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On the nonexistence of higher twistings over a point for Borel cohomology of K-theory

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ABSTRACT

In this note we show that there are no higher twistings for the Borel cohomology theory associated to G-equivariant K-theory over a point and for a compact Lie group G. Therefore, twistings over a point for this theory are classified by the group $H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z})$.

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1. Introduction

The goal of this paper is to show that all the higher twistings for the Borel cohomology theory associated to equivariant K-theory over a point and for a compact Lie group G are trivial.

In general for a non-equivariant multiplicative cohomology theory E^*, where the multiplication is rigid enough in the sense it is represented by an E_∞-ring spectrum, we can consider local coefficients or twistings. This procedure allows us to construct finer invariants out of the theory E^*. The use of local coefficients is a standard tool in algebraic topology, where for example in the case of singular cohomology they arise naturally in the Serre spectral sequence.

Suppose that E is an E_∞-ring spectrum and let $Z = E_0$ be the zero space. Z is an E_∞-ring space (see [1] and [2] for definitions and [2, Corollary 7.5]) and if we write $Z = \coprod_{\alpha\in\pi_0(Z)} Z_\alpha$, then $Z_\otimes = GL_1E = \coprod_{\alpha\in\pi_0(Z)^\times} Z_\alpha$, the space of units, is an infinite loop space by [2, Corollary 7.7]. The space $BZ_\otimes$ classifies the twistings for the theory E^*; this means that for a space X and any map $f : X \to BZ_\otimes$, we have a twisting E^*_f of the theory E^* over X. The groups $E^*_f(X)$ and $E^*_{f'}(X)$ are isomorphic through a possibly non-canonical isomorphism whenever f and f' are homotopic. In this sense we say that twistings of E^* over X are classified by the group $[X, BZ_\otimes]$. As a particular case we can consider non-equivariant K-theory. Twistings for this theory are classified by the spectrum of units $K_\otimes \simeq \mathbb{Z}/2 \times BU_\otimes$, where $BU_\otimes$ is the space BU with the structure of an H-space corresponding to the tensor product of vector bundles. Thus for a CW-complex X, the non-equivariant twistings of complex K-theory over X are classified by the group

$$[X, BK_\otimes].$$

In [3], Segal proved that $BU_\otimes$ is an infinite loop space and in [4], Madsen, Snaith and Tornehave proved that there is a factorization $BU_\otimes = K(\mathbb{Z}, 2) \times BSU_\otimes$ of the respective spectra. We conclude that twistings of K-theory over a space X are

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classified by homotopy classes of maps

$$X \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_{\otimes};$$

that is, we have twistings corresponding to the groups $H^1(X, \mathbb{Z}/2)$, $H^3(X, \mathbb{Z})$ and $bsu^1_{\otimes}(X) = [X, BBSU_{\otimes}]$. We call the twistings corresponding to the group $bsu^1_{\otimes}(X)$ higher twistings. For the equivariant case the situation is more complicated. We do not know what the "right" notion for the spectrum of units for an equivariant spectrum is. In particular, we do not know the group that classifies the more general twistings for equivariant K-theory. However, for the untwisted case, we have the famous Atiyah–Segal completion theorem (see [5]). This theorem says that if G is a compact Lie group acting on X a G-CW-complex, then we have a natural isomorphism

$$K^*_G(X)_{\hat{I}} \cong K^*_G(X \times EG),$$

where I is the augmentation ideal of the representation ring.

The twistings that we consider here are the twistings of the Borel cohomology theory associated to G-equivariant K-theory, where G is a compact Lie group. Hence, for a G-CW-complex X, the twistings for the Borel cohomology theory associated to G-equivariant K-theory are classified by the group

$$H^1_G(X, \mathbb{Z}/2) \times H^3_G(X, \mathbb{Z}) \times bsu^1_{\otimes}(EG \times_G X).$$

In particular, for the case of a point we obtain

$$H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z}) \times bsu^1_{\otimes}(BG).$$

The goal of this paper is to prove that the group $bsu^1_{\otimes}(BG)$ vanishes for a compact Lie group, which means that there are no higher twistings for the Borel cohomology theory associated to G-equivariant K-theory and over a point.

In what follows we will use the following notation.

Notation. We will denote by k the spectrum representing connective complex K-theory and by K the spectrum representing complex K-theory. For a prime p we will denote by $\mathbb{Z}_p$ the ring of p-adic integers. Given a spectrum F and an abelian group G we can introduce G coefficients on F by considering the spectrum $F_G = F \wedge MG$, where MG is a Moore spectrum for the group G. Thus in particular we consider $F_{\mathbb{Z}_p}$ and $F_{\mathbb{Z}/(p^k)}$ for a prime number p. Also in general for a spectrum F and any integer n we can find the $(n-1)$-connected cover of F, which we denote by $F\langle n\rangle$. This is a spectrum together with a map $F\langle n\rangle \to F$ that induces an isomorphism $\pi_k(F\langle n\rangle) \xrightarrow{\cong} \pi_k F$ for $k \geq n$ and such that $\pi_k(F\langle n\rangle) = 0$ for $k < n$. Note that in particular by the periodicity we have $\Sigma^4 k \cong K\langle 4\rangle$.

2. Triviality of $bsu^1_{\otimes}(BG)$

In this section we are going to show that the group $bsu^1_{\otimes}(BG)$ is trivial. This implies that there are no higher twistings for the Borel cohomology theory associated to equivariant K-theory over a point and for a compact Lie group G. In this case, there are only lower twistings; that is, those twistings classified by $H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z})$ as in [6].

Definition 1. We say that a topological group G satisfies the Atiyah–Segal Completion Theorem if we have that $K^0(BG) = R(G)^{\hat{}}_I$ and $K^1(BG) = 0$, where I is the augmentation ideal of the representation ring $R(G)$.

Note that by [5] it follows that this is true for any compact Lie group.

Lemma 2. *Let p be a prime number. If G satisfies the Atiyah–Segal Completion Theorem then $K^5_{\mathbb{Z}_p}(BG) = 0$.*

Proof. Let us define

$$X_k = K \wedge M\mathbb{Z}/(p^k).$$

We have maps

$$X_{k+1} \to X_k$$

coming from the maps $\mathbb{Z}/(p^{k+1}) \to \mathbb{Z}/(p^k)$. Consider

$$X_\infty = \operatorname{holim}_{k\to\infty} X_k.$$

We will start by showing

$$K \wedge M\mathbb{Z}_p \cong X_\infty. \tag{1}$$

We have a map $K \wedge M\mathbb{Z}_p \to X_\infty$ arising from the canonical maps $\mathbb{Z}_p \to \mathbb{Z}/(p^k)$. Let us show that it induces an isomorphism on homotopy groups. By [7, Proposition 6.6] there is a short exact sequence

$$0 \to \pi_n(K) \otimes \mathbb{Z}_p \to \pi_n(K \wedge M\mathbb{Z}_p) \to \operatorname{Tor}^{\mathbb{Z}}_1(\pi_{n-1}(K), \mathbb{Z}_p) \to 0. \tag{2}$$

The group $\mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(K), \mathbb{Z}_p)$ vanishes, as $\mathbb{Z}_p$ is flat as a $\mathbb{Z}$-module. Thus by (2) we have

$$\pi_n(K \wedge M\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

On the other hand to compute $\pi_n(X_\infty)$ we have a short exact sequence

$$0 \to \lim_{k\to\infty}{}^1\, \pi_{n+1}(X_k) \to \pi_n(X_\infty) \to \lim_{k\to\infty} \pi_n(X_k) \to 0. \tag{4}$$

Since $\pi_n(K) = \mathbb{Z}$ or 0 according to whether n is even or odd, then by [7, Proposition 6.6] $\pi_n(X_k) = \mathbb{Z}/(p^k)$ or 0 depending on the parity of n. In any case we have that the map

$$\pi_{n+1}(X_{k+1}) \to \pi_{n+1}(X_k)$$

is onto, so the $\lim^1$ vanishes. Therefore

$$\pi_n(X_\infty) = \begin{cases} \mathbb{Z}_p & \text{if } n \text{ is even,} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and the map

$$K \wedge M\mathbb{Z}_p \to \operatorname*{holim}_{k\to\infty} K \wedge M\mathbb{Z}/(p^k) = X_\infty$$

induces isomorphism on π_*. This shows (1).

Let us show now that

$$K^5_{\mathbb{Z}_p}(BG) = X^5_\infty(BG) = 0. \tag{6}$$

To compute this cohomology we consider the short exact sequence

$$0 \to \lim_{k\to\infty}{}^1\, X^4_k(BG) \to X^5_\infty(BG) \to \lim_{k\to\infty} X^5_k(BG) \to 0. \tag{7}$$

On the other hand, since $X_k = K \wedge M\mathbb{Z}/(p^k)$, by [7, Proposition 6.6] we have a short exact sequence

$$0 \to K^5(BG) \otimes \mathbb{Z}/(p^k) \to X^5_k(BG) \to \mathrm{Tor}_1^{\mathbb{Z}}(K^6(BG), \mathbb{Z}/(p^k)) \to 0. \tag{8}$$

Since G satisfies the Atiyah–Segal completion theorem, we have that

$$K^5(BG) = K^1(BG) = 0$$
$$K^6(BG) = K^0(BG) = R(G)_{\hat{I}}.$$

We know that $R(G)$ is a free, and hence flat $\mathbb{Z}$-module, and $R(G)^{\wedge}_I$ is a flat $R(G)$-module. By change of basis it follows that $R(G)^{\wedge}_I$ is a flat $\mathbb{Z}$-module. Therefore from (7) get that $X^5_k(BG) = 0$.

We also have the exact sequence

$$0 \to K^4(BG) \otimes \mathbb{Z}/(p^k) \to X^4_k(BG) \to \mathrm{Tor}_1^{\mathbb{Z}}(K^5(BG), \mathbb{Z}/(p^k)) \to 0. \tag{9}$$

Since $K^5(BG) = 0$, we conclude from (9) that $X^4_k(BG) = K^4(BG) \otimes \mathbb{Z}/(p^k)$. From here we can see that the maps $X^4_{k+1}(BG) \to X^4_k(BG)$ are surjective and thus the $\lim^1$ term in the short exact sequence (7) vanishes. Since the outer terms in that sequence are zero we see that $K^5_{\mathbb{Z}_p}(BG) = X^5_\infty(BG) = 0$. □

Proposition 3. *Let G be a topological group that satisfies the Atiyah–Segal Completion Theorem. Then $k^5(BG) = 0$ and $k^5_{\mathbb{Z}_p}(BG) = 0$ for every prime p.*

Proof. Both $k^5(BG) = 0$ and $k^5_{\mathbb{Z}_p}(BG) = 0$ are proved in a similar way with obvious modifications. Thus we will show in detail that $k^5_{\mathbb{Z}_p}(BG) = 0$.

By the previous lemma we have that $K^5_{\mathbb{Z}_p}(BG) = 0$. In general for a spectrum F we have the Atiyah–Hirzebruch spectral sequence.

$$E_2^{r,s} = H^r(BG, F^s(*)) \Longrightarrow F^{r+s}(BG).$$

Let us apply this for the cases $F = k_{\mathbb{Z}_p}$ and $F = K_{\mathbb{Z}_p}$. This way we obtain two spectral sequences $\{E_n^{r,s}\}$ and $\{{}^1E_n^{r,s}\}$, respectively.

$$E_2^{r,s} = H^r(BG, k^s_{\mathbb{Z}_p}(*)) \Longrightarrow k^{r+s}_{\mathbb{Z}_p}(BG), \tag{10}$$

$${}^1E_2^{r,s} = H^r(BG, K^s_{\mathbb{Z}_p}(*)) \Longrightarrow K^{r+s}_{\mathbb{Z}_p}(BG). \tag{11}$$

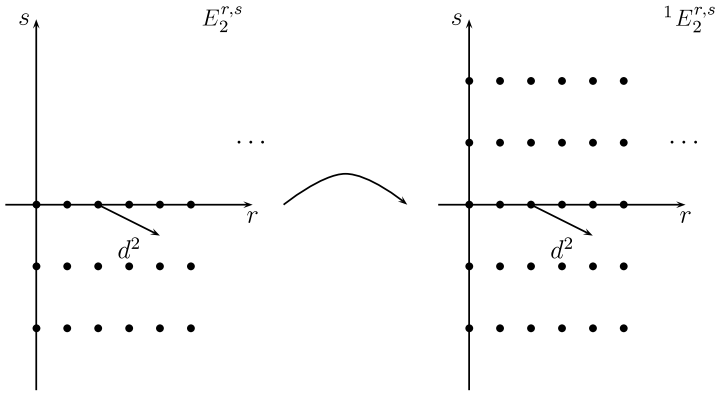


Fig. 1. Spectral sequences E, ${}^1E$.

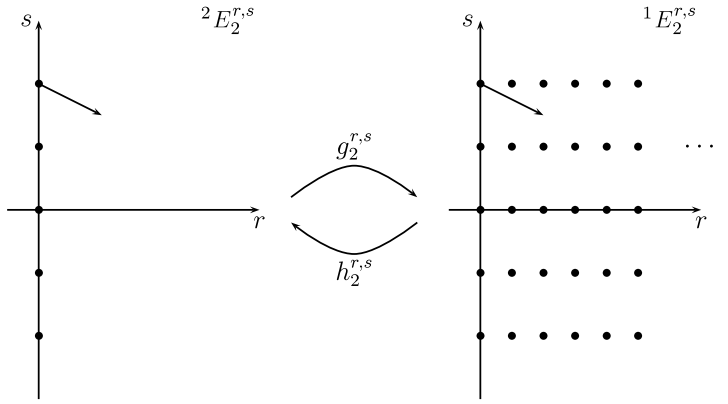


Fig. 2. Spectral sequences ${}^1E$, ${}^2E$.

For the spectrum $k_{\mathbb{Z}_p}$ we know by [7, Proposition 6.6], that $k^n_{\mathbb{Z}_p}(*) = \pi_{-n}(k) \otimes \mathbb{Z}_p = \mathbb{Z}_p$ if $n \leq 0$ and even, and $k^n_{\mathbb{Z}_p}(*) = \pi_{-n}(ku) \otimes \mathbb{Z}_p = 0$ otherwise. For $K_{\mathbb{Z}_p}$ we know that $K^n_{\mathbb{Z}_p}(*) = \pi_{-n}(K_{\mathbb{Z}_p}) = \mathbb{Z}_p$ if n is even and $\pi_n(K_{\mathbb{Z}_p}) = 0$ otherwise. Thus we have that $E_2^{r,s}$ is a fourth quadrant spectral sequence with $E_2^{r,2s} = H^r(BG, \mathbb{Z}_p)$ for $s \leq 0$ and zero otherwise. Similarly, ${}^1E_2^{r,2s} = H^r(BG, \mathbb{Z}_p)$ for $s \in \mathbb{Z}$, and zero otherwise. See Fig. 1.

The spectrum k comes equipped with a map of spectra $k \to K$ inducing an isomorphism on π_n for $n \geq 0$. By smashing with MZ_p we get a map $k_{\mathbb{Z}_p} \to K_{\mathbb{Z}_p}$ also inducing an isomorphism on π_n for $n \geq 0$. This map induces a map of spectral sequences $\{E_n^{r,s}\} \to \{{}^1E_n^{r,s}\}$ as shown in Fig. 1.

We show the result by arguing by contradiction. So assume that $k^5_{\mathbb{Z}_p}(BG) \neq 0$. We know that $K^5_{\mathbb{Z}_p}(BG) = 0$, and we have a map of spectral sequences $\{E_n^{r,s}\} \to \{{}^1E_n^{r,s}\}$. Thus the only way that $k^5_{\mathbb{Z}_p}(BG) \neq 0$ is that one of the differentials that kills elements in total degree 5 in the case $K_{\mathbb{Z}_p}$ fails to do so in the case of $k_{\mathbb{Z}_p}$. Differentials killing elements in total degree 5 must have source of total degree 4. From Fig. 1 we can see at once that the only sources from the $K_{\mathbb{Z}_p}$ case of total degree 4 missing in the $k_{\mathbb{Z}_p}$ case are $H^0(BG, K^4_{\mathbb{Z}_p}(*))$ and $H^2(BG, K^2_{\mathbb{Z}_p}(*))$. We will show that none of these differentials with these sources kill elements of total degree 5 in the case of K, from which we deduce that $k^5_{\mathbb{Z}_p}(BG) = 0$.

Let $*$ be the basepoint of BG and consider the sequence of maps $* \to BG \to *$ factoring the identity $* \to *$. Let us consider now the Atiyah–Hirzebruch spectral sequence applied to the spaces $*$ and BG and the spectrum $K_{\mathbb{Z}_p}$. Then we get a spectral sequence $\{{}^2E_n^{r,s}\}$

$$ {}^2E_2^{r,s} = H^r(*, K^s_{\mathbb{Z}_p}(*)) \Longrightarrow K^{r+s}_{\mathbb{Z}_p}(*) \tag{12} $$

and maps $h_n^{r,s} : {}^1E_n^{r,s} \to {}^2E_n^{r,s}$ and $g_n^{r,s} : {}^2E_n^{r,s} \to {}^1E_n^{r,s}$ of spectral sequences such that $h_n^{r,s} \circ g_n^{r,s} = \mathrm{id}$. (See Fig. 2.)

The maps $h_n^{r,s}$ and $g_n^{r,s}$ and the identity $h_n^{r,s} \circ g_n^{r,s} = \mathrm{id}$ tell us that all the differentials with source $H^0(BG, K^4_{\mathbb{Z}_p}(*))$ for the spectral sequence $\{{}^1E_n^{r,s}\}$ must vanish, as they do for the spectral sequence $\{{}^2E_n^{r,s}\}$.

Now let us study the case of differentials with source $H^2(BG, K^2_{\mathbb{Z}_p}(*))$ for the spectral sequence $\{{}^1E_n^{r,s}\}$. We are going to show that all such differentials are trivial. This is a contradiction and hence the proposition follows.

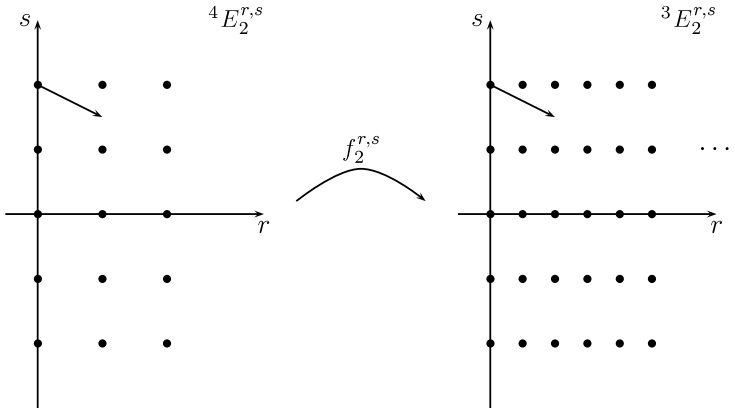


Fig. 3. Spectral sequences ${}^3E$, ${}^4E$.

To investigate these differentials we will first study the differentials for the Atiyah–Hirzebruch spectral sequence for the spectrum K. So we have a spectral sequence $\{{}^3E_n^{r,s}\}$ given by

$$ {}^3E_2^{r,s} = H^r(BG, K^s(*)) \Longrightarrow K^{r+s}(BG). \tag{13} $$

We are going to show first that all the differentials with source $H^2(BG, K^2(*))$ vanish. To show this, notice that

$$ H^2(BG, K^2(*)) = H^2(BG, \mathbb{Z}) = [BG, K(\mathbb{Z}, 2)], $$

and the latter is in a one to one correspondence with isomorphism classes of complex line bundles over BG, so every element in $H^2(BG, K^2(*))$ is the first Chern class of a complex line bundle over BG. Let $\alpha \in H^2(BG, K^2(*))$. Then we can find a map $f : BG \to K(\mathbb{Z}, 2)$ such that $\alpha = f^*(c_1(\gamma_1)) = c_1(f^*\gamma_1)$, where γ_1 is universal line bundle over $K(\mathbb{Z}, 2)$. Let ${}^4E_n^{p,q}$ be the Atiyah–Hirzebruch spectral sequence of the space $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ corresponding to the spectrum K, so that

$$ {}^4E_2^{r,s} = H^r(K(\mathbb{Z}, 2), K^s(*)) \Longrightarrow K^{r+s}(K(\mathbb{Z}, 2)). \tag{14} $$

The ${}^4E_2$-term of this spectral sequence only has terms in the even components and hence the sequence collapse and all the higher differentials are zero.

The map f gives a map of spectral sequences $f_n^{r,s} : {}^4E_n^{r,s} \to {}^3E_n^{r,s}$. By construction we have that $f_2^{2,2}(c_1(\gamma_1)) = \alpha$. Since all the differentials on $\{{}^4E_n^{r,s}\}$ are zero it follows that α vanishes on all the differentials. Since α was arbitrary we see that all the differentials on the spectral sequence $\{{}^3E_n^{p,q}\}$ with source $H^2(BG, K^2(*))$ must vanish (Fig. 3).

Take $i : S \to M\mathbb{Z}_p$ a map representing the unit of $\pi_0 M\mathbb{Z}_p$. This induces a map of spectra $K = K \wedge S \xrightarrow{1\wedge i} K \wedge M\mathbb{Z}_p$. This map induces a map of spectral sequences $j_n^{r,s} : {}^3E_n^{r,s} \to {}^1E_n^{r,s}$. Since each term of the spectral sequence ${}^1E$ is a $\mathbb{Z}_p$-module, by tensoring with $\mathbb{Z}_p$ we get a map of spectral sequences $\tilde{j}_n^{r,s} : {}^3E_n^{r,s} \otimes \mathbb{Z}_p \to {}^1E_n^{r,s}$. Notice that already on the E_2-level this map is an isomorphism because $H_n(BG)$ is finitely generated, and thus by [8, Corollary 56.4] we have a short exact sequence

$$ 0 \to H^r(BG, \mathbb{Z}) \otimes \mathbb{Z}_p \to H^r(BG, \mathbb{Z}_p) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{r+1}(BG), \mathbb{Z}_p) \to 0. \tag{15} $$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module, from (15) we see that

$$ H^r(BG, \mathbb{Z}) \otimes \mathbb{Z}_p \approx H^r(BG, \mathbb{Z}_p), $$

and this isomorphism is precisely the $\tilde{j}$ map. Because the differentials with source $H^2(BG, K^2(*))$ in the spectral sequence $\{{}^1E_n^{r,s}\}$ are all trivial it follows that all the differentials with source $H^2(BG, K^2_{\mathbb{Z}_p}(*))$ are also trivial. □

Definition 4. Given a system of groups

$$ \{G_n\} = \cdots \to G_{n+1} \cdots \to G_2 \to G_1, $$

we say that $\{G_n\}$ satisfies the Mittag-Leffler condition if for every i we can find a $j > i$ such that for every $k > j$

$$ \mathrm{Im}(G_k \to G_i) = \mathrm{Im}(G_j \to G_i). $$

It is well known that if $\{G_n\}$ satisfies the Mittag-Leffler condition then

$$ \lim_{k\to\infty}{}^1 G_k = 0. $$

On the other hand, if each G_k is a countable group, then by [9, Theorem 2] we have that the system $\{G_n\}$ must satisfy the Mittag-Leffler condition.

Suppose now that G is a compact Lie group. Then higher twistings of the Borel cohomology theory associated to G-equivariant K-theory and over a point are classified by

$$bsu^1_{\otimes}(BG) = [BG, BBSU_{\otimes}].$$

We are now able to show that for a compact Lie group this vanishes. We do this in the following theorem.

Theorem 5. *For any compact Lie group G,*

$$bsu^1_{\otimes}(BG) = [BG, BBSU_{\otimes}] = 0.$$

Proof. For every $k \geq 0$ denote by F_k the image of $\coprod_{0\leq n\leq k}(G^n \times \Delta_n)$ in BG. The F_k's form an increasing filtration of BG and since G is compact Lie each F_k is of the homotopy type of a finite CW-complex. Let us denote

$$A_k = k^4(F_k) \quad \text{and} \quad B_k = bsu^0_{\otimes}(F_k).$$

Using the filtration $\{F_k\}$ we get a short exact sequence

$$0 \to \lim_{k\to\infty}{}^1 A_k \to k^5(BG) \to \lim_{k\to\infty} k^5(F_k) \to 0. \tag{16}$$

By Proposition 3 we have that the middle term in (16) vanishes and thus we see that $\lim^1_{k\to\infty} A_k = 0$. By looking at the Atiyah–Hirzebruch spectral sequence, since F_k is of the homotopy type of a finite CW-complex, we see that each A_k and B_k is finitely generated, in particular countable. Therefore the system $\{A_k\}$ satisfies the Mittag-Leffler condition.

On the other hand, by [10, Corollary 1.4] we have that after localization or completion at any prime p, the spectrum $bsu_{\otimes}$ is unique up to equivalence. In our context this means that $K\langle 4\rangle \wedge M\mathbb{Z}_p \simeq bsu_{\otimes} \wedge M\mathbb{Z}_p$ for every prime p. But we have that $bsu_{\otimes} \wedge M\mathbb{Z}_p \simeq \Sigma^4 k \wedge M\mathbb{Z}_p$. Thus for each k we have that

$$A_k \otimes \mathbb{Z}_p = k^4_{\mathbb{Z}_p} \xrightarrow{\simeq} (bsu_{\otimes} \wedge M\mathbb{Z}_p)^0(F_k) = B_k \otimes \mathbb{Z}_p. \tag{17}$$

The outer equalities follow by [7, Proposition 6.6]. Therefore we have a commutative diagram in which the vertical arrows are isomorphisms

$$\begin{array}{ccccccc} \to & A_n \otimes \mathbb{Z}_p & \cdots \to & A_2 \otimes \mathbb{Z}_p & \to & A_1 \otimes \mathbb{Z}_p \\ & \downarrow & & \downarrow & & \downarrow \\ \to & B_n \otimes \mathbb{Z}_p & \cdots \to & B_2 \otimes \mathbb{Z}_p & \to & B_1 \otimes \mathbb{Z}_p. \end{array} \tag{18}$$

Let $i > 0$ be fixed. Since the system $\{A_k\}$ satisfies the Mittag-Leffler property we can find a $j > i$ such that for each $k > j$

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_j \to A_i).$$

The following is a short exact sequence:

$$0 \to \mathrm{Ker}(A_k \to A_i) \to A_k \to \mathrm{Im}(A_k \to A_i) \to 0. \tag{19}$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module we have that

$$0 \to \mathrm{Ker}(A_k \to A_i) \otimes \mathbb{Z}_p \to A_k \otimes \mathbb{Z}_p \to \mathrm{Im}(A_k \to A_i) \otimes \mathbb{Z}_p \to 0 \tag{20}$$

is also exact. This shows that $\mathrm{Im}(A_k \to A_i) \otimes \mathbb{Z}_p = \mathrm{Im}(A_k \otimes \mathbb{Z}_p \to A_i \otimes \mathbb{Z}_p)$ and thus we see that for every $k > j$ and every prime p we have

$$\mathrm{Im}(A_k \otimes \mathbb{Z}_p \to A_i \otimes \mathbb{Z}_p) = \mathrm{Im}(A_j \otimes \mathbb{Z}_p \to A_i \otimes \mathbb{Z}_p).$$

By the diagram (18) we conclude that for every p

$$\mathrm{Im}(B_k \to B_i) \otimes \mathbb{Z}_p = \mathrm{Im}(B_k \otimes \mathbb{Z}_p \to B_i \otimes \mathbb{Z}_p) = \mathrm{Im}(B_j \otimes \mathbb{Z}_p \to B_i \otimes \mathbb{Z}_p) = \mathrm{Im}(B_j \to B_i) \otimes \mathbb{Z}_p.$$

Thus the groups $\mathrm{Im}(B_k \to B_i)$ and $\mathrm{Im}(B_j \to B_i)$ are two finitely generated groups that are equal after tensoring with $\mathbb{Z}_p$. By Lemma 6 we see that

$$\mathrm{Im}(B_k \to B_i) = \mathrm{Im}(B_j \to B_i).$$

We have proved that the system $\{B_k\}$ satisfies the Mittag-Leffler condition and thus

$$\lim_{k\to\infty}{}^1 B_k = \lim_{k\to\infty}{}^1 bsu^0_{\otimes}(F_k) = 0. \tag{21}$$

Using the filtration $\{F_k\}$ for the spectrum $bsu_\otimes$ we get a short exact sequence

$$0 \to \lim_{k\to\infty}{}^1 B_k \to bsu^1_\otimes(BG) \to \lim_{k\to\infty} bsu^1_\otimes(F_k) \to 0. \tag{22}$$

Since the $\lim^1$ part vanishes we get that

$$bsu^1_\otimes(BG) = \lim_{k\to\infty} bsu^1_\otimes(F_k).$$

We show now that the latter vanishes. To see this, note that for every prime p we have a short exact sequence

$$0 \to \lim_{k\to\infty}{}^1 (bsu_\otimes \wedge M\mathbb{Z}_p)^0(F_k) \to (bsu_\otimes \wedge M\mathbb{Z}_p)^1(BG) \to \lim_{k\to\infty} (bsu_\otimes \wedge M\mathbb{Z}_p)^1(F_k) \to 0. \tag{23}$$

The term in the middle of (23) vanishes and hence we see that

$$\lim_{k\to\infty} (bsu_\otimes \wedge M\mathbb{Z}_p)^1(F_k) = 0.$$

But by [7, Proposition 6.6] we have that $(bsu_\otimes \wedge M\mathbb{Z}_p)^1(F_k) = bsu^1_\otimes(F_k) \otimes \mathbb{Z}_p$. Thus for every prime p the map

$$\lim_{k\to\infty} bsu^1_\otimes(F_k) \otimes \mathbb{Z}_p = 0.$$

The proof finishes by using Lemma 7 to see that

$$\lim_{k\to\infty} bsu^1_\otimes(F_k) = 0. \quad \square$$

Lemma 6. *Suppose that A and B are two finitely generated abelian groups with $A \subset B$ and that for every prime p, $A\otimes\mathbb{Z}_p = B\otimes\mathbb{Z}_p$. Then $A = B$.*

Proof. We have a short exact sequence

$$0 \to A \to B \to B/A \to 0.$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module we see that

$$0 \to A \otimes \mathbb{Z}_p \to B \otimes \mathbb{Z}_p \to B/A \otimes \mathbb{Z}_p \to 0$$

is also exact. As $A \otimes \mathbb{Z}_p = B \otimes \mathbb{Z}_p$ we see that $B/A \otimes \mathbb{Z}_p = 0$. This is true for every p. This implies that $B/A = 0$. $\square$

Lemma 7. *Let $\cdots \xrightarrow{f_k} G_k \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} G_2 \xrightarrow{f_1} G_1$ be a system of finitely generated abelian groups such that $\lim_{k\to\infty} G_k \otimes \mathbb{Z}_p = 0$ for all primes p. Then $\lim_{k\to\infty} G_k = 0$.*

Proof. Let $f : \prod_{i\geq 1} G_i \to \prod_{i\geq 1} G_i$ be defined by

$$f(x_1, x_2, \ldots) = (x_1 - f_1(x_2), x_2 - f_2(x_3), \ldots).$$

We want to show that f is injective, as $\lim_{k\to\infty} G_k = \mathrm{Ker}(f)$. Suppose

$$x = (x_1, x_2, \ldots) \in \mathrm{Ker}(f).$$

Then we have that $i_p(x) \in \mathrm{Ker}(f_p) = 0$. Here $i_p : \prod_{i\geq 1} G_i \to \prod_{i\geq 1} G_i\otimes\mathbb{Z}_p$. Thus for each i we have that $x_i \in \mathrm{Ker}(G_k \to G_k\otimes\mathbb{Z}_p)$ for each prime p. Since G_k is finitely generated we have that

$$\bigcap_{p \text{ prime}} \mathrm{Ker}(G_k \to G_k \otimes \mathbb{Z}_p) = 0.$$

Thus $x = 0$. $\square$

Corollary 8. *For a compact Lie group G there are no higher twistings for the Borel cohomology theory associated to G-equivariant K-theory.*

Remark. In general, the group $bsu^1_\otimes(BG)$ does not vanish if G does not satisfy the Atiyah–Segal Completion Theorem. To see this let us consider an odd dimensional sphere S^{2n+1} with $n \geq 2$. By the Kan–Thurston Theorem (see [11]) we know that there is a discrete group G_n and a map $f : BG_n \to S^{2n+1}$ that is a homology equivalence. Since f is a homology equivalence, it follows that $bsu^1_\otimes(BG_n) = bsu^1_\otimes(S^{2n+1})$. (This follows as we get isomorphism in the E_2-term and onward in the Atiyah–Hirzebruch spectral sequence.) Let us show now that $bsu^1_\otimes(S^{2n+1}) \neq 0$. This will prove the proposition. Let p be a prime number. We know that $bsu_\otimes \wedge M\mathbb{Z}_p \simeq bsu_\oplus \wedge M\mathbb{Z}_p \simeq \Sigma^4 k \wedge M\mathbb{Z}_p$, and thus

$$bsu^1_\otimes(S^{2n+1}) \otimes \mathbb{Z}_p = (bsu_\otimes \wedge M\mathbb{Z}_p)^1(S^{2n+1}) = k^5_{\mathbb{Z}_p}(S^{2n+1}) = k^5(S^{2n+1}) \otimes \mathbb{Z}_p.$$

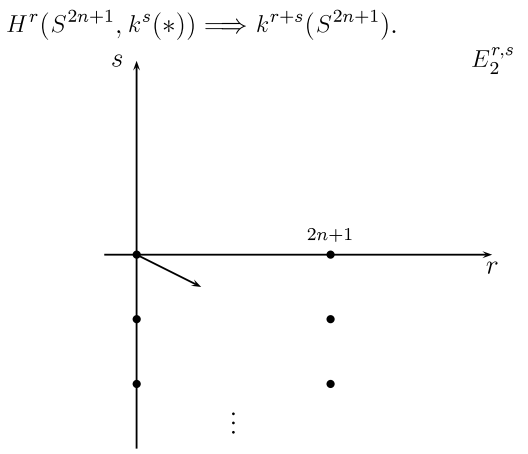


Fig. 4. Atiyah–Hirzebruch spectral sequences for $k^{r+s}(S^{2n+1})$.

Here we used [7, Proposition 6.6] as S^{2n+1} is finite, and also the fact that $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module. Notice that both $bsu^1_\otimes(S^{2n+1})$ and $k^5(S^{2n+1})$ are finitely generated abelian groups. In general, if A is a finitely generated abelian group, $A = 0$ if and only if $A \otimes \mathbb{Z}_p = 0$ for every prime number p. Thus, to show that $bsu^1_\otimes(S^{2n+1}) \neq 0$, we only need to show that $k^5(S^{2n+1}) \neq 0$. To do so we use the Atiyah–Hirzebruch spectral sequence

$$H^r(S^{2n+1}, k^s(*)) \Longrightarrow k^{r+s}(S^{2n+1}).$$

We claim that this spectral sequence collapses on the E_2-term. To see this, we only need to note that the corresponding spectral sequence collapses in the case of K. Since we have a map of spectra $k \to K$ inducing an isomorphism on π_n for $n \geq 0$, the spectral sequence in the case of k also collapses. Since $n \geq 2$, we see $k^5(S^{2n+1}) = \mathbb{Z} \neq 0$ (Fig. 4).

Remark. If G is a compact Lie group and if we consider twistings of the Borel cohomology theory associated with G-equivariant K-theory we encounter higher twistings if we work with spaces more general than a point. For example in the trivial case where $G = \{e\}$ is the trivial group, then for $X = S^{2n+1}$ an odd sphere with $n \geq 2$ we have higher twistings which are classified by the group

$$bsu_\otimes(S^{2n+1}) = \mathbb{Z} \neq 0.$$

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