A}$ to denote the equivalence class of A instead of the standard $[A]$. This will make some of the calculations in Section 4 easier to read.

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Remark 1.2. Before stating the main result of this paper, let us set up a practical terminology. We will frequently use the phrase "a singularity of codimension k". Roughly speaking, this refers to the number of conditions having that singularity imposes on the space of curves. More precisely, it is the expected codimension of the equisingular strata. Hence, a singularity of type A_k, D_k or $E_{k\leq 8}$ is a singularity of codimension k.

The main result of this paper is as follows.

Theorem 1.3. *Let $\mathfrak{X}$ be a singularity of type A_k, D_k or E_k. Denote $N(\mathfrak{X})$ to be the number of degree d curves in $\mathbb{P}^2$ that pass through $d(d+3)/2-k$ generic points and has a singularity of type $\mathfrak{X}$ and codimension k. Then*

$$\begin{aligned}
&N(A_1) = 3(d-1)^2,\\
&N(A_2) = 12(d-1)(d-2), \qquad N(A_3) = 2(25d^2-96d+84),\\
&N(A_4) = 60(d-3)(3d-5), \qquad N(A_5) = 18(35d^2-190d+239),\\
&N(A_6) = 7(316d^2-1935d+2769), \qquad N(A_7) = 12(651d^2-4400d+7002),\\
&N(D_4) = 15(d-2)^2, \qquad N(D_5) = 12(d-2)(7d-19),\\
&N(D_6) = 14(16d^2-87d+114), \qquad N(D_7) = 48(15d^2-92d+135),\\
&N(E_6) = 21(d-3)(4d-9), \qquad N(E_7) = 252d^2-1464d+2079,
\end{aligned}$$

provided $d \geq C_{\mathfrak{X}}$ where

$$C_{A_k} = k, \qquad C_{D_4} = 3, \qquad C_{D_k} = k-2 \quad \text{if } k \geq 5, \qquad C_{E_6} = 3, \qquad C_{E_7} = 4.$$

Remark 1.4. The bound on d is imposed to ensure that the relevant sections we encounter are transverse to zero. In other words, the numbers $N(\mathfrak{X})$ we compute are actually enumerative (meaning that they are counted with a multiplicity of one in the linear system). However, the bound we impose is not necessarily the optimal bound. For example, strictly speaking our formula for $N(A_2)$ is applicable if $d \geq 2$. However, a little bit of thought shows that the formula we have obtained clearly holds even when $d = 1$, since there are no lines with a cusp. In other words, the bound we impose is a sufficient condition to prove transversality; it is not always a necessary condition.

The numbers $N(\mathfrak{X})$ have been previously computed by M. Kazarian in his paper [1] using a completely different method (see Section 2). The method we use to enumerate singular plane curves applies more generally to enumerating curves in any sufficiently ample linear system $L \to X$ where X is a smooth compact complex algebraic surface. For instance, our method also gives a formula for the number of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree (d_1, d_2) that pass through $(d_1+1)(d_2+1)-1-k$ that has a singularity of type $\mathfrak{X}$, where $\mathfrak{X}$ is a singularity of type A_k, D_k or E_k and $k \leq 7$. This is illustrated in our paper [2]. Kazarian's method also generalizes to enumerating curves in a general linear system and he does obtain the results of our paper [2]. Many of these numbers (for plane curves) have also been computed by Dimitry Kerner in his paper [3].

This paper contains the crucial details of computing the degenerate contribution to the Euler class that is needed to prove Theorem 1.3 of this paper and Theorem 1.5 of [2].[2] In order to keep the exposition simple, we have decided to focus on the special case of plane singular curves in this paper and work out the more general case of an arbitrary linear system in [2]. Our method also applies to enumerating singular hypersurfaces, as is illustrated in [4]. The problem of enumerating singular hypersurfaces in $\mathbb{P}^n$ has also been studied by Kerner in [5]. Finally, our method also applies to enumerating curves with more than one singular point, as illustrated in [6] and [2].

In future, we hope to apply this method to enumerate curves with singularities of type A_k, D_k and E_k, even when $k > 7$. One of the challenging aspect of this method is to have a thorough and systematic understanding of the closure of the equisingular strata. We also expect this method to work for other types of singularities not defined in Definition 1.1, provided we are able to find a necessary and sufficient criteria for a curve to have that singularity in terms of vanishing and non vanishing of certain derivatives (see Section 3 for a precise discussion).

The results of this paper are an extension of the topological method employed in [7]. One of the crucial aspects of this method is how we compute the degenerate contribution to the Euler class; we perturb the section *smoothly* (as opposed to *holomorphically*) and count (with a sign) how many zeros are there near the degenerate locus. Hence our method is a *topological* method as opposed to an *algebro-geometric* one. The most conceptual part of applying this method is to describe a neighbourhood of the degenerate locus. This method of computing degenerate contributions continues to play a crucial role when we enumerate curves with two singular points as is illustrated in our paper [6].

Our starting point will be the following well known fact from topology ([8], Proposition 12.8).

Theorem 1.5. *Let $V \longrightarrow X$ be a smooth oriented vector bundle over a compact oriented manifold X and suppose $s : X \longrightarrow V$ is a smooth section that is transverse to zero. Then the Poincaré dual of $[s^{-1}(0)]$ is the Euler class of V. In particular, if the rank of*

[2] Theorem 1.3 is actually a special case of Theorem 1.5 of [2].

V is same as the dimension of X then the signed cardinality of $s^{-1}(0)$ is the Euler class of V, evaluated on the fundamental class of X, i.e.,

$$|\pm s^{-1}(0)| = \langle e(V), [X]\rangle.$$

Remark 1.6. Let X be a compact, complex manifold, V a holomorphic vector bundle and s a holomorphic section that is transverse to the zero section. If the rank of V is same as the dimension of X, then the signed cardinality of $s^{-1}(0)$ is same as its actual cardinality (provided X and V have their natural orientations). In the case of complex vector bundles, the Euler class is also same as the top Chern class.

2. A survey of related results

"Enumerative Geometry of Singular Curves" is a classical subject and has been studied extensively using tools of algebraic geometry. As we explained in the introduction, the crucial aspect of our method is the way we compute the degenerate contribution to the Euler class. We perturb the section *smoothly* (as opposed to holomorphically) and count (with a sign) how many zeros are there in a neighbourhood of the degenerate locus. Hence, the method is *topological* as opposed to *algebro-geometric*.

We now give a brief survey of related results in this area of mathematics. We start by looking at the results of M. Kazarian. In [1] Kazarian has computed the numbers $N(\mathfrak{X})$ using a completely different approach. His method works on the principle that there exists a universal formula (in terms of Chern classes) for the Thom polynomial associated to a given singularity. He then goes on to consider enough special cases to find out what that exact combination is. As an example, suppose there is a polynomial of degree m. To find out what the polynomial is, we simply have to find the value of the polynomial at enough points [1, page 667]. One of the challenges of this method is to prove the existence of such a universal formula. Kazarian has successfully applied this method to solve a large class of enumerative problems; in particular it recovers the result of Theorem 1.3. In fact, he has obtained a formula for the number of degree d-curves (passing through $\frac{d(d+3)}{2} - \kappa$ generic points) and having a singularity of type $\chi_{k_1}, \chi_{k_2}, \ldots, \chi_{k_n}$, provided $\kappa := k_1 + k_2 + \cdots + k_n \leq 7$. Here χ_{k_i} is a singularity of codimension k_i. Kazarian's method also applies to the more general case of enumerating curves in a general linear system that is sufficiently ample; in particular he obtains the results of [2].

Next, let us look at the results of I. Vainsencher. In [9], Vainsencher enumerates curves that have up to six nodes. He also obtains a formula for $N(A_2)$ in [10]. His results are for a general linear system that is sufficiently ample.

Let us now describe the results of S. Kleiman and R. Piene. In [11, Theorem 1.2] they obtain many of the formulas we have obtained in Theorem 1.3; namely $N(A_1)$, $N(D_4)$, $N(D_6)$ and $N(E_7)$. They in fact obtain a formula for enumerating curves that have up to eight simple nodes, or one triple point and up to three simple nodes, or one singularity of type D_6 and up to one simple node. Their results are applicable for a linear system that is suitably ample. By this they mean the following; they consider a line bundle $L \longrightarrow X$ over a smooth projective surface X that is of the form $L := M^{\otimes m} \otimes N$, where m is at least three times the expected codimension of the equisingular strata and where N is spanned by global holomorphic sections. They then make the curves pass through the right number of generic points so that the expected number is finite. The ampleness condition is used to prove that the numbers they compute are indeed enumerative, i.e. each curve appears with a multiplicity of one in this linear system.

To summarize, the results of Kazarian, Vainsencher and Kleiman and Piene are applicable for enumerating singular curves in an arbitrary smooth projective surface (and not just for curves in $\mathbb{P}^2$).

Next, we note that in [12,13] and [14], Z. Ran, L. Caporaso and J. Harris have obtained a formula for the number of curves of degree d in $\mathbb{P}^2$ (through the right number of generic points) having r simple nodes, for any r (provided d is sufficiently larger than r).

The numbers $N(\mathfrak{X})$ have also been computed by Dimitry Kerner for many singularities in his paper [3] (for curves in $\mathbb{P}^2$).

Finally we note that a great deal of progress has been made in proving that universal formulas exist in terms of Chern classes for the number of curves in a sufficiently ample linear system passing through the right number of generic points and having singularities of type $\chi_{k_1}, \chi_{k_2}, \ldots, \chi_{k_n}$. The fact that a universal formula exists was conjectured by Göttsche when the singularities $\chi_{k_1}, \ldots, \chi_{k_n}$ are all simple nodes. Two independent proofs of this conjecture have now been given using methods of BPS calculus and using degeneration methods by Kool, Shende and Thomas [15] and Tzeng [16] respectively. There is also an earlier approach by Liu in [17] and [18]. Recently, Li and Tzeng gave a proof for the existence of universal formulas for any collection of singularities $\chi_{k_1}, \chi_{k_2}, \ldots, \chi_{k_n}$ in [19], thereby generalizing the Göttsche conjecture. A further generalization of the Göttsche conjecture was proved by Rennemo, where he shows that there is a universal polynomial to count hypersurfaces in a sufficiently ample linear system, with any collection of singularities ([20, Proposition 7.8]).

It should be noted that even for r-nodal curves in $\mathbb{P}^2$, it is not at all obvious when the polynomials obtained by the Göttsche conjecture are actually enumerative. They are enumerative if d is large. However, Göttsche conjectured that these polynomials are actually enumerative for all d roughly greater than $\frac{r}{2}$. This has now been proven by Kleiman and Shende in [21].

For an overview of this subject (enumerative geometry of singular curves), we direct the reader to the comprehensive survey article [22] by Kleiman.

3. Necessary and sufficient criteria for a singularity

In this section we state a necessary and sufficient criterion for a curve to have a singularity of type $A_{k\geq 0}$, $D_{k\geq 4}$, E_6 and E_7. Let $f = f(x,y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ and i,j be non-negative integers. We define

$$f_{ij} := \left.\frac{\partial^{i+j} f}{\partial^i x \partial^j y}\right|_{(x,y)=(0,0)}.$$

Let us now define the following directional derivatives, which are functions of f_{ij}:

$$\begin{aligned}
A^f_3 &:= f_{30}, \qquad A^f_4 := f_{40} - \frac{3f_{21}^2}{f_{02}}, \qquad A^f_5 := f_{50} - \frac{10f_{21}f_{31}}{f_{02}} + \frac{15f_{12}f_{21}^2}{f_{02}^2},\\
A^f_6 &:= f_{60} - \frac{15f_{21}f_{41}}{f_{02}} - \frac{10f_{31}^2}{f_{02}} + \frac{60f_{12}f_{21}f_{31}}{f_{02}^2} + \frac{45f_{21}^2f_{22}}{f_{02}^2} - \frac{15f_{03}f_{21}^3}{f_{02}^3} - \frac{90f_{12}^2f_{21}^2}{f_{02}^3},\\
A^f_7 &:= f_{70} - \frac{21f_{21}f_{51}}{f_{02}} - \frac{35f_{31}f_{41}}{f_{02}} + \frac{105f_{12}f_{21}f_{41}}{f_{02}^2} + \frac{105f_{21}^2f_{32}}{f_{02}^2} + \frac{70f_{12}f_{31}^2}{f_{02}^2} + \frac{210f_{21}f_{22}f_{31}}{f_{02}^2}\\
&\quad - \frac{105f_{03}f_{21}^2f_{31}}{f_{02}^3} - \frac{420f_{12}^2f_{21}f_{31}}{f_{02}^3} - \frac{630f_{12}f_{21}^2f_{22}}{f_{02}^3} - \frac{105f_{13}f_{21}^3}{f_{02}^3} + \frac{315f_{03}f_{12}f_{21}^3}{f_{02}^4} + \frac{630f_{12}^3f_{21}^2}{f_{02}^4},\\
A^f_8 &:= f_{80} - \frac{28f_{21}f_{61}}{f_{02}} - \frac{56f_{31}f_{51}}{f_{02}} + \frac{210f_{21}^2f_{42}}{f_{02}^2} + \frac{420f_{21}f_{22}f_{41}}{f_{02}^2} - \frac{210f_{03}f_{21}^2f_{41}}{f_{02}^3} + \frac{560f_{21}f_{31}f_{32}}{f_{02}^2}\\
&\quad - \frac{840f_{13}f_{21}^2f_{31}}{f_{02}^3} - \frac{420f_{21}^3f_{23}}{f_{02}^3} + \frac{1260f_{03}f_{21}^3f_{22}}{f_{02}^4} - \frac{35f_{41}^2}{f_{02}} + \frac{280f_{22}f_{31}^2}{f_{02}^2} - \frac{280f_{03}f_{21}f_{31}^2}{f_{02}^3} - \frac{1260f_{21}^2f_{22}^2}{f_{02}^3}\\
&\quad + \frac{105f_{04}f_{21}^4}{f_{02}^4} - \frac{315f_{03}^2f_{21}^4}{f_{02}^5} + \frac{168f_{21}f_{51}f_{12}}{f_{02}^2} + \frac{280f_{31}f_{41}f_{12}}{f_{02}^2} - \frac{1680f_{21}^2f_{32}f_{12}}{f_{02}^3} - \frac{3360f_{21}f_{22}f_{31}f_{12}}{f_{02}^3}\\
&\quad + \frac{2520f_{03}f_{21}^2f_{31}f_{12}}{f_{02}^4} + \frac{2520f_{13}f_{21}^3f_{12}}{f_{02}^4} - \frac{840f_{21}f_{41}f_{12}^2}{f_{02}^3} + \frac{7560f_{21}^2f_{22}f_{12}^2}{f_{02}^4} - \frac{560f_{31}^2f_{12}^2}{f_{02}^3} - \frac{5040f_{03}f_{21}^3f_{12}^2}{f_{02}^5}\\
&\quad + \frac{3360f_{21}f_{31}f_{12}^3}{f_{02}^4} - \frac{5040f_{21}^2f_{12}^4}{f_{02}^5}
\end{aligned} \tag{3.1}$$

and

$$D^f_6 := f_{40}, \qquad D^f_7 := f_{50} - \frac{5f_{31}^2}{3f_{12}}, \qquad D^f_8 := f_{60} + \frac{5f_{03}f_{31}f_{50}}{3f_{12}^2} - \frac{5f_{31}f_{41}}{f_{12}} - \frac{10f_{03}f_{31}^3}{3f_{12}^3} + \frac{5f_{22}f_{31}^2}{f_{12}^2}. \tag{3.2}$$

We will now state a necessary and sufficient criterion for a curve to have a specific singularity of one of the types $A_{k\geq 0}$, $D_{k\geq 4}$, E_6 and E_7.

Lemma 3.1. *Let $f = f(x,y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ such that $f_{00} = 0$ and $\nabla f|_{(0,0)} \neq 0$. Then the curve has a singularity of type A_0 at the origin (i.e., a smooth point).*

Lemma 3.2. *Let $f = f(x,y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $f_{00} = 0$, $\nabla f|_{(0,0)} = 0$ and $\nabla^2 f|_{(0,0)}$ is non-degenerate. Then the curve has a singularity of type A_1 at the origin.*

Remark 3.3. Lemma 3.1 is also known as the *Implicit Function Theorem* and Lemma 3.2 is also known as the *Morse Lemma.*

We now state the remaining Lemmas, which can be thought of as a continuation of Lemma 3.2.

Lemma 3.4. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $f_{00} = 0$, $\nabla f|_{(0,0)} = 0$ and there exists a non-zero vector $\eta = (v_1, v_2)$ such that at the origin $\nabla^2 f(\eta, \cdot) = 0$, i.e., the Hessian is degenerate. Let $x := v_1 r + v_2 s$, $y := -\overline{v}_2 r + \overline{v}_1 s$ and f_{ij} be the partial derivatives with respect to the new variables x and y. Then, the curve $f^{-1}(0)$ has a singularity of type A_k at the origin (for $2 \leq k \leq 7$) if $f_{02} \neq 0$ and the directional derivatives A^f_i defined in (3.1) are zero for all $i \leq k$ and $A^f_{k+1} \neq 0$.*

Proof. The result follows from the following observation.

Observation 3.5. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $f(0,0)$, $\nabla f|_{(0,0)} = 0$ and there exists a non-zero vector $\eta = (v_1, v_2)$ such that at the origin $\nabla^2 f(v, \cdot) = 0$, i.e., the Hessian is* *degenerate. Let* $x := v_1 r + v_2 s$, $y := -\overline{v}_2 r + \overline{v}_1 s$ *and* f_{ij} *be the partial derivatives with respect to the new variables x and y. If* $f_{02} \neq 0$*, there exists a coordinate chart* (u, v) *centred around the origin in* $\mathbb{C}^2$ *such that*

$$f = \begin{cases} v^2, & \text{or} \\ v^2 + u^{k+1}, & \text{for some } k \geq 2. \end{cases} \tag{3.3}$$

In terms of the new coordinates we have $f_{00} = f_{10} = f_{01} = f_{20} = f_{11} = 0$ and $f_{02} \neq 0$. Here $\partial_x + 0\partial_y = (1, 0)$ is the distinguished direction along which the Hessian is degenerate.

Proof of observation. Let the Taylor expansion of f in the new coordinates be given by

$$f(x, y) = \mathrm{A}_0(x) + \mathrm{A}_1(x)y + \mathrm{A}_2(x)y^2 + \cdots .$$

By our assumption on f, $\mathrm{A}_2(0) \neq 0$. We claim that there exists a holomorphic function $\mathrm{B}(x)$ such that after we make a change of coordinates $y = y_1 + \mathrm{B}(x)$, the function f is given by

$$f = \hat{\mathrm{A}}_0(x) + \hat{\mathrm{A}}_2(x)y_1^2 + \hat{\mathrm{A}}_3(x)y_1^3 + \cdots$$

for some $\hat{\mathrm{A}}_k(x)$ (i.e., $\hat{\mathrm{A}}_1(x) \equiv 0$). To see this, we note that this is possible if $\mathrm{B}(x)$ satisfies the identity

$$\mathrm{A}_1(x) + 2\mathrm{A}_2(x)\mathrm{B} + 3\mathrm{A}_3(x)\mathrm{B}^2 + \cdots \equiv 0. \tag{3.4}$$

Since $\mathrm{A}_2(0) \neq 0$, $\mathrm{B}(x)$ exists by the Implicit Function Theorem.[3] Therefore, we can compute $\mathrm{B}(x)$ as a power series using (3.4) and then compute $\hat{\mathrm{A}}_0(x)$. Hence,

$$f = v^2 + \frac{A_3^f}{3!}x^3 + \frac{A_4^f}{4!}x^4 + \cdots, \quad \text{where } v = \sqrt{(\hat{\mathrm{A}}_2 + \hat{\mathrm{A}}_3 y_1 + \cdots)}\,y_1, \tag{3.5}$$

satisfies (3.3). $\square$

Following the above procedure we find A_i^ρ for $i = 3, \ldots, 7$. In particular,

$$A_3^f = f_{30}, \qquad A_4^f = f_{40} - \frac{3f_{21}^2}{f_{02}}, \qquad A_5^f = f_{50} - \frac{10f_{21}f_{31}}{f_{02}} + \frac{15f_{12}f_{21}^2}{f_{02}^2}.$$

This concludes the proof of the Lemma. $\square$

Lemma 3.6. *Let* $f = f(x, y)$ *be a holomorphic function defined on a neighbourhood of the origin in* $\mathbb{C}$ *such that* $f_{00}, \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$ *and there does not exist a non-zero vector* $\eta = (v_1, v_2)$ *such that at the origin* $\nabla^3 f(\eta, \eta, \cdot) = 0$*. Then the curve* $f^{-1}(0)$ *has a singularity of type* D_4 *at the origin.*

Lemma 3.7. *Let* $f = f(r, s)$ *be a holomorphic function defined on a neighbourhood of the origin in* $\mathbb{C}$ *such that* $f_{00}, \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$ *and there exists a non-zero vector* $\eta = (v_1, v_2)$ *such that at the origin* $\nabla^3 f(\eta, \eta, \cdot) = 0$*. Let* $x := v_1 r + v_2 s$, $y := -\overline{v}_2 r + \overline{v}_1 s$ *and* f_{ij} *be the partial derivatives with respect to the new variables x and y. Then, the curve* $f^{-1}(0)$ *has a singularity of type* D_k *at the origin (for* $5 \leq k \leq 7$*) if and only if* $f_{12} \neq 0$ *and the directional derivatives* D_i^f *defined in (3.2) are zero for all* $i \leq k$ *and* $D_{k+1}^f \neq 0$*.*

As in the case of singularities of type A_k, we see that the proof of the above rests on the following observation which can be proved via a combination of Taylor expansion and Implicit Function Theorem.

Observation 3.8. *Let* $f = f(r, s)$ *be a holomorphic function defined on a neighbourhood of the origin in* $\mathbb{C}$ *such that* $f_{00}, \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$ *and there exists a non-zero vector* $\eta = (v_1, v_2)$ *such that at the origin* $\nabla^3 f(\eta, \eta, \cdot) = 0$*. Let* $x = v_1 r + v_2 s$, $y = -\overline{v}_2 r + \overline{v}_1 s$ *and* f_{ij} *be the partial derivatives with respect to the new variables x and y. If* $f_{12} \neq 0$*, there exists a coordinate chart* (u, v) *centred around the origin in* $\mathbb{C}^2$ *such that*

$$f(u, v) \equiv \begin{cases} v^2 u & \text{or} \\ v^2 u + u^{k-1} & \text{for some } k \geq 5. \end{cases}$$

Lemma 3.9. *Let* $f = f(r, s)$ *be a holomorphic function defined on a neighbourhood of the origin in* $\mathbb{C}$ *such that* $f_{00} = \nabla f|_{(0,0)} = \nabla^2 f|_{(0,0)} = 0$ *and there exists a non-zero vector* $\eta = (v_1, v_2)$ *such that at the origin* $\nabla^3 f(\eta, \eta, \cdot) = 0$*. Let* $x = v_1 r + v_2 s$, $y = -\overline{v}_2 r + \overline{v}_1 s$ *and* f_{ij} *be partial derivatives with respect to the new coordinates, x and y. Then, the curve* $f^{-1}(0)$ *has a singularity of type* E_6 *at the origin if* $f_{12} = 0$ *and* $f_{03} \neq 0, f_{40} \neq 0$*.*

[3] Moreover it is unique if we require $\mathrm{B}(0) = 0$.

Lemma 3.10. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $f_{00}, \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$ and there exists a non-zero vector $\eta = (v_1, v_2)$ such that at the origin $\nabla^3 f(\eta, \eta, \cdot) = 0$. Let $x = v_1 r + v_2 s,\ y = -\overline{v}_2 r + \overline{v}_1 s$. Let f_{ij} be the partial derivatives with respect to the new variables x and y. Then, the curve $f^{-1}(0)$ has a singularity of type E_7 at the origin if $f_{12} = 0, f_{40} = 0$ and $f_{03} \neq 0, f_{31} \neq 0$.*

We omit the proof of these Lemmas. As illustrated in the proof of Lemma 3.4 and Observation 3.8, the basic idea is to write down the Taylor expansion of $f(x,y)$ and make a suitable (locally analytic) change of coordinates.

4. Proof of the main theorem via Euler class computation

4.1. A few basic notations

For the convenience of the reader, we define the various spaces and bundles as and when they get needed as opposed to defining them earlier in one place. We feel this will make our manuscript easier to follow.

First, let us set up some basic notation. We recall the definition of the tautological line bundle $\gamma_{\mathbb{P}^n}$ over $\mathbb{P}^n$. Although this definition is a standard one, we urge the reader to carefully go through the definition, since it will be crucial throughout the paper. The tautological line bundle is defined as follows:

$$\gamma_{\mathbb{P}^n} := \{(l, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in l\} \longrightarrow \mathbb{P}^n.$$

Thus, an element of $\mathbb{P}^n$ is a line l through the origin in $\mathbb{C}^{n+1}$. The phrase "$v \in l$" should now make sense; it means that v is a point in $\mathbb{C}^{n+1}$ that belongs to the line l. If $\pi_{\mathbb{P}^n} : \gamma_{\mathbb{P}^n} \longrightarrow \mathbb{P}^n$ is the projection map, then $\pi_{\mathbb{P}^n}^{-1}(l)$ is the entire line l in $\mathbb{C}^{n+1}$. Hence, we are viewing $\gamma_{\mathbb{P}^n}$ as a sub bundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$.

Next, following the notation in [7] we will denote the space of curves of degree d in $\mathbb{P}^2$ by $\mathcal{D}$. It follows that $\mathcal{D} \cong \mathbb{P}^{\kappa_d}$, where $\kappa_d := d(d+3)/2$. A homogeneous polynomial f, of degree d in 3 variables, induces a holomorphic section of the line bundle $\gamma_{\mathbb{P}^2}^{*d} \longrightarrow \mathbb{P}^2$. If f is non-zero, then we will denote its *equivalence class* in $\mathcal{D}$ by $\tilde{f}$. Similarly, if p is a non-zero vector in $\mathbb{C}^3$, we will denote its equivalence class in $\mathbb{P}^2$ by $\tilde{p}$. As explained earlier, in this paper we will use the symbol $\tilde{A}$ to denote the equivalence class of A instead of the standard $[A]$ since it will make some of the calculations in Section 4 easier to read.

Next, we will denote the tautological line bundle over $\mathcal{D}$ as $\gamma_{\mathcal{D}}$. Let us also denote by $\hat{\gamma}$ the tautological line bundle over $\mathbb{P}T\mathbb{P}^2$, the projectivization of the tangent space of $\mathbb{P}^2$. Following the notation of [7], we denote

$$a := c_1(\gamma_{\mathbb{P}^2}^*) \in H^2(\mathbb{P}^2, \mathbb{Z}), \qquad y := c_1(\gamma_{\mathcal{D}}^*) \in H^2(\mathbb{P}^2, \mathbb{Z}) \quad \text{and} \quad \lambda := c_1(\hat{\gamma}^*) \in H_2(\mathbb{P}T\mathbb{P}^2, \mathbb{Z}).$$

Finally, we will be following a standard abuse of notation: if V is a bundle over M_1, we will also say that V is a bundle over $M_1 \times M_2$. The intended meaning is that we are referring by V the pullback bundle $\pi_1^* V \longrightarrow M_1 \times M_2$, where π_1 is the projection map. Similarly, if α is a cohomology class in M_1, we will also say that α is a cohomology class in $M_1 \times M_2$; the intended meaning being $\pi_1^*\alpha$. Hence, when we say that $\gamma_{\mathbb{P}^2}^* \longrightarrow \mathcal{D} \times \mathbb{P}^2$ is a bundle over $\mathcal{D} \times \mathbb{P}^2$, our intended meaning is $\pi_{\mathbb{P}^2}^* \gamma_{\mathbb{P}^2}^* \longrightarrow \mathcal{D} \times \mathbb{P}^2$. Similarly, when we say that $a \in H^2(\mathcal{D} \times \mathbb{P}^2; \mathbb{Z})$, our intended meaning is $\pi_{\mathbb{P}^2}^* a \in H^2(\mathcal{D} \times \mathbb{P}^2; \mathbb{Z})$.

We will now give a proof for each of the formulas in Theorem 1.3. For the convenience of the reader, we have split the results into several subsections.

4.2. Computing $N(A_1)$

We will now start with the computation of $N(A_1)$.

Theorem 4.1. *Let $N(A_1, n)$ denote the number of degree d curves in $\mathbb{P}^2$ through $\kappa_d - 1 - n$ generic points and having an A_1-singularity at the intersection of n generic lines. Then,*

$$N(A_1, n) = \begin{cases} 3(d-1)^2, & \text{if } n = 0; \\ 3(d-1), & \text{if } n = 1; \\ 1, & \text{if } n = 2; \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

Proof. Let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\kappa_d - 1 - n}$ be $\kappa_d - 1 - n$ generic points and $\mathrm{L}_1, \ldots, \mathrm{L}_n$ be n generic lines in $\mathbb{P}^2$. Define

$$\mathrm{H}_i := \{\tilde{f} \in \mathcal{D} : f(\tilde{p}_i) = 0\}, \qquad \hat{\mathrm{H}}_i := \mathrm{H}_i \times \mathbb{P}^2 \quad \text{and} \quad \hat{\mathrm{L}}_i := \mathcal{D} \times \mathrm{L}_i.$$

Then $N(A_1, n)$ is the cardinality of the set

$$\{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(\tilde{p}) = 0,\ \nabla f|_{\tilde{p}} = 0\} \cap \hat{\mathrm{H}}_1 \cap \ldots \hat{\mathrm{H}}_{\kappa_d - 1 - n} \cap \hat{\mathrm{L}}_1 \cap \ldots \hat{\mathrm{L}}_n. \tag{4.2}$$

Let us now define holomorphic sections of the following bundles:

$$\psi_{\mathcal{A}_0} : \mathcal{D} \times \mathbb{P}^2 \longrightarrow \mathcal{L}_{\mathcal{A}_0} := \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}, \qquad \{\psi_{\mathcal{A}_0}(\tilde{f}, \tilde{p})\}(f) := f(\tilde{p}), \tag{4.3}$$

$$\psi_{\mathcal{A}_1} : \psi_{\mathcal{A}_0}^{-1}(0) \longrightarrow \mathcal{V}_{\mathcal{A}_1} := \gamma_{\mathcal{D}}^* \otimes T^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^{*d}, \qquad \{\psi_{\mathcal{A}_1}(\tilde{f}, \tilde{p})\}(f) := \nabla f|_{\tilde{p}}. \tag{4.4}$$

We will now explain the meaning of (4.3) and (4.4). Although what we are about to explain is a standard fact, we request the reader to carefully go through our explanation since it will prove to be extremely important throughout the paper.[4] Note that $\gamma_{\mathbb{P}^2}^{*d}$ refers to the dth-tensor power of the dual of $\gamma_{\mathbb{P}^2}$, i.e.

$$\gamma_{\mathbb{P}^2}^{*d} := (\gamma_{\mathbb{P}^2}^*)^{\otimes d}.$$

We will denote the fibre at each point $\tilde{p} \in \mathbb{P}^2$ by $\gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Similarly, we will denote the fibre of the tautological bundle at each point $\tilde{f} \in \mathcal{D}$ by $\gamma_{\mathcal{D}}|_{\tilde{f}}$.

Let us now explain an important point; note the difference between $\tilde{f}$ and f. Observe that $\tilde{f}$ is an element of $\mathcal{D}$, while f is an element of the fibre $\gamma_{\mathcal{D}}|_{\tilde{f}}$. We are now ready to explain the meaning of (4.3). In (4.3), the right hand side is an element of the vector space $\gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Secondly, f belongs to the vector space $\gamma_{\mathcal{D}}|_{\tilde{f}}$. Hence $\psi_{A_0}(\tilde{f}, \tilde{p})$ is a linear map from the vector space $\gamma_{\mathcal{D}}|_{\tilde{f}}$, with values in $\gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Hence $\psi_{A_0}(\tilde{f}, \tilde{p})$ is an element of the vector space $\gamma_{\mathcal{D}}|_{\tilde{f}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Hence, $\psi_{A_0}(\tilde{f}, \tilde{p})$ is a section of the line bundle $\mathcal{L}_{A_0} := \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}$.

Next, if $f : \mathbb{P}^2 \longrightarrow \gamma_{\mathbb{P}^2}^{*d}$ is a section, then $\nabla f|_{\tilde{p}}$ is a linear map from $T_{\tilde{p}}\mathbb{P}^2 \longrightarrow \gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Hence, the right hand side of (4.4) is an element of $T_{\tilde{p}}^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Hence $\psi_{\mathcal{A}_1}(\tilde{f}, \tilde{p})$ is a linear map from $\gamma_{\mathcal{D}}|_{\tilde{f}}$ to $T_{\tilde{p}}^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^{*d}|_{\tilde{p}}$. Hence, $\psi_{\mathcal{A}_1}$ is a section of the rank two bundle $\mathcal{V}_{\mathcal{A}_1} := \gamma_{\mathcal{D}}^* \otimes T^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^{*d}$.

We will justify shortly (in Lemmas 4.2 and 4.3) that the sections $\psi_{\mathcal{A}_0}$ and $\psi_{\mathcal{A}_1}$ are transverse to zero if $d \geq 1$. Since the $\kappa_d - 1 - n$ points and n lines are generic, we conclude that the intersection in (4.2) is transverse. Furthermore, if $(\tilde{f}, \tilde{p})$ belongs to the set defined in (4.2), then f has a genuine A_1-singularity at $\tilde{p}$ (as opposed to something more degenerate like a cusp for instance). Since ψ_{A_0} and ψ_{A_1} are holomorphic, we conclude that

$$\begin{aligned} N(A_1, n) &= \langle e(\mathcal{V}_{A_1}) y^{\kappa_d - 1 - n} a^n,\ [\psi_{\mathcal{A}_0}^{-1}(0)] \rangle, \\ &= \langle e(\mathcal{L}_{\mathcal{A}_0}) e(\mathcal{V}_{\mathcal{A}_1}) y^{\kappa_d - 1 - n} a^n,\ [\mathcal{D} \times \mathbb{P}^2] \rangle. \end{aligned} \tag{4.5}$$

The second equality follows from the fact that the Poincaré Dual of $[\psi_{\mathcal{A}_0}^{-1}(0)]$ in $\mathcal{D} \times \mathbb{P}^2$ is $e(\mathcal{L}_{\mathcal{A}_0})$ (using Theorem 1.5). Using the splitting principle, we conclude that

$$e(\mathcal{L}_{\mathcal{A}_0}) e(\mathcal{V}_{\mathcal{A}_1}) = (y + da)((y + da)^2 - 3a(y + da) + 3a^2). \tag{4.6}$$

Eq. (4.1) now follows from (4.5), (4.6) and by extracting the coefficient of $y^{\kappa_d} a^2$ in

$$e(\mathcal{L}_{\mathcal{A}_0}) e(\mathcal{V}_{\mathcal{A}_1}) y^{\kappa_d - 1 - n} a^n. \quad \Box$$

We now prove the two assertions about transversality.

Lemma 4.2. *The section*

$$\psi_{A_0} : \mathcal{D} \times \mathbb{P}^2 \longrightarrow \mathcal{L}_{A_0} := \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \quad \textit{given by} \quad \{\psi_{A_0}(\tilde{f}, \tilde{p})\}(f) := f(\tilde{p})$$

is transverse to the zero section if $d \geq 1$.

Proof. Suppose $(\tilde{f}, \tilde{p}) \in \psi_{\mathcal{A}_0}^{-1}(0)$. Choose homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$ so that $\tilde{p} = [0 : 0 : 1]$ and let

$$\mathcal{U} := \big\{[X : Y : Z] \in \mathbb{P}^2 : Z \neq 0\big\}, \qquad \varphi_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathbb{C}^2, \qquad \varphi_{\mathcal{U}}([X : Y : Z]) = (X/Z, Y/Z).$$

Let us also denote $x := X/Z$ and $y := Y/Z$. In local coordinates, proving transversality is equivalent to showing that the map

$$\mathcal{F}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}, \qquad (f, x, y) \mapsto f(x, y)$$

is transverse to zero at $(f, 0, 0)$ (in the usual calculus sense). Here $\mathcal{F} \cong \mathbb{C}^{\kappa_d + 1}$ denotes the space of polynomials in two variables of degree at most d and $\mathcal{F}^*$ is the subspace of non-zero polynomials.[5] Thus, the section can be rewritten as

$$\psi_{A_0}(f, x, y) = f(x, y) := f_{00} + f_{10}x + f_{01}y + \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \cdots.$$

Since the Jacobian matrix of this map at $(f, 0, 0)$ is $(1 \quad 0 \quad 0 \quad \ldots)$, where the first column is partial derivative with respect to f_{00}, transversality follows. $\Box$

4 The reader is urged to first review the definition of the tautological line bundle as defined in Section 4.1.

5 This can also be thought of as homogeneous polynomials, in 3 variables, of degree d.

Lemma 4.3. *The section*

$$\psi_{A_1} : \psi_{A_0}^{-1}(0) \longrightarrow \mathcal{V}_{A_1} \quad \textit{given by} \quad \{\psi_{A_1}(\tilde{f}, \tilde{p})\}(f) := \nabla f|_{\tilde{p}}$$

is transverse to the zero section if $d \geq 1$.

Proof. Continuing with the setup of Lemma 4.2, transversality is equivalent to showing that the map

$$\mathcal{F}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^3, \qquad (f, x, y) \mapsto \big(f(x, y), f_x(x, y), f_y(x, y)\big)$$

is transverse to zero at $(f, 0, 0)$ (in the usual calculus sense). Since $d \geq 1$,

$$f(x, y) = f_{00} + f_{10}x + f_{01}y + \cdots$$

and the Jacobian at $(f, 0, 0)$ has the identity matrix as the first three columns, where the first three columns are partial derivatives with respect to f_{00}, f_{10} and f_{01}. Hence, transversality follows. □

Remark 4.4. This method is different from Dimitry Kerner's method (cf. [3], page 20). His method is specific to $\mathbb{P}^2$. Our proof of Theorem 4.1 easily generalizes to enumerating curves in a linear system $H^0(X, L)$, provided $L \longrightarrow X$ is a sufficiently ample line bundle. This is shown in our paper [2]. By a slight modification of the proof of Theorem 4.1, we can show that

$$N(A_1) = \langle 3c_1(L)^2 + 2c_1(L)c_1(T^*X) + c_2(TX),\ [X]\rangle.$$

The ampleness condition is required to prove the transversality hypothesis. Kazarian's method also produces the above formula.

4.3. Computing $N(A_2)$

Before proving our formula for $N(A_2)$, let us set up some notation. Let $\hat{\gamma} \longrightarrow \mathbb{P}T\mathbb{P}^2$ denote the tautological bundle over $\mathbb{P}T\mathbb{P}^2$. Given an element $l_{\tilde{p}} \in \mathbb{P}T\mathbb{P}^2$, we denote $\hat{\gamma}|_{l_{\tilde{p}}}$ to be the fibre over $l_{\tilde{p}}$. We now define

$$A_2 := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has an } A_2\text{-singularity at } \tilde{p}\} \quad \text{and}$$

$$\mathcal{P}A_2 := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has an } A_2\text{-singularity at } \tilde{p},\ \nabla^2 f(v, \cdot) = 0\ \forall\, v \in \hat{\gamma}|_{l_{\tilde{p}}}\}.$$

Let us explain in words what this space is. First, we recall that $\mathbb{P}T\mathbb{P}^2$ is the projectivized tangent space of $\mathbb{P}^2$. We denote an element of $\mathbb{P}T\mathbb{P}^2$ as $l_{\tilde{p}}$. The letter l is to remind the reader that an element of $\mathbb{P}T_{\tilde{p}}\mathbb{P}^2$ is actually a *line* through the origin in $T_{\tilde{p}}\mathbb{P}^2$. Hence, the phrase $v \in \hat{\gamma}|_{l_{\tilde{p}}}$ means that v is a tangent vector in $T_{\tilde{p}}\mathbb{P}^2$ that belongs to the line $l_{\tilde{p}}$. Hence, $\mathcal{P}A_2$ is the space of degree d curves $\tilde{f}$ and a marked point $\tilde{p}$ and a marked direction $l_{\tilde{p}}$ such that f has an A_2-singularity at $\tilde{p}$ *and* the Hessian is degenerate along the direction $l_{\tilde{p}}$. That is what the phrase

$$\nabla^2 f|_{\tilde{p}}(v, \cdot) = 0 \quad \forall\, v \in \hat{\gamma}|_{l_{\tilde{p}}}$$

means.[6] Let us now define the following number:

$$N(\mathcal{P}A_2, n, m) := \langle y^{\kappa_d - 2 - n - m} a^n \lambda^m,\ [\overline{\mathcal{P}A_2}]\rangle,$$

where $\lambda := c_1(\hat{\gamma}^*) \in H^2(\mathbb{P}T\mathbb{P}^2, \mathbb{Z})$. There is a one to one correspondence between the elements of A_2 and $\mathcal{P}A_2$. In other words, the projection map from $\mathcal{P}A_2$ to A_2 is one to one. Hence, we conclude that

$$N(A_2, n) = N(\mathcal{P}A_2, n, 0).$$

Next, we will denote the pullback of the tangent bundle $T\mathbb{P}^2$ over $\mathbb{P}T\mathbb{P}^2$ by $\pi^*T\mathbb{P}^2 \longrightarrow \mathbb{P}T\mathbb{P}^2$, where $\pi : \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ denotes the projection map. This is the one place we will not omit writing the pullback map; that is because $\pi^*T\mathbb{P}^2$ actually splits as a sum of line bundles. Note that $\hat{\gamma}$ is a sub bundle of $\pi^*T\mathbb{P}^2$. Hence we can define the quotient bundle $(\pi^*T\mathbb{P}^2/\hat{\gamma}) \longrightarrow \mathbb{P}T\mathbb{P}^2$. Hence

$$\pi^*T\mathbb{P}^2 \approx \hat{\gamma} \oplus \left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right) \longrightarrow \mathbb{P}T\mathbb{P}^2.$$

Let us explain the above splitting more clearly. The tautological line bundle $\hat{\gamma} \longrightarrow \mathbb{P}T\mathbb{P}^2$ is a sub bundle of the pullback of the tangent bundle $\pi^*T\mathbb{P}^2 \longrightarrow \mathbb{P}T\mathbb{P}^2$. More precisely, if $l_{\tilde{p}} \in \mathbb{P}T\mathbb{P}^2$, that means $l_{\tilde{p}}$ can be thought of as a non zero tangent vector at $T_{\tilde{p}}\mathbb{P}^2$ up to scaling. The fibre $\hat{\gamma}|_{l_{\tilde{p}}}$ is precisely all scalar multiples of this non zero tangent vector. More precisely,

$$\hat{\gamma} := \{(l_{\tilde{p}}, v) \in \pi^*T\mathbb{P}^2 : v \in l_{\tilde{p}}\}.$$

[6] Note that if $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}A_2$, then f has a cusp at $\tilde{p}$, but it can also have a singularity at some point other than $\tilde{p}$.

Hence $\hat{\gamma}$ is a sub bundle of $\pi^*T\mathbb{P}^2$. Given a holomorphic bundle V and a holomorphic sub bundle W, the quotient V/W is also a holomorphic sub bundle. And hence V splits as a direct sum of two holomorphic bundles

$$V \approx W \oplus V/W.$$

Hence $\pi^*T\mathbb{P}^2$ splits holomorphically as a direct sum of the tautological bundle $\hat{\gamma}$ and the quotient bundle $\pi^*T\mathbb{P}^2/\hat{\gamma}$.

Both the tautological bundle $\hat{\gamma}$ and the quotient bundle $(\pi^*T\mathbb{P}^2/\hat{\gamma})$ will play an important role in our computations. We are now ready to prove the formula for $N(A_2)$.

Theorem 4.5. *If $d \geq 2$, then*

$$N(\mathcal{P}A_2, n, 0) = 2N(A_1, n) + 2(d-3)N(A_1, n+1)$$
$$N(\mathcal{P}A_2, n, 1) = N(A_1, n) + (2d-9)N(A_1, n+1) + (d^2 - 9d + 18)N(A_1, n+2)$$
$$N(\mathcal{P}A_2, n, m) = -3N(\mathcal{P}A_2, n+1, m-1) - 3N(\mathcal{P}A_2, n+2, m-2) \quad \forall m \geq 2.$$

Note that combining this result with our formula for $N(A_1, n)$ (Theorem 4.1) we obtain the formula for $N(A_2)$ (cf. Theorem 1.3).

Remark 4.6. The number $N(\mathcal{P}A_2, n, m)$ is the signed cardinality of a set when $m \neq 0$. This number can very easily be negative. For example $N(\mathcal{P}A_2, 0, 1) = 10d^2 - 48d + 48$. For $d = 2$ and 3 this turns out to be -8 and -6. We will see shortly why $N(\mathcal{P}A_2, n, m)$ is in general a signed cardinality; this is essentially a consequence of the fact that the bundle $\hat{\gamma}^* \longrightarrow \mathbb{P}T\mathbb{P}^2$ does not admit any non zero holomorphic sections.

Proof. Define the bundle

$$\mathbb{W}_{n,m,2} := \left(\bigoplus_{i=1}^{\kappa_d-2-n-m} \gamma_{\mathcal{D}}^*\right) \oplus \left(\bigoplus_{i=1}^{n} \gamma_{\mathbb{P}^2}^*\right) \oplus \left(\bigoplus_{i=1}^{m} \hat{\gamma}^*\right) \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \tag{4.7}$$

and let $\mathcal{Q} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{W}_{n,m,2}$ be a generic smooth section that is transverse to zero. We remind the reader about the abuse of notation we make: if V is a bundle over M_1, we will also say that V is a bundle over $M_1 \times M_2$, where the intended meaning is $\pi_1^*V \longrightarrow M_1 \times M_2$, where π_1 is the projection map.

Next, we note that unless $m = 0$, the bundle $\mathbb{W}_{n,m,2}$ need not have any holomorphic sections that is transverse to zero. To see why this is so, let us think of $\mathcal{Q}$ as

$$\mathcal{Q} := \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3,$$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ map into $\left(\bigoplus_{i=1}^{\kappa_d-2-n-m} \gamma_{\mathcal{D}}^*\right)$, $\left(\bigoplus_{i=1}^{n} \gamma_{\mathbb{P}^2}^*\right)$ and $\left(\bigoplus_{i=1}^{m} \hat{\gamma}^*\right)$ respectively. The sections $\mathcal{Q}_1$ and $\mathcal{Q}_2$ can be non zero and holomorphic, but $\mathcal{Q}_3$ cannot be a non zero holomorphic section. This is because $\hat{\gamma}^* \longrightarrow \mathbb{P}T\mathbb{P}^2$ does not admit any non zero holomorphic section. And if we take $\mathcal{Q}_3$ to be the zero section, then $\mathcal{Q}$ will not be transverse to zero (even if $\mathcal{Q}_1 \oplus \mathcal{Q}_2$ is transverse to zero). Our purpose of considering $\mathcal{Q}$ is to look at its zero set and compute its intersection number with a given variety of the complementary dimension. In order to do that we have to be able to make sure that we can perturb $\mathcal{Q}$ so that not only is $\mathcal{Q}$ transverse to zero, but the zero set intersects a given variety transversally (in other words $\mathcal{Q}^{-1}(0)$ can be made to intersect the given variety only on the smooth locus of the variety, where it makes sense to talk of transverse intersection). This is possible in general only if we insist that the section $\mathcal{Q}$ is smooth as opposed to holomorphic. This is the reason $N(\mathcal{P}A_2, n, m)$ is in general the signed cardinality of a set when $m \neq 0$ (see Remark 4.6).

Let us now define

$$A_1 := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has an } A_1\text{-singularity at } \tilde{p}\}, \quad \text{and} \quad \hat{A}_1 := \pi^{-1}(A_1),$$

where $\pi : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{D} \times \mathbb{P}^2$ is the projection map. We claim that the closure of A_1 inside $\mathcal{D} \times \mathbb{P}^2$ is given by

$$\overline{A}_1 = \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(\tilde{p}) = 0, \ \ \nabla f_{\tilde{p}} = 0\}.$$

We will prove that shortly (in Lemma 4.7). By Lemma 4.3 we conclude that $\overline{A}_1$ is a smooth complex manifold of dimension $\kappa_d - 1$. Hence $\overline{\hat{A}}_1 = \pi^{-1}(\overline{A}_1)$ is a smooth complex manifold of dimension κ_d. In particular,

$$\overline{\hat{A}}_1 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f(\tilde{p}), \ \ \nabla f|_{\tilde{p}} = 0\}.$$

We now define a section of the following bundle

$$\Psi_{\mathcal{P}A_2} : \overline{\hat{A}}_1 \longrightarrow \mathbb{V}_{\mathcal{P}A_2} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^* \otimes \pi^*T^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^{*d}$$

given by $\{\Psi_{\mathcal{P}A_2}(\tilde{f}, l_{\tilde{p}})\}(f \otimes v) := \nabla^2 f|_{\tilde{p}}(v, \cdot) \quad \forall\, v \in \hat{\gamma}|_{l_{\tilde{p}}}$.

We will show shortly (in Lemma 4.9), that if $d \geq 2$, then this section is transverse to zero. Hence, if $\mathcal{Q}$ is a generic section, then the zeros of the section

$$\Psi_{\mathcal{P}A_2} \oplus \mathcal{Q} : \hat{\overline{A}}_1 \longrightarrow \mathbb{V}_{\mathcal{P}A_2} \oplus \mathbb{W}_{n,m,2},$$

counted with a sign, is our desired number. Hence, we have

$$\begin{aligned} N(\mathcal{P}A_2, n, m) &= \big\langle e(\mathbb{V}_{\mathcal{P}A_2})e(\mathbb{W}_{n,m,2}),\ [\hat{\overline{A}}_1]\big\rangle, \\ &= \big\langle e(\mathbb{V}_{\mathcal{P}A_2})e(\mathbb{W}_{n,m,2}),\ [\pi^{-1}(\overline{A}_1)]\big\rangle. \end{aligned} \tag{4.8}$$

By the splitting principle,

$$e(\mathbb{V}_{\mathcal{P}A_2})e(\mathbb{W}_{n,m,2}) = ((\lambda + y + da)^2 - 3a(\lambda + y + da) + 3a^2)y^{\kappa_d - (n+m+2)}a^n\lambda^m. \tag{4.9}$$

Next we use the fact that for all n_1

$$\begin{aligned} &\big\langle \pi^*(y^{\kappa_d-(n_1+1)}a^{n_1})\lambda,\ [\pi^{-1}(\overline{A}_1)]\big\rangle = \big\langle y^{\kappa_d-(n_1+1)}a^{n_1},\ [\overline{A}_1]\big\rangle \quad \text{and} \\ &\big\langle \pi^*(y^{\kappa_d-n_1}a^{n_1}),\ [\pi^{-1}(\overline{A}_1)]\big\rangle = 0. \end{aligned} \tag{4.10}$$

This follows from [8] (pp. 270). Finally, using the ring structure of $H^*(\mathbb{P}T\mathbb{P}^2; \mathbb{Z})$ (again using [8], pp. 270) we conclude that

$$\lambda^2 + 3a\lambda + 3a^2 = 0. \tag{4.11}$$

Using Eqs. (4.8)–(4.11) we obtain the identities stated at the outset. □

We now prove the closure and transversality claims.

Lemma 4.7. *If $d \geq 2$, then the variety $\overline{A}_1$ is given by*

$$\overline{A}_1 = \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(\tilde{p}) = 0,\ \ \nabla f_{\tilde{p}} = 0\}. \tag{4.12}$$

Proof. First, we note that by Lemma 3.2 we conclude that the space A_1 can be described as

$$A_1 = \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(\tilde{p}) = 0,\ \nabla f|_{\tilde{p}} = 0,\ \det\nabla^2 f|_{\tilde{p}} \neq 0\}. \tag{4.13}$$

Remark 4.8. We said that the space A_1 "can be described as" as opposed to "is described as". This is because by *definition*, the space A_1 is described as

$$A_1 := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 :\ f \text{ has an } A_1\text{-singularity at } \tilde{p}\}.$$

It is using Lemma 3.2 that we can describe the space A_1 as in (4.13).

Let us now continue with the proof of Lemma 4.7. It is clear the left hand side of (4.12) is a subset of the right hand side. We will now prove the converse. Suppose $(\tilde{f}, \tilde{p})$ belongs to the right hand side of (4.12). We need to show that there exists a sequence in the left hand side of (4.12) that converges to $(\tilde{f}, \tilde{p})$. In order to do that, we will use local coordinates. In an affine coordinate chart, let us fix the point $\tilde{p}$ to be $(0, 0)$ and let us assume that the Taylor expansion of f around $\tilde{p}$ is given by

$$f(x, y) = f_{00} + f_{10}x + f_{01}y + \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \cdots.$$

Since $(\tilde{f}, \tilde{p})$ belongs to the right hand side of (4.12), we conclude $f_{00},\ f_{10}$ and f_{01} are equal to zero. If $f_{20}f_{02} - f_{11}^2 \neq 0$ then there is nothing further to prove. Hence, suppose $f_{20}f_{02} - f_{11}^2 = 0$. Then we consider three cases: suppose $f_{20} \neq 0$. Then define the sequence f_n as

$$\begin{aligned} &(f_n)_{ij} := f_{ij} \qquad \text{if } (i,j) \neq (0,2) \quad \text{and} \\ &(f_n)_{02} := \frac{f_{11}^2}{f_{20}} + \frac{1}{n}. \end{aligned}$$

A similar argument holds if $f_{02} \neq 0$. Finally, suppose f_{20} and f_{02} are both zero. Since $f_{20}f_{02} - f_{11}^2 = 0$, this implies that f_{11} is also zero. Hence, construct the sequence given by

$$\begin{aligned} &(f_n)_{ij} := f_{ij} \qquad \text{if } (i,j) \neq (0,2) \ \text{or}\ (2,0) \quad \text{and} \\ &(f_n)_{02} := \frac{1}{n} \quad \text{and} \quad (f_n)_{20} := \frac{1}{n}. \end{aligned}$$

It is easy to see that in all the cases, f_n is a sequence in the left hand side of (4.12) that converges to $(\tilde{f}, \tilde{p})$. □

We now prove the transversality claim.

Lemma 4.9. *The section*

$$\begin{aligned} &\Psi_{\mathcal{P}A_2} : \overline{\hat{A}}_1 \longrightarrow \mathbb{V}_{\mathcal{P}A_2} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^* \otimes \pi^* T^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^{*d} \\ \textit{given by} \quad &\{\Psi_{\mathcal{P}A_2}(\tilde{f}, l_{\tilde{p}})\}(f \otimes v) := \nabla^2 f|_{\tilde{p}}(v, \cdot) \qquad \forall\, v \in \hat{\gamma}|_{l_{\tilde{p}}}. \end{aligned}$$

is transverse to the zero section if $d \geq 2$.

Proof. We continue with the setup of Lemma 4.3, but choose coordinate chart so that

$$\tilde{\mathcal{U}} := \{[a\partial_x,\ b\partial_y] \in \mathbb{P}T\mathbb{P}^2|_{\mathcal{U}} : a \neq 0\}, \qquad \varphi_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \longrightarrow \mathbb{C}^3,\ \ \varphi_{\tilde{\mathcal{U}}}([a\partial_x,\ b\partial_y]) = (x, y, \eta),$$

where $\eta := b/a$. With respect to this coordinate chart and the standard trivialization of the bundles, transversality is equivalent to showing that the map

$$\mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^5, \qquad (f, x, y, \eta) \mapsto \big(f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}\big)$$

is transverse to zero at $(f, 0, 0, 0)$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $5 \times (\kappa_d + 4)$ matrix which has full rank if $d \geq 2$. This is easy to see if the first five columns of the matrix are partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}$ and f_{11}. $\square$

4.4. Computing $N(A_3)$

Let us now consider the following two spaces

$$\begin{aligned} A_3 &:= \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times T\mathbb{P}^2 : f \text{ has a singularity of type } A_3 \text{ at } \tilde{p}\} \quad \text{and} \\ \mathcal{P}A_3 &:= \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } A_3 \text{ at } \tilde{p},\ \nabla^2 f|_{\tilde{p}}(v, \cdot) = 0\ \forall\, v \in \hat{\gamma}_{l_{\tilde{p}}}\}. \end{aligned}$$

Define the following number:

$$N(\mathcal{P}A_3, n, m) := \langle y^{\kappa_d - 3 - n - m} a^n \lambda^m,\ [\overline{\mathcal{P}A_3}]\rangle.$$

Since the projection map from $\mathcal{P}A_3$ to A_3 is one to one, we conclude that when $m = 0$,

$$N(A_3, n) = N(\mathcal{P}A_3, n, 0).$$

We are now ready to prove the formula for $N(A_3)$.

Theorem 4.10. *If $d \geq 3$, then*

$$N(\mathcal{P}A_3, n, m) = N(\mathcal{P}A_2, n, m) + 3N(\mathcal{P}A_2, n, m+1) + dN(\mathcal{P}A_2, n+1, m). \tag{4.14}$$

Note that combining this result with Theorem 4.5 we obtain the formula for $N(A_3)$ as stated in Theorem 1.3.

Proof. Define

$$\mathbb{W}_{n,m,3} := \left(\bigoplus_{i=1}^{\kappa_d - 3 - n - m} \gamma_{\mathcal{D}}^*\right) \oplus \left(\bigoplus_{i=1}^{n} \gamma_{\mathbb{P}^2}^*\right) \oplus \left(\bigoplus_{i=1}^{m} \hat{\gamma}^*\right) \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \tag{4.15}$$

and let $\mathcal{Q} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{W}_{n,m,3}$ be a generic smooth section. Again, unless $m = 0$, the bundle $\mathbb{W}_{n,m,3}$ need not have any holomorphic sections that is transverse to zero. We claim that

$$\overline{\mathcal{P}A}_2 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f(\tilde{p}),\ \nabla f|_{\tilde{p}} = 0,\ \nabla^2 f|_{\tilde{p}}(v, \cdot) = 0\ \forall\, v \in \hat{\gamma}|_{l_{\tilde{p}}}\}. \tag{4.16}$$

The proof of this claim is very similar to the proof of Lemma 4.7, hence we omit it. It is clear that the left hand side of (4.16) is a subset of the right hand side. Proving the converse is very similar to the proof of Lemma 4.7; given an element in the right hand side of (4.16), it is easy to construct a sequence that lies in $\mathcal{P}A_2$ converging to that given element.

Next, we note that this claim in particular implies that the variety $\overline{\mathcal{P}A}_2$ is smooth of dimension $\kappa_d - 2$; this follows from Lemma 4.9.

Let us now define a section of the following line bundle

$$\Psi_{\mathcal{P}A_3} : \overline{\mathcal{P}A}_2 \longrightarrow \mathbb{L}_{\mathcal{P}A_3} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^{*3} \otimes \gamma_{\mathbb{P}^2}^{*d} \tag{4.17}$$

defined by

$$\{\Psi_{\mathcal{P}A_3}(\tilde{f}, l_{\tilde{p}})\}(f \otimes v^{\otimes 3}) := \nabla^3 f|_{\tilde{p}}(v, v, v). \tag{4.18}$$

If $d \geq 3$, then this section is transverse to the zero section. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}$ and f_{30}.

Since $\mathcal{Q}$ is a generic section, the zeros of the section

$$\Psi_{\mathcal{P}A_3} \oplus \mathcal{Q} : \overline{\mathcal{P}A_2} \longrightarrow \mathbb{L}_{\mathcal{P}A_3} \oplus \mathbb{W}_{n,m,3}$$

counted with a sign is $N(\mathcal{P}A_3, n, m)$. Hence

$$N(\mathcal{P}A_3, n, m) = \langle e(\mathbb{L}_{\mathcal{P}A_3}) e(\mathbb{W}_{n,m,3}), \ \ [\overline{\mathcal{P}A_2}]\rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}A_3}) = y + 3\lambda + da,$$

we immediately get (4.14). $\square$

We will be using the following fact in our subsequent computations:

Lemma 4.11. *Let*

$$\Psi_{\mathcal{P}A_3} : \overline{\mathcal{P}A_2} \longrightarrow \mathbb{L}_{\mathcal{P}A_3} := \gamma_{\mathcal{D}}^{*} \otimes \hat{\gamma}^{*3} \otimes \gamma_{\mathbb{P}^2}^{*d}$$

be the section induced by taking the third derivative as defined in (4.18). If $d \geq 4$, then the variety $\overline{\mathcal{P}A_3}$ can be described as $\Psi_{\mathcal{P}A_3}^{-1}(0)$. In particular, the variety $\overline{\mathcal{P}A_3}$ is smooth of dimension $\kappa_d - 3$.

Proof. Let us first set up some terminology. Given an element $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2$, let us fix a non zero vector $v \in \hat{\gamma}|_{l_{\tilde{p}}}$. Furthermore, let us fix a non zero *representative*

$$w \in \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)\Big|_{l_{\tilde{p}}}$$

of the quotient space. Let us make the following abbreviation

$$f_{ij} := \nabla^{i+j} f|_{\tilde{p}}(\underbrace{v, \ldots, v}_{i \text{ times}}, \underbrace{w, \ldots, w}_{j \text{ times}}).$$

With this notation, we note that the space $\mathcal{P}A_3$ can be described as

$$\mathcal{P}A_3 = \left\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_2} : f_{30} = 0, \ \ f_{02} \neq 0, \ \ f_{02}f_{40} - 3f_{21}^2 \neq 0\right\}.$$

This follows from Lemma 3.4. First of all, note that all the three expressions are well defined on the quotient space. In order to see that, we need to show that the expressions are invariant under $w \longrightarrow w + v$. Clearly f_{30} is well defined, since it does not involve w. Next, we note that

$$\begin{aligned}\nabla^2 f_{\tilde{p}}(w + v, w + v) &= \nabla^2 f_{\tilde{p}}(w, w) + 2\nabla^2 f_{\tilde{p}}(v, w) + \nabla^2 f_{\tilde{p}}(v, v)\\ &= \nabla^2 f_{\tilde{p}}(w, w).\end{aligned}$$

The last equality follows from the fact that $\nabla^2 f_{\tilde{p}}(v, w)$ and $\nabla^2 f_{\tilde{p}}(v, v)$ are both zero. This is because $(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_2}$ and hence $\nabla^2 f|_{\tilde{p}}(v, \cdot) = 0$.

Next, we need to show that $f_{02}f_{40} - 3f_{21}^2$ is well defined. To see that, we note that

$$\begin{aligned}\nabla^3 f|_{\tilde{p}}(v, v, w + v) &= \nabla^3 f|_{\tilde{p}}(v, v, v) + \nabla^3 f|_{\tilde{p}}(v, v, w)\\ &= \nabla^3 f|_{\tilde{p}}(v, v, w).\end{aligned}$$

The last equality follows from the fact that $f_{30} = 0$. Hence f_{21} is well defined. Combined with the fact that f_{02} is well defined, we conclude that $f_{02}f_{40} - 3f_{21}^2$ is well defined.

We claim that

$$\overline{\mathcal{P}A_3} = \left\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_2} : f_{30} = 0\right\}. \tag{4.19}$$

Again, it is clear that the left hand side of (4.19) is a subset of its right hand side. To prove the converse, suppose $(\tilde{f}, l_{\tilde{p}})$ belongs to the right hand side of (4.19). Following the setup of Lemma 4.7, let us use local coordinates and fix the point $\tilde{p}$ to be $(0, 0)$. Suppose the Taylor expansion of f around $(0, 0)$ is given by

$$f(x, y) = f_{00} + f_{10}x + f_{01}y + \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \cdots .$$

Since $(\tilde{f}, l_{\tilde{p}})$ belongs to the right hand side of (4.19), we conclude that f_{00}, f_{10}, f_{01}, f_{20}, f_{11} and f_{30} are all 0. If f_{02} and $\left(f_{40} - \frac{3f_{21}^2}{f_{02}}\right)$ are both non zero, then there is nothing to prove. Suppose $f_{02} \neq 0$, but $f_{02}f_{40} - 3f_{21}^2 = 0$. Then define a sequence f_n given by

$$(f_n)_{ij} := f_{ij} \quad \text{if } (i,j) \neq (4,0),$$
$$(f_n)_{40} := \frac{3f_{21}^2}{f_{02}} + \frac{1}{n}.$$

This is a sequence that belongs to $\mathcal{P}A_3$ and converges to $(\tilde{f}, l_{\tilde{p}})$. Next, suppose

$$f_{02} = 0 \quad \text{and} \quad f_{02}f_{40} - 3f_{21}^2 = 0.$$

Then define a sequence given by

$$(f_n)_{ij} := f_{ij} \quad \text{if } (i,j) \neq (4,0),\ (2,1) \text{ or } (0,2),$$
$$(f_n)_{02} := \frac{1}{n}, \qquad (f_n)_{21} := \frac{1}{n} \quad \text{and} \quad (f_n)_{40} := \frac{4}{n}.$$

This is a sequence that belongs to $\mathcal{P}A_3$ and converges to $(\tilde{f}, l_{\tilde{p}})$. This proves the claim. The variety $\overline{\mathcal{P}A_3}$ is smooth because the section $\Psi_{\mathcal{P}A_3}$ is transverse to the zero section (as explained in the proof of Theorem 4.10). $\square$

4.5. Computing $N(D_4)$

Let us consider the spaces D_4 and $\mathcal{P}D_4$ defined as

$$D_4 := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } D_4 \text{ at } \tilde{p}\} \quad \text{and}$$
$$\mathcal{P}D_4 := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } D_4 \text{ at } \tilde{p},\ \nabla^3 f|_{\tilde{p}}(v,v,v) = 0 \text{ if } v \in \hat{\gamma}|_{l_{\tilde{p}}}\}.$$

Define the following number:

$$N(\mathcal{P}D_4, n, m) := \langle y^{\kappa_d - 4 - n - m} a^n \lambda^m,\ [\overline{\mathcal{P}D_4}]\rangle.$$

Note that the projection map $\pi : \mathcal{P}D_4 \longrightarrow D_4$ is three to one because there are *three* distinguished directions along which the third derivative vanishes for an ordinary triple point. Hence, when $m = 0$

$$N(D_4, n) = \frac{1}{3} N(\mathcal{P}D_4, n, 0). \tag{4.20}$$

We are now ready to prove the formula for $N(D_4)$. Let us first define

$$\mathbb{W}_{n,m,k} := \left(\bigoplus_{i=1}^{\kappa_d - k - n - m} \gamma_{\mathcal{D}}^*\right) \oplus \left(\bigoplus_{i=1}^{n} \gamma_{\mathbb{P}^2}^*\right) \oplus \left(\bigoplus_{i=1}^{m} \hat{\gamma}^*\right) \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2. \tag{4.21}$$

Theorem 4.12. *If $d \geq 3$, then*

$$N(\mathcal{P}D_4, n, m) = N(\mathcal{P}A_3, n, m) - 2N(\mathcal{P}A_3, n, m+1) + (d-6)N(\mathcal{P}A_3, n+1, m). \tag{4.22}$$

Note that combining this result with Theorem 4.10 and (4.20), we arrive at the formula for $N(D_4)$ stated in Theorem 1.3.

Proof. Let $\mathbb{W}_{n,m,k}$ be as in (4.21) with $k = 4$ and let $\mathcal{Q} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{W}_{n,m,4}$ be a generic smooth section. We continue with the setup of Theorem 4.10; given an element $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2$, we fix a non zero vector $v \in \hat{\gamma}|_{l_{\tilde{p}}}$ and fix a non zero representative

$$w \in \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)\Big|_{l_{\tilde{p}}}$$

on the quotient space. We also abbreviate

$$f_{ij} := \nabla^{i+j} f|_{\tilde{p}}(\underbrace{v, \dots, v}_{i \text{ times}}, \underbrace{w, \dots, w}_{j \text{ times}}). \tag{4.23}$$

We claim that the space $\mathcal{P}D_4$ can be described as

$$\mathcal{P}D_4 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_3} : f_{02} = 0,\ \ 4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2 \neq 0\}. \tag{4.24}$$

We will now justify why this is so. First of all we note that as before f_{02} and $4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2$ are well defined on the quotient space; the proof follows from a straightforward computation of these directional derivatives by replacing w with $w + v$.

Next, suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_4$. That implies that

$$f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30} = 0 \quad \text{and}$$
$$\beta := f_{30}^2 f_{03}^2 - 6f_{03}f_{12}f_{21}f_{30} + 4f_{12}^3 f_{30} + 4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2 \neq 0.$$

The last condition is saying that the cubic term in the Taylor expansion of f has no repeated roots (which is precisely the non degeneracy condition for an ordinary triple point). Note that β is the discriminant of the cubic term of the Taylor expansion. Let us rewrite the above two equations in the following way

$$f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30} = 0, \tag{4.25}$$
$$f_{02} = 0 \quad \text{and} \quad \beta \neq 0. \tag{4.26}$$

If $(\tilde{f}, l_{\tilde{p}})$ satisfies (4.25), then it means that $(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3$ (this follows from Lemma 4.11). Since $f_{30} = 0$, β simplifies to $4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2$. This proves (4.24).

Next, let us define a section of the following line bundle

$$\Psi_{\mathcal{P}D_4} : \overline{\mathcal{P}A}_3 \longrightarrow \mathbb{L}_{\mathcal{P}D_4} := \gamma_{\mathcal{D}}^* \otimes \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)^{*2} \otimes \gamma_{\mathbb{P}^2}^{*d}$$

defined by

$$\{\Psi_{\mathcal{P}D_4}(\tilde{f}, l_{\tilde{p}})\}(f \otimes w^{\otimes 2}) := \nabla^2 f|_{\tilde{p}}(w, w) = f_{02}. \tag{4.27}$$

If $d \geq 3$, then this section is transverse to the zero section. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}$ and f_{02}.

Since $\mathcal{Q}$ is a generic section, the zeros of the section

$$\Psi_{\mathcal{P}D_4} \oplus \mathcal{Q} : \overline{\mathcal{P}A}_3 \longrightarrow \mathbb{L}_{\mathcal{P}D_4} \oplus \mathbb{W}_{n,m,4}$$

counted with a sign is $N(\mathcal{P}D_4, n, m)$. Hence

$$N(\mathcal{P}D_4, n, m) = \langle e(\mathbb{L}_{\mathcal{P}D_4})e(\mathbb{W}_{n,m,3}), \ [\overline{\mathcal{P}A}_3]\rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}D_4}) = y + 2(-3a - \lambda) + da = y - 2\lambda + (d-6)a,$$

we immediately get (4.14). □

We will be using the following fact in our subsequent computations:

Lemma 4.13. *Let*

$$\Psi_{\mathcal{P}D_4} : \overline{\mathcal{P}A}_3 \longrightarrow \mathbb{L}_{\mathcal{P}D_4} := \gamma_{\mathcal{D}}^* \otimes \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)^{*2} \otimes \gamma_{\mathbb{P}^2}^{*d}$$

be the section as defined in (4.57). If $d \geq 3$, then the variety $\overline{\mathcal{P}D}_4$ can be described as $\Psi_{\mathcal{P}D_4}^{-1}(0)$. Furthermore, the variety $\overline{\mathcal{P}D}_4$ is smooth of dimension $\kappa_d - 4$.

Proof. As shown in the proof of Theorem 4.12, the space $\mathcal{P}D_4$ can be described as

$$\mathcal{P}D_4 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3 : f_{02} = 0, \ \ 4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2 \neq 0\}.$$

We claim that

$$\overline{\mathcal{P}D}_4 = \left\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3 : f_{02} = 0\right\}. \tag{4.28}$$

Again, it is clear that the left hand side of (4.28) is a subset of the right hand side. To prove the converse, suppose $(\tilde{f}, l_{\tilde{p}})$ belongs to the right hand side of (4.28). Following the setup of Lemma 4.7, let us use local coordinates and fix the point $\tilde{p}$ to be $(0, 0)$. Suppose the Taylor expansion of f around $(0, 0)$ is given by

$$f(x, y) = f_{00} + f_{10}x + f_{01}y + \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \cdots.$$

Since $(\tilde{f}, l_{\tilde{p}})$ belongs to the right hand side, we conclude that $f_{00}, \ f_{10}, \ f_{01}, \ f_{20}, \ f_{11}, \ f_{30}$ and f_{02} are all 0. If $4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2$ is non zero, then there is nothing to prove. Hence, assume that $4f_{03}f_{21}^3 - 3f_{12}^2 f_{21}^2 = 0$. This implies that either $4f_{03}f_{21} - 3f_{12}^2 = 0$ or $f_{21} = 0$. Let us first assume that $4f_{03}f_{21} - 3f_{12}^2 = 0$ but $f_{21} \neq 0$. Then define a sequence f_n given by

$$(f_n)_{ij} := f_{ij} \quad \text{if } (i, j) \neq (0, 3),$$
$$(f_n)_{03} := \frac{3f_{12}^2}{f_{21}} + \frac{1}{n}.$$

This is a sequence that belongs to $\mathcal{P}D_4$ and converges to $(\tilde{f}, l_{\tilde{p}})$. Next, suppose $f_{21} = 0$. Denote $\alpha := 4f_{03}f_{21} - 3f_{12}^2$ and define a sequence given by

$$(f_n)_{ij} := f_{ij} \quad \text{if } (i,j) \neq (2,1) \text{ or } (1,2),$$
$$(f_n)_{21} := \frac{1}{n}, \qquad (f_n)_{12} := \sqrt{\frac{1}{3}\Big(\frac{4f_{03}}{n} - \alpha\Big)},$$

where $\sqrt{\ }$ denotes a branch of the square root. This is a sequence that belongs to $\mathcal{P}D_4$ and converges to $(\tilde{f}, l_{\tilde{p}})$. This proves the claim.

The variety $\overline{\mathcal{P}D_4}$ is smooth because the section $\Psi_{\mathcal{P}D_4}$ is transverse to the zero section (as explained in the proof of Theorem 4.12). □

4.6. Computing $N(D_5)$

If $k \geq 5$, let us define the spaces D_k and $\mathcal{P}D_k$ as

$$D_k := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } D_k \text{ at } \tilde{p}\} \quad \text{and}$$
$$\mathcal{P}D_k := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } D_k \text{ at } \tilde{p},\ \nabla^3 f|_{\tilde{p}}(v, v, \cdot) = 0 \text{ if } v \in \hat{\gamma}|_{l_{\tilde{p}}}\}.$$

Define the following number:

$$N(\mathcal{P}D_k, n, m) := \langle y^{\kappa_d - k - n - m} a^n \lambda^m,\ [\overline{\mathcal{P}D_k}]\rangle.$$

Since the projection map $\pi : \mathcal{P}D_k \longrightarrow D_k$ is one to one if $k \geq 5$, we conclude that when $m = 0$,

$$N(\mathcal{P}D_k, n, 0) = N(D_k, n) \quad \forall\, k \geq 5. \tag{4.29}$$

We are now ready to prove the formula for $N(D_5)$.

Theorem 4.14. *If $d \geq 3$ then*

$$N(\mathcal{P}D_5, n, m) = N(\mathcal{P}D_4, n, m) + N(\mathcal{P}D_4, n, m+1) + (d-3)N(\mathcal{P}D_4, n+1, m). \tag{4.30}$$

Note that combining this result with Theorem 4.12 and (4.29), we arrive at the formula for $N(D_5)$ as stated in Theorem 1.3.

Proof. We continue with the setup of the proof of Theorem 4.12; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\big(\pi^* T\mathbb{P}^2/\hat{\gamma}\big)|_{l_{\tilde{p}}}$ respectively while f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,5}$ and $\mathcal{Q}$ be as in (4.15) with $k = 5$. We claim that the space $\mathcal{P}D_5$ can be described as

$$\mathcal{P}D_5 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_4} : f_{21} = 0,\ \ f_{12} \neq 0,\ \ f_{40} \neq 0\}. \tag{4.31}$$

Let us prove this claim. By Lemma 3.7, we conclude that $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$ if and only if

$$f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02} = 0, \tag{4.32}$$
$$f_{30}, f_{21} = 0 \quad \text{and} \tag{4.33}$$
$$f_{12}, f_{40} \neq 0. \tag{4.34}$$

Eq. (4.32) is simply saying that the linear and quadratic terms in the Taylor expansion of f around $\tilde{p}$ are zero. Since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$ we conclude that $\nabla^3 f|_{\tilde{p}}(v, v, \cdot) = 0$. This is equivalent to saying that $f_{30} = 0$ and $f_{21} = 0$. Finally, since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$, we conclude that f_{40} and f_{12} are non zero (using Lemma 3.7 and (3.2)). Hence, $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$ if and only if

$$f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02} = 0, \tag{4.35}$$
$$f_{21} = 0 \quad \text{and} \tag{4.36}$$
$$f_{12}, f_{40} \neq 0. \tag{4.37}$$

Eq. (4.35) holds if and only if $(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_4}$ (this is using Lemma 4.13). Eqs. (4.36) and (4.37) imply (4.31).

Let us now define a section of the following line bundle

$$\Psi_{\mathcal{P}D_5} : \overline{\mathcal{P}D_5} \longrightarrow \mathbb{L}_{\mathcal{P}D_5} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^{*2} \otimes \Big(\pi^* T\mathbb{P}^2/\hat{\gamma}\Big)^* \otimes \gamma_{\mathbb{P}^2}^{*d},$$

given by

$$\{\Psi_{\mathcal{P}D_5}(\tilde{f}, l_{\tilde{p}})\}(f \otimes v^{\otimes 2} \otimes w) := \nabla^3 f|_{\tilde{p}}(v, v, w) = f_{21}. \tag{4.38}$$

If $d \geq 3$, then this section is transverse to the zero section. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}$ and f_{21}.

Since $\mathcal{Q}$ is a generic section, the zeros of the section

$$\Psi_{\mathcal{P}D_5} \oplus \mathcal{Q} : \overline{\mathcal{P}D}_4 \longrightarrow \mathbb{L}_{\mathcal{P}D_5} \oplus \mathbb{W}_{n,m,5}$$

counted with a sign is $N(\mathcal{P}D_5, n, m)$. Hence

$$N(\mathcal{P}D_5, n, m) = \langle e(\mathbb{L}_{\mathcal{P}D_5}) e(\mathbb{W}_{n,m,5}), \ \ [\overline{\mathcal{P}D}_4]\rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}D_5}) = y + 2\lambda + (-3a - \lambda) + \lambda + da = y + \lambda + (d-3)a,$$

we immediately get (4.30). $\square$

We will be using the following fact in our subsequent computations:

Lemma 4.15. *Let*

$$\Psi_{\mathcal{P}D_5} : \overline{\mathcal{P}D}_4 \longrightarrow \mathbb{L}_{\mathcal{P}D_5} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^{*2} \otimes \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)^* \otimes \gamma_{\mathbb{P}^2}^{*d}$$

be the section as defined in (4.38). If $d \geq 4$, then the variety $\overline{\mathcal{P}D}_5$ can be described as $\Psi_{\mathcal{P}D_5}^{-1}(0)$. Furthermore, the variety $\overline{\mathcal{P}D}_5$ is smooth of dimension $\kappa_d - 5$.

Proof. As shown in the proof of Theorem 4.14, the space $\mathcal{P}D_5$ can be described as

$$\mathcal{P}D_5 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_4 : f_{21} = 0, \ \ f_{12} \neq 0, \ \ f_{40} \neq 0\}.$$

We claim that

$$\overline{\mathcal{P}D}_5 = \left\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_4 : f_{21} = 0\right\}. \tag{4.39}$$

Again, it is clear that the left hand side of (4.39) is a subset of the right hand side. The converse follows in the same way we proved Lemmas 4.7 and 4.13. $\square$

4.7. Computing $N(D_6)$ and $N(E_6)$

In Section 4.6, we have defined the spaces $\mathcal{P}D_k$ for all $k \geq 5$. Let us now similarly define the spaces E_k and $\mathcal{P}E_k$ for $k = 6$ and 7:

$$E_k := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } E_k \text{ at } \tilde{p}\} \quad \text{and}$$
$$\mathcal{P}E_k := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } E_k \text{ at } \tilde{p}, \ \nabla^3 f|_{\tilde{p}}(v, v, \cdot) = 0 \text{ if } v \in \hat{\gamma}|_{l_{\tilde{p}}}\}.$$

Define the following number:

$$N(\mathcal{P}E_k, n, m) := \langle y^{\kappa_d - k - n - m} a^n \lambda^m, \ [\overline{\mathcal{P}E}_k]\rangle.$$

Since the projection map $\pi : \mathcal{P}E_k \longrightarrow E_k$ is one to one, we conclude that when $m = 0$,

$$N(\mathcal{P}E_k, n, 0) = N(E_k, n). \tag{4.40}$$

We will now prove the formula for $N(D_6)$ and $N(E_6)$. Since their computations are similar, we will give the proofs simultaneously.

Theorem 4.16. *The numbers $N(\mathcal{P}D_6, n, m)$ and $N(\mathcal{P}E_6, n, m)$ are given by*

$$N(\mathcal{P}D_6, n, m) = N(\mathcal{P}D_5, n, m) + 4N(\mathcal{P}D_5, n, m+1) + dN(\mathcal{P}D_5, n+1, m) \tag{4.41}$$

and

$$N(\mathcal{P}E_6, n, m) = N(\mathcal{P}D_5, n, m) - N(\mathcal{P}D_5, n, m+1) + (d-6)N(\mathcal{P}D_5, n+1, m) \tag{4.42}$$

provided $d \geq 4$ and 3 respectively.

Note that combining this result with Theorem 4.14, (4.29) and (4.40) we arrive at the formulas for $N(D_6)$ and $N(E_6)$ as stated in Theorem 1.3.

Proof. We continue with the setup of the proof of Theorem 4.14; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ and f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,6}$ and $\mathcal{Q}$ be as in (4.15) with $k = 6$. We note that the spaces $\mathcal{P}D_6$ and $\mathcal{P}E_6$ can be described as

$$\mathcal{P}D_6 = \left\{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_5 : f_{40} = 0,\ \ f_{12} \neq 0,\ \ \left(f_{12}f_{50} - \frac{5f_{31}^2}{3}\right) \neq 0\right\}, \tag{4.43}$$

$$\mathcal{P}E_6 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_5 : f_{12} = 0,\ \ f_{31} \neq 0,\ \ f_{40} \neq 0\}. \tag{4.44}$$

This follows from Lemmas 3.7 and 3.9 and using (3.2). Define sections of the following line bundles

$$\Psi_{\mathcal{P}D_6} : \overline{\mathcal{P}D}_5 \longrightarrow \mathbb{L}_{\mathcal{P}D_6} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^{*4} \otimes \gamma_{\mathbb{P}^2}^{*d} \quad \text{and}$$

$$\Psi_{\mathcal{P}E_6} : \overline{\mathcal{P}D}_5 \longrightarrow \mathbb{L}_{\mathcal{P}E_6} := \gamma_{\mathcal{D}}^* \otimes \hat{\gamma}^{*} \otimes \left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)^{*2} \otimes \gamma_{\mathbb{P}^2}^{*d} \tag{4.45}$$

given by

$$\{\Psi_{\mathcal{P}D_6}(\tilde{f}, l_{\tilde{p}})\}(f \otimes v^{\otimes 4}) := \nabla^4 f|_{\tilde{p}}(v, v, v, v) = f_{40} \quad \text{and}$$

$$\{\Psi_{\mathcal{P}E_6}(\tilde{f}, l_{\tilde{p}})\}(f \otimes v^{\otimes 2} \otimes w) := \nabla^3 f|_{\tilde{p}}(v, w, w) = f_{12}. \tag{4.46}$$

If $d \geq 4, 3$, then the sections $\Psi_{\mathcal{P}D_6}$ and $\Psi_{\mathcal{P}E_6}$ are transverse to zero respectively. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}$ and: (a) f_{40} in the first case and (b) f_{12} in the second case.

Since $\mathcal{Q}$ is a generic section, the zeros of the section

$$\Psi_{\mathcal{P}D_6} \oplus \mathcal{Q} : \overline{\mathcal{P}D}_5 \longrightarrow \mathbb{L}_{\mathcal{P}D_6} \oplus \mathbb{W}_{n,m,6} \quad \text{and} \quad \Psi_{\mathcal{P}E_6} \oplus \mathcal{Q} : \overline{\mathcal{P}D}_5 \longrightarrow \mathbb{L}_{\mathcal{P}E_6} \oplus \mathbb{W}_{n,m,6}$$

counted with a sign is $N(\mathcal{P}D_6, n, m)$ and $N(\mathcal{P}E_6, n, m)$ respectively. Hence

$$N(\mathcal{P}D_6, n, m) = \langle e(\mathbb{L}_{\mathcal{P}D_6})e(\mathbb{W}_{n,m,6}),\ \ [\overline{\mathcal{P}D}_5]\rangle \quad \text{and}$$

$$N(\mathcal{P}E_6, n, m) = \langle e(\mathbb{L}_{\mathcal{P}E_6})e(\mathbb{W}_{n,m,6}),\ \ [\overline{\mathcal{P}D}_5]\rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}D_6}) = y + 4\lambda + da \qquad e(\mathbb{L}_{\mathcal{P}E_6}) = y - \lambda + (d-6)a$$

we immediately get (4.41) and (4.42). □

We will be using the following fact in our subsequent computations:

Lemma 4.17. *Let*

$$\Psi_{\mathcal{P}D_6} : \overline{\mathcal{P}D}_5 \longrightarrow \mathbb{L}_{\mathcal{P}D_6} \quad \textit{and} \quad \Psi_{\mathcal{P}E_6} : \overline{\mathcal{P}D}_5 \longrightarrow \mathbb{L}_{\mathcal{P}E_6}$$

be the sections as defined in (4.45). The varieties $\overline{\mathcal{P}D}_6$ and $\overline{\mathcal{P}E}_6$ are smooth of dimension $\kappa_d - 6$ and can be described as $\Psi_{\mathcal{P}D_6}^{-1}(0)$ and $\Psi_{\mathcal{P}E_6}^{-1}(0)$ respectively, provided $d \geq 5$ and 4 respectively.

Proof. As explained in the proof of Theorem 4.16, the spaces $\mathcal{P}D_6$ and $\mathcal{P}E_6$ can be described as in (4.43) and (4.44). We claim that

$$\overline{\mathcal{P}D}_6 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_5 : f_{40} = 0\}, \tag{4.47}$$

$$\overline{\mathcal{P}E}_6 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_5 : f_{12} = 0\}. \tag{4.48}$$

Again, it is clear that the left hand sides of (4.47) and (4.48) are subsets of their respective right hand sides. In both the cases, the converse follows in the same way we prove Lemma 4.7, Lemma 4.13 and Lemma 4.15. The varieties $\overline{\mathcal{P}D}_6$ and $\overline{\mathcal{P}E}_6$ are smooth because the sections $\Psi_{\mathcal{P}D_6}$ and $\Psi_{\mathcal{P}E_6}$ are transverse to zero (as explained in the proof of Theorem 4.16). □

4.8. Computing $N(E_7)$

We are now ready to prove the formula for $N(E_7)$.

Theorem 4.18. *If $d \geq 5$, then*

$$N(\mathcal{P}E_7, n, m) = N(\mathcal{P}D_6, n, m) - N(\mathcal{P}D_6, n, m+1) + (d-6)N(\mathcal{P}D_6, n+1, m). \tag{4.49}$$

Note that combining this result with Theorem 4.16, and (4.40) we arrive at the formula for $N(E_7)$ as stated in Theorem 1.3.

Proof. We continue with the setup of the proof of Theorem 4.16; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ respectively while f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,7}$ and $\mathcal{Q}$ be as in (4.15) with $k=7$. We note that the space $\mathcal{P}E_7$ can be described as

$$\mathcal{P}E_7 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_6 : f_{12}=0,\ \ f_{03}\neq 0,\ \ f_{31}\neq 0\}.$$

This follows from Lemmas 4.17 and 3.10. Define a section of the following line bundle

$$\Psi_{\mathcal{P}E_7} : \overline{\mathcal{P}D}_6 \longrightarrow \mathbb{L}_{\mathcal{P}E_7} := \gamma_{\mathcal{D}}^* \otimes \left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)^* \otimes \hat{\gamma}^{*2} \otimes \gamma_{\mathbb{P}^2}^{*d}$$

defined by

$$\{\Psi_{\mathcal{P}E_7}(\tilde{f}, l_{\tilde{p}})\}(f\otimes v^{\otimes 4}) := \nabla^4 f|_{\tilde{p}}(v, w, w) = f_{12}. \tag{4.50}$$

If $d\geq 4$, then this section is transverse to the zero section. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{40}$ and f_{12}.

Since $\mathcal{Q}$ is a generic section, the zeros of the section

$$\Psi_{\mathcal{P}E_7} \oplus \mathcal{Q} : \overline{\mathcal{P}E}_6 \longrightarrow \mathbb{L}_{\mathcal{P}E_7} \oplus \mathbb{W}_{n,m,7}$$

counted with a sign is $N(\mathcal{P}E_7, n, m)$. Hence

$$N(\mathcal{P}E_7, n, m) = \langle e(\mathbb{L}_{\mathcal{P}E_7}) e(\mathbb{W}_{n,m,7}),\ \ [\overline{\mathcal{P}D}_6]\rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}E_7}) = y - \lambda + (d-6)a,$$

we immediately get (4.49). $\square$

4.9. Computing $N(D_7)$ and $N(A_4)$

We are now ready to prove the formula for $N(D_7)$ and $N(A_4)$. These are the first two examples of non-linear singularities.

Theorem 4.19. *If $d\geq 5$, then*

$$N(\mathcal{P}D_7, n, m) = 2N(\mathcal{P}D_6, n, m) + 4N(\mathcal{P}D_6, n, m+1) + (2d-6)N(\mathcal{P}D_6, n+1, m). \tag{4.51}$$

Note that combining this result with Theorem 4.16, and (4.29) we arrive at the formula for $N(D_7)$ as stated in Theorem 1.5.

Proof. We continue with the setup of the proof of Theorem 4.16; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ respectively and f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,7}$ and $\mathcal{Q}$ be as in (4.15) with $k=7$. We note that the space $\mathcal{P}D_7$ can be described as

$$\mathcal{P}D_7 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_6 :\ \ f_{12}D_7^f = 0,\ \ f_{12}^3 D_8^f \neq 0,\ \ f_{12}\neq 0\},$$

where D_k^f is as defined in (3.2). This follows from Lemma 3.7. Note that $f_{12}D_7^f$ and $f_{12}^3D_8^f$ are defined even when $f_{12}=0$ (as opposed to D_7^f and D_8^f). Define a section of the following line bundle given by

$$\Psi_{\mathcal{P}D_7} : \overline{\mathcal{P}D}_6 \longrightarrow \mathbb{L}_{\mathcal{P}D_7} := \gamma_{\mathcal{D}}^{*2} \otimes \hat{\gamma}^{*6} \otimes \left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)^{*2} \otimes \gamma_{\mathbb{P}^2}^{*2d}$$

defined by

$$\{\Psi_{\mathcal{P}D_7}(\tilde{f}, l_{\tilde{p}})\}(f^{\otimes 2}\otimes v^{\otimes 6}\otimes w^{\otimes 2}) := f_{12}D_7^f = f_{12}f_{50} - \frac{5f_{31}^2}{3}. \tag{4.52}$$

Unlike the previous cases, this section is not transverse to zero everywhere. Let us now define the following two sets

$$M := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_6 : f_{12}\neq 0\} \quad \text{and}$$
$$\mathcal{B} := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_6 : f_{12}=0\}.$$

Restricted to M, the section $\Psi_{\mathcal{P}D_7}$ is transverse to zero if $d\geq 5$. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{12}, f_{40}$ and f_{50}. Note that the condition $f_{12}\neq 0$ is used to achieve transversality, since in the last step we take the partial derivative with respect to f_{50}. The section $\Psi_{\mathcal{P}D_7}$ is *not* transverse to zero on the *whole* of $\overline{\mathcal{P}D}_6$.

Since $\mathcal{Q}$ is a generic section, the signed number of zeros of the section

$$\Psi_{\mathcal{P}D_7} \oplus \mathcal{Q} : \overline{\mathcal{P}D_6} \longrightarrow \mathbb{L}_{\mathcal{P}D_7} \oplus \mathbb{W}_{n,m,7}$$

restricted to M is $N(\mathcal{P}D_7, n, m)$. This section a priori could vanish on $\mathcal{B}$. We claim that this section does not vanish on $\mathcal{B}$. Assuming that claim, we conclude that

$$N(\mathcal{P}D_7, n, m) = \langle e(\mathbb{L}_{\mathcal{P}D_7})e(\mathbb{W}_{n,m,7}), \ [\overline{\mathcal{P}D_6}]\rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}D_7}) = 2y + 4\lambda + (2d-6)a,$$

we immediately get (4.51).

It remains to explain why the section $\Psi_{\mathcal{P}D_7} \oplus \mathcal{Q}$ does not vanish on $\mathcal{B}$. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{B}$ and $\Psi_{\mathcal{P}D_7}(\tilde{f}, l_{\tilde{p}}) = 0$. That means

$$f_{12} = 0 \quad \text{and} \quad \left(f_{12}f_{50} - \frac{5f_{31}^2}{3}\right) = 0 \Longrightarrow f_{31} = 0.$$

Hence, let us define the space

$$S := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_6} : f_{12} = 0, \ \ f_{31} = 0\}.$$

We claim that S is a smooth manifold of dimension $\kappa_d - 8$; the proof again follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{40}, f_{12}, f_{31}$. Next, we note that $\mathcal{Q}$ is a section of a rank $\kappa_d - 7$ bundle. Hence, if $\mathcal{Q}$ is a generic smooth section then $\mathcal{Q}^{-1}(0)$ will not intersect S, since the dimension of S is less than the rank of $\mathbb{W}_{n,m,7}$. Hence $\Psi_{\mathcal{P}D_7} \oplus \mathcal{Q}$ does not vanish on $\mathcal{B}$ if $\mathcal{Q}$ is generic. □

The computation of $N(A_4)$ is almost identical. Before we compute $N(A_4)$, let us make a definition. If $k \geq 2$, then define the spaces A_k and $\mathcal{P}A_k$ as

$$A_k := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } A_k \text{ at } \tilde{p}\} \quad \text{and}$$
$$\mathcal{P}A_k := \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } A_k \text{ at } \tilde{p}, \ \nabla^2 f|_{\tilde{p}}(v, \cdot) = 0 \quad \text{if } v \in \hat{\gamma}|_{l_{\tilde{p}}}\}.$$

Define the following number:

$$N(\mathcal{P}A_k, n, m) := \langle y^{\kappa_d - k - n - m} a^n \lambda^m, \ [\overline{\mathcal{P}A_k}]\rangle.$$

Since the projection map $\pi : \mathcal{P}A_k \longrightarrow A_k$ is one to one we conclude that when $m = 0$,

$$N(\mathcal{P}A_k, n, 0) = N(A_k, n). \tag{4.53}$$

Recall that we had already defined these spaces for $k = 2$ and 3 (see Sections 4.3 and 4.4). We are now ready to prove the formula for $N(A_4)$.

Theorem 4.20. *If $d \geq 4$ then*

$$N(\mathcal{P}A_4, n, m) = 2N(\mathcal{P}A_3, n, m) + 2N(\mathcal{P}A_3, n, m+1) + (2d-6)N(\mathcal{P}A_3, n+1, m). \tag{4.54}$$

Note that combining this result with Theorem 4.10 and (4.53) we arrive at the formula for $N(A_4)$ as stated in Theorem 1.3.

Proof. We continue with the setup of the proof of Theorem 4.10 and Lemma 4.11; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ respectively while f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,4}$ and $\mathcal{Q}$ be as in (4.15) with $k = 4$. We note that the space $\mathcal{P}A_4$ can be described as

$$\mathcal{P}A_4 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_3} : \ \ f_{02}A_4^f = 0, \ \ f_{02}^2 A_5^f \neq 0, \ \ f_{02} \neq 0\},$$

where A_k^f is as defined in (3.1). This follows from Lemma 3.4. Note that $f_{02}A_4^f$ and $f_{02}^2A_5^f$ are defined even when $f_{02} = 0$ (as opposed to A_4^f and A_5^f). Define section of the following line bundle given by

$$\Psi_{\mathcal{P}A_4} : \overline{\mathcal{P}A_3} \longrightarrow \mathbb{L}_{\mathcal{P}A_4} := \gamma_{\mathcal{D}}^{*2} \otimes \hat{\gamma}^{*4} \otimes \left(\pi^*T\mathbb{P}^2/\hat{\gamma}\right)^{*2} \otimes \gamma_{\mathbb{P}^2}^{*2d}$$

defined by

$$\{\Psi_{\mathcal{P}A_4}(\tilde{f}, l_{\tilde{p}})\}(f^{\otimes 2} \otimes v^{\otimes 4} \otimes w^{\otimes 2}) := f_{02}A_4^f = f_{02}f_{40} - 3f_{21}^2. \tag{4.55}$$

This section is *not* transverse to zero everywhere. Let us now define the following two sets

$$M := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3 : f_{02} \neq 0\} \quad \text{and}$$
$$\mathcal{B} := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3 : f_{02} = 0\}.$$

Restricted to M, the section $\Psi_{\mathcal{P}A_4}$ is transverse to zero if $d \geq 4$. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}$ and f_{40}. Note that we required the condition $f_{02} \neq 0$ to achieve transversality because in the last step we compute the derivative with respect to f_{40}.

Since $\mathcal{Q}$ is a generic section, the signed number of zeros of the section

$$\Psi_{\mathcal{P}A_4} \oplus \mathcal{Q} : \overline{\mathcal{P}A}_3 \longrightarrow \mathbb{L}_{\mathcal{P}A_4} \oplus \mathbb{W}_{n,m,4}$$

restricted to M is $N(\mathcal{P}A_4, n, m)$. This section a priori could vanish on $\mathcal{B}$. We claim that this section does not vanish on $\mathcal{B}$. Assuming that claim, we conclude that

$$N(\mathcal{P}A_4, n, m) = \langle e(\mathbb{L}_{\mathcal{P}A_4}) e(\mathbb{W}_{n,m,4}), \ [\overline{\mathcal{P}A}_3] \rangle.$$

Since

$$e(\mathbb{L}_{\mathcal{P}A_4}) = 2y + 2\lambda + (2d - 6)a,$$

we immediately get (4.54).

It remains to explain why the section $\Psi_{\mathcal{P}A_3} \oplus \mathcal{Q}$ does not vanish on $\mathcal{B}$. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{B}$ and $\Psi_{\mathcal{P}A_4}(\tilde{f}, l_{\tilde{p}}) = 0$. That means

$$f_{02} = 0 \quad \text{and} \quad \left(f_{02} f_{40} - 3 f_{21}^2\right) = 0 \Longrightarrow f_{21} = 0.$$

Hence, let us define the space

$$S := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3 : f_{02} = 0, \ \ f_{21} = 0\}.$$

We claim that S is a smooth manifold of dimension $\kappa_d - 5$; the proof again follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}$, and f_{21}. Next, we note that $\mathcal{Q}$ is a section of a rank $\kappa_d - 4$ bundle. Hence, if $\mathcal{Q}$ is a generic smooth section then $\mathcal{Q}^{-1}(0)$ will not intersect S, since the dimension of S is less than the rank of $\mathbb{W}_{n,m,4}$. Hence $\Psi_{\mathcal{P}A_4} \oplus \mathcal{Q}$ does not vanish on $\mathcal{B}$ if $\mathcal{Q}$ is generic. □

4.10. Degenerate contribution to the Euler class: computing $N(A_5)$

We are now ready to prove the formula for $N(A_5)$. This is the first example where we will be computing a degenerate contribution to the Euler class.

Theorem 4.21. *If $d \geq 5$ then*

$$N(\mathcal{P}A_5, n, m) = 3N(\mathcal{P}A_4, n, m) + N(\mathcal{P}A_4, n, m+1) + (3d - 12)N(\mathcal{P}A_4, n+1, m) - 2N(\mathcal{P}D_5, n, m). \tag{4.56}$$

Note that combining this result with Theorems 4.20, 4.14 and (4.53) we arrive at the formula for $N(A_5)$ as stated in Theorem 1.3.

Proof. We continue with the setup of the proof of Theorem 4.20; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ respectively and f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,5}$ and $\mathcal{Q}$ be as in (4.15) with $k = 5$. We note that the space $\mathcal{P}A_5$ can be described as

$$\mathcal{P}A_5 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_4 : \ f_{02}^2 A_5^f = 0, \ f_{02}^3 A_6^f \neq 0, \ f_{02} \neq 0\},$$

where A_k^f is as defined in (3.1). This follows from Lemma 3.4. Define section of the following line bundle given by

$$\Psi_{\mathcal{P}A_5} : \overline{\mathcal{P}A}_4 \longrightarrow \mathbb{L}_{\mathcal{P}A_5} := \gamma_{\mathcal{D}}^{*3} \otimes \hat{\gamma}^{*5} \otimes \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)^{*4} \otimes \gamma_{\mathbb{P}^2}^{*3d}$$

defined by

$$\{\Psi_{\mathcal{P}A_4}(\tilde{f}, l_{\tilde{p}})\}(f^{\otimes 3} \otimes v^{\otimes 5} \otimes w^{\otimes 4}) := f_{02}^2 A_5^f. \tag{4.57}$$

Note that we are being a little bit sloppy with our terminology; since $\overline{\mathcal{P}A}_4$ is not a smooth variety, a little bit of care is needed in saying that $\mathbb{L}_{\mathcal{P}A_5}$ is a "bundle" over $\overline{\mathcal{P}A}_4$. Restricted to the smooth part of the variety, $\mathbb{L}_{\mathcal{P}A_5}$ is a holomorphic vector bundle. On the whole space $\overline{\mathcal{P}A}_4$, $\mathbb{L}_{\mathcal{P}A_5}$ is a topological vector bundle (the transition maps are continuous). Furthermore, restricted to each stratum of the variety $\overline{\mathcal{P}A}_4$, $\mathbb{L}_{\mathcal{P}A_5}$ is a holomorphic bundle. Henceforth we will be using the terminology that L is a bundle over some variety, even when the variety is not smooth.

Let us now define the following two sets

$$M := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_4 : f_{02} \neq 0\} \quad \text{and}$$
$$\mathcal{B} := \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_4 : f_{02} = 0\}.$$

Note that M belongs to the smooth part of the variety $\overline{\mathcal{P}A}_4$. This follows from the fact that the section $\Psi_{\mathcal{P}A_4}$ defined in the proof of Theorem 4.20 is transverse to zero when $f_{02} \neq 0$.

Restricted to M, the section $\Psi_{\mathcal{P}A_5}$ is transverse to zero if $d \geq 5$. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}$ and f_{50}. Note that we required the condition $f_{02} \neq 0$ to achieve transversality because in the last two steps we compute the derivative with respect to f_{40} and f_{50}.

Since $\mathcal{Q}$ is a generic section, the signed number of zeros of the section

$$\Psi_{\mathcal{P}A_5} \oplus \mathcal{Q} : \overline{\mathcal{P}A}_4 \longrightarrow \mathbb{L}_{\mathcal{P}A_5} \oplus \mathbb{W}_{n,m,5}$$

restricted to M is $N(\mathcal{P}A_5, n, m)$. This section *does* vanish on $\mathcal{B}$. We claim that

$$\mathcal{B} = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_4 : f_{02} = 0\} = \overline{\mathcal{P}D}_5. \tag{4.58}$$

Furthermore, we claim that the section $\Psi_{\mathcal{P}A_5} \oplus \mathcal{Q}$ vanishes on each of the points of $\mathcal{B}$ with a multiplicity of 2. Assuming this claim, we conclude that

$$\big\langle e(\mathbb{L}_{\mathcal{P}A_5}) e(\mathbb{W}_{n,m,5}),\ [\overline{\mathcal{P}A}_4] \big\rangle = N(\mathcal{P}A_5, n, m) + 2N(\mathcal{P}D_5, n, m). \tag{4.59}$$

Since

$$e(\mathbb{L}_{\mathcal{P}A_5}) = 3y + \lambda + (3d - 12)a,$$

we immediately get (4.56).

Remark 4.22. Let us take a careful look at Eq. (4.59). First of all we note that although $\overline{\mathcal{P}A}_4$ is not a smooth variety, (4.59) makes perfect sense. This is because, $\overline{\mathcal{P}A}_4$ defines an integer homology class since the singularities of $\overline{\mathcal{P}A}_4$ are of complex codimension one and hence real codimension two. This fact is proven in [23]. The left hand side of (4.59) is the pairing of the cohomology class $e(\mathbb{L}_{\mathcal{P}A_5})$ evaluated on the homology class $\overline{\mathcal{P}A}_5$. By the discussion on pages 4 and 5 of [24], we conclude that the Euler class is the number of zeros in the section we consider on the open part, plus the contribution of the section to the Euler class from the boundary $\mathcal{B}$. The discussion on pages 4 and 5 of [24] is a generalization of Theorem 1.3 where the base space is a variety that is not necessarily smooth.

We now start the proof of (4.58) and the multiplicity claim. We need to show that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_4 : f_{02} = 0\} = \overline{\mathcal{P}D}_5. \tag{4.60}$$

First we note that the left hand side of (4.60) is a subset of its right hand side. To see why that is so, we note that if $(\tilde{f}, l_{\tilde{p}})$ belongs to the left hand side, then $(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3$ and $f_{02}A_4^f = 0$ and $f_{02} = 0$. This implies that

$$(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_3, \qquad f_{02} = 0 \quad \text{and} \quad f_{21} = 0.$$

Hence $(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D}_5$, by (4.39) and (4.28).

We will now show that the right hand side of (4.60) is a subset of its left hand side. We will simultaneously prove the following two statements:

$$\overline{\mathcal{P}A}_4 \supset \mathcal{P}D_5, \tag{4.61}$$
$$\overline{\mathcal{P}A}_5 \cap \mathcal{P}D_5 = \varnothing. \tag{4.62}$$

Since $\overline{\mathcal{P}A}_4$ is a closed set, (4.61) implies that the right hand side of (4.60) is a subset of its left hand side.[7]

Claim 4.23. *Let $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$. Then there exists a solution $(\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{P}A}_3$ near $(\tilde{f}, l_{\tilde{p}})$ to the set of equations*

$$f_{02}(t) \neq 0, \qquad f_{02}(t)A_4^{f(t)} = f_{02}(t)f_{40}(t) - 3f_{21}(t)^2 = 0. \tag{4.63}$$

Moreover, whenever such a solution $(\tilde{f}(t), l_{\tilde{p}}(t))$ is sufficiently close to $(\tilde{f}, l_{\tilde{p}})$ it lies in $\mathcal{P}A_4$, i.e., $A_5^{f(t)} \neq 0$. In particular $(\tilde{f}(t), l_{\tilde{p}}(t))$ does not lie in $\mathcal{P}A_5$.

It is easy to see that Claim 4.23 proves statements (4.61) and (4.62) simultaneously.

[7] Eq. (4.62) is not necessary right now, but it will be needed later.

Proof of Claim 4.23. It is easy to see that the *only* solutions to (4.63) are of the form

$$f_{21}(t) = t, \qquad f_{40}(t) \neq 0, \qquad f_{02}(t) = \frac{3t^2}{f_{40}(t)}, \quad t \neq 0 \,(\text{but small}). \tag{4.64}$$

Note that since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$, we conclude that $f_{40}, f_{12} \neq 0$ (this follows from Lemma 3.7). Hence, if $(\tilde{f}(t), l_{\tilde{p}}(t))$ is sufficiently close to $(\tilde{f}, l_{\tilde{p}})$, we conclude $f_{40}(t), f_{12}(t) \neq 0$. Next, we observe that using (4.64), we get

$$f_{02}(t)^2 A_5^{f(t)} = 15 f_{12}(t) t^2 + O(t^3). \tag{4.65}$$

Since t is small but non zero, $f_{02}(t)^2 A_5^{f(t)}$ is non zero, proving the claim. □

Corollary 4.24. *Let $\mathbb{W}_{n,m,5}$ and $\mathcal{Q}$ be as in (4.15) with $k = 5$. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5 \cap \mathcal{Q}^{-1}(0)$. Then the section*

$$\Psi_{\mathcal{P}A_5} \oplus \mathcal{Q} : \overline{\mathcal{P}A}_4 \longrightarrow \mathbb{L}_{\mathcal{P}A_5} \oplus \mathbb{W}$$

vanishes around $(\tilde{f}, l_{\tilde{p}})$ with a multiplicity of 2.

Proof. First we note that if $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_5$ then $f_{12} \neq 0$. The claim now follows immediately from (4.65) and using the fact that $\mathcal{Q}^{-1}(0)$ intersects $\mathcal{P}D_5$ transversely. □

4.11. *Degenerate contribution to the Euler class: computing $N(A_6)$*

We are now ready to prove the formula for $N(A_6)$.

Theorem 4.25. *If $d \geq 6$ then*

$$N(\mathcal{P}A_6, n, m) = 4N(\mathcal{P}A_5, n, m) + (4d - 18)N(\mathcal{P}A_5, n+1, m) - 4N(\mathcal{P}D_6, n, m) - 3N(\mathcal{P}E_6, n, m). \tag{4.66}$$

Note that combining this result with Theorems 4.21, 4.16 and (4.53) we arrive at the formula for $N(A_6)$ as stated in Theorem 1.3.

Proof. We continue with the setup of the proof of Theorem 4.21; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ respectively and f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,m,6}$ and $\mathcal{Q}$ be as in (4.15) with $k = 6$. We note that the space $\mathcal{P}A_6$ can be described as

$$\mathcal{P}A_5 = \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_5 : \ f_{02}^3 A_6^f = 0, \ f_{02}^4 A_7^f \neq 0, \ f_{02} \neq 0\},$$

where A_k^f is as defined in (3.1). This follows from Lemma 3.4. Define a section of the following line bundle

$$\Psi_{\mathcal{P}A_6} : \overline{\mathcal{P}A}_5 \longrightarrow \mathbb{L}_{\mathcal{P}A_6} := \gamma_{\mathcal{D}}^{*4} \otimes \hat{\gamma}^{*6} \otimes \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)^{*6} \otimes \gamma_{\mathbb{P}^2}^{*4d}$$

given by

$$\{\Psi_{\mathcal{P}A_6}(\tilde{f}, l_{\tilde{p}})\}(f^{\otimes 4} \otimes v^{\otimes 6} \otimes w^{\otimes 6}) := f_{02}^3 A_6^f. \tag{4.67}$$

Let us now define the following two sets

$$\begin{aligned}
M &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_5 : f_{02} \neq 0\},\\
\mathcal{B}_1 &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_5 : f_{02} = 0, \ f_{12} \neq 0\} \quad \text{and}\\
\mathcal{B}_2 &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A}_5 : f_{02} = 0, \ f_{12} = 0\}.
\end{aligned}$$

Note that M belongs to the smooth part of the variety $\overline{\mathcal{P}A}_5$. This follows from the fact that the section $\Psi_{\mathcal{P}A_5}$ defined in the proof of Theorem 4.21 is transverse to zero when $f_{02} \neq 0$.

Restricted to M, the section $\Psi_{\mathcal{P}A_6}$ is transverse to zero if $d \geq 6$. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}, f_{50}$ and f_{60}. Note that we required the condition $f_{02} \neq 0$ to achieve transversality because in the last three steps we compute the derivative with respect to f_{40}, f_{50} and f_{60}.

Since $\mathcal{Q}$ is a generic section, the signed number of zeros of the section

$$\Psi_{\mathcal{P}A_6} \oplus \mathcal{Q} : \overline{\mathcal{P}A}_5 \longrightarrow \mathbb{L}_{\mathcal{P}A_6} \oplus \mathbb{W}_{n,m,6}$$

restricted to M is $N(\mathcal{P}A_6, n, m)$. This section *does* vanish on $\mathcal{B}_1$ and $\mathcal{B}_2$. We claim that

$$\mathcal{B}_1 = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_6} : f_{12} \neq 0\} \quad \text{and} \tag{4.68}$$

$$\mathcal{B}_2 = \overline{\mathcal{P}E_6}. \tag{4.69}$$

Furthermore, we claim that the section $\Psi_{\mathcal{P}A_6} \oplus \mathcal{Q}$ vanishes on each of the points of $\mathcal{B}_1$ and $\mathcal{B}_2$ with multiplicities of 4 and 3 respectively. Assuming that claim, we conclude that

$$\left\langle e(\mathbb{L}_{\mathcal{P}A_6})e(\mathbb{W}_{n,m,6}),\ [\overline{\mathcal{P}A_5}]\right\rangle = N(\mathcal{P}A_6, n, m) + 4N(\mathcal{P}D_6, n, m) + 3N(\mathcal{P}E_6, n, m). \tag{4.70}$$

Since

$$e(\mathbb{L}_{\mathcal{P}A_6}) = 4y + (4d - 18)a,$$

we immediately get (4.66). $\square$

We now start the proof of (4.68), (4.69) and the multiplicity claims. We need to show that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_5} : f_{02} = 0,\ \ f_{12} \neq 0\} = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_6} : f_{12} \neq 0\} \quad \text{and} \tag{4.71}$$

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_5} : f_{02} = 0,\ \ f_{12} = 0\} = \overline{\mathcal{P}E_6}. \tag{4.72}$$

First we note that the left hand side (4.71) is a subset of its right hand side. To see why that is so, first we take a look at (4.62). This says that if an A_5-singularity degenerates and the Hessian is zero, then it has to be more degenerate than a D_5-singularity. Hence, it has to be at least as degenerate as a D_6-singularity. Hence, the left hand side (4.71) is a subset of its right hand side.

Next, we note that the left hand side of (4.72) is a subset of its right hand side. This follows from (4.48). Hence the left hand sides of both (4.62) and (4.72) are subsets of their respective right hand sides.

We will now show the converse; let us start with (4.71). We will simultaneously prove the following statements

$$\overline{\mathcal{P}A_5} \supset \mathcal{P}D_6, \tag{4.73}$$

$$\overline{\mathcal{P}A_6} \cap \mathcal{P}D_6 = \varnothing. \tag{4.74}$$

Since $\overline{\mathcal{P}A_5}$ is a closed set, (4.73) implies that the right hand side of (4.71) is a subset of its left hand side.

Claim 4.26. *Let $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_6$. Then there exists a solution $(\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{P}A_3}$ near $(\tilde{f}, l_{\tilde{p}})$ to the set of equations*

$$f_{02}(t) \neq 0, \qquad f_{02}(t)A_4^{f(t)} = 0, \qquad f_{02}(t)^2 A_5^{f(t)} = 0. \tag{4.75}$$

Moreover, whenever such a solution $(\tilde{f}(t), l_{\tilde{p}}(t))$ is sufficiently close to $(\tilde{f}, l_{\tilde{p}})$ it lies in $\mathcal{P}A_5$, i.e., $A_6^{f(t)} \neq 0$. In particular, $(\tilde{f}(t), l_{\tilde{p}}(t))$ does not lie in $\mathcal{P}A_6$.

It is clear that Claim 4.26 proves (4.73) and (4.74) simultaneously.

Proof of Claim 4.26. We note that the only solutions to (4.75) are of the form

$$f_{02}(t) = t, \qquad f_{21}(t) = \left(\frac{5f_{31}(t) \pm \sqrt{-15f_{12}(t)D_7^{f(t)}}}{15f_{12}(t)}\right)t,$$

$$f_{40}(t) = 3\left(\frac{5f_{31}(t) \pm \sqrt{-15f_{12}(t)D_7^{f(t)}}}{15f_{12}(t)}\right)^2 t, \qquad t \neq 0\,(\text{but small}). \tag{4.76}$$

The second equation comes from solving a quadratic arising from $f_{02}(t)^2 A_5^{f(t)} = 0$ while the third is from solving $f_{02}(t)A_4^{f(t)} = 0$ and using $f_{21}(t)$ from the second equation. Since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_6$ we conclude that $f_{12} \neq 0$ and $D_7^f \neq 0$ (using Lemma 3.7).

Next, to show that $A_6^{f(t)} \neq 0$, we observe

$$f_{02}(t)^3 A_3^{f(t)} = \frac{D_7^{f(t)}}{f_{12}(t)}t^2 + O(t^3) \quad \text{using (4.76), for either choice of } \sqrt{f_{12}D_7^f}\,. \tag{4.77}$$

This proves the claim, since t is small and non zero and $D_7^f \neq 0$.

Corollary 4.27. *Let $\mathbb{W}_{n,m,6}$ and $\mathcal{Q}$ be as in (4.15) with $k = 6$. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_6 \cap \mathcal{Q}^{-1}(0)$. Then the section*

$$\Psi_{\mathcal{P}A_6} \oplus \mathcal{Q} : \overline{\mathcal{P}A_5} \longrightarrow \mathbb{L}_{\mathcal{P}A_6} \oplus \mathbb{W}_{n,m,6}$$

vanishes around $(\tilde{f}, l_{\tilde{p}})$ with a multiplicity of 4.

Proof. First we note that if $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_6$ then $f_{12} \neq 0$ and $D_7^f \neq 0$. The claim now follows immediately from (4.77) and using the fact that $\mathcal{Q}^{-1}(0)$ intersects $\mathcal{P}D_5$ transversely. This is because each branch of $\sqrt{f_{12}D_7^f}$ contributes with a multiplicity of 2. Hence, the total multiplicity is 4. □

Next we will prove that the right hand side of (4.72) is a subset of its left hand side. We will simultaneously prove that

$$\overline{\mathcal{P}A_5} \supset \mathcal{P}E_6, \tag{4.78}$$

$$\overline{\mathcal{P}A_6} \cap \mathcal{P}E_6 = \varnothing. \tag{4.79}$$

Claim 4.28. *Let $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_6$. Then there exists a solution $(\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{P}A_3}$ near $(\tilde{f}, l_{\tilde{p}})$ to the set of equations*

$$f_{02}(t) \neq 0, \qquad f_{02}(t)A_4^{f(t)} = 0, \qquad f_{02}(t)^2 A_5^{f(t)} = 0. \tag{4.80}$$

Moreover, whenever such a solution $(\tilde{f}(t), l_{\tilde{p}}(t))$ is sufficiently close to $(\tilde{f}, l_{\tilde{p}})$, it lies in $\mathcal{P}A_5$, i.e., $A_6^{f(t)} \neq 0$. In particular $(\tilde{f}(t), l_{\tilde{p}}(t))$ does not lie in $\mathcal{P}A_6$.

Note that Claim 4.28 proves (4.78) and (4.79) simultaneously.

Proof of Claim 4.28. It is easy to see that the only solutions to (4.80) are of the form

$$f_{21}(t) = t, \qquad f_{02} = \frac{3t^2}{f_{40}(t)}, \qquad f_{12} = \frac{2f_{31}(t)}{f_{40}(t)}t - \frac{3f_{50}(t)}{5f_{40}(t)^2}t^2, \qquad t \neq 0 \text{ (but small)}. \tag{4.81}$$

Since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_6$ we conclude that $f_{40}, f_{30} \neq 0$.

To show that any such solution satisfies the condition $A_6^{f(t)} \neq 0$, we observe that

$$f_{02}(t)^3 A_6^{f(t)} = -15f_{03}(t)t^3 + \mathrm{O}(t^4) \quad \text{using (4.81)}. \tag{4.82}$$

This proves the claim, since t is small and non zero and $f_{03}(t) \neq 0$. □

Corollary 4.29. *Let $\mathbb{W}_{n,m,6}$ and $\mathcal{Q}$ be as in (4.15) with $k = 6$ Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_6 \cap \mathcal{Q}^{-1}(0)$. Then the section*

$$\Psi_{\mathcal{P}A_6} \oplus \mathcal{Q} : \overline{\mathcal{P}A_5} \longrightarrow \mathbb{L}_{\mathcal{P}A_6} \oplus \mathbb{W}_{n,m,6}$$

vanishes around $(\tilde{f}, l_{\tilde{p}})$ with a multiplicity of 3.

Proof. First we note that if $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_6$ then $f_{03} \neq 0$ and $f_{40} \neq 0$. The claim now follows immediately from (4.82) and using the fact that $\mathcal{Q}^{-1}(0)$ intersects $\mathcal{P}E_6$ transversely. □

4.12. Degenerate contribution to the Euler class: computing $N(A_7)$

We are now ready to prove the formula for $N(A_7)$.

Theorem 4.30. *If $d \geq 7$, then*

$$\begin{aligned} N(\mathcal{P}A_7, n, 0) = {} & 5N(\mathcal{P}A_6, n, 0) - N(\mathcal{P}A_6, n, 1) + (5d - 24)N(\mathcal{P}A_6, n+1, 0) \\ & - 6N(\mathcal{P}D_7, n, 0) - 7N(\mathcal{P}E_7, n, 0). \end{aligned} \tag{4.83}$$

Note that combining this result with Theorems 4.25, 4.19, 4.18 and (4.53) we arrive at the formula for $N(A_7)$ as stated in Theorem 1.3.

Remark 4.31. We are *not* claiming that

$$\begin{aligned} N(\mathcal{P}A_7, n, m) = {} & 5N(\mathcal{P}A_6, n, m) - N(\mathcal{P}A_6, n, m+1) + (5d - 24)N(\mathcal{P}A_6, n+1, m) \\ & - 6N(\mathcal{P}D_7, n, m) - 7N(\mathcal{P}E_7, n, m). \end{aligned}$$

The above equation is only true when $m = 0$. We will see shortly where $m = 0$ plays a role.

Proof. We continue with the setup of the proof of Theorem 4.25; v and w are fixed non zero vectors that belong to $\hat{\gamma}|_{l_{\tilde{p}}}$ and the quotient space $\left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)|_{l_{\tilde{p}}}$ respectively while f_{ij} denotes the directional derivative as defined in (4.23). Let $\mathbb{W}_{n,0,7}$ and $\mathcal{Q}$ be as in (4.15) with $k = 7$ and $m = 0$. We note that the space $\mathcal{P}A_7$ can be described as

$$\mathcal{P}A_7 = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : \; f_{02}^4 A_7^f = 0, \; f_{02}^5 A_8^f \neq 0, \; f_{02} \neq 0\},$$

where A^f_k is as defined in (3.1). This follows from Lemma 3.4. Define a section of the following line bundle

$$\Psi_{\mathcal{P}A_7} : \overline{\mathcal{P}A_6} \longrightarrow \mathbb{L}_{\mathcal{P}A_7} := \gamma_{\mathcal{D}}^{*5} \otimes \hat{\gamma}^{*7} \otimes \left(\pi^* T\mathbb{P}^2/\hat{\gamma}\right)^{*8} \otimes \gamma_{\mathbb{P}^2}^{*5d}$$

given by

$$\{\Psi_{\mathcal{P}A_7}(\tilde{f}, l_{\tilde{p}})\}(f^{\otimes 5} \otimes v^{\otimes 7} \otimes w^{\otimes 8}) := f_{02}^4 A_7^f. \tag{4.84}$$

Let us now define the following three sets

$$\begin{aligned} M &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : f_{02} \neq 0\},\\ \mathcal{B}_1 &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : f_{02} = 0,\ f_{12} \neq 0\},\\ \mathcal{B}_2 &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : f_{02} = 0,\ f_{12} = 0,\ f_{03} \neq 0\} \quad \text{and}\\ \mathcal{B}_3 &:= \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : f_{02} = 0,\ f_{12} = 0,\ f_{03} = 0\}. \end{aligned} \tag{4.85}$$

Note that M belongs to the smooth part of the variety $\overline{\mathcal{P}A_6}$. This follows from the fact that the section $\Psi_{\mathcal{P}A_6}$ defined in the proof of Theorem 4.25 is transverse to zero when $f_{02} \neq 0$.

Restricted to M, the section $\Psi_{\mathcal{P}A_7}$ is transverse to zero if $d \geq 7$. The proof follows from the setup of the proof of Lemma 4.9, by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}, f_{50}, f_{60}$ and f_{70}. Note that we required the condition $f_{02} \neq 0$ to achieve transversality because in the last four steps we compute the derivative with respect to f_{40}, f_{50}, f_{60} and f_{70}.

Since $\mathcal{Q}$ is a generic section, the signed number of zeros of the section

$$\Psi_{\mathcal{P}A_7} \oplus \mathcal{Q} : \overline{\mathcal{P}A_6} \longrightarrow \mathbb{L}_{\mathcal{P}A_7} \oplus \mathbb{W}_{n,0,7}$$

restricted to M is $N(\mathcal{P}A_7, n, 0)$. This section *does* vanish on $\mathcal{B}_1$ and$\mathcal{B}_2$. We claim that

$$\mathcal{B}_1 = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_7} : f_{12} \neq 0\} \quad \text{and} \tag{4.86}$$

$$\mathcal{B}_2 = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}E_7} : f_{12} = 0,\ f_{03} \neq 0\}. \tag{4.87}$$

Furthermore, we claim that the section $\Psi_{\mathcal{P}A_7} \oplus \mathcal{Q}$ vanishes on each of the points of $\mathcal{B}_1$ and $\mathcal{B}_2$ with multiplicities of 6 and 7 respectively. We also claim that the section $\Psi_{\mathcal{P}A_7} \oplus \mathcal{Q}$ does not vanish on $\mathcal{B}_3$; this is where we use that $\mathcal{Q}$ is a section of $\mathbb{W}_{n,0,7}$ as opposed to $\mathbb{W}_{n,m,k}$. Assuming these claims, we conclude that

$$\left\langle e(\mathbb{L}_{\mathcal{P}A_7}) e(\mathbb{W}_{n,0,7}), [\overline{\mathcal{P}A_6}]\right\rangle = N(\mathcal{P}A_7, n, m) + 6N(\mathcal{P}D_7, n, m) + 7N(\mathcal{P}E_7, n, m).$$

Since

$$e(\mathbb{L}_{\mathcal{P}A_7}) = 5y - \lambda + (5d - 24)a,$$

we immediately get (4.83).

We now start the proof of (4.86) and (4.87) and the multiplicity claims. We need to show that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : f_{02} = 0,\ f_{12} \neq 0\} \equiv \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}D_7} : f_{12} \neq 0\} \quad \text{and} \tag{4.88}$$

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}A_6} : f_{02} = 0,\ f_{12} = 0,\ f_{03} \neq 0\} \equiv \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}E_7} : f_{03} \neq 0\}. \tag{4.89}$$

First we note that the left hand side of (4.88) is a subset of its right hand side. To see why that is so, first we take a look at (4.74). This says that if an A_6-singularity degenerates and the Hessian is zero, then it has to be more degenerate than a D_6-singularity. Hence the left hand side of (4.88) has to be a subset of its right hand side.

Next, we note that the left hand side of (4.89) is a subset of its right hand side. To see why that is so, we take a look at (4.79). This says that if an A_6-singularity degenerates and the Hessian is zero and $f_{12} = 0$, then it has to be more degenerate than a E_6-singularity. Hence the left hand side of (4.89) has to be a subset of its right hand side.

Hence the left hand sides of both (4.88) and (4.89) are subsets of their respective right hand sides. We will now start with the proof of the converse.

We will first prove that the right hand side of (4.88) is a subset of its left hand side. We will simultaneously prove

$$\overline{\mathcal{P}A_6} \supset \mathcal{P}D_7, \tag{4.90}$$

$$\overline{\mathcal{P}A_7} \cap \mathcal{P}D_7 = \varnothing. \tag{4.91}$$

Claim 4.32. *Let $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_7$. Then there exists a solution $(\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{P}A_3}$ near $(\tilde{f}, l_{\tilde{p}})$ to the set of equations*

$$f_{02}(t) \neq 0, \qquad f_{02}(t) A_4^{f(t)} = 0, \qquad f_{02}(t)^2 A_5^{f(t)} = 0, \qquad f_{02}(t)^3 A_6^{f(t)} = 0. \tag{4.92}$$

Moreover, whenever such a solution $(\tilde{f}(t), l_{\tilde{p}}(t))$ is sufficiently close to $(\tilde{f}, l_{\tilde{p}})$ it lies in $\mathcal{P}A_6$, i.e., $A_7^{f(t)} \neq 0$. In particular $(\tilde{f}(t), l_{\tilde{p}}(t))$ does not lie in $\mathcal{P}A_7$.

Note that Claim 4.32 proves (4.90) and (4.91) simultaneously.

Proof of Claim 4.32. We claim that the *only* solutions to (4.92) are of the form

$$f_{02}(t) = t^2 + \mathrm{O}(t^4), \tag{4.93}$$

$$f_{21}(t) = \frac{f_{31}(t)}{3f_{12}(t)}t^2 + \sqrt{\frac{\beta(t)}{f_{12}(t)}}t^3 + \mathrm{O}(t^4), \tag{4.94}$$

$$f_{40}(t) = \frac{f_{31}(t)^2}{3f_{12}(t)^2}t^2 + \mathrm{O}(t^3),$$

$$\left(f_{50}(t) - \frac{5f_{31}(t)^2}{3f_{12}(t)}\right) = -15\beta(t)t^2 + \mathrm{O}(t^3), \quad t \neq 0 \text{ (but small)},$$

$$\text{where } \beta(t) := -\frac{f_{03}(t)f_{31}(t)^3}{162f_{12}(t)^4} + \frac{f_{22}(t)f_{31}(t)^2}{18f_{12}(t)^3} - \frac{f_{41}(t)f_{31}(t)}{18f_{12}(t)^2} + \frac{f_{60}(t)}{90f_{12}(t)} \tag{4.95}$$

for just *one* choice of a branch of $\sqrt{\beta(t)}$.[8] We will see shortly that $\beta(t) \neq 0$. The value for f_{40} can be calculated using f_{21}, f_{02} and $f_{02}(t)A_4^{f(t)} = 0$ while the fourth equation follows by using the first three equations and $f_{02}(t)^2A_5^{f(t)} = 0$. Let us now explain how we obtain (4.93) and (4.94). The equation $f_{02}(t)^3A_6^{f(t)} = 0$ is a cubic equation in $f_{21}(t)$, i.e., it is of the form

$$\mathrm{A}_3(f_{02}(t))f_{21}(t)^3 + \mathrm{A}_2(f_{02}(t))f_{21}(t)^2 + \mathrm{A}_1(f_{02}(t))f_{21}(t) + \mathrm{A}_0(f_{02}(t)) = 0.$$

Since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_7$, we conclude that $f_{12}(t) \neq 0$. It follows that as $f_{02}(t)$ goes to zero A_2 remains non zero. Hence, there exists a unique holomorphic function $\mathrm{P}(f_{02}(t))$, of $f_{02}(t)$ (close to the zero function), such that if we make a change of variables

$$f_{21}(t) = \mathrm{H} + \mathrm{P}(f_{02}(t))$$

then our cubic equation becomes

$$\hat{\mathrm{A}}_3(f_{02}(t))\mathrm{H}^3 + \hat{\mathrm{A}}_2(f_{02}(t))\mathrm{H}^2 + \hat{\mathrm{A}}_0(f_{02}(t)) = 0.$$

The argument is same as in the proof Lemma 3.4, where we show the existence of $\mathrm{B}(x)$; it is an application of the Implicit Function Theorem. In fact, we observe that

$$\mathrm{P}(f_{02}) = \frac{1}{3\mathrm{A}_3}\left(-\mathrm{A}_2 + \sqrt{\mathrm{A}_2^2 - 3\mathrm{A}_1\mathrm{A}_3}\right).$$

This is defined even when $\mathrm{A}_3 = 0$ as can be seen by a standard binomial expansion, i.e.,

$$\mathrm{P}(f_{02}(t)) = \frac{f_{31}(t)}{3f_{12}(t)}f_{02}(t) + \mathrm{O}(f_{02}(t)^2) = \frac{f_{31}(t)}{3f_{12}(t)}t^2 + \mathrm{O}(t^4).$$

The other root has the property that $\mathrm{P}(f_{02}(t))$ goes to a non-zero constant as $f_{02}(t)$ goes to zero.

Since $\hat{\mathrm{A}}_2(0) \neq 0$, we can divide out by $\hat{\mathrm{A}}_2(f_{20}(t))$ and get

$$\hat{\hat{\mathrm{A}}}_3(f_{02}(t))\mathrm{H}^3 + \mathrm{H}^2 + \hat{\hat{\mathrm{A}}}_0(f_{02}(t)) = 0. \tag{4.96}$$

By a simple calculation, it is easy to see that

$$\hat{\hat{\mathrm{A}}}_0(f_{02}(t)) = -\frac{\beta(t)}{f_{12}(t)}f_{02}(t)^3 + \mathrm{O}(f_{02}(t)^4).$$

Assuming $\beta(t) \neq 0$ we can make a change of coordinates

$$\hat{f}_{02} = f_{02}(t)\left(\frac{f_{12}(t)\,\hat{\hat{\mathrm{A}}}_0(f_{02}(t))}{-\beta(t)f_{02}(t)^3}\right)^{\frac{1}{3}}, \qquad \hat{\mathrm{H}} = \mathrm{H}(1 + \hat{\hat{\mathrm{A}}}_3(f_{02}(t))\mathrm{H})^{\frac{1}{2}}.$$

Our cubic equation (4.96) now becomes

$$\hat{\mathrm{H}}^2 - \frac{\beta(t)}{f_{12}(t)}\hat{f}_{02}^3 = 0. \tag{4.97}$$

Now, it is easy to see that the *only* small solutions to (4.97) are of the form

$$\hat{\mathrm{H}} = \sqrt{\frac{\beta(t)}{f_{12}(t)}}t^3, \qquad \hat{f}_{02} = t^2$$

8 In other words, choosing the other branch of the square root does not give us any extra solutions.

for just *one* choice of $\sqrt{\beta(t)}$. In other words, by choosing just *one* branch of $\sqrt{\beta(t)}$, we get *all* the possible small solutions of (4.97). By inverting the change of coordinates, $(\mathrm{H}, f_{02}) \longrightarrow (\hat{\mathrm{H}}, \hat{f}_{02})$, we conclude that the *only* small solutions to (4.96) are of the form

$$\mathrm{H} = \sqrt{\frac{\beta(t)}{f_{12}(t)}}t^3 + \mathrm{O}(t^4), \qquad f_{02}(t) = t^2 + \mathrm{O}(t^4).$$

(Note that the transformation $(\mathrm{H}, f_{02}) \longrightarrow (\hat{\mathrm{H}}, \hat{f}_{02})$ is identity to first order, i.e., the Jacobian matrix of this transformation at the origin is the identity matrix.) Combining the expression for $\mathrm{P}(f_{02})$ and H gives us (4.94) and (4.93). It remains to show that $\beta(t) \neq 0$. To see this, note that

$$\beta(t) = \frac{D_8^{f(t)}}{90 f_{12}(t)} - \frac{f_{30}(t) f_{31}(t) D_7^{f(t)}}{54 f_{12}(t)^3}. \tag{4.98}$$

Since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_7$, $D_7^f = 0$ and $D_8^f \neq 0$. Therefore, by (4.98) we conclude $\beta(t) \neq 0$.

To see why any such solution satisfies $A_7^{f(t)} \neq 0$, we simply observe that

$$f_{02}(t)^4 A_7^{f(t)} = 630 f_{12}(t)^2 \beta(t) t^6 + \mathrm{O}(t^7) \quad \text{using (4.95)}. \tag{4.99}$$

This finishes the proof of the claim. □

Corollary 4.33. *Let $\mathbb{W}_{n,0,7}$ and $\mathcal{Q}$ be as in (4.15) with $k = 7$ and $m = 0$. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_7 \cap \mathcal{Q}^{-1}(0)$. Then the section*

$$\Psi_{\mathcal{P}A_7} \oplus \mathcal{Q} : \overline{\mathcal{P}A_6} \longrightarrow \mathbb{L}_{\mathcal{P}A_7} \oplus \mathbb{W}_{n,0,7}$$

vanishes around $(\tilde{f}, l_{\tilde{p}})$ with a multiplicity of 6.

Proof. First we note that if $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}D_7$, then $\beta \neq 0$ (this follows from (4.98)). The claim now follows immediately from (4.95) and using the fact that $\mathcal{Q}^{-1}(0)$ intersects $\mathcal{P}D_7$ transversely. □

Next we will prove that the rhs of (4.89) is a subset of its lhs. We will simultaneously show that

$$\overline{\mathcal{P}A_6} \supset \mathcal{P}E_7, \tag{4.100}$$

$$\overline{\mathcal{P}A_7} \cap \mathcal{P}E_7 = \varnothing. \tag{4.101}$$

Claim 4.34. *Let $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_7$. Then there exists a solution $(\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{P}A_3}$ near $(\tilde{f}, l_{\tilde{p}})$ to the set of equations*

$$f_{02}(t) \neq 0, \qquad f_{02}(t) A_4^{f(t)} = 0, \qquad f_{02}(t)^2 A_5^{f(t)} = 0, \qquad f_{02}(t)^3 A_6^{f(t)} = 0. \tag{4.102}$$

Moreover, whenever such a solution $(\tilde{f}(t), l_{\tilde{p}}(t))$ is sufficiently close to $(\tilde{f}, l_{\tilde{p}})$ it lies in $\mathcal{P}A_6$, i.e., $A_7^{f(t)} \neq 0$. In particular $(\tilde{f}(t), l_{\tilde{p}}(t))$ does not lie in $\mathcal{P}A_7$.

Note that Claim 4.34 proves (4.100) and (4.101) simultaneously.

Proof of Claim 4.34. We claim that the *only* solutions to (4.102) are of the form

$$\begin{aligned} &f_{12}(t) = t, \qquad f_{21}(t) = -\frac{3}{2 f_{03}(t)} t^2 + \mathrm{O}(t^3) \\ &f_{02}(t) = -\frac{9}{4 f_{31}(t) f_{03}(t)} t^3 + \mathrm{O}(t^4), \qquad f_{40}(t) = -\frac{3 f_{31}(t)}{f_{03}(t)} t + \mathrm{O}(t^2), \quad t \neq 0 \text{ (but small)}. \end{aligned} \tag{4.103}$$

Since $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_7$, we conclude that f_{31}, $f_{03} \neq 0$ (this follows from Lemma 3.10). Let us now explain how we obtained the solutions. First, we set $f_{12}(t) = t$. Using $f_{02}(t)^2 A_5^{f(t)} = 0$ we can solve for $\frac{f_{02}(t)}{f_{21}(t)}$ and get

$$\frac{f_{02}(t)}{f_{21}(t)} = \frac{10 f_{31}(t) - \sqrt{100 f_{31}(t)^2 - 60 f_{50}(t) t}}{2 f_{50}(t)} = \frac{3}{2 f_{31}(t)} t + \mathrm{O}(t^2). \tag{4.104}$$

Note that the equality of the first and last terms remains valid even when $f_{50} = 0$. We will justify shortly why we did not choose the other branch of the square root. Plugging in the value of f_{02} from (4.104) in equation $f_{02}(t)^3 A_6^{f(t)} = 0$ and by using the Implicit Function Theorem, we get the expression for $f_{21}(t)$ in (4.103). And now using the value of $f_{21}(t)$ and (4.104) we get the expression for $f_{02}(t)$ in (4.103).

It remains to justify why we did not choose the other branch of the square root. It is easy to see that if we chose the other branch, it would imply that as $f_{02}(t)$ and $f_{21}(t)$ go to zero, the ratio $L_t := \frac{f_{21}(t)}{f_{02}(t)}$ tends to a finite number L, since $f_{31} \neq 0$. Using $f_{03}(t)^3 A_6^{f(t)} = 0$ we can solve for $f_{31}(t)$ as a quadratic equation and get that

$$f_{31}(t) = \frac{30L_t f_{12}(t) \pm \sqrt{10}\sqrt{-15L_t^3 f_{02}(t)f_{03}(t) + 45L_t^2 f_{02}(t)f_{22}(t) - 15L_t f_{02}(t)f_{41}(t) + f_{02}(t)f_{60}(t)}}{10}.$$

It is now easy to see that $f_{31}(t)$ tends to zero as $f_{12}(t)$ and $f_{02}(t)$ tend to zero. This gives us a contradiction, since $f_{31} \neq 0$.

To show that any solution satisfies $A_7^{f(t)} \neq 0$, we simply observe that

$$f_{02}(t)^4 A_7^{f(t)} = -\frac{2835}{16 f_{03}(t)^2} t^7 + O(t^8) \quad \text{using (4.103).} \tag{4.105}$$

This completes the proof of the claim. □

Corollary 4.35. *Let $\mathbb{W}_{n,0,7}$ and $\mathcal{Q}$ be as in (4.15) with $k = 7$ and $m = 0$. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_7 \cap \mathcal{Q}^{-1}(0)$. Then the section*

$$\Psi_{\mathcal{P}A_7} \oplus \mathcal{Q} : \overline{\mathcal{P}A_6} \longrightarrow \mathbb{L}_{\mathcal{P}A_7} \oplus \mathbb{W}_{n,0,7}$$

vanishes around $(\tilde{f}, l_{\tilde{p}})$ with a multiplicity of 6.

Proof. First we note that if $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}E_7$, then $f_{03} \neq 0$. The claim now follows immediately from (4.103) and using the fact that $\mathcal{Q}^{-1}(0)$ intersects $\mathcal{P}E_7$ transversely. □

It remains to show that the section $\Psi_{\mathcal{P}A_7} \oplus \mathcal{Q}$ does not vanish on $\mathcal{B}_3$. First, let us define the spaces

$$S := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(\tilde{p}) = 0,\ \nabla f|_{\tilde{p}} = 0,\ \nabla^2 f|_{\tilde{p}} = 0,\ \nabla^3 f|_{\tilde{p}} = 0\} \quad \text{and} \quad \hat{S} := \pi^{-1}(S),$$

where $\pi : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{D} \times \mathbb{P}^2$ is the projection map. We note that if $d \geq 3$, then S is a smooth manifold of dimension $\kappa_d - 8$; this follows from the setup of Lemma 4.3 by taking partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}, f_{12}$ and f_{03}.

Next, we note that $\mathbb{W}_{n,0,7} \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2$ is the pullback of the bundle over $\mathcal{D} \times \mathbb{P}^2$; this is where we use that $m = 0$. Let us denote the corresponding bundle on $\mathcal{D} \times \mathbb{P}^2$ by $\mathcal{W}_{n,0,7}$. Note that the rank of $\mathcal{W}_{n,0,7}$ is $\kappa_d - 7$. Since the dimension of S is one less than the rank of $\mathcal{W}_{n,0,7}$, the zeros of a generic smooth section of $\mathcal{W} \longrightarrow \mathcal{D} \times \mathbb{P}^2$ will not intersect S. Finally, we note that any section of $\mathbb{W}_{n,0,7} \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2$ is the pullback of a section of $\mathcal{W}_{n,0,7} \longrightarrow \mathcal{D} \times \mathbb{P}^2$ (or said in a different way, any section of $\mathbb{W}_{n,0,7} \longrightarrow \mathcal{D} \times T\mathbb{P}^2$ can be pushed forward to a section of $\mathcal{W}_{n,0,7} \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2$). Hence, the zeros of a generic smooth section of $\mathbb{W}_{n,0,7} \longrightarrow \mathcal{D} \times \mathbb{P}T\mathbb{P}^2$ does not intersect $\hat{S}$. Since $\mathcal{B}_3$ is a subset of $\hat{S}$, we conclude that $\mathcal{Q}^{-1}(0)$ does not intersect $\mathcal{B}_3$.

The fact that $\mathcal{B}_3$ is a subset of $\hat{S}$ is easy to see. If $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{B}_3$, then $f_{02} = 0$; this implies that $\nabla^2 f|_{\tilde{p}} = 0$. Moreover, (4.60) implies that $f_{21} = 0$. Combined with the fact that $f_{12} = 0$ and $f_{03} = 0$ we conclude that $\nabla^3 f|_{\tilde{p}} = 0$. Hence $\mathcal{B}_3$ is a subset of $\hat{S}$. This completes the proof.

Remark 4.36. It is actually true that $\mathcal{B}_3 = \hat{S}$. Since we did not actually need this stronger fact, we did not prove it here. This fact would be needed if we wanted to compute $N(\mathcal{P}A_7, n, m)$ for $m \neq 0$.

Remark 4.37. There is an error in the formula for $N(A_6)$ obtained by Kerner in his published version; he has corrected that in his updated arXiv version. There is also a mistake in the formula for $N(A_7)$ obtained by Kerner in the published version. In the updated arXiv version, there is a mistake in his formula for the cohomology class $[A_7]$ in Proposition 1.2. Kerner agrees that there is indeed some mistake he is making while computing the cohomology class [25].

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