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Causality and Legendrian linking for higher dimensional spacetimes

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ABSTRACT

Let (X^{m+1}, g) be an $(m+1)$-dimensional globally hyperbolic spacetime with Cauchy surface M^m, and let $\widetilde{M}^m$ be the universal cover of the Cauchy surface. Let $\mathcal{N}_X$ be the contact manifold of all future directed unparameterized light rays in X that we identify with the spherical cotangent bundle ST^*M. Jointly with Stefan Nemirovski we showed that when $\widetilde{M}^m$ is **not** a compact manifold, then two points $x, y \in X$ are causally related if and only if the Legendrian spheres $\mathfrak{S}_x, \mathfrak{S}_y$ of all light rays through x and y are linked in $\mathcal{N}_X$.

In this short note we use the contact Bott–Samelson theorem of Frauenfelder, Labrousse and Schlenk to show that the same statement is true for all X for which the integral cohomology ring of a closed $\widetilde{M}$ is **not** the one of the CROSS (compact rank one symmetric space).

If M admits a Riemann metric $\overline{g}$, a point x and a number $\ell > 0$ such that all unit speed geodesics starting from x return back to x in time ℓ, then $(M, \overline{g})$ is called a Y^x_ℓ manifold. Jointly with Stefan Nemirovski we observed that causality in $(M \times \mathbb{R}, \overline{g} \oplus -t^2)$ is **not** equivalent to Legendrian linking. Every Y^x_ℓ-Riemann manifold has compact universal cover and its integral cohomology ring is the one of a CROSS. So we conjecture that Legendrian linking is equivalent to causality if and only if one can **not** put a Y^x_ℓ Riemann metric on a Cauchy surface M.

All manifolds, maps etc. are assumed to be smooth unless the opposite is explicitly stated, and the word *smooth* means C^∞.

1. Introduction

Let M be a not necessarily orientable, connected manifold of dimension $m \geq 2$ and let $\pi_M : ST^*M \to M$ be its spherical cotangent bundle. The manifold ST^*M carries a canonical co-oriented contact structure. An isotopy $\{L_t\}_{t\in[0,1]}$ of Legendrian submanifolds in a co-oriented contact manifold is called respectively *positive, non-negative* if it can be parameterized in such a way that the tangent vectors of all the trajectories of individual points lie in respectively positive, non-negative tangent half-spaces defined by the contact structure.

For the introduction of basic notions from Lorentz geometry we follow our paper [1].

Let (X^{m+1}, g) be an $(m+1)$-dimensional Lorentz manifold and $x \in X$. A nonzero $\mathbf{v} \in T_xX$ is called *timelike, spacelike, non-spacelike (causal) or null (lightlike)* if $g(\mathbf{v}, \mathbf{v})$ is respectively negative, positive, non-positive or zero. An piecewise smooth curve is timelike if all of its velocity vectors are timelike. Null and non-spacelike curves are defined similarly. The Lorentz

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manifold (X, g) has a unique Levi-Civita connection, see for example [2, page 22], so we can talk about timelike and null geodesics. A submanifold $M \subset X$ is *spacelike* if g restricted to TM is a Riemann metric.

All non-spacelike vectors in T_xX form a cone consisting of two hemicones, and a continuous with respect to $x \in X$ choice of one of the two hemicones (if it exists) is called the *time orientation* of (X, g). The vectors from the chosen hemicones are called *future directed.* A time oriented Lorentz manifold is called a *spacetime* and its points are called *events.*

For x in a spacetime (X, g) its *causal future* $J^+(x) \subset X$ is the set of all $y \in X$ that can be reached by a future pointing causal curve from x. The causal past $J^-(x)$ of the event $x \in X$ is defined similarly.

Two events x, y are said to be *causally related* if $x \in J^+(y)$ or $y \in J^+(x)$.

A spacetime is said to be *globally hyperbolic* if $J^+(x) \cap J^-(y)$ is compact for every $x, y \in X$ and if it is *causal,* i.e. it has no closed non-spacelike curves. The classical definition of global hyperbolicity requires (X, g) to be strongly causal rather than just causal, but these two definitions are equivalent, see Bernal and Sanchez [3, Theorem 3.2].

A *Cauchy surface* in (X, g) is a subset such that every inextendible nonspacelike curve $\gamma(t)$ intersects it at exactly one value of t. A classical result is that (X, g) is globally hyperbolic if and only if it has a Cauchy surface, see [4, pages 211–212]. Geroch [5] proved that every globally hyperbolic (X, g) is homeomorphic to a product of $\mathbb{R}$ and a Cauchy surface. Bernal and Sanchez [6, Theorem 1], [7, Theorem 1.1], [8, Theorem 1.2] proved that every globally hyperbolic (X^{m+1}, g) has a smooth spacelike Cauchy surface M^m and that moreover for every smooth spacelike Cauchy surface M there is a diffeomorphism $h : M \times \mathbb{R} \to X$ such that

- **a:** $h(M \times t)$ is a smooth spacelike Cauchy surface for all t,
- **b:** $h(x \times \mathbb{R})$ is a future directed timelike curve for all $x \in M$, and finally
- **c**: $h(M \times 0) = M$ with $h|_{M\times 0} : M \to M$ being the identity map.

For a spacetime X we consider its space of light rays $\mathfrak{N} = \mathfrak{N}_X$. By definition, a point $\boldsymbol{\gamma} \in \mathfrak{N}$ is an equivalence class of inextendible future-directed null geodesics up to an orientation preserving affine reparametrization.

A seminal observation of Penrose and Low [9–11] is that the space $\mathfrak{N}$ has a canonical structure of a contact manifold (see also [12–15]). A contact form α_M on $\mathfrak{N}$ defining that contact structure can be associated to any smooth spacelike Cauchy surface $M \subset X$. Namely, consider the map

$$\iota_M : \mathfrak{N}_X \hookrightarrow T^*M$$

taking $\boldsymbol{\gamma} \in \mathfrak{N}$ represented by a null geodesic $\gamma \subset X$ to the 1-form on M at the point $x = \gamma \cap M$ collinear to $\langle \dot{\gamma}(x), \cdot \rangle|_M$ and having unit length with respect to the induced Riemann metric on M. This map identifies $\mathfrak{N}$ with the unit cosphere bundle ST^*M of the Riemannian manifold M. Then

$$\alpha_M := \iota_M^* \lambda_{\text{can}},$$

where $\lambda_{\text{can}} = \sum p_k dq^k$ is the canonical Liouville 1-form on T^*M.

Remark 1.1 (*Bott–Samelson Type Result of Frauenfelder, Labrousse and Schlenk and its Strengthening*)**.** The contact Bott–Samelson [16] type result of Frauenfelder, Labrousse and Schlenk [17, Theorem 1.13] says that if there is a positive Legendrian isotopy of a fiber S_x of ST^*M to itself, then the universal cover $\widetilde{M}$ of M is compact and has the integral cohomology ring of a CROSS. (This result answers our question with Nemirovski [18, Example 8.3] and compactness of $\widetilde{M}$ was first proved in [18, Corollary 8.1].)

Our result with Nemirovski [19, Proposition 4.5] says that if there is a non-constant non-negative Legendrian isotopy of S_x to itself, then there is a positive Legendrian isotopy of S_x to itself. (Note that this positive Legendrian isotopy generally is not a perturbation of the non-negative non-constant Legendrian isotopy that was assumed.)

So we can somewhat strengthen [17, Theorem 1.13] to say that if there is a non-constant non-negative Legendrian isotopy of S_x to itself, then the universal cover $\widetilde{M}$ of M is compact and has the integral cohomology ring of a CROSS, and in this work we will have to use the strengthened version of the contact Bott–Samelson type theorem.

2. Main results

Let (X, g) be a globally hyperbolic spacetime with Cauchy surface M. For a point $x \in X$ we denote by $\mathfrak{S}_x \subset \mathfrak{N}$ the Legendrian sphere of all (unparameterized, future directed) light rays passing through x.

For two causally unrelated points $x, y \in X$ the Legendrian link $(\mathfrak{S}_x, \mathfrak{S}_y)$ in $\mathcal{N}$ does not depend on the choice of the causally unrelated points. Under the identification $\mathcal{N} = ST^*M$ this link is Legendrian isotopic to the link of sphere-fibers over two points of some (and then any) spacelike Cauchy surface M, see [20, Theorem 8], [1, Lemma 4.3] and [12]. We call a Legendrian link *trivial* if it is isotopic to such a link.

In [1, Theorem A] and [18, Theorem 10.4] we proved the following result. Assume that the universal cover $\widetilde{M}$ of a Cauchy surface of M of X is not compact and events $x, y \in X$ are causally related. Then the Legendrian link $(\mathfrak{S}_x, \mathfrak{S}_y)$ is nontrivial.

In the case where $M = \mathbb{R}^3$ this proved the Legendrian Low conjecture of Natario and Tod [12]. The question to explore relations between causality and linking was motivated by the observations of Low [10,11] and appeared on Arnold's problem lists as a problem communicated by Penrose [21, Problem 8], [22, Problem 1998-21].

In this work we prove the following theorem.

Theorem 2.1. *Assume that two events x, y in a globally hyperbolic spacetime X are causally related and the universal cover $\widetilde{M}$ of a Cauchy surface M of X is compact but does* **not** *have integral cohomology ring of a compact rank one symmetric space (CROSS). Then the Legendrian link $(\mathfrak{S}_x, \mathfrak{S}_y)$ is nontrivial.*

Proof. The beginning of the proof follows the one of our [1, Theorem A].

If x, y are on the same null geodesic then the Legendrian link $(\mathfrak{S}_x, \mathfrak{S}_y)$ has a double point and hence is singular. So we do not have to consider this case.

Without the loss of generality we can assume that $y \in J^+(x)$ and hence $\mathfrak{S}_y$ is connected to $\mathfrak{S}_x$ by a non-negative Legendrian isotopy, see [1, Proposition 4.2].

Suppose that $\mathfrak{S}_x$ and $\mathfrak{S}_y$ are Legendrian unlinked, i.e., that the link $\mathfrak{S}_x, \mathfrak{S}_y$ is Legendrian isotopic to the link $F \sqcup F'$ formed by two different fibers of ST^*M. By the Legendrian isotopy extension theorem, there exists a contactomorphism $\Psi : ST^*M \to ST^*M$ such that $\Psi(\mathfrak{S}_x \sqcup \mathfrak{S}_y) = F \sqcup F'$. Hence, we get a non-negative Legendrian isotopy connecting two different fibers F and F' of ST^*M. But the Legendrian link $F \sqcup F'$ is symmetric, i.e., it is Legendrian isotopic to $F' \sqcup F$. Hence there exists a contactomorphism of ST^*M exchanging the two link components and we also have a non-negative Legendrian isotopy connecting F' to F.

Composing the non-negative Legendrian isotopy from F to F' and from F' to F we get a non-constant non-negative Legendrian isotopy from a fiber of ST^*M to itself.

Finally Remark 1.1 says that $\widetilde{M}$ has the integral cohomology ring of a CROSS. This contradicts to our assumptions. □

Remark 2.2. Let $(M, \overline{g})$ be a Riemann manifold having a point x and a positive number ℓ such that all the unit speed geodesics from x return back to x in time ℓ, then $(M, \overline{g})$ is called a Y^x_ℓ Riemann manifold. The result of Bérard-Bergery [23, Theorem 7.37], [24] says that the universal cover $\widetilde{M}$ in this case is compact and the rational cohomology ring of $\widetilde{M}$ has exactly one generator. Clearly the co-geodesic flow on M gives a positive Legendrian isotopy of the Legendrian sphere fiber of ST^*M over x to itself. So the Bott–Samelson [16] type result of Frauenfelder, Labrousse and Schlenk [17, Theorem 1.13] says more, namely that $\widetilde{M}$ has the integral cohomology ring of a CROSS.

In [18, Example 10.5] we observed that when $(M, \overline{g})$ is a Y^x_ℓ Riemann manifold, then causality in the globally hyperbolic $(M \times \mathbb{R}, \overline{g} \oplus -dt^2)$ is **not** equivalent to Legendrian linking of sphere of light rays. In particular, causality is not equivalent to Legendrian linking in the case where $(M, \overline{g})$ is a metric quotient of the standard sphere. The Thurston Elliptization Conjecture (solved as the part of the Thurston Geometrization Conjecture) by Perelman [25–27] says that every 3-dimensional M whose universal cover is compact (and hence is the sphere by the Poincare conjecture) is a metric quotient of the sphere. **So the results of Theorem 2.1 are new and interesting only for spacetimes of dimension greater than four.**

It is not currently known whether every compact simply connected manifold whose integer cohomology ring is the one of a CROSS admits a Y^x_ℓ Riemann metric. So we make the following conjecture.

Conjecture 2.3. *Assume that a globally hyperbolic spacetime X is such that one can* **not** *put a Y^x_ℓ Riemann metric on its Cauchy surface M. Then two events $x, y \in X$ are causally related if and only if the Legendrian link $(\mathfrak{S}_x, \mathfrak{S}_y)$ is non-trivial.*

In [1, Theorem C] we proved the following result. Assume that a Cauchy surface M has a cover diffeomorphic to an open domain in $\mathbb{R}^m$ and two events $x, y \in X$ are such that $y \in J^+(x)$ but x, y do not belong to a common light geodesic. Then the Legendrian links $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ and $\mathfrak{S}_y \sqcup \mathfrak{S}_x$ are not Legendrian isotopic. (In this case there is a non-negative Legendrian isotopy from $\mathfrak{S}_y$ to $\mathfrak{S}_x$, but there is no such non-negative Legendrian isotopy from $\mathfrak{S}_x$ to $\mathfrak{S}_y$.) One of the key ingredients in the proof is that for such M there is no non-constant non-negative Legendrian isotopy of a fiber of ST^*M to itself. In [18, Corollary 8.1, Remark 8.2] we showed that for the case where the universal cover $\widetilde{M}$ of M is not a compact manifold there is no non-negative Legendrian isotopy of a fiber of ST^*M to itself. So the result and the proof of [1, Theorem C] immediately generalize to this case.

In this work we use the Bott–Samelson [16] type Theorem of Frauenfelder, Labrousse and Schlenk [17, Theorem 1.13] to get the following result.

Theorem 2.4. *Assume that two events x, y in a globally hyperbolic spacetime X are causally related and the universal cover $\widetilde{M}$ of a Cauchy surface M of X is compact but does* **not** *have integral cohomology ring of a compact rank one symmetric space (CROSS). Then the Legendrian link $(\mathfrak{S}_x, \mathfrak{S}_y)$ is not Legendrian isotopic to $(\mathfrak{S}_y, \mathfrak{S}_x)$.*

Proof. The beginning of the proof follows the one of our [1, Theorem C].

Suppose that the links $\mathfrak{S}_x, \mathfrak{S}_y$ and $\mathfrak{S}_y, \mathfrak{S}_x$ are Legendrian isotopic. By the Legendrian isotopy extension theorem, there exists an auto contactomorphism Ψ such that $\Psi(\mathfrak{S}_x, \mathfrak{S}_y) = (\mathfrak{S}_y, \mathfrak{S}_x)$. Without loss of generality we assume that y is in the causal past of x. Let $\{\Lambda_t\}_{t\in[0,1]}$ be a non-negative Legendrian isotopy in ST^*M connecting $\mathfrak{S}_x$ to $\mathfrak{S}_y$ provided by [1, Proposition 4.2]. Then $\{\Psi(\Lambda_t)\}_{t\in[0,1]}$ is a non-negative Legendrian isotopy connecting $\mathfrak{S}_y$ to $\mathfrak{S}_x$. Composing these two isotopies, we obtain a non-constant non-negative Legendrian isotopy connecting $\mathfrak{S}_x$ to itself.

Recall now that $\mathfrak{S}_x$ is Legendrian isotopic to a fiber of ST^*M. By the Legendrian isotopy extension theorem, there exists a contactomorphism Φ taking $\mathfrak{S}_x$ to that fiber. The Legendrian isotopy connecting $\mathfrak{S}_x$ to itself constructed above is taken by Φ to a non-constant non-negative Legendrian isotopy connecting the fiber to itself.

Finally Remark 1.1 says that $\widetilde{M}$ has the integral cohomology ring of a CROSS. This contradicts to our assumptions. □

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