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Lorentzian affine hypersurfaces with an almost symplectic form

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ABSTRACT

In this paper, we study affine hypersurfaces with a Lorentzian second fundamental form additionally equipped with an almost symplectic structure ω. We prove that the rank of the shape operator is at most one if the hypersurface is of dimension at least 6 and $R^k \cdot \omega = 0$ or $\nabla^k \omega = 0$ for some positive integer k.

1. Introduction

Affine hypersurfaces have been widely studied over past decades. Especially, hypersurfaces equipped with some additional structure apart from the induced one. This additional structure can be either induced from the ambient space (see e.g. [1] and [2]) or given in some independent way (see e.g. [3] and [4]). Moreover some interesting results connecting special Kähler and special para-Kähler geometry with some class of affine hyperspheres have been recently obtained in [5] and [6].

In [3], M. Kon studied $2n$-dimensional affine hypersurfaces $f : M \to \mathbb{R}^{2n+1}$ with an almost complex structure J invariant relative to the second fundamental form h (i.e. satisfying $h(JX, JY) = h(X, Y)$ for every $X, Y \in TM$). In other words, (h, J) was almost Hermitian structure on M. For such hypersurfaces, in a natural way, we have an almost symplectic form ω defined by the condition $\omega(X, Y) = h(X, JY)$.

On the other hand, in [4], the author studied affine hypersurfaces $f : M \to \mathbb{R}^{2n+1}$ with a transversal vector field ξ additionally equipped (independently of the induced structure) with an almost symplectic structure ω. When f is non-degenerate, there exists exactly one tensor J_ω of type $(1, 1)$ on M such that

$$\omega(X, J_\omega Y) = h(X, Y),$$

where h is the second fundamental form on M. In this case, J_ω is non-singular and h-antisymmetric. In [4], the following results were obtained:

Theorem 1.1 ([4]). *Let $f : M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface with a transversal vector field ξ and an almost symplectic form ω. Equality $R(X, Y)\omega = 0$ for every $X, Y \in \mathcal{X}(M)$ holds if and only if $\dim M = 2$ and ξ is locally equiaffine or $\dim M \geq 4$ and ∇ is flat.*

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In the case when the second fundamental form is positive definite and the transversal vector field ξ is locally equiaffine, the above theorem generalises to an arbitrary power of R. Namely, we have

Theorem 1.2 ([4]). *Let $f : M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface ($\dim M \geq 4$) with a locally equiaffine transversal vector field ξ and an almost symplectic form ω. Additionally, assume that the second fundamental form is positive definite on M. If $R^l\omega = 0$ for some positive integer l then ∇ is flat.*

As a consequence of the above theorem, it was shown the following

Theorem 1.3 ([4]). *Let $f : M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface ($\dim M \geq 4$) with a locally equiaffine transversal vector field ξ and an almost symplectic form ω. Additionally, assume that the second fundamental form is positive definite on M. If $\nabla^k\omega = 0$ for some positive integer k then ∇ is flat.*

The main purpose of this paper is the study of affine hypersurfaces with the Lorentzian second fundamental form equipped with an almost symplectic structure. In particular, we study the properties of affine hypersurfaces for which there exists an almost symplectic structure ω satisfying condition $R^k \cdot \omega = 0$. In consequence, we obtain some constrains on affine hypersurfaces with the property $\nabla^k\omega = 0$. Note that the affine higher order parallel hypersurfaces were studied by many authors. In particular, L. Vrancken obtained several important results in [7]. However, hypersurfaces studied in [7] are parallel relative to the second fundamental form (i.e. $\nabla^k h = 0$). In this paper, we study the affine hypersurfaces parallel relative to a symplectic form.

In Section 2, we briefly recall the basic formulas of affine differential geometry. We also recall some basic definitions from indefinite linear algebra as well as symplectic geometry that will be used later in this paper.

Section 3 contains the main results of this paper. In this section, we focus on the Lorentzian case. We start with an affine hypersurface with the Lorentzian second fundamental form. We show that if there exists an almost symplectic structure ω satisfying condition $R^k \cdot \omega = 0$ or $\nabla^k\omega = 0$ for some positive integer k then the shape operator must have a very special form. More precisely, we obtain that the rank of the shape operator S must be ≤ 1 provided $\dim M \geq 6$ and the transversal vector field is locally equiaffine.

2. Preliminaries

We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [8]. Let $f : M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then, for any transversal vector field ξ, we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi \tag{2.1}$$

and

$$\mathrm{D}_X \xi = -f_*(SX) + \tau(X)\xi, \tag{2.2}$$

where X, Y are the vector fields tangent to M. It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called *the second fundamental form*, S is a tensor of type $(1, 1)$, called *the shape operator*, and τ is a 1-form, called *the transversal connection form*. The vector field ξ is called *equiaffine* if $\tau = 0$. When $d\tau = 0$ the vector field ξ is called *locally equiaffine*.

When h is non-degenerate then h defines a pseudo-Riemannian metric on M. In this case, we say that the hypersurface or the hypersurface immersion is *non-degenerate*. In this paper, we always assume that f is non-degenerate. We have the following

Theorem 2.1 ([8], *Fundamental Equations*). *For an arbitrary transversal vector field ξ the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X, SY) - h(SX, Y) = 2d\tau(X, Y). \tag{2.6}$$

Eqs. (2.3), (2.4), (2.5), and (2.6) are called the equations of Gauss, Codazzi for h, Codazzi for S and Ricci, respectively.

Let $f : M \to \mathbb{R}^{n+1}$ be an affine hypersurface with a transversal vector field ξ. On M, we define a tensor field C of type $(0, 3)$ by the formula

$$C(X, Y, Z) := (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z)$$

for all $X, Y, Z \in \mathcal{X}(M)$. The tensor field C is called the *cubic form*. It is easy to verify that the cubic form C is symmetric in all three variables.

On the space $\mathbb{R}^{n+1}$, we have the standard volume form determined by the determinant $\det[\cdot]$. If $f : M \to \mathbb{R}^{n+1}$ is an affine hypersurface with a transversal vector field ξ, then on M we define the *induced volume form* by the formula

$$\theta(X_1, \ldots, X_n) := \det[f_*X_1, \ldots, f_*X_n, \xi]$$

for every $X_1, \ldots, X_n \in \mathcal{X}(M)$. The following theorem holds:

Theorem 2.2 ([8]). *If $f : M \to \mathbb{R}^{n+1}$ is an affine hypersurface with a transversal vector field ξ then*

$$\nabla_X \theta = \tau(X)\theta \qquad \text{for every } X \in TM.$$

In particular, the following conditions are equivalent:

(1) $\nabla\theta = 0$,
(2) $\tau = 0$ *(that is ξ is equiaffine).*

Let ω be a non-degenerate 2-form on manifold M. The form ω we call an *almost symplectic structure*. It is easy to see that if a manifold M admits some almost symplectic structure then M is an orientable manifold of even dimension. Structure ω is called a *symplectic structure*, if it is almost symplectic and additionally satisfies $d\omega = 0$. Pair (M, ω) we call *(almost) symplectic manifold*, if ω is (almost) symplectic structure on M.

Recall [9] that affine connection ∇ on an almost symplectic manifold (M, ω) we call an *almost symplectic connection* if $\nabla\omega = 0$. An affine connection ∇ on an almost symplectic manifold (M, ω) we call a *symplectic connection* if it is almost symplectic and torsion-free.

Now, we recall a well-known theorem about Jordan normal form (see e.g. Th. A.2.6 in [10]).

Theorem 2.3 (Jordan). *If $A : V \to V$ is an endomorphism of real finite dimensional vector space V then there exists a basis of V such that the matrix of the endomorphism A in this basis has a form*

$$\begin{bmatrix} L_1 & 0 & \ldots & 0 \\ 0 & L_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots \\ 0 & 0 & \ldots & L_s \end{bmatrix}, \tag{2.7}$$

where L_i is the Jordan block corresponding to the eigenvalue λ_i and given by the formula

$$\begin{bmatrix} \lambda_i & 0 & 0 & \ldots & 0 & 0 \\ 1 & \lambda_i & 0 & \ldots & 0 & 0 \\ 0 & 1 & \lambda_i & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_i & 0 \\ 0 & 0 & 0 & \ldots & 1 & \lambda_i \end{bmatrix} \in M(k_i, k_i, \mathbb{R}), \tag{2.8}$$

when λ_i is real, or by the formula

$$\begin{bmatrix} B_i & 0 & 0 & \ldots & 0 & 0 \\ I & B_i & 0 & \ldots & 0 & 0 \\ 0 & I & B_i & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & B_i & 0 \\ 0 & 0 & 0 & \ldots & I & B_i \end{bmatrix} \in M(2k_i, 2k_i, \mathbb{R}), \tag{2.9}$$

where

$$B_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M(2, 2, \mathbb{R}),$$

when $\lambda_i = \alpha_i + i\beta_i$ ($\beta_i \neq 0$) is complex.

A square matrix P of dimension n is called *the sip matrix (standard involutory permutation)* [10] if it has a form:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.10}$$

Note that P is a non-singular symmetric matrix and $P^2 = I$. In particular, all its eigenvalues are equal ± 1. Moreover, it is easy to verify that we have the following formula for the signature of P:

$$\operatorname{sig} P = \begin{cases} \left(\dfrac{n}{2}, \dfrac{n}{2}\right), & \text{if } n \text{ is even} \\ \left(\dfrac{n+1}{2}, \dfrac{n-1}{2}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{2.11}$$

Theorem 2.4 (*Th. 6.1.5 [10]*). *Let H be a real invertible and symmetric matrix of dimension n. Then, for every square n dimensional and H-selfadjoint matrix A (i.e. $A^T H = HA$) there exists a basis $\{e_1, \dots, e_n\}$ such that*

$$A = J_1 \oplus \cdots \oplus J_t \oplus J_{t+1} \oplus \cdots \oplus J_{t+s}, \tag{2.12}$$

where $J_1, \dots, J_t$ are Jordan blocks of type (2.8) and $J_{t+1}, \dots, J_{t+s}$ are Jordan blocks of type (2.9). Moreover

$$H = \varepsilon_1 P_1 \oplus \cdots \oplus \varepsilon_t P_t \oplus P_{t+1} \oplus \cdots \oplus P_{t+s}, \tag{2.13}$$

where P_j is a sip matrix of dimension equal to dimension of matrix J_j for $j = 1, \dots, t+s$ and $\varepsilon_j = \pm 1$ for $j = 1, \dots, t$. The signs ε_j are determined uniquely by (A, H) up to permutation of signs in the blocks of (2.13) corresponding to the Jordan blocks of A with the same real eigenvalue and the same size.

3. Hypersurfaces with parallel symplectic structure

In this section, we study the properties of affine hypersurfaces equipped with an almost symplectic structure ω satisfying condition $R^k\omega = 0$ (and in particular $\nabla^k\omega = 0$) for some positive integer k.

First, we recall three lemmas from [4].

Lemma 3.1 (*[4]*). *Let T be a tensor of type $(0, p)$ and let ∇ be an affine torsion-free connection. Then, for every $k \geq 1$ and for any $2k + p$ vector fields $X^1_{\pm 1}, \dots, X^k_{\pm 1}, Y_1, \dots, Y_p$ the following identity holds:*

$$\begin{aligned}(R^k \cdot T)&(X^1_1, X^1_{-1}, \dots, X^k_1, X^k_{-1}, Y_1, \dots, Y_p) \\ &= \sum_{a \in \mathcal{J}} \operatorname{sgn} a (\nabla^{2k} T)(X^1_{a(1)}, X^1_{-a(1)}, \dots, X^k_{a(k)}, X^k_{-a(k)}, Y_1, \dots, Y_p),\end{aligned} \tag{3.1}$$

where $\mathcal{J} = \{a : I_k \to \{-1, 1\}\}$ and $\operatorname{sgn} a := a(1) \cdot \dots \cdot a(k)$.

Lemma 3.2 (*[4]*). *Let $f : M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface ($\dim M \geq 4$) with a transversal vector field ξ and an almost symplectic form ω. Let $x \in M$. If there exists a positive integer $2 \leq s \leq 2n$ and a basis $\{e_1, \dots, e_{2n}\}$ of T_xM such that $h(e_i, e_j) = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$ for $i, j = 1, \dots, s$, $h(e_i, e_j) = 0$ for $i = 1, \dots, s$, $j = s+1, \dots, 2n$ and $Se_i = \lambda_i e_i$ for $i = 1, \dots, s$, $\lambda_i \in \mathbb{R}$. Then, for every $k, j = 1, 2, \dots, s$, $k \neq j$ and for every $i = 1, \dots, 2n$, $i \neq j$, $i \neq k$ we have*

$$R^{2l}\omega(e_k, e_j, e_k, e_j, \dots, e_k, e_j, e_k, e_i) = (-1)^l \varepsilon_k^l \varepsilon_j^l \lambda_k^l \lambda_j^l \omega(e_k, e_i) \tag{3.2}$$

for every $l \in \mathbb{N}$.

Lemma 3.3 (*[4]*). *Let $f : M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface ($\dim M \geq 4$) with a transversal vector field ξ and an almost symplectic form ω. Let $x \in M$ and let X, Y, Z_1, Z_2 be vectors from T_xM such that $SX = \lambda X$, $SZ_1 = 0$, $SZ_2 = 0$, $h(Z_1, Z_2) = 0$ and $h(Y, Z_2) = 0$. Then, for every $l \geq 1$ we have*

$$\begin{aligned}R^{2l}&\omega(X, \underbrace{Z_1, Z_2, Z_1, Z_2, \dots, Z_1, Z_2}_{4l}, Y) \\ &= (-1)^l \lambda^{2l} h(Z_1, Z_1)^l h(Z_2, Z_2)^l \omega(X, Y).\end{aligned} \tag{3.3}$$

Before proceeding note that if the second fundamental form is not positive definite then Theorem 1.2 do not hold in general. Indeed, we have the following example [4]:

Example 3.4. Let us consider an immersion

$$f : \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}_+ \ni (x, y, z, t) \mapsto \begin{bmatrix} x \\ y \\ z \\ x^2 - t \\ y^2 + 2zt \end{bmatrix} \in \mathbb{R}^5$$

with a transversal vector field

$$\xi : \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}_+ \ni (x, y, z, t) \mapsto \begin{bmatrix} 0 \\ 0 \\ -t \\ 0 \\ -t^2 \end{bmatrix} \in \mathbb{R}^5.$$

It is easy to see that ξ is equiaffine ($\tau = 0$) and

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} \frac{4z}{t^2} & 0 & 0 & 0 \\ 0 & \frac{2}{t^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{t^2} \\ 0 & 0 & \frac{2}{t^2} & 0 \end{bmatrix}.$$

in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$ on $\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}_+$. From the form of h, it follows that f is non-degenerate. If $z > 0$ then h has signature (3,1) so is Lorentzian. If $z < 0$ then h has signature (2,2). Since S is nonzero, the connection ∇ induced by ξ cannot be flat.

Let ω be an almost symplectic structure given by the following matrix:

$$\omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$. By straightforward computations (see [4] for details), it can be shown that $R^2\omega = 0$.

In a further part of this section, we focus on the Lorentzian case. That is affine hypersurfaces with the Lorentzian second fundamental form for which there exists an almost symplectic structure ω satisfying the condition $R^k\omega = 0$. The above example shows that the condition $R^k\omega = 0$ do not have to necessarily imply flatness of the induced connection ∇. However, we shall show the following

Theorem 3.5. *Let $f : M \to \mathbb{R}^{2n+1}$ ($\dim M \geq 6$) be a non-degenerate affine hypersurface with a locally equiaffine transversal vector field ξ and an almost symplectic form ω. If $R^k\omega = 0$ for some $k \geq 1$ and the second fundamental form is Lorentzian on M (that is has signature $(2n-1, 1)$) then the shape operator S has the rank ≤ 1.*

In order to prove Theorem 3.5, we need several lemmas. In all the below lemmas, we assume that $f : M \to \mathbb{R}^{2n+1}$ is a non-degenerate affine hypersurface with a locally equiaffine transversal vector field ξ and an almost symplectic form ω. About objects ∇, h, S and τ, we assume that they are induced by ξ. If it is not clearly stated otherwise, we assume also that $\dim M \geq 4$ (so it is a weaker assumption than that in Theorem 3.5).

Lemma 3.6. *If h is Lorentzian of signature (2n-1,1) and $d\tau = 0$, then for every point x of manifold M there exists a basis $\{e_1, \ldots, e_{2n}\}$ of T_xM such that S and h in this basis have one of the following forms:*

$$S = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_{2n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{2n} \end{bmatrix} h = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}, \tag{3.4}$$

where $\lambda_1, \ldots, \lambda_{2n} \in \mathbb{R}$;

$$S = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha & \gamma \\ 0 & 0 & \cdots & -\gamma & \beta \end{bmatrix} h = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}, \tag{3.5}$$

where $\lambda_1, \dots, \lambda_{2n-2}, \alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$;

$$S = \begin{bmatrix} \lambda_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \alpha & 0 & 0 \\ 0 & \cdots & 1 & \alpha & 0 \\ 0 & \cdots & 0 & 1 & \alpha \end{bmatrix} h = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \end{bmatrix}, \tag{3.6}$$

where $\lambda_1, \dots, \lambda_{2n-3}, \alpha \in \mathbb{R}$.

Proof. Since $d\tau = 0$ then by the Ricci equation, the operator S is h-selfadjoint. Let $x \in M$. By the virtue of Theorem 2.4, there exists a basis $\{e_1, \dots, e_{2n}\}$ of T_xM such that

$$S = J_1 \oplus \cdots \oplus J_t \oplus J_{t+1} \oplus \cdots \oplus J_{t+s},$$

where $J_1, \dots, J_t$ are the Jordan blocks of type (2.8) and $J_{t+1}, \dots, J_{t+s}$ are the Jordan blocks of type (2.9). Moreover,

$$h = \varepsilon_1 P_1 \oplus \cdots \oplus \varepsilon_t P_t \oplus P_{t+1} \oplus \cdots \oplus P_{t+s},$$

where P_j is a sip matrix for $j = 1, \dots, t+s$. Since the signature of h is $(2n-1, 1)$ and the signature of P_j, by (2.11), has the form $(\dim P_j/2, \dim P_j/2)$ when $\dim P_j$ is even and has the form $((\dim P_j + 1)/2, (\dim P_j - 1)/2)$ when $\dim P_j$ is odd thus $\dim P_j \leq 3$. Moreover, in the above decomposition, there might be at most one j such that $\dim P_j > 1$. Summarising we have the following possibilities:

(1) $\dim P_j = 1$ for every j. In this case, all the Jordan blocks J_j are one dimensional and taking into the account the signature of h, the matrices S and h are as in (3.4).

(2) There exists exactly one $j = j_0$ such that $\dim P_{j_0} = 2$ and for the other j we have $\dim P_j = 1$. In this case, the Jordan blocks J_j are one dimensional for $j \neq j_0$ and the Jordan block J_{j_0} is two dimensional. Then, either J_{j_0} has the form (2.9) and S and h are as follows:

$$S = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & a & b \\ 0 & 0 & \cdots & -b & a \end{bmatrix} h = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{3.7}$$

since all $\varepsilon_j = 1$, or J_{j_0} has the form (2.8). In the latter case, $\varepsilon_{j_0} = \pm 1$ since the signatures of P_{j_0} and $-P_{j_0}$ are the same. For the remaining j, it must be $\varepsilon_j = 1$. In this way, we obtain that S and h have the form given by the following formulas:

$$S = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 1 & a \end{bmatrix} h = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \varepsilon_{j_0} \\ 0 & 0 & \cdots & \varepsilon_{j_0} & 0 \end{bmatrix}. \tag{3.8}$$

Now, in a straightforward way, we can transform (3.8) into (3.5).

(3) There exists exactly one $j = j_0$ such that $\dim P_{j_0} = 3$ and for remaining j we have $\dim P_j = 1$. In this case, the Jordan blocks J_j are one dimensional for $j \neq j_0$ and the Jordan block J_{j_0} is three dimensional and as such must have the form (2.8). Moreover we have $\varepsilon_{j_0} = 1$ since signature of $-P_{j_0}$ is $(1, 2)$. Summarising, S and h can be expressed in the form (3.6).

The proof of the Lemma is completed. □

Lemma 3.7. *If S and h have the form (3.5) then*

$$R(e_{2n-1}, e_{2n})Se_{2n} = -\det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix} e_{2n-1}, \tag{3.9}$$

$$R(e_{2n-1}, e_{2n})Se_{2n-1} = -\det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix} e_{2n}. \tag{3.10}$$

Proof. By (3.5), we have

$$Se_{2n-1} = \alpha e_{2n-1} - \gamma e_{2n}, \quad Se_{2n} = \gamma e_{2n-1} + \beta e_{2n}.$$

Now, the Gauss equation and the form of h imply that

$$\begin{aligned} R(e_{2n-1}, e_{2n})Se_{2n} &= h(e_{2n}, Se_{2n})Se_{2n-1} - h(e_{2n-1}, Se_{2n})Se_{2n} \\ &= -\beta Se_{2n-1} - \gamma Se_{2n} = -(\beta\alpha + \gamma^2)e_{2n-1}, \end{aligned}$$

what proves (3.9). In a similar way, we get

$$\begin{aligned} R(e_{2n-1}, e_{2n})Se_{2n-1} &= h(e_{2n}, Se_{2n-1})Se_{2n-1} - h(e_{2n-1}, Se_{2n-1})Se_{2n} \\ &= \gamma Se_{2n-1} - \alpha Se_{2n} = -(\beta\alpha + \gamma^2)e_{2n}, \end{aligned}$$

what proves (3.10). □

Lemma 3.8. *If S and h are of the form (3.5) then for every $k \geq 1$ we have*

$$\begin{aligned} &R^{2k}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{4k}, e_i, e_{2n}) \\ &\quad = \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_i, e_{2n}) \end{aligned} \tag{3.11}$$

if $i < 2n - 1$,

$$\begin{aligned} &R^{2k+1}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{4k}, e_1, X, e_1, X) \\ &\quad = 4^k\gamma \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_{2n-1}, e_{2n}) \end{aligned} \tag{3.12}$$

for $X = e_{2n-1}$ or $X = e_{2n}$,

$$\begin{aligned} &R^{2k+1}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{4k}, e_1, X, e_1, Y) \\ &\quad = 2 \cdot 4^{k-1}(\alpha - \beta) \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_{2n-1}, e_{2n}), \end{aligned} \tag{3.13}$$

for $X = e_{2n-1}$ and $Y = e_{2n}$ or $X = e_{2n}$ and $Y = e_{2n-1}$.

Proof. First note that, by the Gauss equation, we have

$$\begin{aligned} R(e_{2n-1}, e_{2n})e_{2n-1} &= -Se_{2n}, \\ R(e_{2n-1}, e_{2n})e_{2n} &= -Se_{2n-1}, \\ R(e_{2n-1}, e_{2n})e_i &= 0 \end{aligned}$$

for $i < 2n - 1$. For every $k \geq 1$, we have

$$\begin{aligned} &R^{2k}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{4k}, e_i, e_{2n}) \\ &\quad = -R^{2k-1}\omega(R(e_{2n-1}, e_{2n})e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}) \\ &\qquad - R^{2k-1}\omega(e_{2n-1}, R(e_{2n-1}, e_{2n})e_{2n}, \dots, e_i, e_{2n}) \\ &\qquad \dots \\ &\qquad - R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, \underbrace{R(e_{2n-1}, e_{2n})e_i}_{0}, e_{2n}) \\ &\qquad - R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, R(e_{2n-1}, e_{2n})e_{2n}) \\ &\quad = \gamma R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}) \\ &\qquad - \gamma R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}) \\ &\qquad + \gamma R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}) \\ &\qquad - \gamma R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}) \\ &\qquad \dots \\ &\qquad + R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, Se_{2n-1}) \\ &\quad = R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, Se_{2n-1}), \end{aligned}$$

since in the above sum, terms $1, 3, \dots, 4k-1$ have an opposite sign to terms $2, 4, \dots, 4k$. In a similar way, we get

$$\begin{aligned}
&R^{2k-1}\omega(e_{2n-1}, e_{2n}, \dots, e_i, Se_{2n-1})\\
&\quad = -R^{2k-2}\omega(e_{2n-1}, e_{2n}, \dots, e_i, R(e_{2n-1}, e_{2n})Se_{2n-1})\\
&\quad = \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix} R^{2k-2}\omega(e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}),
\end{aligned}$$

where the last equality is a consequence of Lemma 3.7. In particular, for $k = 1$, we get

$$R^2\omega(e_{2n-1}, e_{2n}, e_{2n-1}, e_{2n}, e_i, e_{2n}) = \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}\omega(e_i, e_{2n}).$$

Now, assume that the formula (3.11) is true for some $k \geq 1$. Then, for $k+1$, we get

$$\begin{aligned}
&R^{2k+2}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{4k+4}, e_i, e_{2n})\\
&\quad = \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix} R^{2k}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{4k}, e_i, e_{2n})\\
&\quad = \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^{k+1} \omega(e_i, e_{2n}).
\end{aligned}$$

Now, by the induction principle, the formula (3.11) holds for any $k \geq 1$. Proofs for (3.12) and (3.13) we do in parallel. In a similar way, like in the proof of (3.11), one may note that for every X, Y and for any $l \geq 1$ we have

$$\begin{aligned}
&R^{l+1}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{2l}, e_1, X, e_1, Y)\\
&\quad = -R^{l}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{2l-2}, e_1, R(e_{2n-1}, e_{2n})X, e_1, Y)\\
&\qquad - R^{l}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{2l-2}, e_1, X, e_1, R(e_{2n-1}, e_{2n})Y)
\end{aligned}$$

Let us denote

$$A_l(X, Y) := R^{l}\omega(\underbrace{e_{2n-1}, e_{2n}, \dots, e_{2n-1}, e_{2n}}_{2l-2}, e_1, X, e_1, Y)$$

for $l \geq 1$. Then, the last equality can be rewritten as follows:

$$A_{l+1}(X, Y) = -A_l(R(e_{2n-1}, e_{2n})X, Y) - A_l(X, R(e_{2n-1}, e_{2n})Y). \tag{3.14}$$

When $X, Y \in \{e_{2n-1}, e_{2n}\}$ then directly from the form of S and h as well as the formula (3.14) and the Gauss equation, we obtain the following four recursive formulas:

$$A_{l+1}(e_{2n-1}, e_{2n-1}) = 2\gamma A_l(e_{2n-1}, e_{2n-1}) + \beta A_l(e_{2n}, e_{2n-1}) + \beta A_l(e_{2n-1}, e_{2n}), \tag{3.15}$$

$$A_{l+1}(e_{2n-1}, e_{2n}) = \beta A_l(e_{2n}, e_{2n}) + \alpha A_l(e_{2n-1}, e_{2n-1}), \tag{3.16}$$

$$A_{l+1}(e_{2n}, e_{2n-1}) = \alpha A_l(e_{2n-1}, e_{2n-1}) + \beta A_l(e_{2n}, e_{2n}), \tag{3.17}$$

$$A_{l+1}(e_{2n}, e_{2n}) = -2\gamma A_l(e_{2n}, e_{2n}) + \alpha A_l(e_{2n}, e_{2n-1}) + \alpha A_l(e_{2n-1}, e_{2n}). \tag{3.18}$$

Since for $X, Y \in \{e_{2n-1}, e_{2n}\}$, we have

$$A_1(X, Y) = -\omega(R(e_1, X)e_1, Y) - \omega(e_1, R(e_1, X)Y) = \omega(SX, Y),$$

so we also obtain the following:

$$\begin{aligned}
A_1(e_{2n-1}, e_{2n-1}) &= \gamma\omega(e_{2n-1}, e_{2n}),\\
A_1(e_{2n-1}, e_{2n}) &= \alpha\omega(e_{2n-1}, e_{2n}),\\
A_1(e_{2n}, e_{2n-1}) &= -\beta\omega(e_{2n-1}, e_{2n}),\\
A_1(e_{2n}, e_{2n}) &= \gamma\omega(e_{2n-1}, e_{2n}).
\end{aligned}$$

Now, using the above formulas and the recursive formulas (3.15)–(3.18), we can compute

$$
\begin{aligned}
A_2(e_{2n-1}, e_{2n-1}) &= (2\gamma^2 - \beta^2 + \alpha\beta)\omega(e_{2n-1}, e_{2n}),\\
A_2(e_{2n-1}, e_{2n}) &= \gamma(\beta + \alpha)\omega(e_{2n-1}, e_{2n}),\\
A_2(e_{2n}, e_{2n-1}) &= \gamma(\beta + \alpha)\omega(e_{2n-1}, e_{2n}),\\
A_2(e_{2n}, e_{2n}) &= (-2\gamma^2 + \alpha^2 - \alpha\beta)\omega(e_{2n-1}, e_{2n}),
\end{aligned}
$$

and

$$A_3(e_{2n-1}, e_{2n-1}) = 4\gamma(\gamma^2 + \alpha\beta)\omega(e_{2n-1}, e_{2n}), \tag{3.19}$$

$$A_3(e_{2n-1}, e_{2n}) = 2(\alpha - \beta)(\gamma^2 + \alpha\beta)\omega(e_{2n-1}, e_{2n}), \tag{3.20}$$

$$A_3(e_{2n}, e_{2n-1}) = 2(\alpha - \beta)(\gamma^2 + \alpha\beta)\omega(e_{2n-1}, e_{2n}), \tag{3.21}$$

$$A_3(e_{2n}, e_{2n}) = 4\gamma(\gamma^2 + \alpha\beta)\omega(e_{2n-1}, e_{2n}). \tag{3.22}$$

Moreover, (3.16) and (3.17) imply that $A_l(e_{2n}, e_{2n-1}) = A_l(e_{2n-1}, e_{2n})$ for $l \geq 2$. Thus, for $l \geq 2$, we have

$$A_{l+1}(e_{2n-1}, e_{2n-1}) = 2\gamma A_l(e_{2n-1}, e_{2n-1}) + 2\beta A_l(e_{2n-1}, e_{2n}), \tag{3.23}$$

$$A_{l+1}(e_{2n-1}, e_{2n}) = \alpha A_l(e_{2n-1}, e_{2n-1}) + \beta A_l(e_{2n}, e_{2n}), \tag{3.24}$$

$$A_{l+1}(e_{2n}, e_{2n-1}) = \alpha A_l(e_{2n-1}, e_{2n-1}) + \beta A_l(e_{2n}, e_{2n}), \tag{3.25}$$

$$A_{l+1}(e_{2n}, e_{2n}) = -2\gamma A_l(e_{2n}, e_{2n}) + 2\alpha A_l(e_{2n-1}, e_{2n}). \tag{3.26}$$

The formulas (3.19)–(3.22) imply that (3.12) as well as (3.13) hold for $k = 1$. Assume now that (3.12) and (3.13) hold for some $k \geq 1$. By the formulas (3.23)–(3.26), we obtain

$$
\begin{aligned}
A_{2k+3}(e_{2n-1}, e_{2n-1}) &= 2\gamma A_{2k+2}(e_{2n-1}, e_{2n-1}) + 2\beta A_{2k+2}(e_{2n-1}, e_{2n})\\
&= 2\gamma(2\gamma A_{2k+1}(e_{2n-1}, e_{2n-1}) + 2\beta A_{2k+1}(e_{2n-1}, e_{2n}))\\
&\quad + 2\beta(\alpha A_{2k+1}(e_{2n-1}, e_{2n-1}) + \beta A_{2k+1}(e_{2n}, e_{2n}))\\
&= (4\gamma^2 + 2\alpha\beta + 2\beta^2)A_{2k+1}(e_{2n-1}, e_{2n-1})\\
&\quad + 4\beta\gamma A_{2k+1}(e_{2n-1}, e_{2n}),
\end{aligned}
$$

since $A_{2k+1}(e_{2n-1}, e_{2n-1}) = A_{2k+1}(e_{2n}, e_{2n})$ by (3.12) (by assumption it holds for k). Now, using the formulas (3.12) and (3.13), we obtain

$$
\begin{aligned}
A_{2k+3}(e_{2n-1}, e_{2n-1}) &= (4\gamma^2 + 2\alpha\beta + 2\beta^2)4^k\gamma \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_{2n-1}, e_{2n})\\
&\quad + 4\gamma\beta 2 \cdot 4^{k-1}(\alpha - \beta)\det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_{2n-1}, e_{2n})\\
&= 4^{k+1}\gamma \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^{k+1} \omega(e_{2n-1}, e_{2n}).
\end{aligned}
$$

In a similar way, we get

$$
\begin{aligned}
A_{2k+3}(e_{2n-1}, e_{2n}) &= \alpha A_{2k+2}(e_{2n-1}, e_{2n-1}) + \beta A_{2k+2}(e_{2n}, e_{2n})\\
&= \alpha(2\gamma A_{2k-1}(e_{2n-1}, e_{2n-1}) + 2\beta A_{2k-1}(e_{2n-1}, e_{2n}))\\
&\quad + \beta(-2\gamma A_{2k-1}(e_{2n}, e_{2n}) + 2\alpha A_{2k-1}(e_{2n-1}, e_{2n}))\\
&= (2\alpha\gamma - 2\beta\gamma)A_{2k-1}(e_{2n-1}, e_{2n-1}) + 4\alpha\beta A_{2k-1}(e_{2n-1}, e_{2n})\\
&= 2 \cdot 4^k(\alpha - \beta)\det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^{k+1} \omega(e_{2n-1}, e_{2n})
\end{aligned}
$$

and

$$
\begin{aligned}
A_{2k+3}(e_{2n}, e_{2n}) &= -2\gamma A_{2k+2}(e_{2n}, e_{2n}) + 2\alpha A_{2k+2}(e_{2n-1}, e_{2n})\\
&= -2\gamma(-2\gamma A_{2k+1}(e_{2n}, e_{2n}) + 2\alpha A_{2k+1}(e_{2n-1}, e_{2n}))\\
&\quad + 2\alpha(\alpha A_{2k+1}(e_{2n-1}, e_{2n-1}) + \beta A_{2k+1}(e_{2n}, e_{2n}))\\
&= (4\gamma^2 + 2\alpha\beta + 2\alpha^2)A_{2k+1}(e_{2n-1}, e_{2n-1}) - 4\alpha\gamma A_{2k+1}(e_{2n-1}, e_{2n})\\
&= 4^{k+1}\gamma \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^{k+1} \omega(e_{2n-1}, e_{2n}) = A_{2k+3}(e_{2n-1}, e_{2n-1}).
\end{aligned}
$$

Because we also have $A_{2k+3}(e_{2n}, e_{2n-1}) = A_{2k+3}(e_{2n-1}, e_{2n})$ so, in consequence, we have proven that the formulas (3.12) and (3.13) hold for $k + 1$. Now, by the induction principle, these formulas hold for every $k \geq 1$. The proof is completed. □

Lemma 3.9. *If S and h are of the form (3.6) then for every $k \geq 1$, $i \neq 2n-2$, $i < 2n$ we have*

$$R^k\omega(\underbrace{e_{2n-2}, e_{2n}, \ldots, e_{2n-2}, e_{2n}}_{2k}, e_i, e_{2n}) = \alpha^k\omega(e_i, e_{2n}) \tag{3.27}$$

and

$$R^k\omega(\underbrace{X_1, e_{2n}, \ldots, X_k, e_{2n}}_{2k}, e_i, e_{2n}) = 0, \tag{3.28}$$

where $X_s \in \{e_{2n-2}, e_{2n-1}\}$ for $s = 1, \ldots, k$ and for some s we have $X_s = e_{2n-1}$. If additionally we assume that $\omega(e_{2n-1}, e_{2n}) = 0$ then the following formulas hold:

$$R^k\omega(\underbrace{X_1, e_{2n}, \ldots, X_k, e_{2n}}_{2k}, e_{2n-2}, e_{2n}) = 0, \tag{3.29}$$

where $X_s \in \{e_{2n-2}, e_{2n-1}\}$ for $s = 1, \ldots, k$ and $k \geq 1$.

$$R^k\omega(e_{2n-1}, e_{2n}, \underbrace{e_{2n-2}, e_{2n}, \ldots, e_{2n-2}, e_{2n}}_{2k-2}, e_{2n-2}, e_{2n-1}) = (-1)^{k-1}\alpha^k\omega(e_{2n-2}, e_{2n}) \tag{3.30}$$

for $k \geq 2$.

$$R^k\omega(\underbrace{X_1, e_{2n}, \ldots, X_k, e_{2n}}_{2k}, e_{2n-2}, e_{2n-1}) = 0, \tag{3.31}$$

where $X_s \in \{e_{2n-2}, e_{2n-1}\}$ for $s = 1, \ldots, k$, $k \geq 2$ and for some $s \neq 1$ we have $X_s = e_{2n-1}$.

Proof. (3.6) and the Gauss equation imply

$$\begin{aligned} R(e_{2n-2}, e_{2n})e_{2n-2} &= Se_{2n-2} = \alpha e_{2n-2} + e_{2n-1}, \\ R(e_{2n-2}, e_{2n})e_{2n} &= -Se_{2n} = -\alpha e_{2n}. \end{aligned}$$

For $i < 2n-2$ or $i = 2n-1$ we have

$$R(e_{2n-2}, e_{2n})e_i = 0.$$

Moreover

$$R(e_{2n-1}, e_{2n})e_{2n} = 0, \quad R(e_{2n-1}, e_{2n})e_{2n-1} = -Se_{2n} = -\alpha e_{2n}$$

and

$$R(e_{2n-1}, e_{2n})e_{2n-2} = Se_{2n-1} = \alpha e_{2n-1} + e_{2n}.$$

By straightforward computations, we get

$$\begin{aligned} R\omega(e_{2n-2}, e_{2n}, e_i, e_{2n}) = {}& -\omega(R(e_{2n-2}, e_{2n})e_i, e_{2n}) \\ & - \omega(e_i, R(e_{2n-2}, e_{2n})e_{2n}) = \alpha\omega(e_i, e_{2n}) \end{aligned}$$

and

$$\begin{aligned} R\omega(e_{2n-1}, e_{2n}, e_i, e_{2n}) = {}& -\omega(R(e_{2n-1}, e_{2n})e_i, e_{2n}) \\ & - \omega(e_i, R(e_{2n-1}, e_{2n})e_{2n}) = 0. \end{aligned}$$

Thus the formulas (3.27) and (3.28) hold for $k = 1$. Assume now that these formulas hold for some $k \geq 1$. Then,

$$\begin{aligned} & R^{k+1}\omega(\underbrace{e_{2n-2}, e_{2n}, \ldots, e_{2n-2}, e_{2n}}_{2k+2}, e_i, e_{2n}) \\ & \quad = -\underbrace{R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \ldots, e_i, e_{2n})}_{0} \\ & \qquad - \underbrace{R^k\omega(e_{2n-2}, e_{2n}, e_{2n-1}, e_{2n}, \ldots, e_i, e_{2n})}_{0} \\ & \qquad \ldots \\ & \qquad + \alpha R^k\omega(e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n}, \ldots, e_i, e_{2n}) \\ & \quad = \alpha^{k+1}\omega(e_i, e_{2n}), \end{aligned}$$

where the last equality is an immediate consequence of (3.27) and (3.28). We also have

$$
\begin{aligned}
&R^{k+1}\omega(\underbrace{X_0, e_{2n}, X_1, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_i, e_{2n})\\
&\quad = -R^k\omega(R(X_0, e_{2n})X_1, e_{2n}, \dots, X_k, e_{2n}, e_i, e_{2n})\\
&\qquad - R^k\omega(X_1, R(X_0, e_{2n})e_{2n}, \dots, X_k, e_{2n}, e_i, e_{2n})\\
&\qquad \dots\\
&\qquad - R^k\omega(X_1, e_{2n}, \dots, R(X_0, e_{2n})X_k, e_{2n}, e_i, e_{2n})\\
&\qquad - R^k\omega(X_1, e_{2n}, \dots, X_k, R(X_0, e_{2n})e_{2n}, e_i, e_{2n})\\
&\qquad - R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, R(X_0, e_{2n})e_i, e_{2n})\\
&\qquad - R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, e_i, R(X_0, e_{2n})e_{2n}).
\end{aligned}
$$

Note, that summand containing $R(X_0, e_{2n})e_i$ is equal to zero. Indeed, if $X_0 = e_{2n-2}$ then $R(X_0, e_{2n})e_i = 0$, if $X_0 = e_{2n-1}$ then $R(X_0, e_{2n})e_i = 0$ for $i < 2n - 2$. This is because e_i is orthogonal to X_0 and e_{2n}. If $i = 2n - 1$ then $R(X_0, e_{2n})e_i = -\alpha e_{2n}$ and zeroing comes from antisymmetry of $R^k\omega$ relative to the last two variables. In order to show that also all other summands of the above sum are equal to zero we need to consider the following two cases:

(1) $X_0 = e_{2n-1}$. In this case, all summands containing $R(X_0, e_{2n})e_{2n}$ are equal to zero since $R(X_0, e_{2n})e_{2n} = 0$. Summands with $R(X_0, e_{2n})e_{2n-1}$ are also equal to zero because $R(X_0, e_{2n})e_{2n-1} = -\alpha e_{2n}$ and $R^k\omega$ is antisymmetric relative to s and $s + 1$ variable for $s = 1, 3, \dots, 2k + 1$. Whereas, if for some s we have $X_s = e_{2n-1}$ then $R(X_0, e_{2n})X_s = \alpha e_{2n-1} + e_{2n}$ and in particular we have

$$
\begin{aligned}
&R^k\omega(\dots, R(X_0, e_{2n})X_s, e_{2n}, \dots, e_i, e_{2n})\\
&\quad = \alpha R^k\omega(\dots, e_{2n-1}, e_{2n}, \dots, e_i, e_{2n}) = 0
\end{aligned}
$$

by (3.28).

(2) $X_0 = e_{2n-2}$. In this case, summands with $R(X_0, e_{2n})e_{2n-1}$ are equal to zero since $R(X_0, e_{2n})e_{2n-1} = 0$. For summands containing $R(X_0, e_{2n})e_{2n}$, we have

$$R^k\omega(\dots, R(X_0, e_{2n})e_{2n}, \dots) = -\alpha R^k\omega(\dots, e_{2n}, \dots) = 0,$$

since $R(X_0, e_{2n})e_{2n} = -\alpha e_{2n}$ and the last equality can be deduced from the fact that there exists $s > 0$ such that $X_s = e_{2n-1}$ and from the formula (3.28). Since $R(X_0, e_{2n})e_{2n-2} = \alpha e_{2n-2} + e_{2n-1}$ thus, in case of summands containing $R(X_0, e_{2n})e_{2n-2}$, we have

$$
\begin{aligned}
&R^k\omega(\dots, R(X_0, e_{2n})e_{2n-2}, e_{2n}, \dots)\\
&\quad = \alpha R^k\omega(\dots, e_{2n-2}, e_{2n}, \dots)\\
&\qquad + R^k\omega(\dots, e_{2n-1}, e_{2n}, \dots) = 0.
\end{aligned}
$$

This is because both summands on the right hand side contain at some position e_{2n-1} so we can use (3.28).

In this way, we have shown that

$$R^{k+1}\omega(\underbrace{X_0, e_{2n}, X_1, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_i, e_{2n}) = 0.$$

Now, by the induction principle, the formulas (3.27) and (3.28) hold for every $k \geq 1$.

In order to prove (3.29) note that

$$
\begin{aligned}
&R\omega(e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n})\\
&\quad = -\omega(R(e_{2n-2}, e_{2n})e_{2n-2}, e_{2n}) - \omega(e_{2n-2}, R(e_{2n-2}, e_{2n})e_{2n})\\
&\quad = -\omega(e_{2n-1}, e_{2n}) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&R\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n})\\
&\quad = -\omega(R(e_{2n-1}, e_{2n})e_{2n-2}, e_{2n}) - \omega(e_{2n-2}, R(e_{2n-1}, e_{2n})e_{2n})\\
&\quad = -\alpha\omega(e_{2n-1}, e_{2n}) = 0.
\end{aligned}
$$

That is, the formula (3.29) holds for $k = 1$. Now, assume that it holds for some $k \geq 1$. Then,

$$
\begin{aligned}
&R^{k+1}\omega(X_0, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n})\\
&\quad = -R^k\omega(R(X_0, e_{2n})X_1, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n})
\end{aligned}
$$

$$
\begin{aligned}
&- R^k\omega(X_1, R(X_0, e_{2n})e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n}) \\
&\dots \\
&- R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, R(X_0, e_{2n})e_{2n-2}, e_{2n}) \\
&- R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, R(X_0, e_{2n})e_{2n}).
\end{aligned}
$$

If $X_0 = e_{2n-2}$ then

$$R(X_0, e_{2n})e_{2n} = -\alpha e_{2n}$$

and

$$
\begin{aligned}
R(X_0, e_{2n})X_s &= 0 && \text{for } X_s = e_{2n-1}, \\
R(X_0, e_{2n})X_s &= \alpha e_{2n-2} + e_{2n-1} && \text{for } X_s = e_{2n-2}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&R^{k+1}\omega(X_0, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n}) \\
&\quad = -R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, R(X_0, e_{2n})e_{2n-2}, e_{2n}) \\
&\quad = -R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, e_{2n-1}, e_{2n}),
\end{aligned}
$$

since all remaining summands can be expressed as a linear combination of summands of the form (3.29) and by assumption they all are equal to zero. Note also that the last summand in the above equality is equal to zero either by virtue of (3.27) (and assumption $\omega(e_{2n-1}, e_{2n}) = 0$) or by virtue of (3.28).

If $X_0 = e_{2n-1}$ then

$$R(X_0, e_{2n})e_{2n} = 0$$

and

$$
\begin{aligned}
R(X_0, e_{2n})X_s &= -\alpha e_{2n} && \text{dla } X_s = e_{2n-1}, \\
R(X_0, e_{2n})X_s &= \alpha e_{2n-1} + e_{2n} && \text{dla } X_s = e_{2n-2}.
\end{aligned}
$$

Now, in the similar way like for $X_0 = e_{2n-2}$, we obtain

$$
\begin{aligned}
&R^{k+1}\omega(X_0, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n}) \\
&\quad = -R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, R(X_0, e_{2n})e_{2n-2}, e_{2n}) \\
&\quad = -\alpha R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, e_{2n-1}, e_{2n}) = 0
\end{aligned}
$$

by virtue of (3.27) or (3.28). In this way, we have shown that the formula (3.29) holds for $k+1$ thus by the induction principle this formula holds for every $k \geq 1$.

In order to prove (3.30) and (3.31) note first that we have

$$
\begin{aligned}
&R^2\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1}) \\
&\quad = -\alpha^2\omega(e_{2n-2}, e_{2n}) - \alpha \underbrace{\omega(e_{2n-1}, e_{2n})}_{0}, \\
&R^2\omega(e_{2n-2}, e_{2n}, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1}) \\
&\quad = \alpha R\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1}) \\
&\qquad - \alpha R\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1}) = 0, \\
&R^2\omega(e_{2n-1}, e_{2n}, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1}) \\
&\quad = -\alpha^2\omega(e_{2n-1}, e_{2n}) = 0.
\end{aligned}
$$

Thus, the formulas (3.30)–(3.31) hold for $k = 2$. Assume now that they hold for some $k \geq 2$. Then,

$$
\begin{aligned}
&R^{k+1}\omega(e_{2n-1}, e_{2n}, \underbrace{e_{2n-2}, e_{2n}, \dots, e_{2n-2}, e_{2n}}_{2k}, e_{2n-2}, e_{2n-1}) \\
&\quad = -\alpha R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \dots, e_{2n-2}, e_{2n-1}) \\
&\qquad - \alpha R^k\omega(e_{2n-2}, e_{2n}, e_{2n-1}, e_{2n}, \dots, e_{2n-2}, e_{2n-1}) \\
&\qquad \dots \\
&\qquad - \alpha R^k\omega(e_{2n-2}, e_{2n}, \dots, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1}) \\
&\qquad + R^k\omega(e_{2n-2}, e_{2n}, \dots, e_{2n-1}, e_{2n})
\end{aligned}
$$

$$+\,\alpha R^k\omega(e_{2n-2}, e_{2n}, \dots, e_{2n-2}, e_{2n})$$
$$= (-1)^k\alpha^{k+1}\omega(e_{2n-2}, e_{2n}),$$

since all summands but the first one are equal to zero by the virtue of (3.31), (3.27) (case $i = 2n-1$) and (3.29). Thus, the formula (3.30) holds for $k+1$. In a similar way, we compute

$$\begin{aligned}
&R^{k+1}\omega(\underbrace{X_0, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_{2n-2}, e_{2n-1})\\
&= -R^k\omega(R(X_0, e_{2n})X_1, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad - R^k\omega(X_1, R(X_0, e_{2n})e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad \dots\\
&\quad - R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, R(X_0, e_{2n})e_{2n-2}, e_{2n-1})\\
&\quad - R^k\omega(X_1, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, R(X_0, e_{2n})e_{2n-1}).
\end{aligned}$$

By assumption, there exists $s \geq 1$ such that $X_s = e_{2n-1}$.

Let $X_0 = e_{2n-2}$. We have the following possibilities:

(1) $X_1 \neq e_{2n-1}$.
(2) $X_1 = e_{2n-1}$ and for some $s > 1$, we have $X_s = e_{2n-1}$.
(3) $X_1 = e_{2n-1}$ and $X_2 = \dots = X_k = e_{2n-2}$.

In the case (1) or (2), the above sum (similarly like in the proof of (3.29)) is a linear combination of summands of the form (3.31) so, by assumption, is equal to zero. In the case (3), we obtain

$$\begin{aligned}
&R^{k+1}\omega(\underbrace{X_0, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_{2n-2}, e_{2n-1})\\
&= k\alpha R^k\omega(e_{2n-1}, e_{2n}, \dots, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad - (k-1)\alpha R^k\omega(e_{2n-1}, e_{2n}, \dots, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad - R^k\omega(e_{2n-1}, e_{2n}, e_{2n-1}, e_{2n}, \dots, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad \dots\\
&\quad - R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \dots, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad - \alpha R^k\omega(e_{2n-1}, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n-1})\\
&= -R^k\omega(e_{2n-1}, e_{2n}, e_{2n-1}, e_{2n}, \dots, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad \dots\\
&\quad - R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \dots, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1}) = 0,
\end{aligned}$$

where the last equality is a consequence of the formula (3.31) for k. In this way, we have shown that if $X_0 = e_{2n-2}$ then

$$R^{k+1}\omega(\underbrace{X_0, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_{2n-2}, e_{2n-1}) = 0.$$

In case of $X_0 = e_{2n-1}$ we have

$$\begin{aligned}
&R^{k+1}\omega(\underbrace{X_0, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_{2n-2}, e_{2n-1})\\
&= -R^k\omega(R(X_0, e_{2n})X_1, e_{2n}, X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad - R^k\omega(X_1, e_{2n}, R(X_0, e_{2n})X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad \dots\\
&\quad - R^k\omega(X_1, e_{2n}, X_2, e_{2n}, \dots, R(X_0, e_{2n})X_k, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad + R^k\omega(X_1, e_{2n}, X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-1}, e_{2n})\\
&\quad + \alpha R^k\omega(X_1, e_{2n}, X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n}),
\end{aligned}$$

since $R(X_0, e_{2n})e_{2n} = 0$ and $R(X_0, e_{2n})e_{2n-1} = -\alpha e_{2n}$. Now, in a similar way like in the previous case, we have the following possibilities:

(1) For some $s > 1$, we have $X_s = e_{2n-1}$.
(2) $X_1 = e_{2n-1}$ and $X_2 = \dots = X_k = e_{2n-2}$.

In case of (1) by assumption, we have that all summands of the above sum (except the last two) are equal to zero. In case of (2), we obtain

$$\begin{aligned}
&R^{k+1}\omega(\underbrace{X_0, e_{2n}, \dots, X_k, e_{2n}}_{2k+2}, e_{2n-2}, e_{2n-1})\\
&\quad - R^k\omega(e_{2n-1}, e_{2n}, R(X_0, e_{2n})e_{2n-2}, e_{2n}, \dots, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad \dots\\
&\quad - R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \dots, R(X_0, e_{2n})e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad + R^k\omega(X_1, e_{2n}, X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-1}, e_{2n})\\
&\quad + \alpha R^k\omega(X_1, e_{2n}, X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-2}, e_{2n})\\
&= -\alpha R^k\omega(e_{2n-1}, e_{2n}, e_{2n-1}, e_{2n}, \dots, e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad \dots\\
&\quad - \alpha R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \dots, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad - R^k\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n}, \dots, R(X_0, e_{2n})e_{2n-2}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad + R^k\omega(X_1, e_{2n}, X_2, e_{2n}, \dots, X_k, e_{2n}, e_{2n-1}, e_{2n}).
\end{aligned}$$

Also, in this case, all summands of the above sum (except the last two) are equal to zero.

Now it is enough to note that zeroing of the last two summands is an immediate consequence of (3.27), (3.28) and (3.29). Summarising we have shown that the formula (3.31) holds for $k+1$. Now, by the induction principle, the formulas (3.30) and (3.31) hold for every $k \geq 2$. The proof of the lemma is completed. □

Lemma 3.10. *If S and h are of the form (3.6) and $\alpha = 0$ then for every $k \in \mathbb{N}$, $k \geq 1$ the following equalities hold:*

$$R^k\omega(\underbrace{e_{2n-1}, e_{2n-2}, \dots, e_{2n-1}, e_{2n-2}}_{2k}, e_i, e_{2n-1}) = k!\omega(e_i, e_{2n-1}) \tag{3.32}$$

for $i = 1, \dots, 2n-2$ and

$$\begin{aligned}
&R^k\omega(\underbrace{e_{2n-1}, e_{2n-2}, \dots, e_{2n-1}, e_{2n-2}}_{2k-2}, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&= (k-1)!\omega(e_{2n-1}, e_{2n}).
\end{aligned} \tag{3.33}$$

Proof. The form of S and h and the Gauss equation imply that

$$R(e_{2n-1}, e_{2n-2})e_{2n-1} = -Se_{2n-2} = -e_{2n-1}$$

and

$$R(e_{2n-1}, e_{2n-2})e_i = 0$$

for $i = 1, \dots, 2n-2$. Additionally, note that

$$\begin{aligned}
R\omega(e_{2n-1}, e_{2n-2}, e_i, e_{2n-1}) &= -\omega(R(e_{2n-1}, e_{2n-2})e_i, e_{2n-1})\\
&\quad - \omega(e_i, R(e_{2n-1}, e_{2n-2})e_{2n-1})\\
&= \omega(e_i, e_{2n-1}),
\end{aligned}$$

that is the formula (3.32) holds for $k = 1$. Assume now that the formula (3.32) holds for some $k \geq 1$. Then, we have

$$\begin{aligned}
&R^{k+1}\omega(\underbrace{e_{2n-1}, e_{2n-2}, \dots, e_{2n-1}, e_{2n-2}}_{2k+2}, e_i, e_{2n-1})\\
&\quad = (k+1)R^k\omega(e_{2n-1}, e_{2n-2}, \dots, e_{2n-1}, e_{2n-2}, e_i, e_{2n-1})\\
&\quad = (k+1)!\omega(e_i, e_{2n-1}).
\end{aligned}$$

Now, by the induction principle, the formula (3.32) holds for every $k \geq 1$. In order to prove (3.33) note first that

$$R(e_{2n-1}, e_{2n})e_{2n-2} = Se_{2n-1} = e_{2n}$$

and

$$R(e_{2n-1}, e_{2n})e_{2n-1} = -Se_{2n} = 0.$$

Thus,

$$\begin{aligned}
&R\omega(e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\quad = -\omega(R(e_{2n-1}, e_{2n})e_{2n-2}, e_{2n-1}) - \omega(e_{2n-2}, R(e_{2n-1}, e_{2n})e_{2n-1})\\
&\quad = \omega(e_{2n}, e_{2n-1}),
\end{aligned}$$

that is the formula (3.33) holds for $k = 1$. Assume now that it holds for some $k \geq 1$. Then,

$$\begin{aligned}
&R^{k+1}\omega(\underbrace{e_{2n-1}, e_{2n-2}, \ldots, e_{2n-1}, e_{2n-2}}_{2k}, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\qquad = k \cdot R^k\omega(\underbrace{e_{2n-1}, e_{2n-2}, \ldots, e_{2n-1}, e_{2n-2}}_{2k-2}, e_{2n-1}, e_{2n}, e_{2n-2}, e_{2n-1})\\
&\qquad = k(k-1)!\omega(e_{2n-1}, e_{2n}) = k!\omega(e_{2n-1}, e_{2n}).
\end{aligned}$$

Now, by the induction principle, the formula (3.33) holds for ever $k \geq 1$. The proof is concluded. □

From Lemmas 3.6, 3.9 and 3.10, we obtain the following lemma:

Lemma 3.11. *Let $f : M \to \mathbb{R}^{2n+1}$ be an affine hypersurface with a Lorentzian second fundamental form h and with an almost symplectic form ω. If $R^k\omega = 0$ for some $k \geq 1$ then for every point $x \in M$ there exists a basis $e_1, \ldots, e_{2n}$ of T_xM such that the shape operator S and the second fundamental form h can be expressed in this basis either in the form (3.4) or in the form (3.5).*

Proof. Without loss of generality, we may assume that $k \geq 2$. By Lemma 3.6, S and h can be expressed in the form (3.4), (3.5) or (3.6). We will show that the case (3.6) is impossible. Since ω is non-degenerate, there exists $i < 2n$ such that $\omega(e_i, e_{2n}) \neq 0$. For $i = 1, \ldots, 2n-3$ and for $i = 2n-1$, since $R^k\omega = 0$, we have that $\alpha^k\omega(e_i, e_{2n}) = 0$ by virtue of Lemma 3.9. Now, it follows that $\alpha = 0$. If $\omega(e_i, e_{2n}) \neq 0$ only for $i = 2n-2$, then in particular we have that $\omega(e_{2n-1}, e_{2n}) = 0$. Now, using again Lemma 3.9 (formula (3.30)), we obtain

$$(-1)^{k-1}\alpha^k\omega(e_{2n-2}, e_{2n}) = 0.$$

So again we get $\alpha = 0$. The above implies that we can use Lemma 3.10. The formulas (3.32) and (3.33) immediately imply that $\omega(e_i, e_{2n-1}) = 0$ for $i = 1, \ldots, 2n$, what contradicts the assumption that ω is non-degenerate. Summarising the case (3.6) is impossible. □

Proof of Theorem 3.5. Let $x \in M$ and let $\{e_1, \ldots, e_{2n}\}$ be the basis from Lemma 3.11. If S and h are of the form (3.4), then in the same way as in the proof of Theorem 1.2 (see [4] for details), we obtain that S is equal to zero thus $\operatorname{rank} S_x = 0$. Assume now that S and h have the form (3.5). If $R^k\omega = 0$ then $R^{2k}\omega = 0$ and $R^{2k+1}\omega = 0$. Denote

$$W := \det\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}.$$

Since ω is non-degenerate, we can find $i < 2n$ such that $\omega(e_i, e_{2n}) \neq 0$. If $i < 2n-1$ then by virtue of Lemma 3.8 (formula (3.11)) $W = 0$. Finally, if $\omega(e_{2n-1}, e_{2n}) \neq 0$ then again using Lemma 3.8 (formula (3.12)), we obtain $W = 0$. By assumption, we have $\dim M \geq 6$. Without loss of generality (rearranging vectors of the basis if needed) we may assume that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of S are nonzero and $\lambda_{p+1} = \cdots = \lambda_{2n-2} = 0$.

By virtue of Lemma 3.2 in a similar way like in the proof of Theorem 1.2 (see [4] for details), we show that $\lambda_3 = \lambda_4 = 0$. Now, by Lemma 3.3 (applied to $X = e_1, Z_1 = e_3, Z_2 = e_4, Y = e_i$ if $i \neq 4$ and applied to $X = e_1, Z_1 = e_4, Z_2 = e_3, Y = e_i$ if $i \neq 3$), we obtain $\lambda_1 = 0$. Thus, we have $\lambda_1 = \cdots = \lambda_{2n-2} = 0$. Summarising

$$\operatorname{rank} S_x = \operatorname{rank}\begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix} \leq 1. \quad \square$$

As a consequence of Theorem 3.5, we have the following:

Theorem 3.12. *Let $f : M \to \mathbb{R}^{2n+1}$ ($\dim M \geq 6$) be a non-degenerate affine hypersurface with a locally equiaffine transversal vector field ξ and an almost symplectic form ω. If $\nabla^k\omega = 0$ for some $k \geq 1$ and the second fundamental form is Lorentzian on M (that is has signature $(2n-1, 1)$) then the shape operator S has the rank ≤ 1.*

Proof. If $\nabla^k\omega = 0$ for some k then, of course, we have that also $\nabla^{2k}\omega = 0$ and now by Lemma 3.1, we get $R^k\omega = 0$. Now, thesis is an immediate consequence of Theorem 3.5. □

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