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Normalization and prescribed extrinsic scalar curvature on lightlike hypersurfaces

C. Atindogbé *

Institut de Mathematiques et de Sciences Physiques, Université d'Abomey-Calavi, 01 BP 613 Porto-Novo, Bénin

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In a recent paper [C. Atindogbé, Scalar curvature on lightlike hypersurfaces, Appl. Sci. 11 (2009) 9–18], the present author considered the concept of extrinsic (induced) scalar curvature on lightlike hypersurfaces. This scalar quantity has been studied on lightlike hypersurfaces equipped with a given normalization. But a very important problem was left open: *How to characterize the set of all normalizations admitting a prescribed extrinsic scalar curvature?* In this paper, we provide various responses to this question, supported by examples.

1. Introduction

The scalar curvature is one of the most important concepts in (semi-) Riemannian geometry and its connected areas such as general relativity, astrophysics, cosmology, etc. Lightlike hypersurfaces have the outstanding aspect that, due to the degeneracy of the metric, the induced connection is not a Levi-Civita connection (except the totally geodesic case). Also, as the induced $(0, 2)$ induced Ricci tensor is not symmetric in general, one fails to construct this scalar quantity in the usual way. But the lightlike geometry of hypersurfaces would be incomplete if this gap were not to be filled. In [1], an attempt has been made in a very limited class of lightlike hypersurfaces in Lorentzian signature called lightlike hypersurfaces of genus zero. The present author extended in [2] the construction of this extrinsic scalar curvature on all lightlike hypersurfaces equipped with a given normalization. But due to the non-uniqueness of the latter, the following question seems naturally to arise: Given a specific normalization with induced extrinsic scalar curvature, say R, how can one characterize the set of all normalizations with this prescribed extrinsic scalar curvature R? It is our purpose in this paper to consider various aspects of this question.

It is well known that the normal bundle $TM^{\perp}$ of the lightlike hypersurface M^{n+1} of a semi-Riemannian manifold $\overline{M}^{n+2}$ is a rank 1 vector subbundle of the tangent bundle TM. A complementary bundle of $TM^{\perp}$ in TM is a rank n nondegenerate distribution over M, called a *screen distribution* of M, denoted by $S(TM)$, such that

$$TM = S(TM) \oplus_{\mathrm{Orth}} TM^{\perp}, \tag{1.1}$$

where $\oplus_{\mathrm{Orth}}$ denotes the orthogonal direct sum. Existence of $S(TM)$ is secured provided M be paracompact. A lightlike hypersurface with a specific screen distribution is denoted by $(M, g, S(TM))$. We know [3] that for such a triplet, there exists a unique rank 1 vector subbundle $\mathrm{tr}(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate

* Tel.: +229 90 05 33 26; fax: +229 20 22 24 55.
E-mail addresses: atincyr@imsp-uac.org, atincyr@gmail.com.

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neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $\mathrm{tr}(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \qquad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \Gamma(ST(M)|_{\mathcal{U}}). \tag{1.2}$$

Then $T\overline{M}$ is decomposed as follows:

$$T\overline{M}|_M = TM \oplus \mathrm{tr}(TM) = \{TM^{\perp} \oplus \mathrm{tr}(TM)\} \oplus_{\mathrm{Orth}} S(TM). \tag{1.3}$$

We call $\mathrm{tr}(TM)$ a *(null) transversal vector bundle* along M. In fact, from (1.2) and (1.3) one shows that, conversely, a choice of a transversal bundle $\mathrm{tr}(TM)$ determines uniquely the screen distribution $S(TM)$. A vector field N as in (1.2) is called a *null transversal vector field* of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution and a unique radical vector field, say ξ, satisfying (1.2). Whence, from now on, by a *normalized lightlike hypersurface* we mean a triplet (M, g, N), where g is the induced metric on M along with a null transversal vector field N. In fact, in the case where the ambient manifold $\overline{M}$ has a Lorentzian signature, at an arbitrary point x in M, a real lightlike cone C_x is invariantly defined in the (ambient) tangent space $T_x\overline{M}$ and is tangent to M along a generator emanating from x. This generator is exactly the radical fiber $\Delta_x = T_xM^{\perp}$. Each null vector field $N, x \longmapsto N_x \in C_x \setminus \Delta_x$ determines a normalization of M. Let (M, g, N) be a normalized lightlike hypersurface. A null vector field $\widetilde{N}$ is a normalizing field for (M, g) if and only if $\widetilde{N} = \phi N + \zeta$, for some nowhere vanishing $\phi \in C^{\infty}(M)$ and $\zeta \in \Gamma(TM)$. A change in normalization $N \longrightarrow \widetilde{N} = \phi N + \zeta$ is called *isotropic scaling* (from N) if $\zeta = 0$, that is $\widetilde{N} = \phi N$. In such a change of normalization the screen distribution corresponding to the null transversal vector field N is preserved while there is an "homothetic" scaling in the radical vector field $\widetilde{\xi} = \frac{1}{\phi}\xi$. It is called *tangential scaling* (from N) if $\phi = 1$, that is $\widetilde{N} = N + \zeta$. Here a change in screen distribution occurs and the null vector fields N and $\widetilde{N}$ are dual to the same radical vector field $\xi \in TM^{\perp}$, i.e $\langle \widetilde{N}, \xi\rangle = \langle N, \xi\rangle = 1$ as in (1.2). The general case $\widetilde{N} = \phi N + \zeta$ is called *mixed scaling* (from N).

Now, on a normalized lightlike hypersurface (M, g, N), the local Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{1.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{1.5}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{1.6}$$

$$\nabla_X \xi = -\overset{\star}{A}_{\xi} X - \tau(X)\xi, \tag{1.7}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$, ∇ and $\overset{\star}{\nabla}$ denote the Levi-Civita connection on $(\overline{M}, \overline{g})$, the induced connection on M and the connection on the screen distribution $S(TM)$ respectively, P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). The $(0, 2)$ tensors B and C are the local second fundamental forms on TM and $S(TM)$ respectively, $\overset{\star}{A}_{\xi}$ the local shape operator on $S(TM)$ and τ a 1-form on TM defined by

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi).$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$ [4]. As per [3, page 83], the second fundamental form B of M is independent of the choice of a screen distribution and satisfies for all $X, Y \in \Gamma(TM)$

$$B(X, \xi) = 0, \quad \text{and} \quad B(X, Y) = g(\overset{\star}{A}_{\xi} X, Y).$$

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Recall the following Gauss–Codazzi equations [3, p. 93]

$$\langle \overline{R}(X, Y)Z, \xi\rangle = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z), \tag{1.8}$$

$$\langle \overline{R}(X, Y)Z, PW\rangle = \langle R(X, Y)Z, PW\rangle + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW), \tag{1.9}$$

$$\langle \overline{R}(X, Y)\xi, N\rangle = \langle R(X, Y)\xi, N\rangle = C(Y, \overset{\star}{A}_{\xi} X) - C(X, \overset{\star}{A}_{\xi} Y) - 2d\tau(X, Y), \quad \forall X, Y, Z, W \in \Gamma(TM|_{\mathcal{U}}). \tag{1.10}$$

Finally, we recall from [5] the following results. Consider on M a normalizing pair $\{\xi, N\}$ satisfying (1.2) and define the one-form

$$\eta(\bullet) = \overline{g}(N, \bullet).$$

For all $X \in \Gamma(TM)$, $X = PX + \eta(X)\xi$ and $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Now, we define $\flat$ by

$$\flat : \Gamma(TM) \longrightarrow \Gamma(T^*M)$$

$$X \longmapsto X^{\flat} = g(X, \bullet) + \eta(X)\eta(\bullet). \tag{1.11}$$

Clearly, such a $\flat$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and can be used to generalize the usual nondegenerate theory. In the latter case, $\Gamma(S(TM))$ coincides with $\Gamma(TM)$, and as a consequence the 1-form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\sharp$ denote the inverse of the isomorphism $\flat$ given by (1.11).

For $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^\flat$ (resp. $\omega^\sharp$) is called the dual 1-form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. It follows from (1.11) that if ω is a 1-form on M, we have for $X \in \Gamma(TM)$,

$$\omega(X) = g(\omega^\sharp, X) + \omega(\xi)\eta(X).$$

Define a (0, 2)-tensor $\widetilde{g}$ by

$$\widetilde{g}(X, Y) = X^\flat(Y), \quad \forall X, Y \in \Gamma(TM).$$

Clearly, $\widetilde{g}$ defines a non-degenerate metric on M which plays an important role in defining the usual differential operators *gradient*, *divergence*, *Laplacian* with respect to degenerate metric g on lightlike hypersurfaces ([5] for details). Also, observe that $\widetilde{g}$ coincides with g if the latter is non-degenerate. The (0, 2)-tensor $g^{[\cdot,\cdot]}$, inverse of $\widetilde{g}$ is called *the pseudo-inverse of* g. With respect to the quasi orthonormal local frame field $\{\partial_0 := \xi, \partial_1, \dots, \partial_n, N\}$ adapted to the decompositions (1.1) and (1.3) we have

$$\begin{aligned} &\widetilde{g}(\xi, \xi) = 1, \qquad \widetilde{g}(\xi, X) = \eta(X), \\ &\widetilde{g}(X, Y) = g(X, Y) \quad \forall X, Y \in \Gamma(S(TM)), \end{aligned} \tag{1.12}$$

and the following is proved [5].

Proposition 1.1. (α) *For any smooth function $f : \mathcal{U} \subset M \to \mathbb{R}$ we have*

$$\mathrm{grad}^g f = g^{[\alpha\beta]} f_\alpha \partial_\beta \quad \textit{where } f_\alpha = \frac{\partial f}{\partial x^\alpha}\ \partial_\beta = \frac{\partial}{\partial x^\beta}\ \alpha, \beta = 0, \dots, n.$$

(β) *For any vector field X on $\mathcal{U} \subset M$*

$$\mathrm{div}^g X = \sum_{\alpha=0}^{n} \varepsilon_\alpha \widetilde{g}(\nabla_{X_\alpha} X, X_\alpha);\ \varepsilon_0 = 1$$

(γ) *for a smooth function f defined on $\mathcal{U} \subset M$ we have*

$$\Delta^g f = \sum_{\alpha=0}^{n} \varepsilon_\alpha \widetilde{g}(\nabla_{X_\alpha} \mathrm{grad}^g f, X_\alpha).$$

In particular, ρ being an endomorphism (resp. a symmetric bilinear form) on $(M, g, S(TM))$, we have

$$\mathrm{tr}\,\rho = \mathrm{trace}_g\,\rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]} \widetilde{g}(\rho(\partial_\alpha), \partial_\beta)$$

$$\left(\text{resp. } \mathrm{trace}_g\,\rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]} \rho_{\alpha\beta}\right).$$

All manifolds will be assumed connected, paracompact and smooth.

2. Some technical results

The following lemma gives the behaviour of the induced geometric objects described above with respect to a change of normalization $N \longrightarrow \widetilde{N} = \phi N + \zeta$.

Lemma 2.1. *Let $\{\xi, N\}$ be a normalizing pair as in (1.2) and consider the change of normalization $\widetilde{N} = \phi N + \zeta$ with corresponding radical vector field $\widetilde{\xi}$. Then,*

(a) $\widetilde{\xi} = \frac{1}{\phi}\xi$,

(b) $2\phi\eta(\zeta) + \|\zeta\|^2 = 0$,

(c) $B^{\widetilde{N}}(X, Y) = \frac{1}{\phi} B^N(X, Y)$,

(d) $\widetilde{P}Y = PY - \frac{1}{\phi} g(\zeta, Y)\xi$,

(e) $C^{\widetilde{N}}(X, PY) = \phi C^N(X, PY) - g(\nabla_X \zeta, PY) + \left[\tau^N(X) + \frac{X\cdot\phi}{\phi} + \frac{1}{\phi} B^N(\zeta, X)\right] g(\zeta, Y)$,

(f) $\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{\phi} B^N(X, Y)\zeta$,

(g) $\tau^{\widetilde{N}} = \tau^N + d\ln|\phi| + \frac{1}{\phi} B^N(\zeta, \cdot)$,

(h) $A_{\widetilde{N}} = \phi A_N - \nabla_{.}\zeta + \left[\tau^N + d\ln|\phi| + \frac{1}{\phi} B^N(\zeta, \cdot)\right]\zeta$,

(i) $\overset{\star}{A}_{\widetilde{\xi}} = \frac{1}{\phi} \overset{\star}{A}_\xi - \frac{1}{\phi^2} B^N(\zeta, \cdot)\xi$,

for all tangent vector fields X and Y.

Proof. The first two relations in items (a) and (b) are immediate consequences of relations $\overline{g}(\widetilde{N},\widetilde{\xi}) = 1$, $\overline{g}(\widetilde{N},\widetilde{N}) = 0$ and $\dim(TM^{\perp}) = 1$. Writing Gauss (resp. Weingarten) formulas for both pairs $\{\xi, N\}$ and $\{\widetilde{\xi},\widetilde{N}\}$, we obtain, by identifications, the relations in items (c) and (f) (resp. (g) and (h)). Now let $Y \in \Gamma(TM)$, we have

$$\begin{aligned} Y &= \widetilde{P}Y + \widetilde{\eta}(Y)\widetilde{\xi} \\ &= \widetilde{P}Y + \widetilde{\eta}(Y)\left(\frac{1}{\phi}\xi\right) \\ &= \widetilde{P}Y + \frac{1}{\phi}\widetilde{\eta}(Y)\xi. \end{aligned}$$

Then,

$$\begin{aligned} \widetilde{P}Y &= Y - \frac{1}{\phi}\widetilde{\eta}(Y)\xi \\ &= Y - \frac{1}{\phi}\overline{g}(\widetilde{N}, Y)\xi \\ &= Y - \frac{1}{\phi}\overline{g}(\phi N + \zeta, Y)\xi \\ &= Y - \frac{1}{\phi}[\phi\eta(Y) + g(\zeta, Y)]\xi \\ &= Y - \eta(Y)\xi - \frac{1}{\phi}g(\zeta, Y)\xi \\ &= PY - \frac{1}{\phi}g(\zeta, Y)\xi \end{aligned}$$

and the item (d) is derived. By using the definition of $C^{\widetilde{N}}$, we have

$$\begin{aligned} C^{\widetilde{N}}(X, \widetilde{P}Y) &= \overline{g}(A_{\widetilde{N}}X, \widetilde{P}Y) \\ &\overset{(h)}{=} \overline{g}\left(\phi A_N X - \nabla_X\zeta + \left[\tau^N(X) + X\cdot(\ln|\phi|) + \frac{1}{\phi}B^N(\zeta, X)\right]\zeta, \widetilde{P}Y\right) \\ &\overset{(d)}{=} \overline{g}\left(\phi A_N X - \nabla_X\zeta + \left[\tau^N(X) + X\cdot(\ln|\phi|) + \frac{1}{\phi}B^N(\zeta, X)\right]\zeta, PY - \frac{1}{\phi}g(\zeta, Y)\xi\right) \\ &\overset{(d)}{=} \overline{g}(\phi A_N X - \nabla_X\zeta + [\tau^N(X) + X\cdot(\ln|\phi|) + \frac{1}{\phi}B^N(\zeta, X)]\zeta, PY) \\ &= C^{\widetilde{N}}(X, PY) = \phi C^N(X, PY) - g(\nabla_X\zeta, PY) + \left[\tau^N(X) + \frac{X\cdot\phi}{\phi} + \frac{1}{\phi}B^N(\zeta, X)\right]g(\zeta, Y) \end{aligned}$$

which establishes relation (e). Finally, we have

$$\widetilde{\nabla}_X\widetilde{\xi} = -\overset{\star}{A}_{\widetilde{\xi}}X - \tau^{\widetilde{N}}(X)\widetilde{\xi}.$$

But using (f) we get

$$\begin{aligned} \widetilde{\nabla}_X\widetilde{\xi} &= \nabla_X\widetilde{\xi} - \frac{1}{\phi}B^N(X, \widetilde{\xi})\zeta \\ &= \nabla_X\widetilde{\xi} \\ &\overset{(a)}{=} \nabla_X\left(\frac{1}{\phi}\xi\right) \\ &= -\frac{X\cdot(\phi)}{\phi^2}\xi + \frac{1}{\phi}(-\overset{\star}{A}_{\xi}X - \tau^N(X)\xi) \\ &= -\left[\frac{X\cdot(\phi)}{\phi^2} + \frac{1}{\phi}\tau^N(X)\right]\xi - \frac{1}{\phi}\overset{\star}{A}_{\xi}X. \end{aligned}$$

Identifying the above two expression of $\widetilde{\nabla}_X\widetilde{\xi}$, we get

$$-\overset{\star}{A}_{\widetilde{\xi}}X = \frac{1}{\phi}\overset{\star}{A}_{\xi}X + \left[\frac{X\cdot(\phi)}{\phi^2} + \frac{1}{\phi}\tau^N(X)\right]\xi - \frac{1}{\phi}\tau^{\widetilde{N}}(X)\xi$$

$$= \frac{1}{\phi}\overset{\star}{A}_\xi X + \left[\frac{X\cdot(\phi)}{\phi^2} + \frac{1}{\phi}\tau^N(X)\right]\xi - \frac{1}{\phi}\left[\tau^N(X) + \frac{X\cdot(\phi)}{\phi} + \frac{1}{\phi}B^N(\zeta,X)\right]\xi$$
$$= \frac{1}{\phi}\overset{\star}{A}_\xi X - \frac{1}{\phi^2}B^N(\zeta,X)\xi,$$

and we obtain the relation in (i), which completes the proof. □

Now it seems noteworthy to clearly state the relationship between the Ricci $(0,2)$-tensors induced by a normalizing null vector field N and the one induced by a change in the normalization $\widetilde{N} = \phi N + \zeta$. We have

Proposition 2.1. *Let (M, g, N) be a normalized lightlike hypersurface with induced Ricci $(0,2)$- tensor Ric and $\widetilde{N} = \phi N + \zeta$ a change in normalization on M. Then, $\widetilde{\mathrm{Ric}}$ being the Ricci $(0,2)$-tensor induced by $\widetilde{N}$,*

$$\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) + \frac{1}{\phi}B(X,Y)\left[\tau^N(\zeta) + \frac{\zeta(\phi)}{\phi} + \frac{1}{\phi}B(\zeta,\zeta) - \mathrm{div}_g(\zeta)\right]$$
$$+ \frac{1}{\phi}(\nabla^N_X\zeta, Y) - \frac{1}{\phi}B(\zeta,Y)\left[\tau^N(X) + \frac{X(\phi)}{\phi} + \frac{1}{\phi}B(\zeta,X)\right] + \frac{1}{\phi}\langle\overline{R}(X,\zeta)Y,\xi\rangle. \tag{2.1}$$

Proof. This basically follows from the Proposition 2.1, and the fact that [2]

$$\widetilde{\mathrm{Ric}}(X,Y) = \overline{\mathrm{R}}\mathrm{ic}(X,Y) + \widetilde{B}(X,Y)\mathrm{tr}\,A_{\widetilde{N}} - g(A_{\widetilde{N}}X, \overset{\star}{A}_{\widetilde{\xi}}\,Y) - \widetilde{\eta}\left(\overline{R}(\widetilde{\xi},Y)X\right).$$

Note that by item (h) in Lemma 2.1 and Proposition 1.1,

$$\mathrm{tr}\,A_{\widetilde{N}} = \phi\,\mathrm{tr}\,A_N - \mathrm{div}_g(\zeta) + \tau^N(\zeta) + \frac{\zeta\cdot\phi}{\phi} + \frac{1}{\phi}B(\zeta,\zeta).$$

Also,

$$\widetilde{\eta} = \phi\eta + \langle\zeta,\cdot\rangle.$$

Then,

$$\widetilde{\eta}\left(\overline{R}(\widetilde{\xi},Y)X\right) = \eta\left(\overline{R}(\xi,Y)X\right) - \frac{1}{\phi}\left\langle\overline{R}(X,\zeta)Y,\xi\right\rangle. \quad \square$$

Corollary 2.1. *Let (M, g, N) be a normalized lightlike hypersurface with induced Ricci tensor Ric. If the normalization data (ϕ,ζ) preserves the induced Ricci tensor R then,*

$$\nabla^N_\zeta\zeta - \mathrm{div}_g(\zeta)\zeta \in \mathrm{Ker}\,B^N. \tag{2.2}$$

Proof. Follows immediately from Proposition 2.1 in which one sets $\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y)$ for all X, Y tangent to M and taking $X = \zeta$. □

3. Normalizations with prescribed extrinsic scalar curvature

Recall that the induced Ricci tensor on a normalized lightlike hypersurface (M^{n+1}, g, N) of a $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$ is not symmetric in general, and in [2] we introduced a *symmetrized induced Ricci* tensor $\mathrm{Ric}^{\mathrm{sym}}$ on M as follows.

$$\mathrm{Ric}^{\mathrm{sym}}(X,Y) = \frac{1}{2}\left[\mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X)\right], \tag{3.1}$$

where $\mathrm{Ric}(X,Y) = \mathrm{trace}\{Z \longmapsto R(Z,X)Y\}$. Using the usual general relativity notation

$$R_{(\alpha\beta)} = \frac{1}{2}\left[R_{\alpha\beta} + R_{\beta\alpha}\right] \tag{3.2}$$

where we have shortened $R_{\alpha\beta} := \mathrm{Ric}_{\alpha\beta}$.

In the sequel, for a change in normalization $N \longrightarrow \widetilde{N} = \phi N + \zeta$, we define

$$\Theta(\phi,\zeta) = \tau^N(\zeta) + \frac{\zeta(\phi)}{\phi} + \frac{1}{\phi}B(\zeta,\zeta) - \mathrm{div}_g\,\zeta \tag{3.3}$$

and

$$K_\zeta = \overset{\star}{A}_\xi \circ \nabla^N_{\cdot}\zeta \tag{3.4}$$

Many further results will be derived from the following.

Theorem 3.1. *Let a normalized lightlike hypersurface (M, g, N) and a change in normalization $\widetilde{N} = \phi N + \zeta$ be given. Then, the extrinsic scalar curvature $\widetilde{R}$ with respect to $\widetilde{N}$ is given by:*

$$\widetilde{R} = R + \frac{1}{\phi}\left(\mathrm{tr}\ \overset{\star}{A}_\xi\right)\Theta(\phi,\zeta) + \frac{1}{\phi}\mathrm{tr}_g K_\zeta - \frac{1}{\phi}\left[\tau^N\left(\overset{\star}{A}_\xi\ \zeta\right) + \frac{\left(\overset{\star}{A}_\xi\ \zeta\right)\cdot(\phi)}{\phi} + \frac{1}{\phi}\|\ \overset{\star}{A}_\xi\ \zeta\|^2\right] - \frac{1}{\phi}\overline{\mathrm{Ric}}(\zeta,\xi). \tag{3.5}$$

Proof. The symmetrized Ricci formula expresses by using Proposition 2.1 that,

$$\begin{aligned}\widetilde{\mathrm{Ric}}^{\mathrm{sym}}(X,Y) =\ &\mathrm{Ric}^{\mathrm{sym}}(X,Y) + \frac{1}{\phi}B^N(X,Y)\Theta(\phi,\zeta) + \frac{1}{\phi}\left[B^N(\nabla_X\zeta,Y) + B^N(\nabla_Y\zeta,X)\right]\\ &- \frac{1}{2\phi}\left[B^N(\zeta,Y)\left[\tau^N(X) + \frac{X(\phi)}{\phi} + \frac{1}{\phi}B^N(\zeta,X)\right] + B^N(\zeta,X)\left[\tau^N(Y) + \frac{Y(\phi)}{\phi} + \frac{1}{\phi}B^N(\zeta,Y)\right]\right]\\ &+ \frac{1}{2\phi}\left[\langle\overline{R}(X,\zeta)Y,\xi\rangle + \langle\overline{R}(Y,\zeta)X,\xi\rangle\right].\end{aligned}$$

With respect to a g-quasi orthogonal frame $(\partial_0 = \xi, \partial_\alpha)$, $\alpha = 1,\ldots,n$ with $S(TM) = \mathrm{span}\{\partial_\alpha\}$, we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{(\alpha\beta)} =\ &\mathrm{Ric}_{(\alpha\beta)} + \frac{1}{\phi}B^N_{\alpha\beta}\Theta(\phi,\zeta) + \frac{1}{\phi}\left[B^N\left(\nabla^N_{\partial_\alpha}\zeta,\partial_\beta\right) + B^N\left(\nabla_{\partial_\beta}\zeta,\partial_\alpha\right)\right]\\ &- \frac{1}{2\phi}\left[B^N(\zeta,\partial_\beta)\left[\tau^N(\partial_\alpha) + \frac{\partial_\alpha(\phi)}{\phi} + \frac{1}{\phi}B^N(\zeta,\partial_\alpha)\right] + B^N(\zeta,\partial_\alpha)\left[\tau^N(\partial_\beta) + \frac{\partial_\beta(\phi)}{\phi} + \frac{1}{\phi}B^N(\zeta,\partial_\beta)\right]\right]\\ &+ \frac{1}{2\phi}\left[\langle\overline{R}(\partial_\alpha,\zeta)\partial_\beta,\xi\rangle + \langle\overline{R}(\partial_\beta,\zeta)\partial_\alpha,\xi\rangle\right].\end{aligned}$$

Hence, taking the trace from both sides leads to

$$\begin{aligned}\widetilde{R} = R &+ \frac{1}{\phi}\left(\mathrm{tr}_g\ \overset{\star}{A}_\xi\right)\Theta(\phi,\zeta) + \frac{1}{\phi}\mathrm{tr}_g K_\zeta - \frac{1}{2\phi}\left[\tau^N\left(\overset{\star}{A}_\xi\ \zeta\right) + \frac{\left(\overset{\star}{A}_\xi\ \zeta\right)\cdot(\phi)}{\phi}\right.\\ &+ \left.\frac{1}{\phi}\widetilde{g}(\overset{\star}{A}_\xi\ \zeta, \overset{\star}{A}_\xi\ \zeta) + \tau^N(\overset{\star}{A}_\xi\ \zeta) + \frac{\left(\overset{\star}{A}_\xi\ \zeta\right)\cdot(\phi)}{\phi} + \frac{1}{\phi}\widetilde{g}(\overset{\star}{A}_\xi\ \zeta, \overset{\star}{A}_\xi\ \zeta)\right] + \frac{1}{2\phi}\left[-\overline{\mathrm{Ric}}(\zeta,\xi) - \overline{\mathrm{Ric}}(\zeta,\xi)\right],\end{aligned}$$

which reduces to the required expression (3.5) using the fact that $\overset{\star}{A}_\xi\ \zeta$ is screen-valued and then $\widetilde{g}(\overset{\star}{A}_\xi\ \zeta, \overset{\star}{A}_\xi\ \zeta) = \|\ \overset{\star}{A}_\xi\ \zeta\|^2_g$. $\square$

We derive immediately the following results on our initial open problem:

Corollary 3.1. *Let (M, g, N) be a normalized lightlike hypersurface with extrinsic scalar curvature R. Then, a normalization data (ϕ, ζ) from N (i.e. a change in normalization $N \longrightarrow \widetilde{N} = \phi N + \zeta$) preserves the scalar quantity R if and only if*

$$\left[\tau^N + \frac{\mathrm{d}\phi}{\phi} + \frac{1}{\phi}B^N(\cdot,\zeta)\right]\left(\left(\mathrm{tr}\ \overset{\star}{A}_\xi\right)\zeta - \overset{\star}{A}_\xi\ \zeta\right) + \mathrm{tr}_g K_\zeta - \left(\mathrm{tr}\ \overset{\star}{A}_\xi\right)\mathrm{div}_g\ \zeta - \overline{\mathrm{Ric}}(\zeta,\xi) = 0. \tag{3.6}$$

Proof. This is a straightforward use of relation (3.5) with the requirement $\widetilde{R} = R$ and noting that $\Theta(\phi,\zeta) = \tau^N(\zeta) + \frac{\zeta(\phi)}{\phi} + \frac{1}{\phi}B(\zeta,\zeta) - \mathrm{div}_g\ \zeta$. $\square$

Corollary 3.2 (*Isotropic Scaling*)**.** *Let R be the (induced) extrinsic scalar curvature on the data (M, g, N). Then, any isotropic scaling $\widetilde{N} = \phi N$ preserves the scalar quantity R.*

Proof. Immediate from (3.6) using $\zeta = 0$. $\square$

Now, using $\phi = 1$ in relation (3.6) we derive the following

Corollary 3.3 (*Tangential Scaling*)**.** *Let R be the (induced) extrinsic scalar curvature on the data (M, g, N). Then, a tangential scaling $\widetilde{N} = N + \zeta$ preserves the scalar quantity R if and only if.*

$$\left[\tau^N + B^N(\cdot, \zeta)\right]\left(\left(\mathrm{tr}\ \overset{\star}{A}_\xi\right)\zeta - \overset{\star}{A}_\xi\ \zeta\right) + \mathrm{tr}_g K_\zeta - \left(\mathrm{tr}\ \overset{\star}{A}_\xi\right)\mathrm{div}_g\ \zeta - \overline{\mathrm{Ric}}(\zeta, \xi) = 0. \quad \square \tag{3.7}$$

Corollary 3.4 (*Mixed Scaling*)**.** *Let R be the induced extrinsic scalar curvature on the data (M, g, N) in a semi-Riemannian Einstein manifold $(\overline{M}, \overline{g})$. Then the mixed scaling normalization (and subsequently a choice of a screen distribution) $N \longrightarrow \widetilde{N} = \phi N + \zeta$ preserves the scalar quantity R if any one of the following items holds:*

(i) $(\mathrm{tr}\ \overset{\star}{A}_\xi) \in \mathrm{spec}\left(\overset{\star}{A}_\xi\right)$ *with ∇^N-parallel eigenvector field ζ.*
(ii) $\mathrm{tr}\ \overset{\star}{A}_\xi = 0$ *and $\zeta \in \mathrm{Ker}\ \overset{\star}{A}_\xi$ is ∇^N-parallel.*
(iii) $\phi = \mathrm{cte} \neq 0$ *and the ∇^N-parallel vector field ζ satisfies $B^N(\zeta, \cdot) = -\lambda\tau^N$.*
(iv) *(M, g) is totally geodesic.*

Proof. As the ambient manifold is assumed to be Einstein, we have $\overline{\mathrm{Ric}}(\zeta, \xi) = \lambda\overline{g}(\zeta, \xi) = \lambda g(\zeta, \xi) = 0$. Then, each item from (i) to (iv) is a sufficient condition for (3.6). $\square$

4. Some examples

In this section the reader will be provided some examples for which we examine preservation of the given extrinsic scalar curvature under various (re-)normalizations.

4.1. The lightcone

Beyond all physical considerations, the null cone $\wedge_0^{n+1} \subset \mathbb{R}_1^{n+2}$ is one of the most important manifolds with lightlike metric. In fact, as we known from [6], the null cone is, up to a homogeneous Riemannian factor, the only homogeneous lightlike manifold on which a Lie group with finite center acts faithfully, isometrically and non-properly.

Consider the null cone $\wedge_0^{n+1}$ at the origin in $\mathbb{R}_1^{n+2}$. Set $\langle , \rangle_{\wedge_0^{n+1}}$ the induced metric and $N_0 = \frac{1}{2t^2}\left\{-t\partial_t + \sum_{a=1}^{n+1} x^a\partial_a\right\}$, a normalizing null vector field on it. We know from [2] that $R_0 = \frac{n^2-n}{2t^2}$ is the (induced) extrinsic scalar curvature on the data $\left(\wedge_0^{n+1}, \langle , \rangle_{\wedge_0^{n+1}}, N_0\right)$. As we know from Corollary 3.4, very selective are data preserving R_0.

We first examine tangential scaling from N_0 preserving our extrinsic scalar curvature R_0. In fact, let $\widetilde{N} = N_0 + \zeta$ be such a null vector field along $\wedge_0^{n+1}$. We have from Corollary 3.3,

$$\left[\tau^{N_0} + B^{N_0}(\cdot, \zeta)\right]\left(\left(\mathrm{tr}\ \overset{\star}{A}_{\xi_0}\right)\zeta - \overset{\star}{A}_{\xi_0}\ \zeta\right) + \mathrm{tr}_g K_\zeta - \left(\mathrm{tr}\ \overset{\star}{A}_{\xi_0}\right)\mathrm{div}_g\ \zeta = 0,$$

as $\overline{\mathrm{Ric}}(\zeta, \xi_0) = 0$, where $\xi_0 = t\partial_t + \sum_{a=1}^{n+1} x^a\partial_a$ represents the position vector field in $\mathbb{R}_1^{n+2}$. But, by straightforward calculation, we get $\tau^{N_0} = -\overline{g}(\cdot, N_0) = -\eta_0$. Now P_0, being the projection morphism onto the screen distribution associated to N_0, we have $\overset{\star}{A}_{\xi_0} X = -P_0 X$ and then $\mathrm{tr}(\overset{\star}{A}_{\xi_0}) = -n$. Thus,

$$\begin{aligned}\tau^{N_0}\left((\mathrm{tr}\ \overset{\star}{A}_{\xi_0})\zeta - \overset{\star}{A}_{\xi_0}\ \zeta\right) &= -\eta_0\left[\mathrm{tr}(\overset{\star}{A}_{\xi_0})(P_0\zeta + \eta_0(\zeta)\xi_0) + P_0\zeta\right] \\ &= n\eta_0(\zeta).\end{aligned}$$

But

$$\begin{aligned}B^{N_0}\left((\mathrm{tr}\ \overset{\star}{A}_{\xi_0})\zeta - \overset{\star}{A}_{\xi_0}\ \zeta, \zeta\right) &= B^{N_0}\left(-n\zeta + P_0\zeta, \zeta\right) \\ &= (-n+1)B^{N_0}(P_0\zeta, P_0\zeta) \\ &= (-n+1)\langle\overset{\star}{A}_{\xi_0}\ \zeta, P_0\zeta\rangle \\ &= (n-1)\langle P_0\zeta, P_0\zeta\rangle \\ &= (n-1)\|\zeta\|^2.\end{aligned}$$

Also, by direct calculation, we have

$$\begin{aligned}\mathrm{tr}_g K_\zeta &= -\mathrm{div}_g\ \zeta + \eta_0(\nabla^{N_0}_{\xi_0}\zeta) \\ &= -\mathrm{div}_g\ \zeta + \xi_0 \cdot (\eta_0(\zeta)) + \eta_0(\zeta),\end{aligned}$$

as $\left(\wedge_0^{n+1}, \langle,\rangle_{\wedge_0^{n+1}}, N_0\right)$ has a conformal screen [7] from which we infer $C^{N_0}(\xi_0, P\zeta) = \frac{1}{2t^2}B^{N_0}(\xi_0, P\zeta) = 0$. Hence a tangential scaling from N_0 preserves the extrinsic scalar curvature R_0 if and only if

$$(n+1)\eta_0(\zeta) + \xi_0 \cdot (\eta_0(\zeta)) + (n-1)\|\zeta\|^2 + (n-1)\mathrm{div}_g\, \zeta = 0,$$

which is

$$(n-3)\|\zeta\|^2 - \xi_0 \cdot (\|\zeta\|^2) + 2(n-1)\mathrm{div}_g\, \zeta = 0, \tag{4.1}$$

using item (ii) in Lemma 2.1 with $\varphi = 1$. In particular if $n \geq 2$ and the tangent vector field ζ is required to have a constant length, say l, the normalizing data $(1, \zeta)$ from N_0 preserves R_0 if and only if the following partial differential equation,

$$\mathrm{div}_g\, \zeta = \frac{n-3}{2(n-1)} l^2 \tag{4.2}$$

holds.

4.2. Cutting the pseudosphere S_1^3

Consider the unit pseudosphere S_1^3 of the Minkowski space $\mathbb{R}_1^4$, given by $-t^2 + x^2 + y^2 + z^2 = 1$. Cut S_1^3 by the hyperplane $t - x = 0$ of $\mathbb{R}_1^4$ and obtain a lightlike surface Σ of S_1^3. It is an easy matter to verify that a normalization of Σ is given by the null transversal vector field

$$N = -\frac{1}{2}\left[(1+t^2)\partial_t + (t^2-1)\partial_x + 2ty\partial_y + 2tz\partial_z\right]$$

along Σ.

Let R^Σ denote the extrinsic scalar curvature induced by N. It is our purpose to show that R^Σ is invariant under mixed scaling. In fact we show that Σ is totally geodesic in the pseudosphere S_1^3 and use Corollary 3.4, as S_1^3 is Einstein. Note that the rank 1 screen distribution corresponding to the normalization with the above N is spanned by the spacelike vector field $W = z\partial_y - y\partial_z$. Also, the radical distribution of Σ is spanned by the null (characteristic) vector field $\xi = \partial_t + \partial_x$ with $\langle N, \xi\rangle = 1$. Now, as $\overset{\star}{A}_\xi\, \xi = 0$, it is sufficient to show that $\overset{\star}{A}_\xi\, W = 0$. Let D and ∇ denote the flat connection on $\mathbb{R}_1^4$ and the induced connection on the pseudosphere S_1^3 respectively. We have,

$$\begin{aligned} -\overset{\star}{A}_\xi\, W - \tau(W)\xi &= \nabla_W \xi = D_W \xi = zD_{\partial_y}\xi - yD_{\partial_z}\xi \\ &= zD_{\partial_y}(\partial_t + \partial_x) - yD_{\partial_z}(\partial_t + \partial_x) \\ &= 0. \end{aligned}$$

Hence, $\overset{\star}{A}_\xi\, W = 0$ and Σ is totally geodesic in S_1^3 and the claim follows item (iv) in Corollary 3.4.

Remark. A similar interesting example is obtained when considering (M, g) to be a black hole event horizon in a C^∞ Lorentzian manifold $(\overline{M}, \overline{g})$ satisfying a natural hypothesis, using the well known regularity and area theorem by Chrusciel et al. [8]. Let Σ_a, $a = 1, 2$, be two achronal C^2 embedded spacelike hypersurfaces, $S_a = \Sigma_a \cap M$ and M_{12} the part of M between S_1 and S_2. If S_1 belongs to the past of S_2 with $\mathrm{area}(S_1) = \mathrm{area}(S_2)$ then, M_{12} is a totally geodesic lightlike hypersurface and the same item (iv) in Corollary 3.4 ensures that every extrinsic scalar curvature is preserved by mixed scaling from a given normalizing.

4.3. Special level hypersurfaces in the Robertson–Walker spacetime

Robertson–Walker spacetime is the solution of the Einstein field equations arising from the idea that our 4-dimensional universe is spatially homogeneous as it admits a 6-parameter group of isometries whose surfaces of transitivity are spacelike hypersurfaces of constant curvature. The spacetime metric is then given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + S^2(t)\mathrm{d}\Sigma^2,$$

where $\mathrm{d}\Sigma^2$ is the metric of the spacelike hypersurface Σ with spherical symmetries and constant curvature $c = 1, -1$ or 0. In the local spherical coordinate system (r, θ, φ),

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + f(r)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right),$$

where $f(r) = \begin{cases} \sin r & \text{if } c = 1 \\ \sinh r & \text{if } c = -1 \\ r & \text{if } c = 0. \end{cases}$

Now consider two null coordinates u and v such that $u = t + r$ and $v = t - r$. Then, the above spacetime metric transforms to

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + f^2(r)\mathrm{d}\Omega^2,$$

with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. It follows that $\{u = \text{constant}\}$ and $\{v = \text{constant}\}$ are lightlike hypersurfaces of the Robertson–Walker spacetime.

Let

$$M_{v_0} = (M, g, v = v_0 = \text{constant})$$

and

$$M_{u_0} = (M, g, u = u_0 = \text{constant})$$

be a pair of these lightlike hypersurfaces, D^v and D^u the 1-dimensional distributions generated by the null vector fields ∂_v and ∂_u respectively in $\overline{M}$.

Obviously, D^u and D^v are null transversal vector bundle to M^{v_0} and M^{u_0}, respectively.

The intersection $M^{u_0} \cap M^{v_0}$ is topologically a 2-sphere S^2. Actually, the transversal null vector ∂_v induces a normalization on M^{u_0} whose corresponding screen distribution is integrable with leaves diffeomorphic to S^2. We let R_v denote the extrinsic scalar curvature induced on the triplet (M^{u_0}, g, ∂_v). Assume ζ is a tangent vector field to M^{u_0}, parallel along null orbits of ∂_u. Then in virtue of item (ii) in Corollary 3.4, the tangential scaling $\widetilde{N} = \partial_v + \zeta$ preserves the extrinsic scalar curvature R_v if and only if M^{u_0} has zero mean curvature.

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