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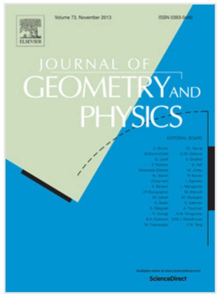

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Maximal surface equation on a Riemannian 2-manifold with finite total curvature

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Abstract

The differential equation of maximal surfaces on a complete Riemannian 2-manifold with finite total curvature is studied. Uniqueness theorems that widely extend the classical Calabi-Bernstein's theorem in non-parametric version, as well as previous results on complete maximal graphs into Lorentzian warped products, are given. All entire solutions of maximal equation in certain natural Lorentzian warped product, as well as non-existence results, are provided.

1 Introduction

The classical Calabi-Bernstein's theorem for maximal surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, in non-parametric version, states that the only entire solutions to the maximal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \quad |Du|<1 \tag{1}$$

on the Euclidean plane $\mathbb{R}^2$ are affine functions.

A maximal surface in $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. The term spacelike means that the induced metric from the ambient Lorentzian metric is a Riemannian metric on the surface. The terminology maximal comes from a variational problem, since theses surfaces locally maximize area among all nearby surfaces having the same boundary. Besides their mathematical interest, maximal surfaces and, more generally, spacelike surfaces with constant mean curvature are also important in General Relativity (see, for instance, [14]).

A singular fact in Lorentzian products (or warped products) in contrast to the case of graph into complete Riemannian products (or warped products) is that an entire spacelike graph

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in the Lorentzian case is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete (see [1, Section 4]).

The theorem aforementioned is a relevant uniqueness result, which was first proved by Calabi [6] and later extended for maximal hypersurfaces in $\mathbb{L}^n$ by Cheng and Yau [7]. It can also be stated in terms of the local complex representation of the surface [11], [8]. Even two different types of direct simple proofs are given in [17] and [18].

In [12], the authors give new examples of non-parametric Calabi-Bernstein type problems for warped Lorentzian products, whose warping function is non-locally constant and its fiber is the Euclidean plane. Obviously the Calabi-Bernstein theorem is not included in this case. A new version of non-parametric Calabi-Bernstein type theorem in the case of a Lorentzian product $\mathbb{R} \times F$, where F denotes a Riemannian 2-manifold, with non-negative curvature and positive at some point, has been given in [1] and [2]. Recently, another Calabi-Bernstein type results in the more general ambient of a warped Lorentzian product are given in [4] and [5].

In this work we deal with entire solutions of the maximal surface equation on a complete 2-dimensional Riemannian manifold with finite total curvature. Recall that a complete Riemannian surface has finite total curvature if the integral of the absolute value of its Gaussian curvature is finite (Section 2.4). In fact, consider the following nonlinear elliptic differential equation, in divergence form:

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}\left(2 + \frac{\mid Du \mid^2}{f(u)^2}\right) \tag{E.1}$$

$$\mid Du \mid < f(u) \tag{E.2}$$

where f is a smooth real-valued function defined on an open interval I of the real line $\mathbb{R}$, the unknown u is a function defined on a domain Ω of a non-compact complete Riemannian surface (F, g) with finite total curvature, $u(\Omega) \subseteq I$, D and div denote the gradient and the divergence of (F, g) and $\mid Du \mid^2 := g(Du, Du)$. The constraint (E.2) is the ellipticity condition. We are mainly interested in uniqueness and non-existence results for entire solutions (i.e. defined on all F) of equation (E).

The solutions of (E) are the extremals under interior variations for the functional

$$u \longmapsto \int f(u)\sqrt{f(u)^2 - \mid Du \mid^2}\, dA,$$

where dA is the area element for the Riemannian metric g, which acts on functions u such that $u(\Omega) \subseteq I$ and $\mid Du \mid < f(u)$.

This variational problem naturally arise from Lorentzian geometry. In order to see this, consider the product manifold $M := I \times F$ endowed with the Lorentzian metric

$$\langle\, , \,\rangle = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g), \tag{2}$$

where π_I and π_F denote the projections from M onto I and F, respectively. The Lorentzian manifold $(M = I \times_f F, \langle\, , \,\rangle)$ is a warped product, in the sense of [15, p. 204], with base $(I, -dt^2)$, fiber (F, g) and warping function f. Any warped product $I \times_f F$ possesses an infinitesimal timelike conformal symmetry (see Subsection 2.1) which is an important tool in this paper.

For each $u \in C^\infty(\Omega)$, $u(\Omega) \subseteq I$, the induced metric on Ω from the Lorentzian metric (2), via its graph $\Sigma_u = \{(u(p), p) : p \in \Omega\}$ in M, is written as follows

$$g_u = -du^2 + f(u)^2 g,$$

and it is positive definite, i.e. Riemannian, if and only if u satisfies $\mid Du \mid < f(u)$ everywhere on Ω. When g_u is Riemannian, $f(u)\sqrt{f(u)^2 - \mid Du \mid^2}\, dA$ is the area element of (Ω, g_u). Therefore (E.1) of (E) is the Euler-Lagrange equation for the area functional, its solutions are spacelike graphs of zero mean curvature in M, and this equation is called the maximal surface equation in M.

If we denote by N the unit normal vector field N on Σ_u such that $\langle N, \partial_t\rangle \geq 1$ on Σ_u, where $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$, then

$$N = \frac{-f(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}} \left(1, \frac{1}{f(u)^2} Du\right),$$

and the hyperbolic angle θ between $-\partial_t$ and N is given by

$$\langle N, \partial_t\rangle = \cosh\theta = \frac{f(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}.$$

Observe that when $I = \mathbb{R}$, $F = \mathbb{R}^2$ and $f = 1$, the equation (E) is the maximal surface equation in $\mathbb{L}^3$. Of course, the Euclidean plane $\mathbb{R}^2$ has finite total curvature, but note that any complete Riemannian surface whose curvature is non-negative out a compact set has finite total curvature (see section 2.4). On the other hand, examples of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature are known (see, [9]). Examples in a different ambient space can be seen in [16].

In this work, we give new results on uniqueness and non-existence of solutions of the equation (E) on a complete Riemannian surface with finite total curvature. Our results widely extend and improve the non-parametric following results, [12, Th. A and Th. B], [1, Th. 4.3 and Cor. 4.4], [2, Cor. 8], [4, Th. 6.2 and Th. 6.3] and [5, Th. 4.2 and Cor. 4.3]. Moreover, the Calabi-Bernstein's Theorem is included as a particular case (see Section 4).

2 Preliminaries

2.1 The infinitesimal timelike conformal symmetry

Let f be a positive smooth function defined on an open interval I of $\mathbb{R}$ and (F, g) a Riemannian surface. Consider a warped product $M = I \times_f F$ endowed with the Lorentzian metric (2). The unit timelike vector field $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$ determines a time-orientation on M. We consider the vector field $\xi := f(\pi_I)\,\partial_t$, which is timelike and, from the relationship between the Levi-Civita connections of M and those of the base and the fiber [15, Cor. 7.35], satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I)\, X \tag{3}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the Levi-Civita connection of the metric (2). Thus, ξ is conformal with $\mathcal{L}_\xi \langle\, ,\, \rangle = 2\, f'(\pi_I)\, \langle\, ,\, \rangle$ and its metrically equivalent 1-form is closed.

2.2 The null convergence energy condition

We are interested in graphs immersed in Lorentzian warped product spaces satisfying certain natural energy condition, which turns out to have an expression in terms of the curvature of its fiber (F, g) and the warping function f.

Recall that a Lorentzian manifold obeys the null convergence condition (NCC) if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for any null vector Z, i.e. $Z \neq 0$ such that $\langle Z, Z\rangle = 0$. It is easy to check that a Lorentzian warped product space $I \times_f F$ with a 2-dimensional fiber obeys NCC if and only if

$$\frac{K^F(\pi_F)}{f^2} - (\log f)'' \geq 0, \tag{4}$$

where K^F denotes the Gaussian curvature of the fiber.

Recall that any solution to the Einstein's equation obeys the NCC condition.

2.3 The restriction of the warping function on a spacelike surface

Let $x : S \longrightarrow M$ be a (connected) spacelike surface in M; that is, x is an immersion and induces a Riemannian metric on the 2-dimensional manifold S from the Lorentzian metric (2). As usual, we agree to represent the induced metric with the same symbol as the one used in (2). Then the time-orientability of M allows us to consider $N \in \mathfrak{X}^{\perp}(S)$ as the only, globally defined, unit timelike normal vector field on S in the same time-orientation of $-\partial_t$. Thus, from the wrong way Cauchy-Schwarz inequality, (see [15, Prop. 5.30], for instance) we have $\langle N, \partial_t\rangle \geq 1$ and $\langle N, \partial_t\rangle = 1$ at a point p if and only if $N(p) = -\partial_t(p)$. In fact, $\langle N, \partial_t\rangle = \cosh\theta$, where θ is the hyperbolic angle, at each point, between the unit timelike vectors $-\partial_t$ and N. From now on, we will refer to θ as the hyperbolic angle between S and ∂_t. We will call spacelike slice to a spacelike surface x such that $\pi_I \circ x$ is a constant. A spacelike surface is a spacelike slice if and only if it is orthogonal to ∂_t or, equivalently, orthogonal to ξ. Denote by $\partial_t^{\top} := \partial_t + \langle N, \partial_t\rangle N$ the tangential component of ∂_t on S. It is not difficult to see that

$$\nabla\tau = -\partial_t^{\top}, \tag{5}$$

where $\nabla\tau$ is the gradient of $\tau := \pi_I \circ x$ on $(S, \langle\,,\rangle)$. Now, from the Gauss formula, taking into account $\xi^{\top} = f(\tau)\,\partial_t^{\top}$ and (5), the Laplacian of τ in $(S, \langle\,,\rangle)$ is given by

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\Big\{2+ \mid \nabla\tau \mid^2 \Big\} - 2H\,\langle N, \partial_t\rangle, \tag{6}$$

where $f(\tau) := f \circ \tau$, $f'(\tau) := f' \circ \tau$ and the function $H := -(1/2)\,\mathrm{trace}(A)$ is called the mean curvature of S relative to N, where A is the shape operator associated to N. A spacelike surface with constant H is called a constant mean curvature spacelike surface. Note that, with our choice of N, the shape operator of the spacelike slice $t = t_0$ is $A = (f'(t_0)/f(t_0))\,I$ and $H = -f'(t_0)/f(t_0)$. Therefore the spacelike slices are totally umbilical constant mean curvature surfaces. When the spacelike slice is maximal, then it is totally geodesic. From (6), it is easy to obtain

$$\Delta f(\tau) = -2\,\frac{f'(\tau)^2}{f(\tau)} + f(\tau)(\log f)''(\tau) \mid \nabla\tau \mid^2 - 2f'(\tau)H\,g(N, \partial_t). \tag{7}$$

On the other hand, in [4, For. 13] is computed the Gaussian curvature K of a spacelike surface $(S, \langle\,,\rangle)$ in M. In particular, if S is maximal we get

$$K = \frac{f'(\tau)^2}{f(\tau)^2} + \left\{\frac{K^F(\pi_F)}{f(\tau)^2} - (\log f)''(\tau)\right\} \mid \nabla\tau \mid^2 + \frac{K^F(\pi_F)}{f(\tau)^2} + \frac{1}{2}\mathrm{trace}(A^2), \tag{8}$$

where K^F denotes the Gaussian curvature of the fiber (F, g). If we consider the function $\langle\xi, N\rangle$ on $(S, \langle\,,\rangle)$, it is easy to see [4, For. 16,17,18] that

$$\Delta\langle N, \xi\rangle = \left\{\frac{K^F(\pi_F)}{f(\tau)^2} - (\log f)''(\tau)\right\} \mid \partial_t^T \mid^2 \langle N, \xi\rangle + \mathrm{trace}(A^2)\langle N, \xi\rangle, \tag{9}$$

or equivalently

$$\Delta\langle N, \xi\rangle = \left\{K - \frac{f'(\tau)^2}{f(\tau)^2} - \frac{K^F(\pi_F)}{f(\tau)^2} + \frac{1}{2}\mathrm{trace}(A^2)\right\}\langle N, \xi\rangle. \tag{10}$$

2.4 Curvature and parabolicity

Recall that a complete Riemannian surface (Σ, g_Σ) is parabolic if each non-negative superharmonic function on Σ must be constant. On the other hand, a complete Riemannian surface (Σ, g_Σ) is said to have finite total curvature if the negative part of its Gaussian curvature is integrable. More precisely, if $K(p)$, $p \in \Sigma$, denotes the Gaussian curvature on Σ, then Σ has finite total curvature if

$$\int_\Sigma \max(0, -K)\, dA_\Sigma < \infty, \tag{11}$$

where the integral is defined with a compact exhaustion procedure.

It is well know (see [13]) that a complete Riemannian surface (Σ, g_Σ) with finite total curvature is parabolic.

Huber [10] showed that if $\int_\Sigma \max(0, -K)\, dA_\Sigma < \infty$ then Σ must be conformally equivalent to a compact Riemannian surface with finite punctures. Moreover, the Cohn-Vossen inequality,

$$\int_\Sigma K dA_\Sigma \leq 2\pi\mathcal{X}(\Sigma),$$

is satisfied. Since $K = \max\{K, 0\} - \max\{-K, 0\} := K_+ - K_-$, the Cohn-Vossen inequality implies that

$$\int_\Sigma K_+\, dA_\Sigma \leq \int_\Sigma K_-\, dA_\Sigma + 2\pi\mathcal{X}(\Sigma).$$

On the other hand, Huber's Theorem asserts that the right hand side is finite. Hence,

$$\int_\Sigma \max(0, -K)\, dA_\Sigma < \infty$$

implies

$$\int_\Sigma |K|\, dA_\Sigma < \infty.$$

This fact justifies the term total curvature.

3 On the Gaussian curvature of certain conformal graphs

Let (F, g) be a 2-dimensional (non-compact) complete Riemannian manifold and let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function. For each $u \in C^\infty(F)$ such that $u(F) \subseteq I$ we can consider its graph $\Sigma_u = \{(u(p), p) : p \in F\}$ in the Lorentzian warped product $M = I \times_f F$. The graph of u inherits a metric $(\Sigma_u, \langle\,,\rangle)$ from M, represented on F by $g_u := -du^2 + f(u)^2 g$, which is Riemannian if and only if u satisfies $g(Du, Du) < f(u)^2$ everywhere on F, where Du denotes the gradient of u in (F, g). In this case, the graph is a spacelike surface. Note that $\tau(u(p), p) = u(p)$ for any $p \in F$, and so τ and u can be naturally identified by the isometry between $(\Sigma_u, \langle\,,\rangle)$ and (F, g_u). Analogously, the differential operators ∇ and Δ in $(\Sigma_u, \langle\,,\rangle)$ can be identified with those ones ∇_u and Δ_u in (F, g_u).

On the manifold F we consider the following Riemannian metric

$$g'_u := f(u)^2 \cosh^2\theta \, g_u, \tag{12}$$

where

$$f(u)\cosh\theta = \frac{f(u)^2}{\sqrt{f(u)^2 - \mid Du \mid^2}}$$

and $\mid Du \mid^2 := g(Du, Du)$. Therefore, if $\epsilon := \mathrm{Inf}(f) > 0$ we get the following inequality

$$L' \geq \epsilon^2 L,$$

where L' and L denote the lengths of a curve in F with respect to g'_u and g, respectively. Consequently, g'_u is complete whenever g is complete.

Now, suppose that $\mathrm{Sup} f(u) < \infty$. Put $\lambda = \mathrm{Sup} f(u)$ and consider the new Riemannian metric

$$g^*_u := (f(u)\cosh\theta + \lambda)^2 g_u \tag{13}$$

on F.

The completeness of the metric (12) assures that g^*_u is also complete. Moreover, it has the advantage over g'_u that we can control its Gaussian curvature under reasonable assumptions. In order to concrete this assertion, let K^*_u and K_u denote the Gaussian curvatures of the Riemannian metrics g^*_u and g_u, respectively. From (13) and using the relation between Gaussian curvatures for conformal changes (see [3, Ch.1, Section J], we have

$$K_u - (f(u)\cosh\theta + \lambda)^2 K^*_u = \Delta_u \log(f(u)\cosh\theta + \lambda). \tag{14}$$

Lemma 3.1 *Suppose that (F, g) is complete, with finite total curvature. If $\mathrm{Inf} f > 0$, $\mathrm{Sup} f < \infty$ and $(\log f)''(u) \leq 0$, then the complete Riemannian surface (F, g^*_u) has finite total curvature.*

Proof. Taking into account (8) and (9), we have

$$\Delta_u \log(f(u)\cosh\theta + \lambda) \leq \frac{1}{f(u)\cosh\theta + \lambda}\left\{\Big(K_u - \frac{K^F}{f(u)^2}\Big) f(u)\cosh\theta + \Big(K_u - \cosh^2\theta \frac{K^F}{f(u)^2}\Big)\lambda\right\}$$

$$\leq K_u + \frac{f(u)\cosh\theta + \lambda\cosh^2\theta}{f(u)^2 (f(u)\cosh\theta + \lambda)^3}.$$

Making use of (14) and since the Riemannian area elements of the metrics g and g_u^* satisfy

$$dA_u^* = \frac{(f(u)\cosh\theta + \lambda)^2 f(u)^2}{\cosh\theta}\, dA, \tag{15}$$

we have

$$\int_F \max(-K_u^*, 0)\, dA_u^* \leq \int_F \max(-K^F, 0)\, \frac{f(u)\cosh\theta + \lambda\cosh^2\theta}{f(u)\cosh^2\theta + \lambda\cosh\theta}\, dA < \infty.$$

where we have also used that $f(u)$ is bounded with $\mathrm{Inf} f(u) > 0$. $\square$

Lemma 3.2 *Suppose that (F, g) is complete, with finite total curvature. If* $\mathrm{Inf} f > 0$, $\mathrm{Sup} f < \infty$ *and the inequality* $\frac{K^F}{f(u)^2} - (\log f)''(u) \geq 0$ *holds on F, then the complete Riemannian surface (F, g_u^*) has finite total curvature.*

Proof. Again, from (8) and (9) we get,

$$\begin{aligned}\Delta_u \log(f(u)\cosh\theta + \lambda) &\leq \frac{1}{f(u)\cosh\theta + \lambda}\left\{\left(K_u - \frac{K^F}{f(u)^2}\right) f(u)\cosh\theta + \left(K_u - \frac{K^F}{f(u)^2}\right)\lambda\right\} \\ &\leq K_u - \frac{K^F}{f(u)^2}.\end{aligned}$$

Using (14) and the equality (15), we obtain

$$\int_F \max(-K_u^*, 0)\, dA_u^* \leq \int_F \max(-K^F, 0)\frac{1}{\cosh\theta}\, dA < \int_F \max(-K^F, 0)\, dA < \infty.$$

$\square$

4 Main results

Under a convexity assumption on the warping function we obtain (compare with [12, Th. A]),

Theorem 4.1 *If the smooth function f is non-locally constant,* $\mathrm{Inf} f > 0$, $(\log f)'' \leq 0$ *and the constant function $u = u_0$ is a solution to the equation* (E) *on a complete Riemannian surface (F, g), with finite total curvature, then it is the only solution to this equation.*

Proof. Since $(\log f)'' \leq 0$ and $f'(u_0) = 0$, the function f has a global maximum at u_0 and this is the only zero of f'. We can apply the Lemma 3.1 and as a consequence we know that the complete Riemannian surface (F, g_u^*) has finite total curvature. On the other hand, if u is an entire solution to (E), from (7) we obtain

$$\Delta_u f(u) = -2\frac{f'(u)^2}{f(u)} + f(u)(\log f)''(u)|\nabla_u u|^2 \leq 0. \tag{16}$$

Taking into account the invariance of superharmonic functions by conformal changes of metric, the positive function $f(u)$ is superharmonic on the parabolic Riemannian surface (F, g_u^*). Hence, since f is non-locally constant, we conclude that u must be constant.

$\square$

The main assumptions in Theorem 4.1 cannot be removed, as shows the following example,

Example 4.2 Consider the hyperbolic plane of Gauss curvature -1, $\mathbb{H}^2$, in $\mathbb{L}^3 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ and let $f :]0, \infty[\longrightarrow \mathbb{R}$ be the function defined by $f(t) = t$. If $\Omega = \{(x, y, z) \in \mathbb{L}^3 : z > 0, x^2 + y^2 - z^2 < 0, \}$, then it is easy to check that the map $\phi :]0, \infty[\times_f \mathbb{H}^2 \longrightarrow \Omega$, $\phi(t, (x, y, z)) = (tx, ty, tz)$ is an isometry. Hence, for each $z_0 \in]0, \infty[$, $S_{z_0} := \phi^{-1}(\Omega \cap \{z = z_0\})$ is a maximal surface in $]0, \infty[\times_f \mathbb{H}^2$. It can be easily proved that S_{z_0} is the graph on $]0, \infty[\times_f \mathbb{H}^2$ of the function $u_{z_0} : \mathbb{H}^2 \longrightarrow \mathbb{R}$, $u_{z_0}(x, y, z) = \frac{z_0}{z}$. Thus, we have found a family of (entire) non constant solutions to (E).

Remark 4.3 Note that for any analytical and non-constant function, the non-locally constant assumption on f is satisfied.

If we do not ask the function f to be non-locally constant, the previous theorem is false as shows the Calabi-Bernstein theorem. Nevertheless, we can enunciate the following result

Corollary 4.4 *If* $\text{Inf} > 0$, $(\log f)'' \leq 0$ *and the constant function* $u = u_0$ *is a solution to Equation* (E) *on a complete Riemannian surface* (F, g)*, with finite total curvature, then it is the only solution to this equation, which is bounded from above or from below.*

Proof. Let u an entire solution which is bounded from above or from below. As in the previous proof, $f(u)$ is constant and so $\Delta_u f(u) = 0$. Taking into account the signs of the terms in (16), we obtain $f'(u) = 0$. From (6), the function u must be harmonic for the metric g_u. Now, from the invariance of harmonic functions by conformal changes of metric and the boundedness of u, the result holds.

□

We have also the following non-existence result (compare with [12, Th. B]),

Theorem 4.5 *Let* f *be a smooth non-locally constant function, with* $\text{Inf} f > 0$, $\text{Sup} f < \infty$ *and* $(\log f)'' \leq 0$*. If* f' *does not vanishes at any point, then there exists no entire solution to Equation* (E)*.*

Proof. Any solution must be constant $u = u_0$, and as a direct consequence $H = \frac{-f'(u_0)}{f(u_0)} \neq 0$, which is a contradiction.

□

Now, consider the Lorentzian warped product $M = I \times_f F$, where (F, g) is a complete Riemannian surface, with finite total curvature. Suppose that M obeys the NCC, i.e., satisfies the inequality (4).

Let Σ_u an entire spacelike graph in M. From (9) and (10), the Laplacian of the positive function $f(u) \cosh \theta$ in (F, g_u) is given by

$$\Delta_u(f(u) \cosh \theta) = \Big\{ \frac{K^F}{f(u)^2} - (\log f)''(u) \Big\} |\nabla_u u|^2 f(u) \cosh \theta + \frac{1}{2} \text{trace}(A^2) \cosh \theta \tag{17}$$

or equivalently

$$\Delta_u(f(u) \cosh \theta) = \Big\{ K_u - \frac{f'(u)^2}{f(u)^2} - \frac{K^F}{f(u)^2} + \frac{1}{2} \text{trace}(A^2) \Big\} f(u) \cosh \theta. \tag{18}$$

Therefore, under the NCC, we have

$$\Delta_u(f(u)\cosh\theta) \geq 0.$$

□

We obtain the following result, which extends and improves [1, Th. 4.3 and Cor. 4.4], [2, Cor. 8], [4, Th. 6.2 and Th. 6.3] and [5, Th. 4.2 and Cor. 4.3],

Theorem 4.6 *Let $M = I \times_f F$ a Lorentzian warped product, with fiber (F, g) a complete Riemannian surface, which has finite total curvature and whose warping function satisfies* $\mathrm{Inf} f > 0$ *and* $\mathrm{Sup} f < \infty$*. If M obeys the NCC, then any entire graph $(\Sigma_u, \langle\, , \,\rangle)$ must be totally geodesic. Moreover, if there exists a point $p \in F$ such that $\frac{K^F(p)}{f(u(p))^2} - (\log f)''(u(p)) > 0$, then u is constant.*

Proof. From Lemma 3.2, we have that (F, g_u^*) is complete with finite total curvature.

Consider the function $\frac{1}{f(u)\cosh\theta}$ on (F, g_u). Then

$$\Delta_u\left(\frac{1}{f(u)\cosh\theta}\right) = -\frac{1}{f(u)^2\cosh^2\theta}\Delta_u(f(u)\cosh\theta) + 2\frac{|\nabla_u(f(u)\cosh\theta)|^2}{(f(u)^3\cosh^3\theta)} \leq 0.$$

Again, taking into account the invariance of superharmonic functions by conformal changes of metric, we get a positive superharmonic function on the complete parabolic Riemannian surface (F, g_u^*) and as a consequence the function $f(u)\cosh\theta$ must be constant. Thus, from the second term of (17) we obtain that the graph $(\Sigma_u, \langle\, , \,\rangle)$ is totally geodesic. On the other hand, if moreover there exists a point $p \in F$ such that $\frac{K^F(p)}{f(u(p))^2} - (\log f)''(u(p)) > 0$, taking into account the first addend of (17), then there exists an open neighborhood of $(p, u(p))$ in Σ_u which is contained in the complete spacelike graph $u = u_0$, with $f'(u_0) = 0$. As $(\Sigma_u, \langle\, , \,\rangle)$ is entire and totally geodesic, it must be to coincide with the totally geodesic spacelike slice $t = u_0$.

□

Corollary 4.7 *(The classical Calabi-Bernstein's Theorem) The only entire solutions to Equation* (E) *on the Euclidean plane when $f = 1$ are the affine functions.*

Proof. It is enough to observe that for any solution u the function $\cosh\theta$ must be constant.

□

We can enunciate the following uniqueness and non-existence result

Theorem 4.8 *Let (F, g) be a complete Riemannian surface with finite total curvature and let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function satisfying* $\mathrm{Inf} f > 0$, $\mathrm{Sup}\, f < \infty$, $\mathrm{Inf}\, K^F \geq f(\log f)''$ *(or $K^F \geq \mathrm{Sup}\{f(\log f)''\}$), being the previous inequality strict at some point of F. Then the only solutions to Equation* (E) *on F, are the constant functions $u = u_0$ such that $f'(u_0) = 0$.*

Remark 4.9 Observe that when in the previous theorem the function $f$satisfies $(\log f)'' \leq 0$ and there is $u_0 \in I$ such that $f'(u_0) = 0$, then the assumption $\mathrm{Inf} f > 0$ can be removed and $u = u_0$ is the only solution to Equation (E).

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