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Generalized Lie bialgebroids and Jacobi structures

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Abstract

The notion of a generalized Lie bialgebroid (a generalization of the notion of a Lie bialgebroid) is introduced in such a way that a Jacobi manifold has associated a canonical generalized Lie bialgebroid. As a kind of converse, we prove that a Jacobi structure can be defined on the base space of a generalized Lie bialgebroid. We also show that it is possible to construct a Lie bialgebroid from a generalized Lie bialgebroid and, as a consequence, we deduce a duality theorem. Finally, some special classes of generalized Lie bialgebroids are considered: triangular generalized Lie bialgebroids and generalized Lie bialgebras. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Roughly speaking, a Lie algebroid over a manifold M is a vector bundle A over M such that its space of sections $\Gamma(A)$ admits a Lie algebra structure $[\![,]\!]$ and, moreover, there exists a bundle map ρ from A to TM which provides a Lie algebra homomorphism from $(\Gamma(A), [\![,]\!])$ into the Lie algebra of vector fields $\mathfrak{X}(M)$ (see [24,27]). Lie algebroids are a natural generalization of tangent bundles and real Lie algebras of finite dimension. But, there are many other interesting examples, for instance, the cotangent bundle T^*M of any Poisson manifold M possesses a natural Lie algebroid structure [1,2,7,30]. In fact, there is a one-to-one correspondence between Lie algebroid structures on a vector bundle A and linear Poisson structures on the dual bundle A^* (see [2,3]). An important class of Lie algebroids

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are the so-called Lie bialgebroids. This is a Lie algebroid A such that the dual vector bundle A^* also carries a Lie algebroid structure which is compatible in a certain way with that on A (see [16,25]). If M is a Poisson manifold, then the pair (TM, T^*M) is a Lie bialgebroid. As a kind of converse, it was proved in [25] that the base space of a Lie bialgebroid is a Poisson manifold. Apart from the pair (TM, T^*M) (M being a Poisson manifold), other interesting examples of Lie bialgebroids are Lie bialgebras [5]. A Lie bialgebra is a Lie bialgebroid such that the base space is a single point and there is a one-to-one correspondence between Lie bialgebras and connected simply connected Poisson Lie groups (see [5,17,23,30]). We remark that a connected simply connected abelian Poisson Lie group is isomorphic to the dual space of a real Lie algebra endowed with the usual linear Poisson structure (the Lie–Poisson structure). Moreover, Poisson Lie groups are closely related with quantum groups (see [6]).

As it is well known, a Jacobi structure on a manifold M is a 2-vector Λ and a vector field E on M such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$, where $[\,,\,]$ is the Schouten–Nijenhuis bracket [22]. If (M, Λ, E) is a Jacobi manifold one can define a bracket of functions, the Jacobi bracket, in such a way that the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a local Lie algebra in the sense of Kirillov [15]. Conversely, a local Lie algebra structure on $C^\infty(M, \mathbb{R})$ induces a Jacobi structure on M [9,15]. Jacobi manifolds are natural generalizations of Poisson manifolds. However, very interesting manifolds like contact and locally conformal symplectic (l.c.s.) manifolds are also Jacobi and they are not Poisson. In fact, a Jacobi manifold admits a generalized foliation whose leaves are contact or l.c.s. manifolds (see [4,9,15]). If M is an arbitrary manifold, the vector bundle $TM \times \mathbb{R} \to M$ possesses a natural Lie algebroid structure. Moreover, if M is a Jacobi manifold then the 1-jet bundle $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure [14] (for a Jacobi manifold the vector bundle T^*M is not, in general, a Lie algebroid). However, the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not, in general, a Lie bialgebroid (see [31]).

On the other hand, in [12], we studied Jacobi structures on the dual bundle A^* to a vector bundle A such that the Jacobi bracket of linear functions is again linear and the Jacobi bracket of a linear function and the constant function 1 is a basic function. We proved that a Lie algebroid structure on A and a 1-cocycle $\phi_0 \in \Gamma(A^*)$ induce a Jacobi structure on A^* which satisfies the above conditions. Moreover, we showed that this correspondence is a bijection. We also consider two interesting examples: (i) for an arbitrary manifold M, the Lie algebroid $A = TM \times \mathbb{R}$ and the 1-cocycle $\phi_0 = (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A^*)$, we prove that the resultant linear Jacobi structure on $T^*M \times \mathbb{R}$ is just the canonical contact structure and (ii) for a Jacobi manifold M, the Lie algebroid $A^* = T^*M \times \mathbb{R}$ and the 1-cocycle $X_0 = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A)$, we deduce that the corresponding linear Jacobi structure $(\Lambda_{(TM\times\mathbb{R}, X_0)}, E_{(TM\times\mathbb{R}, X_0)})$ on $TM \times \mathbb{R}$ is given by

$$\Lambda_{(TM\times\mathbb{R}, X_0)} = \Lambda^{\mathrm{c}} + \frac{\partial}{\partial t} \wedge E^{\mathrm{c}} - t\left(\Lambda^{\mathrm{v}} + \frac{\partial}{\partial t} \wedge E^{\mathrm{v}}\right), \qquad E_{(TM\times\mathbb{R}, X_0)} = E^{\mathrm{v}},$$

where Λ^{c} (resp. Λ^{v}) is the complete (resp. vertical) lift to TM of Λ and E^{c} (resp. E^{v}) is the complete (resp. vertical) lift to TM of E. This Jacobi structure was introduced in [11] and it is the Jacobi counterpart to the tangent Poisson structure first used in [28] (see also [3,8]).

Therefore, for a Jacobi manifold M, it seems reasonable to consider the pair $((A = TM \times \mathbb{R}, \phi_0 = (0, 1)), (A^* = T^*M \times \mathbb{R}, X_0 = (-E, 0)))$ instead of the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$.

In fact, we prove, in this paper, that the Lie algebroids $TM \times \mathbb{R}$ and $T^*M \times \mathbb{R}$ and the 1-cocycles ϕ_0 and X_0 satisfy some compatibility conditions. These results suggest us to introduce, in a natural way, the definition of a generalized Lie bialgebroid. The aim of this paper is to discuss some relations between generalized Lie bialgebroids and Jacobi structures.

The paper is organized as follows. In Section 2, we recall several definitions and results about Jacobi manifolds and Lie algebroids which will be used in the sequel. In Section 3 we introduce the definition of a generalized Lie bialgebroid as a pair $((A, \phi_0), (A^*, X_0))$, where A is a Lie algebroid, A^* (the dual bundle to A) is also a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ (resp. $X_0 \in \Gamma(A)$) is a 1-cocycle of A (resp. A^*). In addition, the Lie algebroids A, A^* and the 1-cocycles ϕ_0, X_0 must satisfy some compatibility conditions. If ϕ_0 and X_0 are zero, we recover the notion of a Lie bialgebroid. Moreover, if (M, Λ, E) is a Jacobi manifold, the pair $((TM \times \mathbb{R}, (0, 1)),(T^*M \times \mathbb{R}, (-E, 0)))$ is a generalized Lie bialgebroid. In fact, extending a result of [25] for Lie bialgebroids, we prove that the base space of a generalized Lie bialgebroid is a Jacobi manifold (Theorem 3.7). It is well known that the product of a Jacobi manifold with $\mathbb{R}$, endowed with the Poissonization of the Jacobi structure, is a Poisson manifold (see [22] and Section 2.2). We show a similar result for generalized Lie bialgebroids. Namely, we prove that if $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid over M then it is possible to define a Lie bialgebroid structure on the dual pair of vector bundles $(A \times \mathbb{R}, A^* \times \mathbb{R})$ over $M \times \mathbb{R}$, in such a way that the induced Poisson structure on $M \times \mathbb{R}$ is just the Poissonization of the Jacobi structure on M (Theorem 3.9). Using this result, we deduce that the generalized Lie bialgebroids satisfy a duality theorem, that is, if $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid, so is $((A^*, X_0), (A, \phi_0))$.

In Section 4, we prove that it is possible to obtain a generalized Lie bialgebroid from a Lie algebroid $(A, [\![\, ,]\!], \rho)$, a 1-cocycle ϕ_0 on it and a bisection $P \in \Gamma(\wedge^2 A)$ satisfying $[\![P, P]\!]_{\phi_0} = [\![P, P]\!] - 2P \wedge i_{\phi_0} P = 0$ (Theorem 4.1). This type of generalized Lie bialgebroids are called triangular. Examples of triangular generalized Lie bialgebroids are the triangular Lie bialgebroids in the sense of [25] and the generalized Lie bialgebroid associated with a Jacobi structure. In Section 5, we study generalized Lie bialgebras, i.e., generalized Lie bialgebroids over a single point. Using the results of Section 4, we deduce that generalized Lie bialgebras can be obtained from algebraic Jacobi structures on a Lie algebra. This fact allows us to give examples of Lie groups whose Lie algebras are generalized Lie bialgebras. The study of this type of Lie groups is the subject of the paper [13].

Finally, the paper closes with two appendices. In the first one, we show some facts about the differential calculus on Lie algebroids in the presence of a 1-cocycle and in the second one we obtain, from a Lie algebroid A over M and a 1-cocycle $\phi_0 \in \Gamma(A^*)$, two Lie algebroid structures on the pull-back $\pi_1^* A$ of A over the canonical projection $\pi_1 : M \times \mathbb{R} \to M$. The notations, definitions and results in these appendices will be used throughout the paper (particularly, in Sections 3 and 4). On the other hand, we must note that most of the results in these appendices are a consequence of the application of standard techniques (see [10,24]).

Notation. Throughout this paper, we will use the following notation. If M is a differentiable manifold of dimension n, we will denote by $C^\infty(M, \mathbb{R})$ the algebra of C^∞ real-valued functions on M, by $\Omega^k(M)$ the space of k-forms, by $\mathcal{V}^k(M)$ the space of k-vectors, with $k \geq 2$, by $\mathfrak{X}(M)$ the Lie algebra of vector fields, by δ the usual differential on $\Omega^*(M) =$

$\oplus_k \Omega^k(M)$ and by $[\,,\,]$ the Schouten–Nijenhuis bracket [1,30]. Moreover, if $A \to M$ is a vector bundle over M, $\Gamma(A)$ (resp. $\Gamma(A^*)$) is the space of sections of A (resp. of the dual bundle $A^* \to M$) and $P \in \Gamma(\wedge^2 A)$ then $\#_P : \Gamma(A^*) \to \Gamma(A)$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by

$$\beta(\#_P(\alpha)) = P(\alpha, \beta) \quad \text{for}\, \alpha, \beta \in \Gamma(A^*). \tag{1.1}$$

We also denote by $\#_P : A^* \to A$ the corresponding bundle map.

Identifications. We will also use the following identifications. Let $A \to M$ be a vector bundle over M. Then, it is clear that $A \times \mathbb{R}$ is the total space of a vector bundle over M. Moreover, the space $\Gamma(\wedge^r(A \times \mathbb{R}))$ of the sections of the vector bundle $\wedge^r(A \times \mathbb{R}) \to M$ can be identified with $\Gamma(\wedge^r A) \oplus \Gamma(\wedge^{r-1} A)$ in such a way that

$$\begin{aligned}&(P, Q)((\alpha_1, f_1), \ldots, (\alpha_r, f_r))\\ &\quad = P(\alpha_1, \ldots, \alpha_r) + \sum_{i=1}^{r}(-1)^{i+1} f_i Q(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_r)\end{aligned} \tag{1.2}$$

for $(P, Q) \in \Gamma(\wedge^r A) \oplus \Gamma(\wedge^{r-1} A)$ and $(\alpha_i, f_i) \in \Gamma(A^*) \oplus C^\infty(M, \mathbb{R})$ with $i \in \{1, \ldots, r\}$. Under this identification, the contraction and the exterior product are given by

$$\begin{aligned}&i_{(\alpha,\beta)}(P, Q) = (i_\alpha P + i_\beta Q, (-1)^k i_\alpha Q),\\ &(P, Q) \wedge (P', Q') = (P \wedge P', Q \wedge P' + (-1)^r P \wedge Q')\end{aligned} \tag{1.3}$$

for $(P', Q') \in \Gamma(\wedge^{r'} A) \oplus \Gamma(\wedge^{r'-1} A)$ and $(\alpha, \beta) \in \Gamma(\wedge^k A^*) \oplus \Gamma(\wedge^{k-1} A^*)$.

2. Jacobi manifolds and Lie algebroids

2.1. Lie algebroids and Lie bialgebroids

A *Lie algebroid* A over a manifold M is a vector bundle A over M together with a Lie algebra structure on the space $\Gamma(A)$ of the global cross-sections of $A \to M$ and a bundle map $\rho : A \to TM$, called the *anchor map*, such that if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules induced by the anchor map then,

1. $\rho : (\Gamma(A), [\![\,,\,]\!]) \to (\mathfrak{X}(M), [\,,\,])$ is a Lie algebra homomorphism and
2. for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$, one has

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y.$$

The triple $(A, [\![\,,\,]\!], \rho)$ is called a *Lie algebroid over M* (see [24,27]).

A real Lie algebra of finite dimension is a Lie algebroid over a point. Another trivial example of a Lie algebroid is the triple $(TM, [\,,\,], Id)$, where M is a differentiable manifold, TM is its tangent bundle and $Id : TM \to TM$ is the identity map.

If A is a Lie algebroid, the Lie bracket on $\Gamma(A)$ can be extended to the so-called *Schouten bracket* $[\![\,,\,]\!]$ on the space $\Gamma(\wedge^* A) = \oplus_k \Gamma(\wedge^k A)$ of multi-sections of A in such a way that $(\oplus_k \Gamma(\wedge^k A), \wedge, [\![\,,\,]\!])$ is a graded Lie algebra (see [30]).

On the other hand, imitating the usual differential on the space $\Omega^*(M)$, we define the *differential of the Lie algebroid* A, $\mathrm{d} : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, as follows. For $\omega \in \Gamma(\wedge^k A^*)$ and $X_0, \dots, X_k \in \Gamma(A)$,

$$\mathrm{d}\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \rho(X_i)(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)) + \sum_{i<j} (-1)^{i+j} \omega([\![X_i, X_j]\!], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k).$$

Moreover, since $\mathrm{d}^2 = 0$, we have the corresponding cohomology spaces. This cohomology is the *Lie algebroid cohomology with trivial coefficients* (see [24]).

Using the above definitions, it follows that a 1-cochain $\phi \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi[\![X, Y]\!] = \rho(X)(\phi(Y)) - \rho(Y)(\phi(X)) \quad \text{for all } X, Y \in \Gamma(A). \tag{2.1}$$

On the other hand, recall that a *Lie bialgebroid* [25] (see also [16]) is a dual pair (A, A^*) of vector bundles equipped with Lie algebroid structures $([\![,]\!], \rho)$ and $([\![,]\!]_*, \rho_*)$, respectively, such that the differential d_* of A^* satisfies

$$\mathrm{d}_*[\![X, Y]\!] = [\![X, \mathrm{d}_* Y]\!] - [\![Y, \mathrm{d}_* X]\!] \quad \text{for } X, Y \in \Gamma(A).$$

An interesting example is the following. If (M, Λ) is a Poisson manifold then $(T^*M, [\![,]\!]_\Lambda, \#_\Lambda)$ is a Lie algebroid, where $[\![,]\!]_\Lambda$ is the bracket of 1-forms defined by (see [1,2,7,30])

$$\begin{aligned} &[\![,]\!]_\Lambda : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \\ &[\![\alpha, \beta]\!]_\Lambda = \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)). \end{aligned} \tag{2.2}$$

Note that, for this algebroid, the differential is the operator $\mathrm{d}_* = -[\Lambda, \cdot]$, which was introduced by Lichnerowicz in [21] to define the Poisson cohomology. Moreover, if on TM we consider the trivial Lie algebroid structure, the pair (TM, T^*M) is a Lie bialgebroid (see [25]).

2.2. Jacobi manifolds

A *Jacobi structure* on M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad [E, \Lambda] = 0. \tag{2.3}$$

The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f, g\} = \Lambda(\delta f, \delta g) + fE(g) - gE(f)$$

for all $f, g \in C^\infty(M, \mathbb{R})$. In fact, the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a *local Lie algebra* in the sense of Kirillov (see [15]), that is, the mapping $\{,\} : C^\infty(M, \mathbb{R}) \times$

$C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ is $\mathbb{R}$-bilinear, skew-symmetric, satisfies the Jacobi identity and is a first-order differential operator on each of its arguments, with respect to the ordinary multiplication of functions, i.e.,

$$\{f_1f_2,g\} = f_1\{f_2,g\} + f_2\{f_1,g\} - f_1f_2\{1,g\} \quad \text{for } f_1, f_2, g \in C^\infty(M,\mathbb{R}).$$

Conversely, a structure of local Lie algebra on $C^\infty(M,\mathbb{R})$ defines a Jacobi structure on M (see [9,15]). Note that if the vector field E identically vanishes then (M,Λ) is a Poisson manifold. Jacobi and Poisson manifolds were introduced by Lichnerowicz [21,22] (see also [1,4,20,30,32]).

Remark 2.1. Let (Λ,E) be a Jacobi structure on a manifold M and consider on the product manifold $M\times\mathbb{R}$ the 2-vector $\tilde{\Lambda}$ given by

$$\tilde{\Lambda} = \mathrm{e}^{-t}\left(\Lambda + \frac{\partial}{\partial t}\wedge E\right),$$

where t is the usual coordinate on $\mathbb{R}$. Then, $\tilde{\Lambda}$ defines a Poisson structure on $M\times\mathbb{R}$ (see [22]). The manifold $M\times\mathbb{R}$ endowed with the structure $\tilde{\Lambda}$ is called the *Poissonization of the Jacobi manifold* (M,Λ,E).

If M is a differentiable manifold then the triple $(TM\times\mathbb{R},[\,,\,],\pi)$ is a Lie algebroid over M, where $\pi: TM\times\mathbb{R}\to TM$ is the canonical projection over the first factor and $[\,,\,]$ is the bracket given by (see [24,26])

$$[(X,f),(Y,g)] = ([X,Y], X(g) - Y(f)) \tag{2.4}$$

for $(X,f),(Y,g)\in\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})\cong\Gamma(TM\times\mathbb{R})$. In this case, the dual bundle to $TM\times\mathbb{R}$ is $T^*M\times\mathbb{R}$ and the spaces $\Gamma(\wedge^k(TM\times\mathbb{R}))$ and $\Gamma(\wedge^r(T^*M\times\mathbb{R}))$ can be identified with $\mathcal{V}^k(M)\oplus\mathcal{V}^{k-1}(M)$ and $\Omega^r(M)\oplus\Omega^{r-1}(M)$ (see (1.2)). Under these identifications, the differential $\tilde{\delta}$ and the Schouten bracket $[\,,\,]$ of this Lie algebroid are

$$\tilde{\delta}(\alpha,\beta) = (\delta\alpha, -\delta\beta), \tag{2.5}$$

$$[(P,Q),(P',Q')] = ([P,P'], (-1)^{k+1}[P,Q'] - [Q,P']) \tag{2.6}$$

for $(\alpha,\beta)\in\Omega^r(M)\oplus\Omega^{r-1}(M)$, $(P,Q)\in\mathcal{V}^k(M)\oplus\mathcal{V}^{k-1}(M)$ and $(P',Q')\in\mathcal{V}^{k'}(M)\oplus\mathcal{V}^{k'-1}(M)$.

On the other hand, a Jacobi manifold (M,Λ,E) has an associated Lie algebroid $(T^*M\times\mathbb{R}, [\![\,,\,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$, where $[\![\,,\,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)}$ are defined by

$$\begin{aligned}[\![(\alpha,f),(\beta,g)]\!]_{(\Lambda,E)} = (&\mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \delta(\Lambda(\alpha,\beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha\\ &-i_E(\alpha\wedge\beta), \Lambda(\beta,\alpha) + \#_\Lambda(\alpha)(g) - \#_\Lambda(\beta)(f)\\ &+fE(g) - gE(f)),\end{aligned} \tag{2.7}$$

$$\tilde{\#}_{(\Lambda,E)}(\alpha,f) = \#_\Lambda(\alpha) + fE$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$, $\mathcal{L}$ being the Lie derivative operator (see [14]). If d_* is the differential of this algebroid and $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$, we have (see [18,19])

$$d_*(P, Q) = (-[\Lambda, P] + kE \wedge P + \Lambda \wedge Q, [\Lambda, Q] - (k-1)E \wedge Q + [E, P]). \tag{2.8}$$

Remark 2.2. If (M, Λ, E) is a Jacobi manifold and on $TM \times \mathbb{R}$ (resp. $T^*M \times \mathbb{R}$) we consider the Lie algebroid structure $([,], \pi)$ (resp. $([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$) then the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not, in general, a Lie bialgebroid (see [31]).

3. Generalized Lie bialgebroids

Let A be a vector bundle over M and A^* the dual bundle to A. Suppose that $([\![,]\!], \rho)$ (resp. $([\![,]\!]_*, \rho_*)$) is a Lie algebroid structure on A (resp. A^*) and that $\phi_0 \in \Gamma(A^*)$ (resp. $X_0 \in \Gamma(A)$) is a 1-cocycle in the corresponding Lie algebroid cohomology complex with trivial coefficients. Then, we will use the following notation (see Appendix A):

- d (resp. d_*) is the differential of $(A, [\![,]\!], \rho)$ (resp. $(A^*, [\![,]\!]_*, \rho_*)$).
- d_{ϕ_0} (resp. d_{*X_0}) is the ϕ_0-differential (resp. X_0-differential) of A (resp. A^*).
- $\mathcal{L}$ (resp. $\mathcal{L}_*$) is the Lie derivative of A (resp. A^*).
- $\mathcal{L}_{\phi_0}$ (resp. $\mathcal{L}_{*X_0}$) is the ϕ_0-Lie derivative (resp. X_0-Lie derivative).
- $[\![,]\!]_{\phi_0}$ (resp. $[\![,]\!]_{*X_0}$) is the ϕ_0-Schouten bracket (resp. X_0-Schouten bracket) on $(A, [\![,]\!], \rho)$ (resp.$(A^*, [\![,]\!]_*, \rho_*)$).
- ρ_{ϕ_0} (resp. ρ_{*X_0}) is the representation of A (resp. A^*) on the trivial vector bundle $M \times \mathbb{R} \to M$ given by (A.1).
- $(\rho, \phi_0) : \Gamma(A) \to \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ (resp. $(\rho_*, X_0) : \Gamma(A^*) \to \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$) is the homomorphism of $C^\infty(M, \mathbb{R})$-modules given by (A.7) and $(\rho, \phi_0)^* : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \Gamma(A^*)$ (resp. $(\rho_*, X_0)^* : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \Gamma(A)$) is the adjoint operator of (ρ, ϕ_0) (resp. (ρ_*, X_0)).

3.1. *Generalized Lie bialgebroids and Jacobi structures on the base space*

Suppose that M is a differentiable manifold and let $([,], \pi)$ be the Lie algebroid structure on the vector bundle $TM \times \mathbb{R} \to M$. As we know (see Appendix A), the 1-cochain $\phi_0 = (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle. Thus, we can consider the ϕ_0-Lie derivative $\mathcal{L}_{\phi_0} = \mathcal{L}_{(0,1)}$ and the ϕ_0-Schouten bracket $[,]_{\phi_0} = [,]_{(0,1)}$.

Now, assume that (Λ, E) is a Jacobi structure on M and denote by $([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ the Lie algebroid structure on $T^*M \times \mathbb{R} \to M$ and by d_* the differential of this Lie algebroid. From (2.3) and (2.8), it follows that $X_0 = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$ is a 1-cocycle.

Using (1.3), (2.6), (2.8), (A.2) and (A.9), we also have that the X_0-differential $d_{*X_0} = d_{*(-E,0)}$ of the Lie algebroid $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ is given by

$$d_{*(-E,0)}(P, Q) = -[(\Lambda, E), (P, Q)]_{(0,1)} \tag{3.1}$$

for $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. Note that $\mathrm{d}_{*(-E,0)}$ is just the cohomology operator of the 1-*differentiable Chevalley–Eilenberg cohomology complex of M* (see [9,22]). Moreover, compare Eq. (3.1) with the expression of the differential of the Lie algebroid associated with a Poisson manifold (see Section 2.1).

For the above Lie algebroids and 1-cocycles, we deduce

Proposition 3.1.

1. *If* $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$, *then*

$$\mathrm{d}_{*(-E,0)}[(X, f), (Y, g)] = [(X, f), \mathrm{d}_{*(-E,0)}(Y, g)]_{(0,1)} - [(Y, g), \mathrm{d}_{*(-E,0)}(X, f)]_{(0,1)}.$$

2. *If* $\mathcal{L}_*$ *denotes the Lie derivative on the Lie algebroid* $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ and $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \cong \Gamma(\wedge^k(TM \times \mathbb{R}))$, *then*

$$(\mathcal{L}_{*(-E,0)})_{(0,1)}(P, Q) + (\mathcal{L}_{(0,1)})_{(-E,0)}(P, Q) = 0.$$

Proof.

1. It follows from (3.1), (A.11) and (A.14).
2. Using (1.3), (2.6), (3.1), (A.3), (A.9) and (A.15), we have that

$$\begin{aligned}&(\mathcal{L}_{*(-E,0)})_{(0,1)}(P, Q) + (\mathcal{L}_{(0,1)})_{(-E,0)}(P, Q)\\ &\quad = \mathrm{d}_{*(-E,0)}(Q, 0) + i_{(0,1)}(-[\Lambda, P] + (k-1)E \wedge P + \Lambda \wedge Q, [\Lambda, Q]\\ &\qquad -(k-2)E \wedge Q + [E, P]) - ([E, P], [E, Q]) = 0.\end{aligned}$$ □

Now, let $(A, [\![,]\!], \rho)$ be a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ a 1-cocycle. Assume also that the dual bundle A^* admits a Lie algebroid structure $([\![,]\!]_*, \rho_*)$ and that $X_0 \in \Gamma(A)$ is a 1-cocycle.

Suggested by Proposition 3.1, we introduce the following definition.

Definition 3.2. The pair $((A, \phi_0), (A^*, X_0))$ is said to be a generalized Lie bialgebroid over M if for all $X, Y \in \Gamma(A)$ and $P \in \Gamma(\wedge^k A)$

$$\mathrm{d}_{*X_0}[\![X, Y]\!] = [\![X, \mathrm{d}_{*X_0} Y]\!]_{\phi_0} - [\![Y, \mathrm{d}_{*X_0} X]\!]_{\phi_0}, \tag{3.2}$$

$$(\mathcal{L}_{*X_0})_{\phi_0} P + (\mathcal{L}_{\phi_0})_{X_0} P = 0. \tag{3.3}$$

Using (A.1), (A.4), (A.10), (A.12) and (A.15), we obtain that (3.3) holds if and only if

$$\phi_0(X_0) = 0, \qquad \rho(X_0) = -\rho_*(\phi_0), \tag{3.4}$$

$$(\mathcal{L}_*)_{\phi_0} X + [\![X_0, X]\!] = 0 \quad \text{for } X \in \Gamma(A). \tag{3.5}$$

Note that (3.4) and (3.5) follow applying (3.3) to $P = f \in C^\infty(M, \mathbb{R}) = \Gamma(\wedge^0 A)$ and $P = X \in \Gamma(A)$.

Example 3.3.

1. In the particular case when $\phi_0 = 0$ and $X_0 = 0$, (3.2) and (3.3) are equivalent to the condition $d_*[\![X, Y]\!] = [\![X, d_*Y]\!] - [\![Y, d_*X]\!]$. Thus, the pair $((A, 0), (A^*, 0))$ is a generalized Lie bialgebroid if and only if the pair (A, A^*) is a Lie bialgebroid.
2. Let (M, Λ, E) be a Jacobi manifold. From Proposition 3.1, we deduce that the pair $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)))$ is a generalized Lie bialgebroid.

Next, we will show that if $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid over M, then M carries an induced Jacobi structure. First, we will prove some results.

Proposition 3.4. *Let $((A, \phi_0), (A^*, X_0))$ be a generalized Lie bialgebroid. Then,*

$$(\mathcal{L}_{*X_0})_{d_{\phi_0} f} X = [\![X, d_{*X_0} f]\!] \tag{3.6}$$

for $X \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$.

Proof. Using (A.2) and the derivation law on Lie algebroids, we obtain that

$$\begin{aligned} d_{*X_0}([\![X, fY]\!]) = {} & (d_{*X_0} f) \wedge [\![X, Y]\!] + f d_{*X_0}[\![X, Y]\!] - fX_0 \wedge [\![X, Y]\!] \\ & + d_{*X_0}(\rho(X)(f)) \wedge Y + \rho(X)(f) d_{*X_0} Y - \rho(X)(f) X_0 \wedge Y \end{aligned}$$

for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$.

On the other hand, from (3.2), (A.2), (A.12) and (A.15), we deduce that

$$\begin{aligned} d_{*X_0}([\![X, fY]\!]) &= (\mathcal{L}_{\phi_0})_X (d_{*X_0}(fY)) - (\mathcal{L}_{\phi_0})_{fY}(d_{*X_0}(X)) \\ &= ((\mathcal{L}_{\phi_0})_X (d_{*X_0} f)) \wedge Y + (d_{*X_0} f) \wedge (\mathcal{L}_{\phi_0})_X Y \\ &\quad -\phi_0(X)(d_{*X_0} f) \wedge Y + f(\mathcal{L}_{\phi_0})_X (d_{*X_0} Y) + \rho(X)(f) d_{*X_0} Y \\ &\quad -f((\mathcal{L}_{\phi_0})_X X_0 \wedge Y + X_0 \wedge (\mathcal{L}_{\phi_0})_X Y - \phi_0(X) X_0 \wedge Y) \\ &\quad -\rho(X)(f) X_0 \wedge Y - f(\mathcal{L}_{\phi_0})_Y (d_{*X_0} X) - i_{df}(d_{*X_0} X) \wedge Y. \end{aligned}$$

Thus, using again (3.2), it follows that

$$\begin{aligned} d_{*X_0}(\rho(X)(f)) \wedge Y = {} & ((\mathcal{L}_{\phi_0})_X d_{*X_0} f - \phi_0(X) d_{*X_0} f - f(\mathcal{L}_{\phi_0})_X X_0 \\ & + f\phi_0(X) X_0 - i_{df}(d_{*X_0} X)) \wedge Y, \end{aligned}$$

and so

$$\begin{aligned} & d_{*X_0}(\rho(X)(f)) - (\mathcal{L}_{\phi_0})_X d_{*X_0} f + \phi_0(X) d_{*X_0} f + f(\mathcal{L}_{\phi_0})_X X_0 \\ & \quad - f\phi_0(X) X_0 + i_{df}(d_{*X_0} X) = 0, \end{aligned}$$

which, by (3.5), (A.2) and (A.3), implies (3.6). □

Corollary 3.5. *Under the same hypothesis as in Proposition* 3.4, *for all $f, g \in C^\infty(M, \mathbb{R})$ we have*

$$[\![d_{*X_0} g, d_{*X_0} f]\!] = d_{*X_0}(d_{\phi_0} f \cdot d_{*X_0} g).$$

Proof. From (A.3), Proposition 3.4 and since $\mathrm{d}^2_{*X_0} = 0$, we deduce that

$$[\![\mathrm{d}_{*X_0} g, \mathrm{d}_{*X_0} f]\!] = (\mathcal{L}_{*X_0})_{\mathrm{d}_{\phi_0} f}(\mathrm{d}_{*X_0} g) = \mathrm{d}_{*X_0}(\mathrm{d}_{\phi_0} f \cdot \mathrm{d}_{*X_0} g). \qquad \Box$$

Remark 3.6. Using (A.6)–(A.8), it follows that

$$\begin{aligned} \mathrm{d}_{\phi_0} f \cdot \mathrm{d}_{*X_0} g &= \tilde{\delta}_{(0,1)} f \cdot ((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\tilde{\delta}_{(0,1)} g) \\ &= \pi_{(0,1)}(((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\tilde{\delta}_{(0,1)} g), f), \end{aligned}$$

where $\pi_{(0,1)}$ is the representation of the Lie algebroid $(TM \times \mathbb{R}, [\,,\,], \pi)$ on the vector bundle $M \times \mathbb{R} \to M$ given by (A.5).

Now, we will prove the main result of this section.

Theorem 3.7. *Let* $((A, \phi_0), (A^*, X_0))$ *be a generalized Lie bialgebroid. Then, the bracket of functions* $\{,\} : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ *given by*

$$\{f, g\} := \mathrm{d}_{\phi_0} f \cdot \mathrm{d}_{*X_0} g \quad \text{for } f, g \in C^\infty(M, \mathbb{R}),$$

defines a Jacobi structure on M.

Proof. First of all, we need to prove that $\{,\}$ is skew-symmetric. Using (3.4), (A.2) and (A.8), we have that

$$\{f, f\} = \mathrm{d} f \cdot \mathrm{d}_* f = (((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\delta f, 0)) \cdot (\delta f, 0). \tag{3.7}$$

On the other hand, from (3.7) and Corollary 3.5, it follows that $\mathrm{d}_{*X_0}(((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\delta f^2, 0) \cdot (\delta f^2, 0)) = 0$. Then (see (A.2))

$$\begin{aligned} 0 &= (((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\delta f, 0) \cdot (\delta f, 0))(\mathrm{d}_{*X_0} f^2 - f^2 X_0) \\ &= 2f(((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\delta f, 0) \cdot (\delta f, 0))\mathrm{d}_* f. \end{aligned} \tag{3.8}$$

Thus, using (3.8) and (A.8), we deduce that $f(((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\delta f, 0) \cdot (\delta f, 0))^2 = 0$ for all $f \in C^\infty(M, \mathbb{R})$. This implies that

$$(((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\delta f, 0)) \cdot (\delta f, 0) = 0,$$

and, therefore, we conclude that $\{f, f\} = 0$ (see (3.7)) for all $f \in C^\infty(M, \mathbb{R})$, that is, $\{,\}$ is skew-symmetric.

Using (A.2) and since $\mathrm{d}_{\phi_0} 1 = \phi_0$, we deduce that $\{,\}$ is a first-order differential operator on each of its arguments.

Now, let us prove the Jacobi identity. From Corollary 3.5, we have

$$\mathrm{d}_{\phi_0} h \cdot [\![\mathrm{d}_{*X_0} g, \mathrm{d}_{*X_0} f]\!] = \mathrm{d}_{\phi_0} h \cdot \mathrm{d}_{*X_0}(\{f, g\}) = \{h, \{f, g\}\}.$$

Thus, using (A.8) and the fact that (ρ, ϕ_0) is a Lie algebroid homomorphism, we deduce

$$\begin{aligned} &\pi_{(0,1)}([((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\tilde{\delta}_{(0,1)}(g)), ((\rho, \phi_0) \circ (\rho_*, X_0)^*)(\tilde{\delta}_{(0,1)}(f))], h) \\ &\quad = \{h, \{f, g\}\}. \end{aligned}$$

Consequently, since $\pi_{(0,1)}$ is a representation of the Lie algebroid $(TM \times \mathbb{R}, [\,,\,], \pi)$ on the vector bundle $M \times \mathbb{R} \to M$, this implies that (see Remark 3.6)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \qquad \square$$

From (3.4) and (A.2), we have that

$$\{f, g\} = \mathrm{d}f \cdot \mathrm{d}_* g - f\rho(X_0)(g) + g\rho(X_0)(f) \tag{3.9}$$

for $f, g \in C^\infty(M, \mathbb{R})$. Since the differential d is a derivation with respect to the ordinary multiplication of functions we have that the map $(f, g) \mapsto \mathrm{d}f \cdot \mathrm{d}_* g$, for $f, g \in C^\infty(M, \mathbb{R})$, is also a derivation on each of its arguments. Thus, we can define the 2-vector $\Lambda \in \mathcal{V}^2(M)$ characterized by the relation,

$$\Lambda(\delta f, \delta g) = \mathrm{d}f \cdot \mathrm{d}_* g = -\mathrm{d}g \cdot \mathrm{d}_* f \tag{3.10}$$

for $f, g \in C^\infty(M, \mathbb{R})$, and the vector field $E \in \mathfrak{X}(M)$ by

$$E = -\rho(X_0) = \rho_*(\phi_0). \tag{3.11}$$

From (3.9), we obtain that $\{f, g\} = \Lambda(\delta f, \delta g) + fE(g) - gE(f)$ for $f, g \in C^\infty(M, \mathbb{R})$. Therefore, the pair (Λ, E) is the Jacobi structure induced by the Jacobi bracket $\{,\}$.

If (A, A^*) is a Lie bialgebroid, then the pair $((A, 0), (A^*, 0))$ is a generalized Lie bialgebroid and, by Theorem 3.7, a Jacobi structure (Λ, E) can be defined on the base space M. Since $\phi_0 = X_0 = 0$, we deduce that $E = 0$, that is, the Jacobi structure is Poisson, which implies a well-known result (see [25]): given a Lie bialgebroid (A, A^*) over M, the base space M carries an induced Poisson structure.

3.2. Lie bialgebroids and generalized Lie bialgebroids

In this section, we will show a relation between generalized Lie bialgebroids and Lie bialgebroids. For this purpose, we will use the notations and results contained in Appendix B.

Firstly, suppose that A_1 and A_2 are two Lie algebroids on M such that the dual bundles A_1^* and A_2^* are also Lie algebroids. Then, a direct computation shows the following result.

Lemma 3.8. *Let $\Phi : A_1 \to A_2$ be a Lie algebroid isomorphism over the identity such that its adjoint homomorphism $\Phi^* : A_2^* \to A_1^*$ is also a Lie algebroid isomorphism. If (A_1, A_1^*) is a Lie bialgebroid, so is (A_2, A_2^*).*

Next, assume that (M, Λ, E) is a Jacobi manifold. Consider on $A = TM \times \mathbb{R}$ and on $A^* = T^*M \times \mathbb{R}$ the Lie algebroid structures $([\,,\,], \pi)$ and $([\![\,,\,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$, respectively. Then, the pair $((A, \phi_0 = (0, 1)), (A^*, X_0 = (-E, 0)))$ is a generalized Lie bialgebroid.

Now, the map $\Phi : \tilde{A} = A \times \mathbb{R} \to T(M \times \mathbb{R})$ defined by $\Phi((v_{x_0}, \lambda_0), t_0) = v_{x_0} + \lambda_0(\partial/\partial t)_{|t_0}$, for $x_0 \in M$, $v_{x_0} \in T_{x_0}M$ and $\lambda_0, t_0 \in \mathbb{R}$, induces an isomorphism between the vector bundles $A \times \mathbb{R} \to M \times \mathbb{R}$ and $T(M \times \mathbb{R}) \to M \times \mathbb{R}$. Thus, $\tilde{A} = A \times \mathbb{R}$ can be identified with $T(M \times \mathbb{R})$ and, under this identification, the Lie algebroid structure $(\widetilde{[\,,\,]}^{\phi_0}, \tilde{\pi}^{\phi_0})$ (see (B.1)) is just the trivial Lie algebroid structure on $T(M \times \mathbb{R})$.

The pair $([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ and the 1-cocycle $X_0=(-E,0)$ allow us to introduce the Lie algebroid structure $(\hat{[\![,]\!]}^{X_0}_{(\Lambda,E)}, \widehat{\tilde{\#}_{(\Lambda,E)}}^{X_0})$ (see (B.2)) on the vector bundle $\tilde{A}^*=A^*\times\mathbb{R}\to M\times\mathbb{R}$. A long computation, using (2.2), (2.7) and (B.2), shows that

$$[\![(\tilde{\alpha},\tilde{f}),(\tilde{\beta},\tilde{g})]\!]\hat{}^{X_0}_{(\Lambda,E)}=[\![\Phi^*(\tilde{\alpha}+\tilde{f}\,\mathrm{d}t),\Phi^*(\tilde{\beta}+\tilde{g}\,\mathrm{d}t)]\!]\hat{}^{X_0}_{(\Lambda,E)}=[\![\tilde{\alpha}+\tilde{f}\,\mathrm{d}t,\tilde{\beta}+\tilde{g}\,\mathrm{d}t]\!]_{\tilde{\Lambda}},$$

$$\widehat{\tilde{\#}_{(\Lambda,E)}}^{X_0}(\tilde{\alpha},\tilde{f})=\widehat{\tilde{\#}_{(\Lambda,E)}}^{X_0}(\Phi^*(\tilde{\alpha}+\tilde{f}\,\mathrm{d}t))=\#_{\tilde{\Lambda}}(\tilde{\alpha}+\tilde{f}\,\mathrm{d}t)$$

for $\tilde{\alpha},\tilde{\beta}$ time-dependent 1-forms on M and $\tilde{f},\tilde{g}\in C^\infty(M\times\mathbb{R},\mathbb{R})$, where $\tilde{\Lambda}$ is the Poissonization of the Jacobi structure (Λ,E) and $\Phi^*:T^*(M\times\mathbb{R})\to\tilde{A}^*=A^*\times\mathbb{R}$ is the adjoint isomorphism of Φ.

Therefore, $\tilde{A}^*=A^*\times\mathbb{R}$ can be identified with $T^*(M\times\mathbb{R})$ and, under this identification, the Lie algebroid structure $([\![,]\!]^{X_0}_{(\Lambda,E)}, \widehat{\tilde{\#}_{(\Lambda,E)}}^{X_0})$ is just the Lie algebroid structure $([\![,]\!]_{\tilde{\Lambda}},\#_{\tilde{\Lambda}})$ on $T^*(M\times\mathbb{R})$. Consequently, using Lemma 3.8, we deduce that, for this particular case, the pair $(\tilde{A},\tilde{A}^*)$ is a Lie bialgebroid, when we consider on $\tilde{A}$ and $\tilde{A}^*$ the Lie algebroid structures $([,]\tilde{}^{\phi_0},\tilde{\pi}^{\phi_0})$ and $(\hat{[\![,]\!]}^{X_0}_{(\Lambda,E)}, \widehat{\tilde{\#}_{(\Lambda,E)}}^{X_0})$, respectively.

In this section, we generalize the above result for an arbitrary generalized Lie bialgebroid. In fact, we prove the following theorem.

Theorem 3.9. *Let $((A,\phi_0),(A^*,X_0))$ be a generalized Lie bialgebroid and (Λ,E) the induced Jacobi structure over M. Consider on $\tilde{A}=A\times\mathbb{R}$ (resp. $\tilde{A}^*=A^*\times\mathbb{R}$) the Lie algebroid structure $([\![,]\!]\tilde{}^{\phi_0},\tilde{\rho}^{\phi_0})$ (resp. $(\hat{[\![,]\!]}^{X_0}_*,\widehat{\rho_*}^{X_0})$). Then,*

1. *The pair $(\tilde{A},\tilde{A}^*)$ is a Lie bialgebroid over $M\times\mathbb{R}$.*
2. *If $\tilde{\Lambda}$ is the induced Poisson structure on $M\times\mathbb{R}$ then $\tilde{\Lambda}$ is the Poissonization of the Jacobi structure (Λ,E).*

Proof. Denote by $\tilde{\mathrm{d}}$ (resp. $\tilde{\mathrm{d}}_*$, $\tilde{\mathrm{d}}^{\phi_0}$ and $\widehat{\mathrm{d}_*}^{X_0}$) the differential of the Lie algebroid $(\tilde{A},[\![,]\!]\tilde{},\tilde{\rho})$ (resp. $(\tilde{A}^*,[\![,]\!]\tilde{}_*,\tilde{\rho}_*)$, $(\tilde{A},[\![,]\!]\tilde{}^{\phi_0},\tilde{\rho}^{\phi_0})$ and $(\tilde{A}^*,\hat{[\![,]\!]}^{X_0}_*,\widehat{\rho_*}^{X_0})$).

1. If $\tilde{X},\tilde{Y}\in\Gamma(\tilde{A})$ then, using (A.2), (B.1) and (B.4) and the properties of the derivative with respect to the time of a time-dependent multisection, it follows that

$$\begin{aligned}\widehat{\mathrm{d}_*}^{X_0}[\![\tilde{X},\tilde{Y}]\!]\tilde{}^{\phi_0}=\mathrm{e}^{-t}\Bigg(&\tilde{\mathrm{d}}_{*X_0}[\![\tilde{X},\tilde{Y}]\!]\tilde{}+X_0\wedge\left[\!\!\left[\frac{\partial\tilde{X}}{\partial t},\tilde{Y}\right]\!\!\right]\tilde{}+X_0\wedge\left[\!\!\left[\tilde{X},\frac{\partial\tilde{Y}}{\partial t}\right]\!\!\right]\tilde{}\\&+\phi_0(\tilde{X})\frac{\partial}{\partial t}(\tilde{\mathrm{d}}_{*X_0}\tilde{Y})-\phi_0(\tilde{Y})\frac{\partial}{\partial t}(\tilde{\mathrm{d}}_{*X_0}\tilde{X})-\phi_0(\tilde{X})X_0\\&\wedge\frac{\partial\tilde{Y}}{\partial t}+\phi_0(\tilde{Y})X_0\wedge\frac{\partial\tilde{X}}{\partial t}+\phi_0(\tilde{X})X_0\wedge\frac{\partial^2\tilde{Y}}{\partial t^2}-\phi_0(\tilde{Y})X_0\\&\wedge\frac{\partial^2\tilde{X}}{\partial t^2}+\frac{\partial}{\partial t}(\phi_0(\tilde{X}))X_0\wedge\frac{\partial\tilde{Y}}{\partial t}-\frac{\partial}{\partial t}(\phi_0(\tilde{Y}))X_0\wedge\frac{\partial\tilde{X}}{\partial t}\\&+\tilde{\mathrm{d}}_{*X_0}(\phi_0(\tilde{X}))\wedge\frac{\partial\tilde{Y}}{\partial t}-\tilde{\mathrm{d}}_{*X_0}(\phi_0(\tilde{Y}))\wedge\frac{\partial\tilde{X}}{\partial t}\Bigg).\end{aligned}$$

On the other hand, using (3.4), (A.10), (A.12), (B.4) and (B.5) and the fact that $(\partial/\partial t)(\phi_0(\tilde{Z})) = \phi_0(\partial\tilde{Z}/\partial t)$ for $\tilde{Z} \in \Gamma(\tilde{A})$, we have that

$$[\![\tilde{X}, \widehat{\mathrm{d}_*}^{X_0}\tilde{Y}]\!]^{\tilde{}\phi_0} = \mathrm{e}^{-t}\left([\![\tilde{X}, \widetilde{\mathrm{d}_{*X_0}}\tilde{Y}]\!]^{\tilde{}}_{\phi_0} + [\![\tilde{X}, X_0]\!]^{\tilde{}} \wedge \frac{\partial\tilde{Y}}{\partial t} + X_0 \wedge \left[\!\left[\tilde{X}, \frac{\partial\tilde{Y}}{\partial t}\right]\!\right]^{\tilde{}}\right.$$
$$-\phi_0(\tilde{X})X_0 \wedge \frac{\partial\tilde{Y}}{\partial t} + \phi_0(\tilde{X})\frac{\partial}{\partial t}(\widetilde{\mathrm{d}_{*X_0}}\tilde{Y}) + \phi_0(\tilde{X})X_0$$
$$\left.\wedge\frac{\partial^2\tilde{Y}}{\partial t^2} - \frac{\partial\tilde{X}}{\partial t} \wedge i_{\phi_0}(\widetilde{\mathrm{d}_{*X_0}}\tilde{Y}) + \frac{\partial}{\partial t}(\phi_0(\tilde{Y}))\frac{\partial\tilde{X}}{\partial t} \wedge X_0\right).$$

Finally, from (3.2) and (3.5), we deduce that

$$\widehat{\mathrm{d}_*}^{X_0}[\![\tilde{X}, \tilde{Y}]\!]^{\tilde{}\phi_0} = [\![\tilde{X}, \widehat{\mathrm{d}_*}^{X_0}\tilde{Y}]\!]^{\tilde{}\phi_0} - [\![\tilde{Y}, \widehat{\mathrm{d}_*}^{X_0}\tilde{X}]\!]^{\tilde{}\phi_0}.$$

2. If $\tilde{f}, \tilde{g} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$ then, using (3.4), (B.3) and (B.4) and Theorem 3.7, we obtain that

$$\tilde{\Lambda}(\delta\tilde{f}, \delta\tilde{g}) = \tilde{\mathrm{d}}^{\phi_0}\tilde{f} \cdot \widehat{\mathrm{d}_*}^{X_0}\tilde{g} = \mathrm{e}^{-t}\left(\tilde{\mathrm{d}}\tilde{f} \cdot \widetilde{\mathrm{d}_*}\tilde{g} - \frac{\partial\tilde{f}}{\partial t}\rho(X_0)(\tilde{g}) + \frac{\partial\tilde{g}}{\partial t}\rho(X_0)(\tilde{f})\right).$$

On the other hand, from (3.10) and (3.11), we conclude that $\mathrm{e}^{-t}(\Lambda + (\partial/\partial t) \wedge E) = \tilde{\Lambda}$, that is, $\tilde{\Lambda}$ is the Poissonization of (Λ, E). □

Now, we discuss a converse of Theorem 3.9.

Theorem 3.10. *Let $(A, [\![,]\!], \rho)$ be a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ a 1-cocycle. Suppose that $([\![,]\!]_*, \rho_*)$ is a Lie algebroid structure on A^* and that $X_0 \in \Gamma(A)$ is a 1-cocycle. Consider on $\tilde{A} = A \times \mathbb{R}$ (resp. $\tilde{A}^* = A^* \times \mathbb{R}$) the Lie algebroid structure $([\![,]\!]^{\tilde{}\phi_0}, \tilde{\rho}^{\phi_0})$* (resp. $([\![,]\!]^{\widehat{}X_0}_*, \widehat{\rho_*}^{X_0})$). If $(\tilde{A}, \tilde{A}^*)$ *is a Lie bialgebroid then the pair $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid*.

Proof. Let $\{,\}^{\tilde{}}$ be the induced Poisson bracket on $M \times \mathbb{R}$. Then, from (B.3) and (B.4) and Theorem 3.7, it follows that

$$\{\tilde{f}, \tilde{g}\}^{\tilde{}} = \mathrm{e}^{-t}\left(\tilde{\mathrm{d}}\tilde{f} \cdot \widetilde{\mathrm{d}_*}\tilde{g} + \frac{\partial\tilde{g}}{\partial t}\rho(X_0)(\tilde{f}) + \frac{\partial\tilde{f}}{\partial t}\rho_*(\phi_0)(\tilde{g}) + \frac{\partial\tilde{g}}{\partial t}\frac{\partial\tilde{f}}{\partial t}\phi_0(X_0)\right)$$

for $\tilde{f}, \tilde{g} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$. Since $\{,\}^{\tilde{}}$ is skew-symmetric, we have that $\{t, t\}^{\tilde{}} = 0$ which implies that $\phi_0(X_0) = 0$. Moreover, if $f \in C^\infty(M, \mathbb{R})$ then, using that $\{f, t\}^{\tilde{}} = -\{t, f\}^{\tilde{}}$, we conclude that $\rho(X_0) = -\rho_*(\phi_0)$.

Now, from (B.1), (B.4) and (B.5) and since $\widehat{\mathrm{d}_*}^{X_0}[\![X, Y]\!]^{\tilde{}\phi_0} = [\![X, \widehat{\mathrm{d}_*}^{X_0}Y]\!]^{\tilde{}\phi_0} - [\![Y, \widehat{\mathrm{d}_*}^{X_0}X]\!]^{\tilde{}\phi_0}$, we deduce (3.2).

Finally, if $X \in \Gamma(A)$ then, using the computations in the proof of Theorem 3.9 and the fact that

$$\widehat{\mathrm{d}_*}^{X_0}[\![X, \mathrm{e}^t Y]\!]^{\tilde{\phi}_0} = [\![X, \widehat{\mathrm{d}_*}^{X_0}(\mathrm{e}^t Y)]\!]^{\tilde{\phi}_0} - [\![\mathrm{e}^t Y, \widehat{\mathrm{d}_*}^{X_0} X]\!]^{\tilde{\phi}_0}$$

for all $Y \in \Gamma(A)$, we prove that $([\![X_0, X]\!] + (\mathcal{L}_*)_{\phi_0} X) \wedge Y = 0$. But this implies that

$$[\![X_0, X]\!] + (\mathcal{L}_*)_{\phi_0} X = 0. \qquad \square$$

In [25] it was proved that if the pair (A, A^*) is a Lie bialgebroid then the pair (A^*, A) is also a Lie bialgebroid. Using this fact, Lemma 3.8, Theorems 3.9 and 3.10 and since the Lie algebroids $(\tilde{A}, [\![,]\!]^{\tilde{\phi}_0}, \tilde{\rho}^{\phi_0})$ and $(\tilde{A}, [\![,]\!]^{\hat{\phi}_0}, \hat{\rho}^{\phi_0})$ are isomorphic (see Appendix B), we conclude that a similar result holds for generalized Lie bialgebroids.

Theorem 3.11. *If* $((A, \phi_0), (A^*, X_0))$ *is a generalized Lie bialgebroid, so is* $((A^*, X_0), (A, \phi_0))$.

4. Triangular generalized Lie bialgebroids

Let M be a differentiable manifold and $([,], \pi)$ be the Lie algebroid structure on $TM \times \mathbb{R} \to M$. Then, the section $\phi_0 = (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle. Denote by $\tilde{\delta}$ and $\mathcal{L}$ (resp. $\tilde{\delta}_{(0,1)}$ and $\mathcal{L}_{(0,1)}$) the differential and the Lie derivative (resp. the ϕ_0-differential and the ϕ_0-Lie derivative) of the Lie algebroid $(TM \times \mathbb{R}, [,], \pi)$ and by $[,]_{(0,1)}$ the ϕ_0-Schouten bracket (see Section 2.2 and Appendix A).

Suppose that $(\Lambda, E) \in \mathcal{V}^2(M) \oplus \mathfrak{X}(M) \cong \Gamma(\wedge^2(TM \times \mathbb{R}))$. From (2.3), (2.6) and (A.9), we have that

$$(\Lambda, E) \text{ is a Jacobi structure on } M \Leftrightarrow [(\Lambda, E), (\Lambda, E)]_{(0,1)} = 0. \tag{4.1}$$

Moreover, if (Λ, E) is a Jacobi structure on M and $([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ is the Lie algebroid structure on $T^*M \times \mathbb{R}$ then a long computation, using (1.2), (1.3) (A.3) and (A.6), shows that the Lie bracket $[\![,]\!]_{(\Lambda,E)}$ and the anchor map $\tilde{\#}_{(\Lambda,E)}$ can be written using the homomorphism $\#_{(\Lambda,E)} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R}) \to \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$ and the operators $\mathcal{L}_{(0,1)}$ and $\tilde{\delta}_{(0,1)}$ as follows:

$$\begin{aligned} [\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} &= (\mathcal{L}_{(0,1)})_{\#_{(\Lambda,E)}(\alpha,f)}(\beta, g) - (\mathcal{L}_{(0,1)})_{\#_{(\Lambda,E)}(\beta,g)}(\alpha, f) \\ &\quad -\tilde{\delta}_{(0,1)}((\Lambda, E)((\alpha, f), (\beta, g))) \\ &= i_{\#_{(\Lambda,E)}(\alpha,f)}(\tilde{\delta}_{(0,1)}(\beta, g)) - i_{\#_{(\Lambda,E)}(\beta,g)}(\tilde{\delta}_{(0,1)}(\alpha, f)) \\ &\quad +\tilde{\delta}_{(0,1)}((\Lambda, E)((\alpha, f), (\beta, g))), \end{aligned} \tag{4.2}$$

$$\tilde{\#}_{(\Lambda,E)} = \pi \circ \#_{(\Lambda,E)}.$$

Compare Eq. (2.2) with the above expression of the Lie algebroid bracket $[\![,]\!]_{(\Lambda,E)}$.

Now, let $(A, [\![,]\!], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma(A^*)$ a 1-cocycle. Moreover, let $P \in \Gamma(\wedge^2 A)$ be a bisection satisfying

$$[\![P, P]\!]_{\phi_0} = 0.$$

We shall discuss what happens on the dual bundle $A^* \to M$. Eq. (4.2) suggests us to introduce the bracket $[\![\,,\,]\!]_{*P}$ on $\Gamma(A^*)$ defined by

$$\begin{aligned}[\![\phi,\psi]\!]_{*P} &= (\mathcal{L}_{\phi_0})_{\#_P(\phi)}\psi - (\mathcal{L}_{\phi_0})_{\#_P(\psi)}\phi - \mathrm{d}_{\phi_0}(P(\phi,\psi)) \\ &= i_{\#_P(\phi)}\mathrm{d}_{\phi_0}\psi - i_{\#_P(\psi)}\mathrm{d}_{\phi_0}\phi + \mathrm{d}_{\phi_0}(P(\phi,\psi))\end{aligned} \tag{4.3}$$

for $\phi, \psi \in \Gamma(A^*)$.

Theorem 4.1. *Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid over M, $\phi_0 \in \Gamma(A^*)$ a 1-cocycle and $P \in \Gamma(\wedge^2 A)$ a bisection of $A \to M$ satisfying $[\![P,P]\!]_{\phi_0} = 0$. Then,*

1. *The dual bundle $A^* \to M$ together with the bracket defined in* (4.3) *and the bundle map $\rho_{*P} = \rho \circ \#_P : A^* \to TM$ is a Lie algebroid.*
2. *$X_0 = -\#_P(\phi_0) \in \Gamma(A)$ is a 1-cocycle.*
3. *The pair $((A,\phi_0),(A^*,X_0))$ is a generalized Lie bialgebroid.*

Proof. If we consider the Lie algebroid structure $([\![\,,\,]\!]\tilde{}^{\phi_0}, \tilde\rho^{\phi_0})$ on $\tilde A = A \times \mathbb{R} \to M \times \mathbb{R}$ and the bisection $\tilde P = \mathrm{e}^{-t}P \in \Gamma(\wedge^2 \tilde A)$ then, from (B.1) and (B.3) and Theorem A.3, it follows that $[\![\tilde P, \tilde P]\!]\tilde{}^{\phi_0} = 0$. Thus, using the results of Mackenzie and Xu [25], we deduce that the vector bundle $\tilde A^* \to M \times \mathbb{R}$ admits a Lie algebroid structure $([\![\,,\,]\!]\bar{}, \bar\rho)$, where $\bar\rho : \tilde A^* \to T(M \times \mathbb{R})$ is the bundle map given by $\bar\rho = \tilde\rho^{\phi_0} \circ \#_{\tilde P}$ and $[\![\,,\,]\!]\bar{}$ is the bracket on $\Gamma(\tilde A^*)$ defined by

$$[\![\tilde\phi,\tilde\psi]\!]\bar{} = i_{\#_{\tilde P}(\tilde\phi)}\tilde{\mathrm{d}}^{\phi_0}\tilde\psi - i_{\#_{\tilde P}(\tilde\psi)}\tilde{\mathrm{d}}^{\phi_0}\tilde\phi + \tilde{\mathrm{d}}^{\phi_0}(\tilde P(\tilde\phi,\tilde\psi)) \quad \text{for } \tilde\phi,\tilde\psi \in \Gamma(\tilde A^*),$$

where $\tilde{\mathrm{d}}^{\phi_0}$ is the differential of the Lie algebroid $(\tilde A, [\![\,,\,]\!]\tilde{}^{\phi_0}, \tilde\rho^{\phi_0})$.

Now, if $[\![\,,\,]\!]\tilde{}_{*P}$ denotes the natural extension of the bracket $[\![\,,\,]\!]_{*P}$ to $\Gamma(\tilde A^*)$, we will prove that

$$[\![\tilde\phi,\tilde\psi]\!]\hat{}^{X_0}_{*P} = \mathrm{e}^{-t}\left([\![\tilde\phi,\tilde\psi]\!]\tilde{}_{*P} + \tilde\phi(X_0)\left(\frac{\partial\tilde\psi}{\partial t} - \tilde\psi\right) - \tilde\psi(X_0)\left(\frac{\partial\tilde\phi}{\partial t} - \tilde\phi\right)\right) = [\![\tilde\phi,\tilde\psi]\!]\bar{}. \tag{4.4}$$

If $\tilde{\mathrm{d}}$ is the differential of the Lie algebroid $(\tilde A, [\![\,,\,]\!]\tilde{}, \tilde\rho)$ (see Appendix B) then, from (4.3) and (A.2) and the definition of X_0, we have that

$$\begin{aligned}[\![\tilde\phi,\tilde\psi]\!]\hat{}^{X_0}_{*P} = \mathrm{e}^{-t}\Bigg(& i_{\#_P(\tilde\phi)}\tilde{\mathrm{d}}\tilde\psi - i_{\#_P(\tilde\psi)}\tilde{\mathrm{d}}\tilde\phi + \tilde{\mathrm{d}}(P(\tilde\phi,\tilde\psi)) \\ &+ P(\tilde\psi,\tilde\phi)\phi_0 + (i_{\#_P(\tilde\phi)}\phi_0)\frac{\partial\tilde\psi}{\partial t} - (i_{\#_P(\tilde\psi)}\phi_0)\frac{\partial\tilde\phi}{\partial t}\Bigg).\end{aligned}$$

Using the fact that $\partial/\partial t(P(\tilde{\phi},\tilde{\psi})) = i_{\#_P(\tilde{\phi})}(\partial\tilde{\psi}/\partial t) - i_{\#_P(\tilde{\psi})}(\partial\tilde{\phi}/\partial t)$, we obtain that

$$[\![\tilde{\phi},\tilde{\psi}]\!]\hat{\ }^{X_0}_{*P} = \mathrm{e}^{-t}\left(i_{\#_P(\tilde{\phi})}\left(\tilde{\mathrm{d}}\tilde{\psi} + \phi_0 \wedge \frac{\partial\tilde{\psi}}{\partial t}\right) - i_{\#_P(\tilde{\psi})}\left(\tilde{\mathrm{d}}\tilde{\phi} + \tilde{\phi}_0 \wedge \frac{\partial\tilde{\phi}}{\partial t}\right)\right.$$
$$\left. + \tilde{\mathrm{d}}(P(\tilde{\phi},\tilde{\psi})) - (P(\tilde{\phi},\tilde{\psi}))\phi_0 + \frac{\partial}{\partial t}(P(\tilde{\phi},\tilde{\psi}))\phi_0\right).$$

Thus, from (B.3), we deduce (4.4).

On the other hand, using (B.1) and (B.2), it follows that

$$\bar{\rho}(\tilde{\phi}) = \mathrm{e}^{-t}\left(\widetilde{\rho_{*P}}(\tilde{\phi}) + X_0(\tilde{\phi})\frac{\partial}{\partial t}\right) = \widehat{\rho_{*P}}^{X_0}(\tilde{\phi})$$

for $\tilde{\phi} \in \Gamma(\tilde{A}^*)$, where $\widetilde{\rho_{*P}}$ is the natural extension to $\Gamma(\tilde{A}^*)$ of ρ_{*P}. Therefore, from Proposition B.2, we prove (1) and (2).

Now, if we consider on $\tilde{A}$ (resp. $\tilde{A}^*$) the Lie algebroid structure $([\![,]\!]\tilde{\ }^{\phi_0}, \tilde{\rho}^{\phi_0})$ (resp. $([\![,]\!]\hat{\ }^{X_0}_{*P}, \widehat{\rho_{*P}}^{X_0})$) then the pair $(\tilde{A},\tilde{A}^*)$ is a Lie bialgebroid (see [25]). Consequently, using Theorem 3.10, we conclude that $((A,\phi_0),(A^*,X_0))$ is a generalized Lie bialgebroid. □

Let $(A,[\![,]\!],\rho)$ be a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ a 1-cocycle. Suppose that $([\![,]\!]_*,\rho_*)$ is a Lie algebroid structure on A^* and that $X_0 \in \Gamma(A)$ is a 1-cocycle. Moreover, assume that $((A,\phi_0),(A^*,X_0))$ is a generalized Lie bialgebroid. Then, the pair $((A,\phi_0),(A^*,X_0))$ is said to be a *triangular generalized Lie bialgebroid* if there exists $P \in \Gamma(\wedge^2 A)$ such that $[\![P,P]\!]_{\phi_0} = 0$ and

$$[\![,]\!]_* = [\![,]\!]_{*P}, \qquad \rho_* = \rho_{*P}, \qquad X_0 = -\#_P(\phi_0).$$

Note that a triangular generalized Lie bialgebroid $((A,\phi_0),(A^*,X_0))$ such that $\phi_0 = 0$ is just a *triangular Lie bialgebroid* (see [25]). On the other hand, if (M,Λ,E) is a Jacobi manifold then, using (4.1) and (4.2), we deduce that the pair $((TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(-E,0)))$ is a triangular generalized Lie bialgebroid.

5. Generalized Lie bialgebras

In this section, we will study generalized Lie bialgebroids over a point.

Definition 5.1. A generalized Lie bialgebra is a generalized Lie bialgebroid over a point, that is, a pair $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$, where $(\mathfrak{g},[,]^{\mathfrak{g}})$ is a real Lie algebra of finite dimension such that the dual space $\mathfrak{g}^*$ is also a Lie algebra with Lie bracket $[,]^{\mathfrak{g}^*}$, $X_0 \in \mathfrak{g}$ and $\phi_0 \in \mathfrak{g}^*$ are 1-cocycles on $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively, and

$$\mathrm{d}_{*X_0}[X,Y]^{\mathfrak{g}} = [X,\mathrm{d}_{*X_0}Y]^{\mathfrak{g}}_{\phi_0} - [Y,\mathrm{d}_{*X_0}X]^{\mathfrak{g}}_{\phi_0}, \tag{5.1}$$

$$\phi_0(X_0) = 0, \tag{5.2}$$

$$i_{\phi_0}(\mathrm{d}_*X) + [X_0,X]^{\mathfrak{g}} = 0 \tag{5.3}$$

for all $X,Y \in \mathfrak{g}$, d_* being the differential on $(\mathfrak{g}^*,[,]^{\mathfrak{g}^*})$.

Remark 5.2. In the particular case when $\phi_0 = 0$ and $X_0 = 0$, we recover the concept of a *Lie bialgebra*, that is, a dual pair $(\mathfrak{g}, \mathfrak{g}^*)$ of Lie algebras such that $\mathrm{d}_*[X, Y]^{\mathfrak{g}} = [X, \mathrm{d}_* Y]^{\mathfrak{g}} - [Y, \mathrm{d}_* X]^{\mathfrak{g}}$ for $X, Y \in \mathfrak{g}$ (see [5]).

Next, we give different methods to obtain generalized Lie bialgebras.

Proposition 5.3. *Let* $(\mathfrak{h}, [\,,\,]^{\mathfrak{h}})$ *be a Lie algebra,* $r \in \wedge^2\mathfrak{h}$ *and* $\bar{X}_0 \in \mathfrak{h}$ *such that*

$$[r, r]^{\mathfrak{h}} - 2\bar{X}_0 \wedge r = 0, \qquad [\bar{X}_0, r]^{\mathfrak{h}} = 0. \tag{5.4}$$

Then, if $\mathfrak{g} = \mathfrak{h} \times \mathbb{R}$, *the pair* $((\mathfrak{g}, (0, 1)), (\mathfrak{g}^*, (-\bar{X}_0, 0)))$ *is a generalized Lie bialgebra.*

Proof. Consider on $\mathfrak{g}$ the Lie bracket $[\,,\,]^{\mathfrak{g}}$ given by

$$[(X, \lambda), (Y, \mu)]^{\mathfrak{g}} = ([X, Y]^{\mathfrak{h}}, 0) \tag{5.5}$$

for $(X, \lambda), (Y, \mu) \in \mathfrak{g}$. One easily follows that $\phi_0 = (0, 1) \in \mathfrak{h}^* \times \mathbb{R} \cong \mathfrak{g}^*$ is a 1-cocycle. On the other hand, the space $\wedge^2\mathfrak{g} = \wedge^2(\mathfrak{h} \times \mathbb{R})$ can be identified with the product $\wedge^2\mathfrak{h} \times \mathfrak{h}$ (see (1.2)) and, using (1.3), (5.4) and (5.5) and Theorem A.3, we have that $P = (r, \bar{X}_0) \in \wedge^2\mathfrak{h} \times \mathfrak{h} \cong \wedge^2\mathfrak{g}$ satisfies $[P, P]^{\mathfrak{g}}_{\phi_0} = 0$.

Therefore, from Theorem 4.1, we deduce that there exists a Lie bracket on $\mathfrak{g}^*$ and the pair $((\mathfrak{g}, (0, 1)), (\mathfrak{g}^*, -\#_P(0, 1)))$ is a generalized Lie bialgebra. Moreover, we have that $\#_P(0, 1) = (\bar{X}_0, 0)$ (see (1.1) and (1.3)). □

Remark 5.4.

1. Note that this method of finding generalized Lie bialgebras is related to find algebraic Jacobi structures.
2. Using (1.2), (1.3) and (4.3) it follows that the Lie bracket $[\,,\,]^{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is given by

$$\begin{aligned}[(\alpha, \lambda), (\beta, \mu)]^{\mathfrak{g}^*} = (&\mathrm{coad}_{\#_r(\alpha)}\beta - \mathrm{coad}_{\#_r(\beta)}\alpha - i_{\bar{X}_0}(\alpha \wedge \beta) - \mu\, \mathrm{coad}_{\bar{X}_0}\alpha \\ &+\lambda\, \mathrm{coad}_{\bar{X}_0}\beta, -r(\alpha, \beta))\end{aligned} \tag{5.6}$$

   for $(\alpha, \lambda), (\beta, \mu) \in \mathfrak{g}^*$, where $\mathrm{coad} : \mathfrak{h} \times \mathfrak{h}^* \to \mathfrak{h}^*$ is the coadjoint representation of $\mathfrak{h}$ over $\mathfrak{h}^*$ defined by $(\mathrm{coad}_X\alpha)(Y) = -\alpha[X, Y]^{\mathfrak{h}}$ for $X, Y \in \mathfrak{h}$ and $\alpha \in \mathfrak{h}^*$.

Corollary 5.5. *Let* $(\mathfrak{h}, [\,,\,]^{\mathfrak{h}})$ *be a Lie algebra and* $\mathcal{Z}(\mathfrak{h})$ *the center of* $\mathfrak{h}$. *If* $r \in \wedge^2\mathfrak{h}$, $\bar{X}_0 \in \mathcal{Z}(\mathfrak{h})$ *and*

$$[r, r]^{\mathfrak{h}} - 2\bar{X}_0 \wedge r = 0,$$

then the pair $((\mathfrak{h}, 0), (\mathfrak{h}^*, -\bar{X}_0))$ *is a generalized Lie bialgebra.*

Proof. Using Proposition 5.3, we have that $((\mathfrak{g} = \mathfrak{h} \times \mathbb{R}, (0, 1)), (\mathfrak{g}^* = \mathfrak{h}^* \times \mathbb{R}, (-\bar{X}_0, 0)))$ is a generalized Lie bialgebra. Furthermore, from (5.6) and since $\bar{X}_0 \in \mathcal{Z}(\mathfrak{h})$, we deduce that the Lie bracket on $\mathfrak{g}^*$ is given by

$$[(\alpha, \lambda), (\beta, \mu)]^{\mathfrak{g}^*} = (\mathrm{coad}_{\#_r(\alpha)}\beta - \mathrm{coad}_{\#_r(\beta)}\alpha - i_{\bar{X}_0}(\alpha \wedge \beta), -r(\alpha, \beta)) \tag{5.7}$$

for $(\alpha,\lambda), (\beta,\mu) \in \mathfrak{g}^*$. Then $\mathfrak{h}$ and $\mathfrak{h}^*$ are Lie algebras, where the Lie bracket on $\mathfrak{h}^*$ is defined by

$$[\alpha,\beta]^{\mathfrak{h}^*} = \mathrm{coad}_{\#_r(\alpha)}\beta - \mathrm{coad}_{\#_r(\beta)}\alpha - i_{\bar{X}_0}(\alpha\wedge\beta) \tag{5.8}$$

for $\alpha,\beta \in \mathfrak{h}^*$. Moreover, using (5.5), (5.7) and (5.8) and the fact that $((\mathfrak{g},(0,1)), (\mathfrak{g}^*,(-\bar{X}_0,0)))$ is a generalized Lie bialgebra, we conclude that $((\mathfrak{h},0),(\mathfrak{h}^*,-\bar{X}_0))$ is a generalized Lie bialgebra. □

Remark 5.6. From (5.8) and Corollary 5.5, we deduce a well-known result (see [5]): if $(\mathfrak{h},[\,,\,]^{\mathfrak{h}})$ is a Lie algebra, $r\in\wedge^2\mathfrak{h}$ is a solution of the classical Yang–Baxter equation (that is, $[r,r]^{\mathfrak{h}}=0$) and on $\mathfrak{h}^*$ we consider the bracket defined by

$$[\alpha,\beta]^{\mathfrak{h}^*} = \mathrm{coad}_{\#_r(\alpha)}\beta - \mathrm{coad}_{\#_r(\beta)}\alpha \quad \text{for } \alpha,\beta\in\mathfrak{h}^*,$$

then the pair $(\mathfrak{h},\mathfrak{h}^*)$ is a Lie bialgebra.

Examples 5.7.

1. Let $(\mathfrak{h},[\,,\,]^{\mathfrak{h}})$ be the Lie algebra of the Heisenberg group $H(1,1)$. Then, $\mathfrak{h}=\langle\{e_1,e_2,e_3\}\rangle$ and

$$[e_1,e_2]^{\mathfrak{h}} = e_3, \quad e_3\in\mathcal{Z}(\mathfrak{h}).$$

If we take $r=e_1\wedge e_2$ and $\bar{X}_0=-e_3$, we have that $[r,r]^{\mathfrak{h}}-2\bar{X}_0\wedge r=0$ and thus, using Corollary 5.5, we conclude that $((\mathfrak{h},0),(\mathfrak{h}^*,e_3))$ is a generalized Lie bialgebra. Note that r and $\bar{X}_0$ induce the canonical left-invariant contact structure of $H(1,1)$.

2. Denote by $\mathfrak{h}=\mathfrak{su}(2)=\{A\in gl(2,\mathbb{C})/\bar{A}^T=-A, \text{trace } A=0\}$ the Lie algebra of the special unitary group $SU(2)$ and by σ_1,σ_2 and σ_3 the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Then, the matrices $\{e_1=\frac{1}{2}\mathrm{i}\sigma_1, e_2=\frac{1}{2}\mathrm{i}\sigma_2, e_3=\frac{1}{2}\mathrm{i}\sigma_3\}$ form a basis of $\mathfrak{h}=\mathfrak{su}(2)$ and if $[\,,\,]^{\mathfrak{h}}$ is the Lie bracket on $\mathfrak{h}$, we have that

$$[e_1,e_2]^{\mathfrak{h}}=-e_3,\qquad [e_1,e_3]^{\mathfrak{h}}=e_2,\qquad [e_2,e_3]^{\mathfrak{h}}=-e_1.$$

Since $r=e_1\wedge e_2$ and $\bar{X}_0=e_3$ satisfy equations (5.4), $((\mathfrak{su}(2)\times\mathbb{R},(0,1)),(\mathfrak{su}(2)^*\times\mathbb{R},(-\bar{X}_0,0)))$ is a generalized Lie bialgebra. Note that if $[\,,\,]^{\mathfrak{su}(2)\times\mathbb{R}}$ is the Lie bracket on $\mathfrak{su}(2)\times\mathbb{R}$ then, from (5.5), it follows that $(\mathfrak{su}(2)\times\mathbb{R},[\,,\,]^{\mathfrak{su}(2)\times\mathbb{R}})$ is just the Lie algebra of the unitary group $U(2)$. Moreover, r and $\bar{X}_0$ induce a left-invariant contact structure on $SU(2)$.

3. Let $Gl(2,\mathbb{R})$ be the general linear group and $\mathfrak{h}=gl(2,\mathbb{R})$ its Lie algebra. A basis of $\mathfrak{h}$ is given by the following matrices:

$$e_1=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad e_2=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad e_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad e_4=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

If $[\,,\,]^{\mathfrak{h}}$ is the Lie bracket on $\mathfrak{h}$, we have that

$$[e_1, e_2]^{\mathfrak{h}} = e_3, \qquad [e_1, e_3]^{\mathfrak{h}} = -2e_1, \qquad [e_2, e_3]^{\mathfrak{h}} = 2e_2, \qquad e_4 \in \mathcal{Z}(\mathfrak{h}).$$

Therefore, if $r = e_1 \wedge e_3 + (e_1 - \frac{1}{2}e_3) \wedge e_4$ and $\bar{X}_0 = -e_4$, we deduce that

$$[r, r]^{\mathfrak{h}} - 2\bar{X}_0 \wedge r = 0.$$

Consequently, $((\mathfrak{h}, 0), (\mathfrak{h}^*, -\bar{X}_0))$ is a generalized Lie bialgebra (see Corollary 5.5).

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Appendix A. Differential calculus on Lie algebroids in the presence of a 1-cocycle

Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma(A^*)$ be a 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients. Using (2.1), we can define a representation ρ_{ϕ_0} of A on the trivial vector bundle $M \times \mathbb{R} \to M$ given by

$$\rho_{\phi_0}(X)f = \rho(X)(f) + \phi_0(X)f \tag{A.1}$$

for $X \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$. Thus, one can consider the standard cohomology complex associated with the vector bundle $M \times \mathbb{R} \to M$ and the representation ρ_{ϕ_0} (see [24]). The cohomology operator $\mathrm{d}_{\phi_0} : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ of this complex will be called the ϕ_0-*differential* of A. If d is the differential of the Lie algebroid $(A, [\![\,,\,]\!], \rho)$ then we have

$$\mathrm{d}_{\phi_0}\omega = \mathrm{d}\omega + \phi_0 \wedge \omega \quad \text{for } \omega \in \Gamma(\wedge^k A^*). \tag{A.2}$$

Remark A.1. If ϕ_0 is a closed 1-form on a manifold M then ϕ_0 is a 1-cocycle for the trivial Lie algebroid $(TM, [\,,\,], Id)$ and we can consider the operator d_{ϕ_0}. Some results about the cohomology defined by d_{ϕ_0} were obtained in [9,18,29]. These results were used in the study of locally conformal Kähler and l.c.s. structures.

If $k \geq 0$ and $X \in \Gamma(A)$, we can also define the Lie derivative (associated with the representation ρ_{ϕ_0}) with respect to X, $(\mathcal{L}_{\phi_0})_X : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$, as follows (see [24]):

$$(\mathcal{L}_{\phi_0})_X = \mathrm{d}_{\phi_0} \circ i_X + i_X \circ \mathrm{d}_{\phi_0}, \tag{A.3}$$

where i_X is the usual contraction by X. It is called the ϕ_0-*Lie derivative* with respect to X. A direct computation proves that

$$(\mathcal{L}_{\phi_0})_X\omega = \mathcal{L}_X\omega + \phi_0(X)\omega, \tag{A.4}$$

$\mathcal{L}$ being the usual Lie derivative of the Lie algebroid $(A, [\![\,,\,]\!], \rho)$.

Remark A.2.

1. If we consider the Lie algebroid $(TM \times \mathbb{R}, [\,,\,], \pi)$ (see Section 2.2), using (2.5), we deduce that $\phi_0 = (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle. Thus, we have the corresponding representation $\pi_{(0,1)}$ of $TM \times \mathbb{R}$ on the vector bundle $M \times \mathbb{R} \to M$ which, in this case, is defined by
$$\pi_{(0,1)}((X, f), g) = X(g) + fg \tag{A.5}$$
for $(X, f) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ and $g \in C^\infty(M, \mathbb{R})$. From (1.2), (2.5) and (A.2), we obtain that the ϕ_0-differential $\tilde{\delta}_{\phi_0} = \tilde{\delta}_{(0,1)}$ is given by
$$\tilde{\delta}_{(0,1)}(\alpha, \beta) = (\delta\alpha, \alpha - \delta\beta) \tag{A.6}$$
for $(\alpha, \beta) \in \Omega^k(M) \oplus \Omega^{k-1}(M) \cong \Gamma(\wedge^k(T^*M \times \mathbb{R}))$.
2. Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma(A^*)$ be a 1-cocycle. The homomorphism of $C^\infty(M, \mathbb{R})$-modules $(\rho, \phi_0) : \Gamma(A) \to \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ given by
$$X \mapsto (\rho(X), \phi_0(X)) \tag{A.7}$$
induces a Lie algebroid homomorphism over the identity between the Lie algebroids $(A, [\![\,,\,]\!], \rho)$ and $(TM \times \mathbb{R}, [\,,\,], \pi)$. Moreover, if $(\rho, \phi_0)^* : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \Gamma(A^*)$ is the adjoint homomorphism of (ρ, ϕ_0), then $(\rho, \phi_0)^*(0, 1) = \phi_0$. As a consequence, for $f \in C^\infty(M, \mathbb{R})$,
$$(\rho, \phi_0)^*(\delta f, f) = (\rho, \phi_0)^*(\tilde{\delta}_{(0,1)} f) = \mathrm{d}_{\phi_0} f, \qquad (\rho, \phi_0)^*(\delta f, 0) = \mathrm{d}f. \tag{A.8}$$

In [1], a skew-symmetric Schouten bracket was defined for two multilinear maps of a commutative associative algebra $\mathfrak{F}$ over $\mathbb{R}$ with unit as follows. Let $\mathcal{P}$ and $\mathcal{P}'$ be skew-symmetric multilinear maps of degree k and k', respectively, and $f_1, \dots, f_{k+k'-1} \in \mathfrak{F}$. If A is any subset of $\{1, 2, \dots, (k + k' - 1)\}$, let A' denote its complement and $|A|$ the number of elements in A. If $|A| = l$ and the elements in A are $\{i_1, \dots, i_l\}$ in increasing order, let us write f_A for the ordered k-uple $(f_{i_1}, \dots, f_{i_l})$. Furthermore, we write ε_A for the sign of the permutation which rearranges the elements of the ordered $(k + k' - 1)$-uple (A', A), in the original order. Then, the Schouten bracket of $\mathcal{P}$ and $\mathcal{P}'$, $[\mathcal{P}, \mathcal{P}']_{(0,1)}$, is the skew-symmetric multilinear map of degree $k + k' - 1$ given by
$$\begin{aligned}&[\mathcal{P}, \mathcal{P}']_{(0,1)}(f_1, \dots, f_{k+k'-1})\\ &\quad = \sum_{|A|=k'} \varepsilon_A \mathcal{P}(\mathcal{P}'(f_A), f_{A'}) + (-1)^{kk'} \sum_{|B|=k} \varepsilon_B \mathcal{P}'(\mathcal{P}(f_B), f_{B'}).\end{aligned}$$

One can prove that if $\mathcal{P}$ and $\mathcal{P}'$ are first-order differential operators on each of its arguments, so is $[\mathcal{P}, \mathcal{P}']_{(0,1)}$. In particular, if M is a differentiable manifold and $\mathfrak{F} = C^\infty(M, \mathbb{R})$, we know that a k-linear skew-symmetric first-order differential operator can be identified with a k-section of $TM \times \mathbb{R} \to M$. Under this identification, an easy computation, using (1.3) and (2.6), shows that
$$\begin{aligned}[(P, Q), (P', Q')]_{(0,1)} &= [(P, Q), (P', Q')] + (-1)^{k+1}(k - 1)(P, Q)\\ &\quad \wedge(i_{(0,1)}(P', Q')) - (k'-1)(i_{(0,1)}(P, Q)) \wedge (P', Q')\end{aligned} \tag{A.9}$$

for $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and $(P', Q') \in \mathcal{V}^{k'}(M) \oplus \mathcal{V}^{k'-1}(M)$, where $[\,,\,]$ is the Schouten bracket of the Lie algebroid $(TM \times \mathbb{R}, [\,,\,], \pi)$.

Suggested by (A.9), we prove the following result.

Theorem A.3. *Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ a 1-cocycle. Then, there exists a unique operation $[\![\,,\,]\!]_{\phi_0} : \Gamma(\wedge^k A) \times \Gamma(\wedge^{k'} A) \to \Gamma(\wedge^{k+k'-1} A)$ such that*

$$[\![X, f]\!]_{\phi_0} = \rho_{\phi_0}(X)(f), \qquad [\![X, Y]\!]_{\phi_0} = [\![X, Y]\!], \tag{A.10}$$

$$[\![P, P']\!]_{\phi_0} = (-1)^{kk'}[\![P', P]\!]_{\phi_0}, \tag{A.11}$$

$$\begin{aligned}[\![P, P' \wedge P'']\!]_{\phi_0} = {} & [\![P, P']\!]_{\phi_0} \wedge P'' + (-1)^{k'(k+1)} P' \wedge [\![P, P'']\!]_{\phi_0} \\ & -(i_{\phi_0} P) \wedge P' \wedge P''\end{aligned} \tag{A.12}$$

for $f \in C^\infty(M, \mathbb{R})$, $X, Y \in \Gamma(A)$, $P \in \Gamma(\wedge^k A)$, $P' \in \Gamma(\wedge^{k'} A)$ and $P'' \in \Gamma(\wedge^{k''} A)$. This operation is given by the general formula

$$[\![P, P']\!]_{\phi_0} = [\![P, P']\!] + (-1)^{k+1}(k-1) P \wedge (i_{\phi_0} P') - (k'-1)(i_{\phi_0} P) \wedge P'. \tag{A.13}$$

Furthermore, it satisfies the graded Jacobi identity

$$\begin{aligned}&(-1)^{kk''}[\![[\![P, P']\!]_{\phi_0}, P'']\!]_{\phi_0} + (-1)^{k'k''}[\![[\![P'', P]\!]_{\phi_0}, P']\!]_{\phi_0} \\ &\quad +(-1)^{kk'}[\![[\![P', P'']\!]_{\phi_0}, P]\!]_{\phi_0} = 0.\end{aligned} \tag{A.14}$$

Proof. We define the operation $[\![\,,\,]\!]_{\phi_0} : \Gamma(\wedge^k A) \times \Gamma(\wedge^{k'} A) \to \Gamma(\wedge^{k+k'-1} A)$ by (A.13). Using (A.13) and properties of the Schouten bracket of multi-sections of A, we deduce (A.10)–(A.12). To prove the graded Jacobi identitity (A.14), we proceed by induction on k using the properties of the Schouten bracket and the fact that $i_{\phi_0}([\![\bar{P}, \bar{P}']\!]) = -[\![i_{\phi_0}\bar{P}, \bar{P}']\!] + (-1)^{\bar{k}+1}[\![\bar{P}, i_{\phi_0}\bar{P}']\!]$ for $\bar{P} \in \Gamma(\wedge^{\bar{k}} A)$ and $\bar{P}' \in \Gamma(\wedge^{\bar{k}'} A)$. □

The operation $[\![\,,\,]\!]_{\phi_0}$ is called the ϕ_0-*Schouten bracket* of $(A, [\![\,,\,]\!], \rho)$. Now, if $X \in \Gamma(A)$ and $P \in \Gamma(\wedge^k A)$, we can define the ϕ_0-*Lie derivative* of P by X as follows:

$$(\mathcal{L}_{\phi_0})_X(P) = [\![X, P]\!]_{\phi_0}. \tag{A.15}$$

Using (A.4), (A.13) and (A.15) and relation (15) in [25], we obtain that

$$(\mathcal{L}_{\phi_0})_X(i_\omega P) = i_P((\mathcal{L}_{\phi_0})_X \omega) + i_\omega((\mathcal{L}_{\phi_0})_X P) + (k-1)\phi_0(X) i_\omega P \tag{A.16}$$

for $\omega \in \Gamma(\wedge^k A^*)$, $P \in \Gamma(\wedge^k A)$ and $X \in \Gamma(A)$.

Appendix B. Time-dependent sections of a Lie algebroid

Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid over M, $\phi_0 \in \Gamma(A^*)$ be a 1-cocycle and $\pi_1 : M \times \mathbb{R} \to M$ be the canonical projection over the first factor. We consider the map $\cdot : \Gamma(A) \times C^\infty$

$(M \times \mathbb{R}, \mathbb{R}) \to C^\infty(M \times \mathbb{R}, \mathbb{R})$ given by

$$X \cdot \tilde{f} = \rho(X)(\tilde{f}) + \phi_0(X)\frac{\partial \tilde{f}}{\partial t}.$$

It is easy to prove that $\cdot$ is an action of A on $M \times \mathbb{R}$ in the sense of [10] (see Definition 2.3 in [10]). Thus, if $\pi_1^* A$ is the pull-back of A over π_1 then the vector bundle $\pi_1^* A \to M \times \mathbb{R}$ admits a Lie algebroid structure $(\tilde{[\![,]\!]}^{\phi_0}, \tilde{\rho}^{\phi_0})$ (see Theorem 2.4 in [10]). For the sake of simplicity, when the 1-cocycle ϕ_0 is zero, we will denote by $(\tilde{[\![,]\!]}, \tilde{\rho})$ the resultant Lie algebroid structure on $\pi_1^* A \to M \times \mathbb{R}$. On the other hand, it is clear that the vector bundles $\pi_1^* A \to M \times \mathbb{R}$ and $\tilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ are isomorphic and that the space of sections $\Gamma(\tilde{A})$ of $\tilde{A} \to M \times \mathbb{R}$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, we have that $\tilde{[\![\tilde{X}, \tilde{Y}]\!]}(x, t) = [\![\tilde{X}_t, \tilde{Y}_t]\!](x)$ and that $\tilde{\rho}(\tilde{X})(x, t) = \rho(\tilde{X}_t)(x)$ for $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$ and $(x, t) \in M \times \mathbb{R}$ (see [10]). In addition,

$$\tilde{[\![\tilde{X}, \tilde{Y}]\!]}^{\phi_0} = \tilde{[\![\tilde{X}, \tilde{Y}]\!]} + \phi_0(\tilde{X})\frac{\partial \tilde{Y}}{\partial t} - \phi_0(\tilde{Y})\frac{\partial \tilde{X}}{\partial t}, \qquad \tilde{\rho}^{\phi_0}(\tilde{X}) = \tilde{\rho}(\tilde{X}) + \phi_0(\tilde{X})\frac{\partial}{\partial t}, \tag{B.1}$$

where$(\partial \tilde{X}/\partial t) \in \Gamma(\tilde{A})$ denotes the derivative of $\tilde{X}$ with respect to the time.

Remark B.1. Note that if $[\![,]\!]$ is a bracket on the space $\Gamma(A)$, $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a homomorphism of $C^\infty(M, \mathbb{R})$-modules and ϕ_0 is a section of the dual bundle A^* then we can consider the bracket $\tilde{[\![,]\!]}^{\phi_0}$ on $\Gamma(\tilde{A})$ and the homomorphism of $C^\infty(M \times \mathbb{R}, \mathbb{R})$-modules $\tilde{\rho}^{\phi_0} : \Gamma(\tilde{A}) \to \mathfrak{X}(M \times \mathbb{R})$ given by (B.1). Moreover, if the triple $(\tilde{A}, \tilde{[\![,]\!]}^{\phi_0}, \tilde{\rho}^{\phi_0})$ is a Lie algebroid over $M \times \mathbb{R}$, it is easy to prove that $([\![,]\!], \rho)$ is a Lie algebroid structure on $A \to M$ and that ϕ_0 is a 1-cocycle.

Now, let $\Psi : \tilde{A} \to \tilde{A}$ be the isomorphism of vector bundles over the identity defined by $\Psi(v, t) = (e^t v, t)$ for $(v, t) \in A \times \mathbb{R} = \tilde{A}$. Using Ψ and the Lie algebroid structure $(\tilde{[\![,]\!]}^{\phi_0}, \tilde{\rho}^{\phi_0})$, one can introduce a new Lie algebroid structure $(\hat{[\![,]\!]}^{\phi_0}, \hat{\rho}^{\phi_0})$ on the vector bundle $\tilde{A} \to M \times \mathbb{R}$ in such a way that the Lie algebroids $(\tilde{A}, \tilde{[\![,]\!]}^{\phi_0}, \tilde{\rho}^{\phi_0})$ and $(\tilde{A}, \hat{[\![,]\!]}^{\phi_0}, \hat{\rho}^{\phi_0})$ are isomorphic. We have that

$$\hat{[\![\tilde{X}, \tilde{Y}]\!]}^{\phi_0} = e^{-t}\left(\tilde{[\![\tilde{X}, \tilde{Y}]\!]} + \phi_0(\tilde{X})\left(\frac{\partial \tilde{Y}}{\partial t} - \tilde{Y}\right) - \phi_0(\tilde{Y})\left(\frac{\partial \tilde{X}}{\partial t} - \tilde{X}\right)\right),$$

$$\hat{\rho}^{\phi_0}(\tilde{X}) = e^{-t}\left(\tilde{\rho}(\tilde{X}) + \phi_0(\tilde{X})\frac{\partial}{\partial t}\right) \tag{B.2}$$

for all $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$ (this Lie algebroid structure was considered in [12]).

In conclusion, we have proved the following.

Proposition B.2. *Let $A \to M$ be a vector bundle over a manifold M. Suppose that $[\![,]\!] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is a bracket on the space $\Gamma(A)$, that $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a homomorphism of $C^\infty(M, \mathbb{R})$-modules and that ϕ_0 is a section of the dual bundle A^*. If $\tilde{[\![,]\!]}^{\phi_0} : \Gamma(\tilde{A}) \times \Gamma(\tilde{A}) \to \Gamma(\tilde{A})$ and $\tilde{\rho}^{\phi_0} : \Gamma(\tilde{A}) \to \mathfrak{X}(M \times \mathbb{R})$ (respectively, $\hat{[\![,]\!]}^{\phi_0}$:*

$\Gamma(\tilde{A}) \times \Gamma(\tilde{A}) \to \Gamma(\tilde{A})$ and $\hat{\rho}^{\phi_0} : \Gamma(\tilde{A}) \to \mathfrak{X}(M \times \mathbb{R}))$ are the bracket on $\Gamma(\tilde{A})$ and the homomorphism of $C^\infty(M \times \mathbb{R}, \mathbb{R})$-modules given by (B.1) (*respectively,* (B.2)) *then the following conditions are equivalent:*

1. *The triple $(A, [\![,]\!], \rho)$ is a Lie algebroid and ϕ_0 is a 1-cocycle.*
2. *The triple $(\tilde{A}, [\![,]\!]\hat{}^{\phi_0}, \hat{\rho}^{\phi_0})$ is a Lie algebroid.*
3. *The triple $(\tilde{A}, [\![,]\!]\tilde{}^{\phi_0}, \tilde{\rho}^{\phi_0})$ is a Lie algebroid.*

Remark B.3. Let $(A, [\![,]\!], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma(A^*)$ a 1-cocycle. If $\tilde{\mathrm{d}}$ (resp. $\tilde{\mathrm{d}}^{\phi_0}$ and $\hat{\mathrm{d}}^{\phi_0}$) is the differential of the Lie algebroid $(\tilde{A}, [\![,]\!]\tilde{}, \tilde{\rho})$ (resp. $(\tilde{A}, [\![,]\!]\tilde{}^{\phi_0}, \tilde{\rho}^{\phi_0})$ and $(\tilde{A}, [\![,]\!]\hat{}^{\phi_0}, \hat{\rho}^{\phi_0})$) and $[\![,]\!]\tilde{}^{\phi_0}$ is the Schouten bracket of the Lie algebroid $(\tilde{A}, [\![,]\!]\tilde{}^{\phi_0}, \tilde{\rho}^{\phi_0})$ then

$$\tilde{\mathrm{d}}^{\phi_0}\tilde{f} = \tilde{\mathrm{d}}\tilde{f} + \frac{\partial \tilde{f}}{\partial t}\phi_0, \qquad \tilde{\mathrm{d}}^{\phi_0}\tilde{\phi} = \tilde{\mathrm{d}}\tilde{\phi} + \phi_0 \wedge \frac{\partial \tilde{\phi}}{\partial t}, \tag{B.3}$$

$$\hat{\mathrm{d}}^{\phi_0}\tilde{f} = \mathrm{e}^{-t}\left(\tilde{\mathrm{d}}\tilde{f} + \frac{\partial \tilde{f}}{\partial t}\phi_0\right), \qquad \hat{\mathrm{d}}^{\phi_0}\tilde{\phi} = \mathrm{e}^{-t}\left(\tilde{\mathrm{d}}_{\phi_0}\tilde{\phi} + \phi_0 \wedge \frac{\partial \tilde{\phi}}{\partial t}\right), \tag{B.4}$$

$$[\![\tilde{X}, \tilde{P}]\!]\tilde{}^{\phi_0} = [\![\tilde{X}, \tilde{P}]\!]\tilde{}_{\phi_0} + \phi_0(\tilde{X})\left(\tilde{P} + \frac{\partial \tilde{P}}{\partial t}\right) - \frac{\partial \tilde{X}}{\partial t} \wedge i_{\phi_0}\tilde{P} \tag{B.5}$$

for $\tilde{f} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$, $\tilde{\phi} \in \Gamma(\tilde{A}^*) = \Gamma(A^* \times \mathbb{R})$, $\tilde{X} \in \Gamma(\tilde{A})$ and $\tilde{P} \in \Gamma(\wedge^2 \tilde{A})$.

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