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Representations and cohomology of n-ary multiplicative Hom–Nambu–Lie algebras

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ABSTRACT

The aim of this paper is to provide cohomologies of n-ary Hom–Nambu–Lie algebras governing central extensions and one parameter formal deformations. We generalize to n-ary algebras the notions of derivation and representation introduced by Sheng for Hom–Lie algebras. Also we show that a cohomology of n-ary Hom–Nambu–Lie algebras could be derived from the cohomology of Hom–Leibniz algebras.

0. Introduction

The first instances of n-ary algebras in Physics appeared with a generalization of the Hamiltonian mechanics proposed in 1973 by Nambu [1]. More recent motivation comes from string theory and M-branes involving naturally an algebra with ternary operation called Bagger–Lambert algebra which gives impulse to a significant development. It was used in [2] as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance, and an SO(8) R-symmetry that acts on the eight transverse scalars. On the other hand in the study of supergravity solutions describing M2-branes ending on M5-branes, the Lie algebra appearing in the original Nahm equations has to be replaced with a generalization involving ternary bracket in the lifted Nahm equations; see [3]. For other applications in Physics see [4–6].

The algebraic formulation of Nambu mechanics is due to Takhtajan [7,8] while the abstract definition of n-ary Nambu algebras or n-ary Nambu–Lie algebras (when the bracket is skew-symmetric) was given by Filippov in 1985; see [9]. The Leibniz n-ary algebras were introduced and studied in [10]. For deformation theory and cohomologies of n-ary algebras of Lie type, we refer to [11–15,8]. For more general works involving operads and where deformation theory is described using a certain differential graded Lie algebras see [16–18]. For general theory see also [19–23].

The general Hom-algebra structures arose first in connection with quasi-deformation and discretizations of Lie algebras of vector fields. These quasi-deformations lead to quasi-Lie algebras, a generalized Lie algebra structure in which the skew-symmetry and Jacobi conditions are twisted. For Hom–Lie algebras, Hom–associative algebras, Hom–Lie superalgebras,

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Hom-bialgebras ... see [24–29]. Generalizations of n-ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced in [30]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type such as n-ary Nambu algebras, n-ary Nambu–Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type such as n-ary totally associative and n-ary partially associative algebras. See also [31–33].

The aim of this paper is to provide cohomologies of n-ary Hom–Nambu–Lie algebras governing central extensions and one parameter formal deformations. In the first section we summarize the definitions of n-ary Hom–Nambu (resp. Hom–Nambu–Lie) algebras. We deal mostly in this paper with subclasses of these algebras called multiplicative n-ary Hom–Nambu (resp. Hom–Nambu–Lie) algebras. In Section 2, we extend to n-ary algebras the notions of derivation and representation introduced for Hom–Lie algebras in [34]; see also [35]. In Section 3, we show that for an n-ary Hom–Nambu–Lie algebra $\mathcal{N}$, the space $\wedge^{n-1}\mathcal{N}$ carries a structure of Hom–Leibniz algebra. Section 4 is dedicated to central extensions, we provide a cohomology adapted to central extensions of n-ary multiplicative Hom–Nambu–Lie algebras. In Section 5, we define a cohomology which is suitable for the study of one parameter formal deformations of n-ary Hom–Nambu–Lie algebras. In the last section we show that the cohomology of n-ary Hom–Nambu–Lie algebras with values in the algebra may be derived from the cohomology of Hom–Leibniz algebras. To this end we generalize to twisted situation the process used by Daletskii and Takhtajan [7] for the classical case.

1. *n*-ary Hom–Nambu algebras

Throughout this paper, we will for simplicity of exposition assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero, even though for most of the general definitions and results in the paper this assumption is not essential.

1.1. Definitions

In this section, we recall the definitions of n-ary Hom–Nambu algebras and n-ary Hom–Nambu–Lie algebras, introduced in [30] by Ataguema, Makhlouf and Silvestrov. They correspond to a generalized version by twisting of n-ary Nambu algebras and Nambu–Lie algebras which are also called Filippov algebras. In this paper we deal with a subclass of n-ary Hom–Nambu algebras called multiplicative n-ary Hom–Nambu algebras.

Definition 1.1. An *n-ary Hom–Nambu* algebra is a triple $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ consisting of a vector space $\mathcal{N}$, an n-linear map $[\cdot, \dots, \cdot] : \mathcal{N}^n \longrightarrow \mathcal{N}$ and a family $\widetilde{\alpha} = (\alpha_i)_{1\le i\le n-1}$ of linear maps $\alpha_i : \mathcal{N} \longrightarrow \mathcal{N}$, satisfying

$$\begin{aligned}&\big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\big]\\ &\quad= \sum_{i=1}^{n}\big[\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)\big],\end{aligned} \tag{1.1}$$

for all $(x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, $(y_1, \dots, y_n) \in \mathcal{N}^n$.

Identity (1.1) is called *Hom–Nambu identity*.

Let $x = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, $\widetilde{\alpha}(x) = (\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1})) \in \mathcal{N}^{n-1}$ and $y \in \mathcal{N}$. We define an adjoint map $\mathrm{ad}(x)$ as a linear map on $\mathcal{N}$, such that

$$\mathrm{ad}(x)(y) = [x_1, \dots, x_{n-1}, y]. \tag{1.2}$$

Then the Hom–Nambu identity (1.1) may be written in terms of adjoint map as

$$\mathrm{ad}(\widetilde{\alpha}(x))([x_n, \dots, x_{2n-1}]) = \sum_{i=n}^{2n-1}[\alpha_1(x_n), \dots, \alpha_{2n-i-1}(x_{i-1}), \mathrm{ad}(x)(x_i), \alpha_{2n-i}(x_{i+1}), \dots, \alpha_{n-1}(x_{2n-1})].$$

Remark 1.2. When the maps $(\alpha_i)_{1\le i\le n-1}$ are all identity maps, one recovers the classical n-ary Nambu algebras. The Hom–Nambu Identity (1.1), for $n = 2$, corresponds to Hom–Jacobi identity (see [27]), which reduces to Jacobi identity when $\alpha_1 = \mathrm{id}$.

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ and $(\mathcal{N}', [\cdot, \dots, \cdot]', \widetilde{\alpha}')$ be two n-ary Hom–Nambu algebras where $\widetilde{\alpha} = (\alpha_i)_{i=1,\dots,n-1}$ and $\widetilde{\alpha}' = (\alpha'_i)_{i=1,\dots,n-1}$. A linear map $f : \mathcal{N} \to \mathcal{N}'$ is an n-ary Hom–Nambu algebra *morphism* if it satisfies

$$\begin{aligned}&f([x_1, \dots, x_n]) = [f(x_1), \dots, f(x_n)]'\\ &f \circ \alpha_i = \alpha'_i \circ f \quad \forall i = 1, \dots, n-1.\end{aligned}$$

Definition 1.3. An n-ary Hom–Nambu algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ where $\widetilde{\alpha} = (\alpha_i)_{1\le i\le n-1}$ is called *n-ary Hom–Nambu–Lie* algebra if the bracket is skew-symmetric that is

$$[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = \mathrm{Sgn}(\sigma)[x_1, \dots, x_n], \quad \forall \sigma \in \mathcal{S}_n \text{ and } \forall x_1, \dots, x_n \in \mathcal{N} \tag{1.3}$$

where $\mathcal{S}_n$ stands for the permutation group of n elements.

In what follows we deal with a particular class of n-ary Hom–Nambu–Lie algebras which we call n-ary multiplicative Hom–Nambu–Lie algebras.

Definition 1.4. An *n-ary multiplicative Hom–Nambu algebra* (resp. *n-ary multiplicative Hom–Nambu–Lie algebra*) is an n-ary Hom–Nambu algebra (resp. n-ary Hom–Nambu–Lie algebra) $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ with $\widetilde{\alpha} = (\alpha_i)_{1\le i\le n-1}$ where $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ and satisfying

$$\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)], \quad \forall x_1, \dots, x_n \in \mathcal{N}. \tag{1.4}$$

For simplicity, we will denote the n-ary multiplicative Hom–Nambu algebra as $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ where $\alpha : \mathcal{N} \to \mathcal{N}$ is a linear map. Also by misuse of language an element $x \in \mathcal{N}^n$ refers to $x = (x_1, \dots, x_n)$ and $\alpha(x)$ denotes $(\alpha(x_1), \dots, \alpha(x_n))$.

The following theorem gives a way to construct n-ary multiplicative Hom–Nambu algebras (resp. Hom–Nambu–Lie algebras) starting from classical n-ary Nambu algebras (resp. Nambu–Lie algebras) and algebra endomorphisms.

Theorem 1.5 ([30]). *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be an n-ary Nambu algebra (resp. n-ary Nambu–Lie algebra) and let $\rho : \mathcal{N} \to \mathcal{N}$ be an n-ary Nambu (resp. Nambu–Lie) algebra endomorphism. Then $(\mathcal{N}, \rho \circ [\cdot, \dots, \cdot], \rho)$ is a n-ary multiplicative Hom–Nambu algebra (resp. n-ary multiplicative Hom–Nambu–Lie algebra).*

2. Representations of Hom–Nambu–Lie algebras

In this section we extend the representation theory of Hom–Lie algebras introduced in [34,35] to the n-ary case. We denote by $\mathrm{End}(\mathcal{N})$ the linear group of operators on the $\mathbb{K}$-vector space $\mathcal{N}$. Sometimes it is considered as a Lie algebra with the commutator brackets.

2.1. Derivations of n-ary Hom–Nambu–Lie algebras

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be an n-ary multiplicative Hom–Nambu–Lie algebra.
We denote by α^k the k-times composition of α. In particular, we set $\alpha^{-1} = 0$ and $\alpha^0 = \mathrm{id}$.

Definition 2.1. For any $k \ge -1$, we call $D \in End(\mathcal{N})$ an *α^k-derivation* of the n-ary multiplicative Hom–Nambu–Lie algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ if

$$[D, \alpha] = 0 \quad (\text{i.e. } D \circ \alpha = \alpha \circ D), \tag{2.1}$$

and for $x \in \mathcal{N}^n$

$$D[x_1, \dots, x_n] = \sum_{i=1}^{n} [\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]. \tag{2.2}$$

We denote by $\mathrm{Der}_{\alpha^k}(\mathcal{N})$ the set of α^k-derivations of the n-ary multiplicative Hom–Nambu–Lie algebra $\mathcal{N}$. Notice that we obtain trivial derivations for $k = -1$ and classical derivations for $k = 0$.

For $x \in \mathcal{N}^{n-1}$ satisfying $\alpha(x) = x$ and $k \ge -1$, we define the map $\mathrm{ad}_k(x) \in \mathrm{End}(\mathcal{N})$ by

$$\mathrm{ad}_k(x)(y) = [x_1, \dots, x_{n-1}, \alpha^k(y)] \quad \forall y \in \mathcal{N}. \tag{2.3}$$

Then

Lemma 2.2. *The map $\mathrm{ad}_k(x)$ is an α^{k+1}-derivation, that we call inner α^{k+1}-derivation.*

We denote by $\mathrm{Inn}_{\alpha^k}(\mathcal{N})$ the $\mathbb{K}$-vector space generated by all inner α^{k+1}-derivations.

For any $D \in \mathrm{Der}_{\alpha^k}(\mathcal{N})$ and $D' \in \mathrm{Der}_{\alpha^{k'}}(\mathcal{N})$ we define their commutator $[D, D']$ as usual $[D, D'] = D \circ D' - D' \circ D$. Set $\mathrm{Der}(\mathcal{N}) = \bigoplus_{k\ge -1} \mathrm{Der}_{\alpha^k}(\mathcal{N})$ and $\mathrm{Inn}(\mathcal{N}) = \bigoplus_{k\ge -1} \mathrm{Inn}_{\alpha^k}(\mathcal{N})$.

Lemma 2.3. *For any $D \in \mathrm{Der}_{\alpha^k}(\mathcal{N})$ and $D' \in \mathrm{Der}_{\alpha^{k'}}(\mathcal{N})$, where $k + k' \ge -1$, we have*

$$[D, D'] \in \mathrm{Der}_{\alpha^{k+k'}}(\mathcal{N}).$$

Proof. Let $x_i \in \mathcal{N},\ 1 \le i \le n$, $D \in \mathrm{Der}_{\alpha^k}(\mathcal{N})$ and $D' \in \mathrm{Der}_{\alpha^{k'}}(\mathcal{N})$, then

$$\begin{aligned} D\circ D'([x_1,\dots,x_n]) &= \sum_{i=1}^{n} D([\alpha^{k'}(x_1),\dots,D'(x_i),\dots,\alpha^{k'}(x_n)]) \\ &= \sum_{i=1}^{n}[\alpha^{k+k'}(x_1),\dots,D\circ D'(x_i),\dots,\alpha^{k+k'}(x_n)] \\ &\quad + \sum_{i<j}^{n}[\alpha^{k+k'}(x_1),\dots,\alpha^{k}(D'(x_i)),\dots,\alpha^{k'}(D(x_j)),\dots,\alpha^{k+k'}(x_n)] \\ &\quad + \sum_{i>j}^{n}[\alpha^{k+k'}(x_1),\dots,\alpha^{k'}(D(x_j)),\dots,\alpha^{k}(D'(x_i)),\dots,\alpha^{k+k'}(x_n)]. \end{aligned}$$

The second and the third terms in $[D, D']$ are symmetrical, then

$$\begin{aligned} [D,D']([x_1,\dots,x_n]) &= (D\circ D' - D'\circ D)([x_1,\dots,x_n]) \\ &= \sum_{i=1}^{n}[\alpha^{k+k'}(x_1),\dots,(D\circ D'-D'\circ D)(x_i),\dots,\alpha^{k+k'}(x_n)] \\ &= \sum_{i=1}^{n}[\alpha^{k+k'}(x_1),\dots,[D,D'](x_i),\dots,\alpha^{k+k'}(x_n)], \end{aligned}$$

which yields that $[D, D'] \in \mathrm{Der}_{\alpha^{k+k'}}(\mathcal{N})$. □

Moreover since the bracket is closed we have:

Proposition 2.4. *The pair* $(\mathrm{Der}(\mathcal{N}), [\cdot,\cdot])$*, where the bracket is the usual commutator, defines a Lie algebra and* $\mathrm{Inn}(\mathcal{N})$ *constitutes an ideal of it.*

Proof. $(\mathrm{Der}(\mathcal{N}), [\cdot,\cdot])$ is a Lie algebra by using Lemma 2.3. We show that $\mathrm{Inn}(\mathcal{N})$ is an ideal. Let $\mathrm{ad}_k(x) = [x_1,\dots, x_{n-1}, \alpha^{k-1}(\cdot)]$ be an inner α^k-derivation on $\mathcal{N}$ and $D \in \mathrm{Der}_{\alpha^{k'}}(\mathcal{N})$ for $k \ge -1$ and $k' \ge -1$ where $k + k' \ge -1$. Then

$$[D, \mathrm{ad}_k(x)] \in \mathrm{Der}_{\alpha^{k+k'}}(\mathcal{N})$$

and for any $y \in \mathcal{N}$

$$\begin{aligned} [D,\mathrm{ad}_k(x)](y) &= D([x_1,\dots,x_{n-1},\alpha^{k-1}(y)]) - [x_1,\dots,x_{n-1},\alpha^{k-1}(D(y))], \\ &= D([\alpha^{k}(x_1),\dots,\alpha^{k}(x_{n-1}),\alpha^{k-1}(y)]) - [\alpha^{k+k'}(x_1),\dots,\alpha^{k+k'}(x_{n-1}),\alpha^{k-1}(D(y))], \\ &= \sum_{i\le n-1}[\alpha^{k+k'}(x_1),\dots,D(\alpha^{k}(x_i)),\dots,\alpha^{k+k'}(x_{n-1}),\alpha^{k+k'-1}(y)], \\ &= \sum_{i\le n-1}[x_1,\dots,D(x_i),\dots,x_{n-1},\alpha^{k+k'-1}(y)], \\ &= \sum_{i\le n-1}\mathrm{ad}_{k+k'}(x_1\wedge\cdots\wedge D(x_i)\wedge\cdots\wedge x_{n-1})(y). \end{aligned}$$

Therefore $[D, \mathrm{ad}_k(x)] \in \mathrm{Inn}_{\alpha^{k+k'}}(\mathcal{N})$. □

2.2. Representations of n-ary Hom–Nambu–Lie algebras

In this section we introduce and study the representation of n-ary multiplicative Hom–Nambu–Lie algebras.

Definition 2.5. A representation of an n-ary multiplicative Hom–Nambu–Lie algebra $(\mathcal{N}, [\cdot,\dots,\cdot], \alpha)$ on a vector space V is a skew-symmetric multilinear map $\rho : \mathcal{N}^{n-1} \longrightarrow \mathrm{End}(V)$, satisfying for $x, y \in \mathcal{N}^{n-1}$ the identity

$$\rho(\alpha(x))\circ\rho(y) - \rho(\alpha(y))\circ\rho(x) = \sum_{i=1}^{n-1}\rho(\alpha(x_1),\dots,\mathrm{ad}(y)(x_i),\dots,\alpha(x_{n-1}))\circ\nu \tag{2.4}$$

where ν is an endomorphism of V.

Two representations (ρ, V) and (ρ', V') of $\mathcal{N}$ are *equivalent* if there exists $f : V \to V'$ an isomorphism of vector space such that $f(x\cdot y) = x \cdot' f(y)$ where $x \cdot y = \rho(x)(y)$ and $x \cdot' y = \rho'(x)(y)$ for $x \in \mathcal{N}^{n-1}$ and $y \in V$.

Example 2.6. Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be an n-ary multiplicative Hom–Nambu–Lie algebra. The map ad defined in (1.2) is a representation, where the operator ν is the twist map α. Identity (2.4) is equivalent to Hom–Nambu identity. It is called adjoint representation.

3. From *n*-ary Hom–Nambu–Lie algebra to Hom–Leibniz algebra

In the context of Hom–Lie algebras one gets the class of Hom–Leibniz algebras (see [27]). Following the standard Loday's conventions for Leibniz algebras, a Hom–Leibniz algebra is a triple $(A, [\cdot, \cdot], \alpha)$ consisting of a vector space A, a bilinear map $[\cdot, \cdot] : A \times A \to A$ and a linear map $\alpha : A \to A$ with respect to $[\cdot, \cdot]$ satisfying

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]]. \tag{3.1}$$

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a n-ary multiplicative Hom–Nambu–Lie algebra. On $\wedge^{n-1}\mathcal{N}$ which is the set of elements $x_1 \wedge \cdots \wedge x_{n-1}$ that are skew-symmetric in their arguments, for $x = x_1 \wedge \cdots \wedge x_{n-1} \in \wedge^{n-1}\mathcal{N},\ y = y_1 \wedge \cdots \wedge y_{n-1} \in \wedge^{n-1}\mathcal{N},\ z \in \mathcal{N}$, we define

- a linear map $L : \wedge^{n-1}\mathcal{N} \longrightarrow \mathrm{End}(\mathcal{N})$ by
$$L(x) \cdot z = [x_1, \dots, x_{n-1}, z], \tag{3.2}$$
and extending it linearly to all elements of $\wedge^{n-1}\mathcal{N}$. Notice that $L(x) \cdot z = \mathrm{ad}(x)(z)$.
- a linear map $\tilde{\alpha} : \wedge^{n-1}\mathcal{N} \longrightarrow \wedge^{n-1}\mathcal{N}$ by $\tilde{\alpha}(x) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_{n-1})$,
- a bilinear map $[\,,\,]_\alpha : \wedge^{n-1}\mathcal{N} \times \wedge^{n-1}\mathcal{N} \longrightarrow \wedge^{n-1}\mathcal{N}$ by

$$[x, y]_\alpha = L(x) \bullet_\alpha y = \sum_{i=0}^{n-1} \big(\alpha(y_1), \dots, L(x) \cdot y_i, \dots, \alpha(y_{n-1})\big). \tag{3.3}$$

We denote by $\mathcal{L}(\mathcal{N})$ the space $\wedge^{n-1}\mathcal{N}$ and call it the fundamental set.

Lemma 3.1. *The map L satisfies*

$$L([x, y]_\alpha) \cdot \alpha(z) = L(\alpha(x)) \cdot \big(L(y) \cdot z\big) - L(\alpha(y)) \cdot \big(L(x) \cdot z\big) \tag{3.4}$$

for all $x,\ y \in \mathcal{L}(\mathcal{N}),\ z \in \mathcal{N}$.

Proposition 3.2. *The triple* $\big(\mathcal{L}(\mathcal{N}), [\,,\,]_\alpha, \alpha\big)$ *is a Hom–Leibniz algebra.*

Proof. Let $x = x_1 \wedge \cdots \wedge x_{n-1},\ y = y_1 \wedge \cdots \wedge y_{n-1}$ and $u = u_1 \wedge \cdots \wedge u_{n-1} \in \mathcal{L}(\mathcal{N})$, the Leibniz identity (3.1) can be written

$$\big[[x, y]_\alpha, \alpha(u)\big]_\alpha = [\alpha(x), [y, u]_\alpha\big]_\alpha - [\alpha(y), [x, u]_\alpha\big]_\alpha \tag{3.5}$$

and equivalently for $v \in \mathcal{N}$

$$\Big(L\big(L(x)\bullet_\alpha y\big)\bullet_\alpha \tilde{\alpha}(u)\Big) \cdot (v) = \Big(L(\alpha(x))\bullet_\alpha\big(L(y)\bullet_\alpha u\big)\Big) \cdot (v) - \Big(L(\alpha(y))\bullet_\alpha\big(L(x)\bullet_\alpha u\big)\Big) \cdot (v). \tag{3.6}$$

Let us compute first $\Big(L(\tilde{\alpha}(x))\bullet_\alpha\big(L(y)\bullet_\alpha u\big)\Big)$. This is given by

$$\begin{aligned}\Big(L(\alpha(x))\bullet_\alpha\big(L(y)\bullet_\alpha u\big)\Big) &= \sum_{i=0}^{n-1} L(\alpha(x))\bullet_\alpha\big(\alpha(u_1), \dots, L(y) \cdot u_i, \dots, \alpha(u_{n-1})\big)\\ &= \sum_{i=0}^{n-1}\sum_{i\neq j, j=0}^{n-1} \big(\alpha^2(u_1), \dots, \alpha(L(x) \cdot u_j), \dots, \alpha(L(y) \cdot u_i), \dots, \alpha^2(u_{n-1})\big)\\ &\quad + \sum_{i=0}^{n-1} \big(\alpha^2(u_1), \dots, L(\tilde{\alpha}(x)) \cdot (L(y) \cdot u_i), \dots, \alpha^2(u_{n-1})\big).\end{aligned}$$

The right-hand side of (3.6) is skew-symmetric in x, y. Hence,

$$\begin{aligned}&\Big(L(\alpha(x))\bullet_\alpha\big(L(y)\bullet_\alpha u\big)\Big) - \Big(L(\alpha(y))\bullet_\alpha\big(L(x)\bullet_\alpha u\big)\Big)\\ &= \sum_{i=0}^{n-1} (\alpha^2(u_1), \dots, \{L(\alpha(x)) \cdot (L(y) \cdot u_i) - L(\alpha(y)) \cdot (L(x) \cdot u_i)\}, \dots, \alpha^2(u_{n-1})\big).\end{aligned} \tag{3.7}$$

On the other hand, using definition (3.3), we find

$$\Big(L\big(L(x)\bullet_\alpha y\big)\bullet_\alpha \tilde{\alpha}(u)\Big)$$
$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\big(\alpha^2(u_1),\ldots,\alpha^2(u_{i-1}),[\alpha(y_1),\ldots,L(x)\cdot y_j,\ldots,\alpha(y_{n-1}),\alpha(u_i)],\alpha^2(u_{i+1}),\ldots,\alpha^2(u_{n-1})\big)$$
$$= \sum_{i=0}^{n-1}\big(\alpha^2(u_1),\ldots,\alpha^2(u_{i-1}),[x,y]_\alpha\cdot\alpha(u_i),\alpha^2(u_{i+1}),\ldots,\alpha^2(u_{n-1})\big). \tag{3.8}$$

Identity (3.1) holds by using Lemma 3.1. □

Remark 3.3. We obtain a similar result if we consider the space $T\mathcal{N} = \otimes^n \mathcal{N}$ instead of $\mathcal{L}(\mathcal{N})$.

Remark 3.4. For $n = 2$ the map $L : \mathcal{L}(\mathcal{N}) \longrightarrow \mathrm{End}(\mathcal{N})$ defines a representation of $\mathcal{L}(\mathcal{N})$ on $\mathcal{N}$.
One should set $\nu = \alpha$ and check

$$L(\alpha(x))\cdot\alpha(u) = \alpha(L(x)\cdot u) \tag{3.9}$$
$$L([x,y]_\alpha)\cdot\alpha(u) = L(\alpha(x))(y)\cdot u - L(\alpha(y))(x)\cdot u. \tag{3.10}$$

Indeed (3.9) and (3.10) are equivalent to

$$[\alpha(x),\alpha(y)] = \alpha([x,y]), \tag{3.11}$$
$$[[x,y],\alpha(u)] = [[\alpha(x),y],u] - [[\alpha(y),x],u]. \tag{3.12}$$

According to [34,35] it corresponds to the adjoint representation of a Hom–Lie algebra.

4. Central extensions and scalar cohomology of *n*-ary Hom–Nambu–Lie algebras

4.1. Central extensions of n-ary multiplicative Hom–Nambu–Lie algebras

Let $(\mathcal{N},[\cdot,\ldots,\cdot],\alpha)$ be an n-ary multiplicative Hom–Nambu–Lie algebra.

Definition 4.1. We define a central extension $\tilde{\mathcal{N}}$ of $\mathcal{N}$ by adding a new generator e which is central and by modifying the bracket as follows: for all $\tilde{x}_i = x_i + a_i e$, $a_i \in \mathbb{K}$ and $1 \le i \le n$ we have

$$[\tilde{x}_1,\ldots,\tilde{x}_n]_{\tilde{\mathcal{N}}} = [x_1,\ldots,x_n] + \varphi(x_1,\ldots,x_n)e, \tag{4.1}$$
$$\beta(\tilde{x}_i) = \alpha(x_i) + \lambda(x_i)e, \tag{4.2}$$
$$[\tilde{x}_1,\ldots,\tilde{x}_{n-1},e]_{\tilde{\mathcal{N}}} = 0, \tag{4.3}$$

where $\lambda : \mathcal{N} \to \mathbb{K}$ is a linear map.

We make the following observations:

- Clearly, φ has to be an n-linear and skew-symmetric map, $\varphi \in \wedge^{n-1}\mathcal{N}^* \wedge \mathcal{N}^*$, where $\mathcal{N}^*$ is the dual of $\mathcal{N}$. It will be identified with a 1-cochain.
- The new bracket for the $\tilde{x}_i \in \tilde{\mathcal{N}}$ has to satisfy the Hom–Nambu identity. This leads to a condition on φ when one of the vector involved is e.
- Since e is central then the Hom–Nambu identity has no restriction on λ. For $\tilde{x}_i = x_i + a_i e \in \tilde{\mathcal{N}}$, $\tilde{y}_i = y_i + b_i e \in \tilde{\mathcal{N}}$, $1 \le i \le n$, we have

$$\big[\beta(\tilde{x}_1),\ldots,\beta(\tilde{x}_{n-1}),[\tilde{y}_1,\ldots,\tilde{y}_n]_{\tilde{\mathcal{N}}}\big]_{\tilde{\mathcal{N}}} = \sum_{i=1}^{n-1}\big[\beta(\tilde{y}_1),\ldots,\beta(\tilde{y}_{i-1}),[\tilde{x}_1,\ldots,\tilde{x}_{n-1},y_i]_{\tilde{\mathcal{N}}}.\beta(\tilde{y}_{i+1}),\ldots,\beta(\tilde{y}_n)\big]_{\tilde{\mathcal{N}}}.$$

  Using (4.1) and the Hom–Nambu identity for the original Hom–Nambu–Lie algebra, one gets

$$\varphi\big(\alpha(x_1),\ldots,\alpha(x_{n-1}),[y_1,\ldots,y_n]\big) - \sum_{i=1}^{n-1}\varphi\big(\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,\ldots,x_{n-1},y_i],\alpha(y_{i+1}),\ldots,\alpha(y_n)\big) = 0. \tag{4.4}$$

- The previous equation may be written as

$$\delta^1\varphi(x,y,z) = 0$$

where $x = x_1 \otimes \cdots \otimes x_{n-1} \in \mathcal{N}^{\otimes n-1}$, $y = y_1 \otimes \cdots \otimes y_{n-1} \in \mathcal{N}^{\otimes n-1}$, $z = y_n \in \mathcal{N}$.
We provide below the condition that characterizes $\varphi \in \wedge^{n-1}\mathcal{N}^* \wedge \mathcal{N}^*$, $\varphi : x \wedge z \to \varphi(x,z)$ as a 1-cocycle. It is seen now why it becomes natural to call φ a 1-cochain (rather than a 2-cochain, as is stated in the Hom–Lie cohomology case in [36]).
The number of elements of $\mathcal{L}(\mathcal{N})$ in the argument of a cochain determines its order. As we shall see shortly, an arbitrary p-cochain takes $p(n-1)+1$ arguments in $\mathcal{N}$. A 0-cochain is an element of $\mathcal{N}^*$.

4.2. Scalar cohomology of multiplicative Hom–Nambu–Lie algebras

Let us now construct the cohomology complex relevant for central extensions of multiplicative Hom–Nambu–Lie algebras. Since $\mathcal{N}$ does not act on $\varphi(x,z)$, it will be the cohomology of multiplicative Hom–Nambu–Lie algebras for the trivial action.

Definition 4.2. We define an arbitrary p-cochain as an element $\varphi \in \wedge^{n-1}\mathcal{N}^* \otimes \cdots \otimes \wedge^{n-1}\mathcal{N}^* \wedge \mathcal{N}^*$,

$$\varphi : \mathcal{L}(\mathcal{N}) \otimes \cdots \otimes \mathcal{L}(\mathcal{N}) \wedge \mathcal{N} \longrightarrow \mathbb{K}$$
$$(x_1, \ldots, x_p, z) \longmapsto \varphi(x_1, \ldots, x_p, z).$$

We denote the set of p-cochains with values in $\mathbb{K}$ by $C^p(\mathcal{N}, \mathbb{K})$.

Condition (4.4) guarantees the consistency of φ according to (4.1) with the Hom–Nambu identity (1.1). Then

$$\delta^1\varphi(x,y,z) = \varphi\big(\alpha(x), L(y)\cdot z\big) - \varphi\big(\alpha(y), L(x)\cdot z\big) - \varphi\big([x,y]_\alpha, \alpha(z)\big) = 0, \tag{4.5}$$

where $L(x)\cdot z$ and $[x,y]_\alpha$ are defined in (3.2) and (3.3). It is now straightforward to extend (4.5) to a whole cohomology complex; $\delta^p\varphi$ will be a $(p+1)$-cochain taking one more argument of $\mathcal{L}(\mathcal{N})$ than φ. This is done by means of the following

Definition 4.3. Let $\varphi \in C^p(\mathcal{N}, \mathbb{K})$ be a p-cochain on a multiplicative n-ary Hom–Nambu–Lie algebra $\mathcal{N}$. A coboundary operator δ^p on an arbitrary p-cochain is given by

$$\begin{aligned}\delta^p\varphi(x_1,\ldots,x_{p+1},z) = &\sum_{1\le i<j\le p+1}(-1)^i\varphi\big(\alpha(x_1),\ldots,\hat{x}_i,\ldots,[x_i,x_j]_\alpha,\ldots,\alpha(x_{p+1}),\alpha(z)\big)\\ &+\sum_{i=1}^{p+1}(-1)^i\varphi\big(\alpha(x_1),\ldots,\hat{x}_i,\ldots,\alpha(x_{p+1}),L(x_i)\cdot z\big)\end{aligned} \tag{4.6}$$

where $x_1,\ldots,x_{p+1} \in \mathcal{L}(\mathcal{N})$, $z \in \mathcal{N}$ and $\hat{x}_i$ means that x_i is omitted.

Proposition 4.4. *Let $\varphi \in C^p(\mathcal{N}, \mathbb{K})$ be a p-cochain, then*

$$\delta^{p+1}\circ\delta^p(\varphi) = 0.$$

Proof. Let φ be a p-cochain, $(x_i)_{1\le i\le p} \in \mathcal{L}(\mathcal{N})$ and $z \in \mathcal{N}$, we can write δ^p and $\delta^{p+1}\circ\delta^p$ as

$$\delta^p = \delta^p_1 + \delta^p_2 \quad \text{and} \quad \delta^{p+1}\circ\delta^p = \eta_{11}+\eta_{12}+\eta_{21}+\eta_{22}$$

where $\eta_{ij} = \delta^{p+1}_i \circ \delta^p_j$, $1 \le i,j \le 2$, and

$$\delta^p_1\varphi(x_1,\ldots,x_{p+1},z) = \sum_{1\le i<j}^{p+1}(-1)^i\varphi\big(\alpha(x_1),\ldots,\hat{x}_i,\ldots,[x_i,x_j]_\alpha,\ldots,\alpha(x_{p+1}),\alpha(z)\big)$$
$$\delta^p_2\varphi(x_1,\ldots,x_{p+1},z) = \sum_{i=1}^{p+1}(-1)^i\varphi\big(\alpha(x_1),\ldots,\hat{x}_i,\ldots,\alpha(x_{p+1}),L(x_i)\cdot z\big).$$

Let us compute first $\eta_{11}\varphi(x_1,\ldots,x_{p+1},z)$. This is given by

$$\begin{aligned}&\eta_{11}(\varphi)(x_1,\ldots,x_{p+1},z)\\ &\quad=\sum_{1\le i<k<j}^{p+1}(-1)^{i+k}\varphi\big(\alpha^2(x_1),\ldots,\widehat{x_i},\ldots,\widehat{\alpha(x_k)},\ldots,[\alpha(x_k),[x_i,x_j]_\alpha]_\alpha,\ldots,\alpha^2(x_{p+1}),\alpha^2(z)\big)\\ &\qquad+\sum_{1\le i<k<j}^{p+1}(-1)^{i+k-1}\varphi\big(\alpha^2(x_1),\ldots,\widehat{\alpha(x_i)},\ldots,\widehat{x_k},\ldots,[\alpha(x_i),[x_k,x_j]_\alpha]_\alpha,\ldots,\alpha^2(x_{p+1}),\alpha^2(z)\big)\\ &\qquad+\sum_{1\le i<k<j}^{p+1}(-1)^{i+k-1}\varphi\big(\alpha^2(x_1),\ldots,\widehat{x_i},\ldots,\widehat{[x_i,x_k]_\alpha},\ldots,[[x_i,x_k]_\alpha,\alpha(x_j)]_\alpha,\ldots,\alpha^2(x_{p+1}),\alpha^2(z)\big).\end{aligned}$$

Whence applying the Hom–Leibniz identity (3.5) to $x_i, x_j, x_k \in \mathcal{L}(\mathcal{N})$, we find $\eta_{11} = 0$.

On the other hand we have

$$\eta_{21}(\varphi)(x_1, \ldots, x_{p+1}, z) + \eta_{12}(\varphi)(x_1, \ldots, x_{p+1}, z)$$
$$= \sum_{1\le i<j}^{p+1} (-1)^{i-1} \varphi\big(\alpha^2(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{[x_i, x_j]_\alpha}, \ldots, \alpha^2(x_{p+1}), L([x_i, x_j]_\alpha) \cdot \alpha(z)\big)$$

and

$$\eta_{22}(\varphi)(x_1, \ldots, x_{p+1}, z) = \sum_{1\le i<j}^{p+1} (-1)^{i} \varphi\big(\alpha^2(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{\alpha(x_j)}, \ldots, \alpha^2(x_{p+1}), \big(L(\alpha(x_i)) \cdot (L(x_j) \cdot z)\big)\big)$$
$$+ \sum_{1\le i<j}^{p+1} (-1)^{i-1} \varphi\big(\alpha^2(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \widehat{x_j}, \ldots, \alpha^2(x_{p+1}), \big(L(\alpha(x_j)) \cdot (L(x_i) \cdot z)\big)\big).$$

Then applying Lemma 3.1 to $x_i, x_j \in \mathcal{L}(\mathcal{N})$ and $z \in \mathcal{N}$, $\eta_{12} + \eta_{21} + \eta_{22} = 0$.

Which ends the proof. □

Definition 4.5. The space of p-cocycles is defined by

$$Z^p(\mathcal{N}, \mathbb{K}) = \{\varphi \in C^p(\mathcal{N}, \mathbb{K}) : \delta^p\varphi = 0\},$$

and the space of p-coboundaries is defined by

$$B^p(\mathcal{N}, \mathbb{K}) = \{\psi = \delta^{p-1}\varphi : \varphi \in C^{p-1}(\mathcal{N}, \mathbb{K})\}.$$

Lemma 4.6. $B^p(\mathcal{N}, \mathbb{K}) \subset Z^p(\mathcal{N}, \mathbb{K})$.

Definition 4.7. We call pth cohomology group the quotient

$$H^p(\mathcal{N}, \mathbb{K}) = \frac{Z^p(\mathcal{N}, \mathbb{K})}{B^p(\mathcal{N}, \mathbb{K})}.$$

Example 4.8. We consider the simple n-ary Nambu–Lie algebras $(\mathcal{N}, [\cdot, \ldots, \cdot])$ defined with respect to a basis $\{e_i\}_{i=1,\ldots,n+1}$ by

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{i+1}\varepsilon_i e_i \quad \text{or} \quad [e_{i_1}, \ldots, e_{i_n}] = (-1)^n \sum_{i=1}^{n+1} \varepsilon_i \epsilon^{i}_{i_1,\ldots,i_n} e_i \tag{4.7}$$

where $\varepsilon_i = \pm 1$ (no sum over the i of the ε_i factors) just introduce signs that affect the different terms of the sum in i and using Filippov's notation; see [13,9,37].

Let $\alpha : \mathcal{N} \to \mathcal{N}$ be a morphism of n-ary Nambu–Lie algebras. Then Theorem 1.5 leads to $\mathcal{N}_\alpha = (\mathcal{N}, [\cdot, \ldots, \cdot]_\alpha, \tilde{\alpha} = (\alpha, \ldots, \alpha))$ is a Hom–Nambu–Lie algebra where the bracket $[\cdot, \ldots, \cdot]_\alpha$ is given by

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_\alpha = (-1)^{i+1}\varepsilon_i \alpha(e_i) \quad \text{or} \quad [e_{i_1}, \ldots, e_{i_n}]_\alpha = (-1)^n \sum_{i=1}^{n+1} \varepsilon_i \epsilon^{i}_{i_1,\ldots,i_n} \alpha(e_i). \tag{4.8}$$

Following [13] and by slightly changing the proof, we obtain:

Proposition 4.9. *Any 1-cochain of the $(n+1)$-dimensional n-ary Hom–Nambu–Lie algebra $\mathcal{N}_\alpha$ defined by (4.8) is a 1-coboundary (and thus a trivial 1-cocycle).*

Proof. Let $\varphi \in C^1(\mathcal{N}, \mathbb{K})$ be a 1-cochain on $\mathcal{N}_\alpha$, φ is determined by its coordinates $\varphi_{i_1,\ldots,i_n} = \varphi(e_{i_1}, \ldots, e_{i_n})$. We now show that, in fact a 1-cochain on $\mathcal{N}_\alpha$ is a 1-coboundary, that is there exists a 0-cochain ϕ such that

$$\varphi_{i_1,\ldots,i_n} = -\phi([e_{i_1}, \ldots, e_{i_n}]_\alpha) = -\sum_{k=1}^{n+1} \varepsilon_k \epsilon^{k}_{i_1,\ldots,i_n} \phi_k, \tag{4.9}$$

where $\phi_k = \phi \circ \alpha(e_k)$.

Given φ then we may consider the 0-cochain ϕ given by

$$\phi_k = -\frac{\varepsilon_k}{n!} \sum_{i_1,\ldots,i_n}^{n+1} \epsilon_k^{i_1,\ldots,i_n} \varphi_{i_1,\ldots,i_n}. \tag{4.10}$$

Indeed, property (4.9) is satisfied:

$$
\begin{aligned}
-\phi([e_{i_1},\ldots,e_{i_n}]_\alpha) &= -\sum_{k=1}^{n+1}\varepsilon_k\epsilon^k_{i_1,\ldots,i_n}\phi_k\\
&= \sum_{k=1}^{n+1}\epsilon^k_{i_1,\ldots,i_n}\frac{\varepsilon_k^2}{n!}\sum_{j_1,\ldots,j_n}^{n+1}\epsilon_k^{j_1,\ldots,j_n}\varphi_{j_1,\ldots,j_n}\\
&= \frac{1}{n!}\sum_{j_1,\ldots,j_n}^{n+1}\epsilon^{j_1,\ldots,j_n}_{i_1,\ldots,i_n}\varphi_{j_1,\ldots,j_n} = \varphi_{i_1,\ldots,i_n}. \quad \square
\end{aligned}
\tag{4.11}
$$

5. Deformation of *n*-ary Hom–Nambu–Lie algebras

Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $\mathcal{N}[[t]]$ be the set of formal series whose coefficients are elements of the vector space $\mathcal{N}$, ($\mathcal{N}[[t]]$ is obtained by extending the coefficient domain of $\mathcal{N}$ from $\mathbb{K}$ to $\mathbb{K}[[t]]$). Given a $\mathbb{K}-n$-linear map $\varphi : \mathcal{N}\times\cdots\times\mathcal{N}\to\mathcal{N}$, it admits naturally an extension to a $\mathbb{K}[[t]]-n$-linear map $\varphi : \mathcal{N}[[t]]\times\cdots\times\mathcal{N}[[t]]\to\mathcal{N}[[t]]$, that is, if $x_i=\sum_{j\geq 0}a_i^jt^j$, $1\leq i\leq n$ then $\varphi(x_1,\ldots,x_n)=\sum_{j_1,\ldots,j_n\geq 0}t^{j_1+\cdots+j_n}\varphi(a_1^{j_1},\ldots,a_n^{j_n})$. The same holds for linear maps.

Definition 5.1. Let $(\mathcal{N},[\cdot,\ldots,\cdot],\widetilde{\alpha}),\widetilde{\alpha}=(\alpha_i)_{1\leq i\leq n-1}$ be an n-ary Hom–Nambu–Lie algebra. A formal deformation of the n-ary Hom–Nambu–Lie algebra $\mathcal{N}$ is given by a $\mathbb{K}[[t]]-n$-linear map

$$[\cdot,\ldots,\cdot]_t : \mathcal{N}[[t]]\times\cdots\times\mathcal{N}[[t]]\to\mathcal{N}[[t]]$$

of the form $[\cdot,\ldots,\cdot]_t=\sum_{i\geq 0}t^i[\cdot,\ldots,\cdot]_i$ where each $[\cdot,\ldots,\cdot]_i$ is a $\mathbb{K}[[t]]-n$-linear map $[\cdot,\ldots,\cdot]_i:\mathcal{N}\times\cdots\times\mathcal{N}\to\mathcal{N}$ (extended to a $\mathbb{K}[[t]]-n$-linear map), and $[\cdot,\ldots,\cdot]_0=[\cdot,\ldots,\cdot]$ such that for $(x_i)_{1\leq i\leq n-1},\ (y_i)_{1\leq i\leq n}\in\mathcal{N}$

$$
\begin{aligned}
&\big[\alpha_1(x_1),\ldots,\alpha_{n-1}(x_{n-1}),[y_1,\ldots,y_n]_t\big]_t\\
&\quad=\sum_{i=1}^{n-1}\big[\alpha_1(y_1),\ldots,\alpha_{i-1}(y_{i-1}),[x_1,\ldots,x_{n-1},y_i]_t,\alpha_i(y_{i+1}),\ldots,\alpha_{n-1}(y_n)\big]_t.
\end{aligned}
\tag{5.1}
$$

The deformation is said to be of order k if $[\cdot,\ldots,\cdot]_t=\sum_{i=0}^k t^i[\cdot,\ldots,\cdot]_i$ and infinitesimal if $t^2=0$.

The above condition may be written for $x=(x_i)_{1\leq i\leq n-1},\ y=(y_i)_{1\leq i\leq n-1}\in\mathcal{L}(\mathcal{N})$ and by setting $z=y_n$

$$L_t([x,y]_\alpha)\cdot\alpha_n(z)=L_t(\tilde{\alpha}(x))\cdot\big(L_t(y)\cdot z\big)-L_t(\tilde{\alpha}(y))\cdot\big(L_t(x).z\big) \tag{5.2}$$

where $L_t(x)\cdot z=[x_1,\ldots,x_{n-1},z]_t$ and $\tilde{\alpha}(x)=(\alpha_i(x_i))_{1\leq i\leq n-1}$.

Now let $(\mathcal{N},[\cdot,\ldots,\cdot],\alpha)$ be an n-ary multiplicative Hom–Nambu–Lie algebra ($\alpha_1=\cdots=\alpha_n=\alpha$).

Eq. (5.2) is equivalent, if the deformation is infinitesimal and by setting $\psi=[\cdot,\ldots,\cdot]$, to

$$
\begin{aligned}
&\big[\alpha(x_1),\ldots,\alpha(x_{n-1}),\psi(y_1,\ldots,y_n)\big]+\psi\big(\alpha(x_1),\ldots,\alpha(x_{n-1}),[y_1,\ldots,y_n]\big)\\
&\quad=\sum_{i=1}^{n}\big[\alpha(y_1),\ldots,\alpha(y_{i-1}),\psi(x_1,\ldots,x_{n-1},y_i),\alpha(y_{i+1}),\ldots,\alpha(y_n)\big]\\
&\qquad+\sum_{i=1}^{n}\psi\big(\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,\ldots,x_{n-1},y_i],\alpha(y_{i+1}),\ldots,\alpha(y_n)\big).
\end{aligned}
$$

This identity may be viewed as the 1-cocycle condition $\delta^1\psi=0$ for the $\mathcal{N}$-valued cochain ψ. In terms of $x,\ y\in\mathcal{L}(\mathcal{N})$ it may be written, (setting again $y_n=z$), as

$$
\begin{aligned}
\delta^1\psi(x,y,z) &= \psi(\alpha(x),L(y)\cdot z)-\psi(\alpha(y),L(x)\cdot z)-\psi([x_1,x_2]_\alpha,\alpha(z))\\
&\quad+L(\alpha(x))\cdot\psi(y,z)-L(\alpha(y))\cdot\psi(x,z)+\big(\psi(x,\,)\cdot y\big)\bullet_\alpha\alpha(z)
\end{aligned}
\tag{5.3}
$$

where

$$\big(\psi(x,\,)\cdot y\big)\bullet_\alpha\alpha(z)=\sum_{i=0}^{n-1}[\alpha(y_1),\ldots,\psi(x,y_i),\ldots,\alpha(y_{n-1}),\alpha(z)]. \tag{5.4}$$

Definition 5.2. A p-cochain is a $(p+1)$-linear map $\varphi : \mathcal{L}(\mathcal{N}) \otimes \cdots \otimes \mathcal{L}(\mathcal{N}) \wedge \mathcal{N} \longrightarrow \mathcal{N}$, such that

$$\alpha \circ \varphi(x_1, \ldots, x_p, z) = \varphi(\alpha(x_1), \ldots, \alpha(x_p), \alpha(z)).$$

We denote the set of p-cochains by $C^p(\mathcal{N}, \mathcal{N})$.

Definition 5.3. We call, for $p \geq 1$, p-coboundary operator of the multiplicative n-ary Hom–Nambu–Lie algebra $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha)$ the linear map $\delta^p : C^p(\mathcal{N}, \mathcal{N}) \to C^{p+1}(\mathcal{N}, \mathcal{N})$ defined by

$$\begin{aligned}\delta^p \psi(x_1, \ldots, x_p, x_{p+1}, z) = {} & \sum_{1\leq i\leq j}^{p+1} (-1)^i \psi\big(\alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \alpha(x_{j-1}), [x_i, x_j]_\alpha, \ldots, \alpha(x_{p+1}), \alpha(z)\big) \\ & + \sum_{i=1}^{p+1} (-1)^i \psi\big(\alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \alpha(x_{p+1}), L(x_i) \cdot z\big) \\ & + \sum_{i=1}^{p+1} (-1)^{i+1} L(\alpha^p(x_i)) \cdot \psi\big(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}, z\big) \\ & + (-1)^p \big(\psi(x_1, \ldots, x_p, \ ) \cdot x_{p+1}\big) \bullet_\alpha \alpha^p(z)\end{aligned} \tag{5.5}$$

where

$$\big(\psi(x_1, \ldots, x_p, \,) \cdot x_{p+1}\big) \bullet_\alpha \alpha^p(z) = \sum_{i=1}^{n-1} [\alpha^p(x_{p+1}^1), \ldots, \psi(x_1, \ldots, x_p, x_{p+1}^i), \ldots, \alpha^p(x_{p+1}^{n-1}), \alpha^p(z)], \tag{5.6}$$

for $x_i = (x_i^j)_{1\leq j\leq n-1} \in \mathcal{L}(\mathcal{N}),\ 1 \leq i \leq p+1, z \in \mathcal{N}$.

Proposition 5.4. *Let $\psi \in C^p(\mathcal{N}, \mathcal{N})$ be a p-cochain then*

$$\delta^{p+1} \circ \delta^p(\psi) = 0.$$

Proof. Let ψ be a p-cochain, $x_i = (x_i^j)_{1\leq j\leq n-1} \in \mathcal{L}(\mathcal{N}),\ 1 \leq i \leq p+2$ and $z \in \mathcal{N}$. We set

$$\delta^p = \delta_1^p + \delta_2^p + \delta_3^p + \delta_4^p, \quad \text{and} \quad \delta^{p+1} \circ \delta^p = \sum_{i,j=1}^{4} \eta_{ij},$$

when $\eta_{ij} = \delta_i^{p+1} \circ \delta_j^p$ and

$$\begin{aligned}\delta_1^p \psi(x_1, \ldots, x_{p+1}, z) &= \sum_{1\leq i<j}^{p+1} (-1)^i \psi\big(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, [x_i, x_j]_\alpha, \ldots, \alpha(x_{p+1}), \alpha(z)\big) \\ \delta_2^p \psi(x_1, \ldots, x_{p+1}, z) &= \sum_{i=1}^{p+1} (-1)^i \psi\big(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, \alpha(x_{p+1}), L(x_i).z\big) \\ \delta_3^p \psi(x_1, \ldots, x_{p+1}, z) &= \sum_{i=1}^{p+1} (-1)^{i+1} L(\alpha^p(x_i)) \cdot \psi\big(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}, z\big) \\ \delta_4^p \psi(x_1, \ldots, x_{p+1}, z) &= (-1)^p \big(\psi(x_1, \ldots, x_p, \ ) \cdot x_{p+1}\big) \bullet_\alpha \alpha^p(z).\end{aligned}$$

To simplify the notations we replace $L(x) \cdot z$ by $x \cdot z$.

The proof that $\eta_{11} + \eta_{12} + \eta_{21} + \eta_{22} = 0$ is similar to the proof of Proposition 4.4.

On the other hand, we have

$$\begin{aligned}\eta_{13}\psi(x_1, \ldots, x_{p+2}, z) = {} & \sum_{1\leq i<j<k}^{p+2} \Big\{(-1)^{k+i} \alpha^{p+1}(x_k) \cdot \psi(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, [x_i, x_j]_\alpha, \ldots, \widehat{x_k}, \ldots, \alpha(z)) \\ & + (-1)^{j+i} \alpha^{p+1}(x_j) \cdot \psi(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, [x_i, x_k]_\alpha, \ldots, \alpha(z)) \\ & + (-1)^{j+i-1} \alpha^{p+1}(x_i) \cdot \psi(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, [x_j, x_k]_\alpha, \ldots, \alpha(z))\Big\}\end{aligned}$$

$$\eta_{31}\psi(x_1, \ldots, x_{p+2}, z) = -\eta_{13}\psi(x_1, \ldots, x_{p+2}, z) + \sum_{1\leq i<j}^{p+2} (-1)^{i+j} \alpha^p([x_i, x_j]_\alpha) \cdot \alpha\big(\psi(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, z)\big)$$

$$\begin{aligned}\eta_{33}\psi(x_1, \ldots, x_{p+2}, z) = {} & \sum_{1\leq i<j}^{p+2} \Big\{(-1)^{i+j} \alpha^{p+1}(x_i) \cdot \big(\alpha^p(x_j).\big(\psi(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, z)\big)\big) \\ & + (-1)^{i+j-1} \alpha^{p+1}(x_j) \cdot \big(\alpha^p(x_i) \cdot \big(\psi(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, z)\big)\big)\Big\}.\end{aligned}$$

Then, applying Lemma 3.1 to $\alpha^p(x_i)\in\mathcal{L}(\mathcal{N})$, $\alpha^p(x_j)\in\mathcal{L}(\mathcal{N})$ and $\psi(x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,z)\in\mathcal{N}$, we have

$$\eta_{13}+\eta_{33}+\eta_{31}=0.$$

By the same calculation, we can prove that

$$\eta_{23}+\eta_{32}=0.$$

We have also

$$\begin{aligned}
&\eta_{14}\psi(x_1,\ldots,x_{p+2},z)\\
&=(-1)^p\sum_{1\le i<j}^{p+1}(-1)^i\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,[x_i,x_j]_\alpha,\ldots,\alpha(x_p),\alpha(x^k_{p+1})\big),\ldots,\alpha^{p+1}(z)\big]\\
&\quad+(-1)^p\sum_{i=1}^{p+1}(-1)^i\sum_{k,l=1;k\neq l}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\alpha^p(x_i\cdot x^l_{p+2}),\ldots,\psi\\
&\quad\times\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),\ldots,\alpha^{p+1}(z)\big]\\
&\quad+(-1)^p\sum_{i=1}^{p+1}(-1)^i\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),x_i\cdot x^k_{p+2}\big),\ldots,\alpha^{p+1}(z)\big].
\end{aligned}$$

The first term in η_{14} is equal to $-\eta_{41}$. Hence

$$\begin{aligned}
&(\eta_{14}+\eta_{41})\psi(x_1,\ldots,x_{p+2},z)\\
&=\sum_{i=1}^{p+1}(-1)^{p+i}\sum_{k,l=1;k\neq l}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\alpha^p(x_i\cdot x^l_{p+2}),\ldots,\psi\\
&\quad\times\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),\ldots,\alpha^{p+1}(z)\big]\\
&\quad+\sum_{i=1}^{p+1}(-1)^{p+i}\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),x_i\cdot x^k_{p+2}\big),\ldots,\alpha^{p+1}(z)\big],\\
&\eta_{24}\psi(x_1,\ldots,x_{p+2},z)\\
&=\sum_{i=1}^{p+1}\sum_{k=1}^{n-1}(-1)^{p+i}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),\ldots,\alpha^p(x_i\cdot z)\big]\\
&\quad+\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+1}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_p),\alpha(x^k_{p+1})\big),\ldots,\alpha^p(x_{p+2}\cdot z)\big],\\
&\eta_{42}\psi(x_1,\ldots,x_{p+2},z)\\
&=(-1)^{p+1}\sum_{i=1}^{p+1}(-1)^i\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),x_i\cdot x^k_{p+2}\big),\ldots,\alpha^{p+1}(z)\big].
\end{aligned}$$

Hence, $-\eta_{42}$ and the second term of $(\eta_{14}+\eta_{41})$ are equal.

Using the Hom–Nambu identity for any integers $1\le i\le p+1$ and $1\le k\le n-1$

$$\begin{aligned}
&\alpha^{p+1}(x_i)\cdot\big[\alpha^p(x^1_{p+2}),\ldots,\psi\big(x_1,\ldots,\widehat{x_i},\ldots,x_{p+1},x^k_{p+2}\big),\ldots,\alpha^p(z)\big]\\
&=\sum_{l=1;l\neq k}^{n-1}\Big\{\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\alpha^p(x_i\cdot x^l_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),\ldots,\alpha^{p+1}(z)\big]\Big\}\\
&\quad+\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),\ldots,\alpha^p(x_i\cdot z)\big]\\
&\quad+\big[\alpha^{p+1}(x^1_{p+1}),\ldots,\alpha^p(x_i)\cdot\psi\big(x_1,\ldots,\widehat{x_i},\ldots,x_{p+1},x^k_{p+2}\big),\ldots,\alpha^{p+1}(z)\big].
\end{aligned}$$

When we gather the four terms η_{14}, η_{41}, η_{24} and η_{42}, we obtain

$$\begin{aligned}
&(\eta_{14}+\eta_{41}+\eta_{24}+\eta_{42})\psi(x_1,\ldots,x_{p+2},z)\\
&=\sum_{i=1}^{p+1}(-1)^{i+p}\sum_{l=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),\ldots,\alpha^p(x_i)\cdot\psi\big(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),\ldots,\alpha^{p+1}(z)\big]
\end{aligned}$$

$$+(-1)^{p-1}\sum_{i=1}^{p+1}(-1)^i\sum_{k=1}^{n-1}\alpha^{p+1}(x_i)\cdot\left[\alpha^p(x^1_{p+2}),\ldots,\psi\left(x_1,\ldots,\widehat{x_i},\ldots,x_{p+1},x^k_{p+2}\right),\ldots,\alpha^p(z)\right]$$
$$+\sum_{k=1}^{n-1}\left[\alpha^{p+1}(x^1_{p+1}),\ldots,\psi\left(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_p),\alpha(x^k_{p+1})\right),\ldots,\alpha^p(x_{p+2}\cdot z)\right]$$

and

$$\begin{aligned}&\eta_{43}\psi(x_1,\ldots,x_{p+2},z)\\&=\sum_{i=1}^{p+1}(-1)^{p+i}\sum_{k=1}^{n-1}\alpha^{p+1}(x_i)\cdot\left[\alpha^p(x^1_{p+2}),\ldots,\psi\left(x_1,\ldots,\widehat{x_i},\ldots,x_{p+1},x^k_{p+2}\right),\ldots,\alpha^p(z)\right]\\&\quad-\sum_{k=1}^{n-1}\alpha^{p+1}(x_{p+2})\cdot\left[\alpha^p(x^1_{p+1}),\ldots,\psi\left(x_1,\ldots,x_p,x^k_{p+1}\right),\ldots,\alpha^p(z)\right],\\&\eta_{34}\psi(x_1,\ldots,x_{p+2},z)\\&=\sum_{i=1}^{p+1}(-1)^{i+p+1}\sum_{l=1}^{n-1}\left[\alpha^{p+1}(x^1_{p+2}),\ldots,\alpha^p(x_i)\cdot\psi\left(\alpha(x_1),\ldots,\widehat{x_i},\ldots,\alpha(x_{p+1}),\alpha(x^k_{p+2})\right),\ldots,\alpha^{p+1}(z)\right].\end{aligned}$$

Hence

$$\begin{aligned}&(\eta_{14}+\eta_{41}+\eta_{24}+\eta_{42}+\eta_{34}+\eta_{43})\psi(x_1,\ldots,x_{p+2},z)=-\eta_{44}\psi(x_1,\ldots,x_{p+2},z)\\&=-\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\left[\alpha^{p+1}(x^1_{p+2}),\ldots,[\alpha^p(x^1_{p+1}),\ldots,\psi\left(x_1,\ldots,x_p,x^k_{p+1}\right),\ldots,\alpha^p(x^i_{p+2})],\ldots,\alpha^{p+1}(x^{n-1}_{p+2}),\alpha^{p+1}(z)\right]\\&=-\sum_{k=1}^{n-1}\alpha^{p+1}(x_{p+2})\cdot\left[\alpha^p(x^1_{p+1}),\ldots,\psi\left(x_1,\ldots,x_p,x^k_{p+1}\right),\ldots,\alpha^p(z)\right]\\&\quad+\sum_{k=1}^{n-1}\left[\alpha^{p+1}(x^1_{p+1}),\ldots,\psi\left(\alpha(x_1),\ldots,\alpha(x_p),\alpha(x^k_{p+1})\right),\ldots,\alpha^p(x_{p+2}\cdot z)\right].\end{aligned}$$

Then, we have

$$\eta_{14}+\eta_{41}+\eta_{24}+\eta_{42}+\eta_{34}+\eta_{43}+\eta_{44}=0,$$

which ends the proof. □

Definition 5.5. The space of p-cocycles is defined by

$$Z^p(\mathcal{N},\mathcal{N})=\{\varphi\in C^p(\mathcal{N},\mathcal{N}):\delta^p\varphi=0\},$$

and the space of p-coboundaries is defined by

$$B^p(\mathcal{N},\mathcal{N})=\{\psi=\delta^{p-1}\varphi:\varphi\in C^{p-1}(\mathcal{N},\mathcal{N})\}.$$

Lemma 5.6. $B^p(\mathcal{N},\mathcal{N})\subset Z^p(\mathcal{N},\mathcal{N})$.

Definition 5.7. We call the p^{th} cohomology group the quotient

$$H^p(\mathcal{N},\mathcal{N})=\frac{Z^p(\mathcal{N},\mathcal{N})}{B^p(\mathcal{N},\mathcal{N})}.$$

6. Cohomology of *n*-ary Hom algebras induced by cohomology of Hom–Leibniz algebras

In this section we extend to n-ary multiplicative Hom–Nambu–Lie algebras Takhtajan's construction of a cohomology of ternary Nambu–Lie algebras starting from Chevalley–Eilenberg cohomology of binary Lie algebras, (see [7,8,38]). The cohomology of multiplicative Hom–Lie algebras was introduced in [39] and independently in [34].

6.1. Cohomology complex of Hom–Leibniz algebras

The cohomology complex for Leibniz algebras was defined by Loday and Pirashvili in [40]. We extend it to Hom–Leibniz algebras as follows.

Let $(A, [\cdot,\cdot], \alpha)$ be a Hom–Leibniz algebra and $\mathcal{C}_{\mathcal{L}}(A, A)$ be the set of cochains $\mathcal{C}^p_{\mathcal{L}}(A, A) = \mathrm{Hom}(\otimes^p A, A)$ for $p \geq 1$ and $\mathcal{C}^0_{\mathcal{L}}(A, A) = A$. We define a coboundary operator d by $\mathrm{d}\varphi(a) = -[\varphi, a]$ when $\varphi \in \mathcal{C}^0_{\mathcal{L}}(A, A)$ and for $p \geq 1, \varphi \in \mathcal{C}^p_{\mathcal{L}}(A, A), a_1, \ldots, a_{p+1} \in A$

$$\begin{aligned}\mathrm{d}^p\varphi(a_1, \ldots, a_{p+1}) = &\sum_{k=1}^{p}(-1)^{k-1}\big[\alpha^{p-1}(a_k), \varphi(a_1, \ldots, \widehat{a_k}, \ldots, a_{p+1})\big]\\ &+(-1)^{p+1}\big[\varphi(a_1 \otimes \cdots \otimes a_p), \alpha^{p-1}(a_{p+1})\big]\\ &+\sum_{1\leq k<j}^{p+1}(-1)^k\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_{j-1})\otimes[a_k, a_j]\otimes\alpha(a_{j+1})\otimes\cdots\otimes\alpha(a_{p+1})).\end{aligned} \tag{6.1}$$

Notice that we recover the classical case when $\alpha = \mathrm{id}$. The proof that it defines a cohomology complex is similar to that defining a cohomology complex on Hom–Lie algebras; see [39].

6.2. Main theorem

Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-ary Hom–Nambu–Lie algebra and the triple $(\mathcal{L}(\mathcal{N}) = \mathcal{N}^{\otimes n-1}, [\cdot,\cdot]_\alpha, \alpha)$ be the Hom–Leibniz algebra associates to $\mathcal{N}$ where the bracket is defined in (3.3).

Theorem 6.1. *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-ary Hom–Nambu–Lie algebra and $\mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N}) = \mathrm{Hom}(\otimes^p \mathcal{L}(\mathcal{N}) \otimes \mathcal{N}, \mathcal{N})$ for $p \geq 1$ be the set of p-cochains. Let $\Delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{L}}(\mathcal{L}, \mathcal{L})$ be the linear map defined for $p = 0$ by*

$$\Delta\varphi(x_1 \otimes \cdots \otimes x_{n-1}) = \sum_{i=0}^{n-1} x_1 \otimes \cdots \otimes \varphi(x_i) \otimes \cdots \otimes x_{n-1} \tag{6.2}$$

and for $p > 0$ by

$$(\Delta\varphi)(a_1, \ldots, a_{p+1}) = \sum_{i=1}^{n-1} \alpha^{p-1}(x^1_{p+1}) \otimes \cdots \otimes \varphi(a_1, \ldots, a_{n-1} \otimes x^i_{p+1}) \otimes \cdots \otimes \alpha^{p-1}(x^{n-1}_{p+1}), \tag{6.3}$$

where we set $a_j = x^1_j \otimes \cdots \otimes x^{n-1}_j$.

Then there exists a cohomology complex $(\mathcal{C}^\bullet_{\mathcal{N}}(\mathcal{N}, \mathcal{N}), \delta)$ for n-ary Hom–Nambu–Lie algebras such that

$$d \circ \Delta = \Delta \circ \delta.$$

The coboundary map $\delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{N}}(\mathcal{N}, \mathcal{N})$ is defined for $\varphi \in \mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N})$ by

$$\begin{aligned}\delta\varphi(a_1, \ldots, a_p, a_{p+1}, x) = &\sum_{1\leq i\leq j}^{p+1}(-1)^i\varphi\big(\alpha(a_1), \ldots, \widehat{\alpha(a_i)}, \ldots, \alpha(a_{j-1}), [a_i, a_j]_\alpha, \ldots, \alpha(a_{p+1}), \alpha(x)\big)\\ &+\sum_{i=1}^{p+1}(-1)^i\varphi\big(\alpha(a_1), \ldots, \widehat{\alpha(a_i)}, \ldots, \alpha(a_{p+1}), L(a_i).x\big)\\ &+\sum_{i=1}^{p+1}(-1)^{i+1}L(\alpha^p(a_i))\cdot\varphi\big(a_1, \ldots, \widehat{a_i}, \ldots, a_{p+1}, x\big)\\ &+(-1)^p\big(\varphi(a_1, \ldots, a_p,)\cdot a_{p+1}\big)\bullet_\alpha\alpha^p(x),\end{aligned}$$

where

$$\big(\varphi(a_1, \ldots, a_p,)\cdot a_{p+1}\big)\bullet_\alpha\alpha^p(x) = \sum_{i=1}^{n-1}[\alpha^p(x^1_{p+1}), \ldots, \varphi(a_1, \ldots, a_p, x^i_{p+1}), \ldots, \alpha^p(x^{n-1}_{p+1}), \alpha^p(x)]$$

for $a_i \in \mathcal{L}(\mathcal{N}), x \in \mathcal{N}$.

Proof. Let $\varphi \in \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N})$ and $(a_1,\dots,a_{p+1}) \in \mathcal{L}$ where $a_j = x_1^j \otimes \cdots \otimes x_{n-1}^j$.

Then $\Delta\varphi \in \mathcal{C}^{p+1}_{\mathcal{L}}(\mathcal{L},\mathcal{L})$ and according to (6.1) we set $d = d_1 + d_2 + d_3$, where

$$d_1\varphi(a_1,\dots,a_{p+1}) = \sum_{k=1}^{p}(-1)^{k-1}\left[\alpha^{p-1}(a_k), \varphi(a_1,\dots,\widehat{a_k},\dots,a_{p+1})\right]$$

$$d_2\varphi(a_1,\dots,a_{p+1}) = (-1)^{p+1}\left[\varphi(a_1\otimes\cdots\otimes a_p), \alpha^{p-1}(a_{p+1})\right]$$

$$d_3\varphi(a_1,\dots,a_{p+1}) = \sum_{1\le k<j}^{p+1}(-1)^{k}\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_{j-1})\otimes[a_k,a_j]\otimes\alpha(a_{j+1})\otimes\cdots\otimes\alpha(a_{p+1})).$$

By (6.3) we have

$$\begin{aligned}
&d_1\circ\Delta\varphi(a_1,\dots,a_{p+1})\\
&= \sum_{k=1}^{p}(-1)^{k-1}\left[\alpha^{p-1}(a_k), \Delta\varphi(a_1,\dots,\widehat{a_k},\dots,a_{p+1})\right]\\
&= \sum_{k=1}^{p}(-1)^{k-1}\sum_{i=1}^{n-1}\left[\alpha^{p-1}(a_k), \alpha^{p-1}(x_{p+1}^1)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes\alpha^{p-1}(x_{p+1}^{n-1})\right]\\
&= \sum_{k=1}^{p}(-1)^{k-1}\sum_{i>j}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x_{p+1}^j)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1})\\
&\quad+ \sum_{k=1}^{p}(-1)^{k-1}\sum_{j>i}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x_{p+1}^j)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1})\\
&\quad+ \sum_{k=1}^{p}(-1)^{k-1}\sum_{i=1}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1})\\
&= \sum_{k=1}^{p}(-1)^{k-1}\sum_{i>j}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x_{p+1}^j)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1})\\
&\quad+ \sum_{k=1}^{p}(-1)^{k-1}\sum_{j>i}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\\
&\quad\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x_{p+1}^j)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1}) + \Delta\circ\delta_3\circ\varphi(a_1,\dots,a_{p+1})\\
&= \Lambda_1 + \Lambda_2 + \Delta\circ\delta_3\circ\varphi(a_1,\dots,a_{p+1})
\end{aligned}$$

where

$$\Lambda_1 = \sum_{k=1}^{p}(-1)^{k-1}\sum_{i>j}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x_{p+1}^j)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1})$$

$$\Lambda_2 = \sum_{k=1}^{p}(-1)^{k-1}\sum_{j>i}^{n-1}\alpha^p(x_{p+1}^1)\otimes\cdots\otimes\varphi(a_1,\dots,\widehat{a_k},\dots,x_{p+1}^i)\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x_{p+1}^j)\otimes\cdots\otimes\alpha^p(x_{p+1}^{n-1}).$$

Similarly we can prove that

$$d_2\circ\Delta\varphi(a_1,\dots,a_{p+1}) = \Delta\circ\delta_4\varphi(a_1,\dots,a_{p+1})$$

and

$$\begin{aligned}
d_3\circ\Delta\varphi(a_1,\dots,a_{p+1}) &= \sum_{1\le k<j}^{p}(-1)^k\Delta\circ\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_{j-1})\otimes[a_k,a_j]\otimes\alpha(a_{j+1})\otimes\cdots\otimes\alpha(a_{p+1}))\\
&\quad+ \sum_{k=1}^{p+1}(-1)^k\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_p)\otimes[a_k,a_{p+1}])\\
&= \Delta\circ\delta_1\varphi(a_1,\dots,a_{p+1}) + \Delta\circ\delta_2\varphi(a_1,\dots,a_{p+1}) + \Lambda'_1 + \Lambda'_2,
\end{aligned}$$

where $\Lambda'_1 = -\Lambda_1$ and $\Lambda'_2 = -\Lambda_2$.

Finally we have

$$d\circ\Delta = d_1\circ\Delta + d_2\circ\Delta + d_3\circ\Delta = \Delta\circ\delta_3 + \Delta\circ\delta_4 + \Delta\circ\delta_1 + \Delta\circ\delta_2 = \Delta\circ\delta$$

where $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$ as defined in proof 5.4. $\square$

Remark 6.2. If $d^2 = 0$, then $\delta^2 = 0$.

In fact, since $d \circ \Delta = \Delta \circ \delta$, then

$$\Delta \circ \delta^2 = \Delta \circ \delta \circ \delta = d \circ \Delta \circ \delta = d \circ d \circ \Delta = d^2 \circ \Delta = 0.$$

6.3. Ternary Hom–Nambu–Lie algebra case

We consider now the particular case of ternary Hom–Nambu algebras and show how to derive the cohomology of ternary Hom–Nambu algebras.

Let $(\mathcal{N}, [\cdot,\cdot,\cdot], \alpha)$ be a multiplicative ternary Hom–Nambu–Lie algebra. Using Proposition 3.2 the triple $(\mathcal{L}(\mathcal{N}) = \mathcal{N} \otimes \mathcal{N}, [\cdot,\cdot]_\alpha, \alpha)$ where the bracket defined for $x = x_1 \otimes x_2$ and $y = y_1 \otimes y_2$ by

$$[x, y] = [x_1, x_2, y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [x_1, x_2, y_2], \tag{6.4}$$

is a Hom–Leibniz algebra.

We obtain the following corollary of Theorem 6.1.

Proposition 6.3. *Let $(\mathcal{N}, [\cdot,\cdot,\cdot], \alpha)$ be a multiplicative ternary Hom–Nambu–Lie algebra and $\mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N}) = \mathrm{Hom}(\otimes^{2p+1}\mathcal{N}, \mathcal{N})$ for $p \geq 1$ be the p-cochains. Let $\Delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{L}}(\mathcal{L}, \mathcal{L})$ be the linear map defined for $p = 0$ by*

$$\Delta\varphi(x_1 \otimes x_2) = x_1 \otimes \varphi(x_2) + \varphi(x_1) \otimes x_2$$

and for $p > 0$ by

$$(\Delta\varphi)(a_1, \ldots, a_{p+1}) = \alpha^{p-1}(x_{2p+1}) \otimes \varphi(a_1, \ldots, a_p \otimes x_{2p+2}) + \varphi(a_1, \ldots, a_p \otimes x_{2p+1}) \otimes \alpha^{p-1}(x_{2p+2}),$$

where we set $a_j = x_{2j-1} \otimes x_{2j}$.

Then there exists a cohomology complex $(\mathcal{C}^\bullet_{\mathcal{N}}(\mathcal{N}, \mathcal{N}), \delta)$ for ternary Hom–Nambu–Lie algebras such that

$$d \circ \Delta = \Delta \circ \delta.$$

The coboundary map $\delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{N}}(\mathcal{N}, \mathcal{N})$ is defined for $\varphi \in \mathcal{C}^p_{\mathcal{N}}(\mathcal{N}, \mathcal{N})$ by

$$\begin{aligned}\delta\varphi(x_1 \otimes \cdots \otimes x_{2p+1}) = {} & \sum_{j=1}^{p} \sum_{k=2j+1}^{2p+1} (-1)^j \varphi(\alpha(x_1) \otimes \cdots \otimes [x_{2j-1}, x_{2j}, x_k] \otimes \cdots \otimes \alpha(x_{2p+1})) \\ & + \sum_{k=1}^{p} (-1)^{k-1} [\alpha^{p-1}(x_{2k-1}), \alpha^{p-1}(x_{2k}), \varphi(x_1 \otimes \cdots \otimes \widehat{x_{2k-1}} \otimes \widehat{x_{2k}} \otimes \cdots \otimes x_{2p+1})] \\ & + (-1)^{n+1} [\alpha^{p-1}(x_{2p-1}), \varphi(x_1 \otimes \cdots \otimes x_{2p-2} \otimes x_{2p}), \alpha^{p-1}(x_{2p+1})] \\ & + (-1)^{p+1} [\varphi(x_1 \otimes \cdots \otimes x_{2p-1}), \alpha^{p-1}(x_{2p}), \alpha^{p-1}(x_{2p+1})].\end{aligned} \tag{6.5}$$

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