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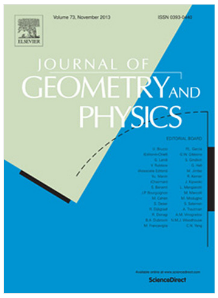

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Riemann-Hilbert problems and soliton solutions for a multi-component cubic-quintic nonlinear Schrödinger equation☆

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Abstract

In this paper, based on the zero curvature equation, an arbitrary order matrix spectral problem is studied and its associated multi-component cubic-quintic nonlinear Schrödinger integrable hierarchy is derived. In order to solve the multi-component cubic-quintic nonlinear Schrödinger system, a class of Riemann-Hilbert problem is proposed with appropriate transformation. Through the special Riemann-Hilbert problem, where the jump matrix is considered to be an identity matrix, the soliton solutions of all integrable equations are explicitly calculated. The specific examples of one-soliton, two-soliton and N-soliton solutions are explicitly presented.

Keywords: Multi-component cubic-quintic nonlinear Schrödinger equation, Integrable hierarchy, Riemann-Hilbert problem, Soliton solution

2010 MSC: 35Q53, 37K10, 35B15

1. Introduction

As a branch of the nonlinear science, soliton theory is indispensable and playing increasingly important role in the nonlinear dynamics. With the development of the soliton theory, there are many methods to solve the solutions of the integrable equations. Among them, the inverse scattering method and the Riemann-Hilbert method are are two widely used and effective methods. The original inverse scattering method is the first method to find the exact solution of the soliton equation. Based on the Gel'fand-Levitan-Marchenko integral equations, the generalized Fourier method is given and the Cauchy problem of the integrable equation is tried [1]. Later, the Riemann-Hilbert method was developed, which greatly simplifies the inverse scattering transform method and provides an equivalent but more straightforward method for solving integrable equations, especially for generating soliton solutions [2]. Based on Riemann-Hilbert problems, a dressing method has also been developed to get soliton solutions through gauge transformations [3, 4, 5], and it has been generalized for Lax operators in the orthogonal and symplectic Lie algebras [6] and further developed in numerous publications [7, 8]. In recent years, a large number of integrable equations have been discussed by solving the associated Riemann-Hilbert problems, such as, the multiple wave interaction equations [2], the coupled Kundu equation [9], the general coupled nonlinear Schrödinger equations [10], the nonlinear Schrödinger-type equation [11], the Harry Dym equation [12], the generalized Sasa-Satsuma equation [13], the Dullin-Gottwald-Holm equation [14], the modified nonlinear Schrödinger equation [15], the mKdV systems [16, 17, 18] and a six-component fourth-order AKNS system [19].

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The integrable Kundu-Eckhaus (KE) equation[20, 21, 22]

$$iq_t + q_{xx} + 2|q|^2q + \delta|q|^4q - 2i\rho_1(|q|^2)_xq = 0$$

is a generalization of the NLS equation with the cubic and quintic nonlinear terms, which contains the higher-order nonlinear terms together with the same second-order linear dispersion term, where i is the unit imaginary number, ρ_1 is a real constant, $q(x,t)$ is a complex function that represents the electromagnetic wave, $\delta = \rho_1^2$ is the quintic nonlinearity coefficient, the last term is a nonlinear term which results from the time-retarded induced Raman process and $2\rho_1$ is the nonlinear dispersion. Compared with the KE equations, the multi-component cubic-quintic nonlinear Schrödinger equations possess soliton solutions with more parametric freedom, which can be expected to model more complex situations in reality.

In this paper, we will study the multi-component cubic-quintic nonlinear Schrödinger equations by the Riemann-Hilbert method. First, We introduce an arbitrary order matrix spectral problem:

$$\varphi_x = Y\varphi, \quad \varphi_t = T^{[n]}\varphi, \quad Y = iA(\lambda) + U(\boldsymbol{u}, \lambda), \quad T^{[n]} = iB^{[n]}(\lambda) + V^{[n]}(\boldsymbol{u}, \lambda), \quad n \geq 1,$$

where λ is a spectral parameter, $\boldsymbol{u}$ is a potential vector, φ is an $(n+1)\times(n+1)$ matrix eigenfunction, $A, B^{[n]}$ are constant commuting $(n+1)\times(n+1)$ matrices, and $U, V^{[n]}$ are $(n+1)\times(n+1)$ matrices. The compatibility condition of Lax pairs is the zero curvature equation

$$Y_{t_n} - T_x^{[n]} + [Y,\ T^{[n]}] = 0,$$

where $[\cdot,\cdot]$ is the matrix commutator.

Second, we adopt the following pair of equivalent matrix spectral problems for studying the Riemann-Hilbert problems,

$$\psi_x = i[A(\lambda),\ \psi] + U(\boldsymbol{u}, \lambda)\psi, \quad \psi_{t_n} = i[B^{[n]}(\lambda),\ \psi] + V^{[n]}(\boldsymbol{u}, \lambda)\psi.$$

with the relation between two matrix φ and ψ:

$$\varphi = e^{-\frac{i}{2}\rho_1\int_{-\infty}^{x}|q|^2d\xi\Lambda}\psi E, \quad E = e^{iA(\lambda)x + iB^{[n]}(\lambda)t_n}.$$

where $\Lambda = \mathrm{diag}(-1, I_n)$. For matrix spectral problems ψ, two bounded analytical matrix eigenfunctions satisfy asymptotic conditions

$$\psi^{\pm}(x, t_n, \lambda) \to I_{n+1}, \quad \text{when} \quad x, t_n \to \pm\infty.$$

Let $\mathbb{C}^+$ represents the upper half-plane: $\mathbb{C}^+ = \{z \in \mathbb{C}|\mathrm{I_m}(z) > 0\}$, and $\mathbb{C}_0^+$ are the closures of $\mathbb{C}^+$. Similarly, $\mathbb{C}^-$ represents the lower half-plane: $\mathbb{C}^- = \{z \in \mathbb{C}|\mathrm{I_m}(z) < 0\}$, and $\mathbb{C}_0^-$ are the closures of $\mathbb{C}^-$. Based on the matrix eigenfunctions $\psi^{\pm}(x, t_n, \lambda)$, we gain two analytical matrix functions $J^{\pm}(x, t_n, \lambda)$, which are analytical in $\lambda \in \mathbb{C}^{\pm}$ and continuous at $\mathbb{C}_0^{\pm}$, respectively. Then we gain a matrix Riemann-Hilbert problem:

$$J^-(x, t_n, \lambda)J^+(x, t_n, \lambda) = G(x, t_n, \lambda), \quad \lambda \in \mathbb{R},$$

with

$$G(x, \lambda) = E(H_1 + H_2S(\lambda))(H_1 + S^{-1}(\lambda)H_2)E^{-1},$$

where S and S^{-1} are scattering matrices. The normalization conditions for the Riemann-Hilbert problems can be obtained by the asymptotic behaviors of the matrix functions $J^{\pm}$ at infinity of λ as

$$J^{\pm} \to I_{n+1}, \quad \lambda \to \infty,$$

which is the canonical normalization conditions

Finally, taking the jump matrix G to be the identity matrix I_{n+1}, the resulting Riemann-Hilbert problems can be normally solved to generate soliton solutions.

In this paper, we focuse on the multi-component cubic-quintic nonlinear Schrödinger (CQNLS) equation and generate its soliton solutions by special associated Riemann-Hilbert problems. As can be seen from the above, obtaining a soliton solution mainly includes two steps [23, 24]: Step 1 is to formulate a Riemann-Hilbert problem with the space variable from a spatial matrix spectral problem, and Step 2 is to compute soliton solutions by solving special associated Riemann-Hilbert problems.

The plan of this paper is as follows. In Section 2, based on zero curvature formulation, we derive the multi-component CQNLS integrable hierarchies and present their bi-Hamiltonian structures. In Section 3, We discuss the analytical properties of matrix eigenfunctions for equivalent spatial matrix spectral problems and establish a class of Riemann-Hilbert problems related to the newly introduced spatial matrix spectral problems. In Section 4, we gain soliton solutions to the multi-component CQNLS integrable hierarchies from special associated Riemann-Hilbert problems. In Section 5, we present one-, two- and N-soliton solutions explicitly and analyze their special properties. In the last section, we give concluding remarks, along with some further questions.

2. Multi-component cubic-quintic nonlinear Schrödinger integrable hierarchies

2.1. Zero curvature formulation

We can use zero curvature formulation to generate integrable hierarchies by choosing a square matrix spectral matrix $Y = Y(\boldsymbol{u}, \lambda)$ from a given matrix loop algebra with a vector potential $\boldsymbol{u}$ and a spectral parameter λ, whose underlying Lie algebra could be either semisimple [25, 26] or nonsemisimple[27]. Assume a series solution

$$T = T(\boldsymbol{u}, \lambda) = \sum_{m=0}^{\infty} T_m \lambda^{-m} = \sum_{m=0}^{\infty} T_m(\boldsymbol{u}) \lambda^{-m} \tag{1}$$

to the corresponding stationary zero curvature equation

$$T_x = [Y,\ T]. \tag{2}$$

When the initial matrix T_0 is fixed, T is uniquely determined. According to the solution T, we construct a series of Lax matrix

$$T^{[n]} = T^{[n]}(\boldsymbol{u}, \lambda) = (\lambda^n T)_+ + \triangle_n, \quad n \geq 1, \tag{3}$$

where the subscript $+$ represents the operation of taking a polynomial part in λ, and $\triangle_n,\ n \geq 0$ are the modification terms. The appropriateness of selecting $\triangle_n$ is required to generate an integrable hierarchy

$$u_{t_n} = K_n(\boldsymbol{u}) = K_n(x, t, \boldsymbol{u}, \boldsymbol{u}_x, \cdots), \quad n \geq 1, \tag{4}$$

from the zero curvature equations

$$Y_{t_n} - T_x^{[n]} + [Y,\ T^{[n]}] = 0, \quad n \geq 1, \tag{5}$$

where the matrices Y and $T^{[n]}$ are Lax pair [28] of the n-th evolution equation in the hierarchy (4). The spatial and temporal matrix spectral problems

$$\varphi_x = Y\varphi = Y(\boldsymbol{u},\lambda)\varphi, \quad \varphi_{t_n} = T^{[n]}\varphi = T^{[n]}(\boldsymbol{u},\lambda)\varphi, \quad n \geq 1, \tag{6}$$

where φ is the matrix eigenfunction, and their compatibility condition is the zero curvature equations (5).

Then, we provide a bi-Hamiltonian structure for analyzing the Liouville integrability of the hierarchy (4)[29]:

$$u_{t_n} = K_n = J_1\frac{\delta \widetilde{H}_{m+1}}{\delta \boldsymbol{u}} = J_2\frac{\delta \widetilde{H}_m}{\delta \boldsymbol{u}}, \quad m \geq 1, \tag{7}$$

where J_1 and J_2 are a Hamiltonian pair and $\dfrac{\delta}{\delta \boldsymbol{u}}$ denotes the variational derivative [30]. The Hamiltonian structures can be usually provided with the help of trace identification [25]:

$$\frac{\delta}{\delta \boldsymbol{u}}\int \operatorname{tr}(T\frac{\partial Y}{\partial \lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}[\lambda^{\gamma}\operatorname{tr}(T\frac{\partial Y}{\partial \boldsymbol{u}})], \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\operatorname{tr}(T^2)|,$$

or more generally, the variational identity [27]:

$$\frac{\delta}{\delta \boldsymbol{u}}\int \langle T, \frac{\partial Y}{\partial \lambda}\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}[\lambda^{\gamma}\langle T, \frac{\partial Y}{\partial \boldsymbol{u}}\rangle], \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle T, T\rangle|,$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra [31]. The bi-Hamiltonian structure ensures unlimited commute Lie symmetries $\{K_n\}_{n=0}^{\infty}$ and conserved quantities $\{\widetilde{H}_m\}_{m=0}^{\infty}$:

$$[K_{n_1}, K_{n_2}] = K'_{n_1}[K_{n_2}] - K'_{n_2}[K_{n_1}] = 0, \tag{8}$$

$$\{\widetilde{\mathcal{H}}_{m_1}, \widetilde{\mathcal{H}}_{m_2}\}_J = \int (\frac{\delta \widetilde{\mathcal{H}}_{m_1}}{\delta u})^T J \frac{\delta \widetilde{\mathcal{H}}_{m_2}}{\delta u}dx = 0, \tag{9}$$

where $n_i, m_i \geq 1, i = 1, 2, J = J_1$ or J_2, and K' denotes the Gateaux derivative of K with respect to $\boldsymbol{u} : K'(\boldsymbol{u})[Y] = \dfrac{\partial}{\partial \varepsilon}|_{\varepsilon=0}K(\boldsymbol{u} + \varepsilon Y, \boldsymbol{u}_x + \varepsilon Y_x, \ldots)$.

It is known that for an evolution equation with a vector potential $\boldsymbol{u}, \widetilde{H} = \int H dx$ is a conserved functional iff $\dfrac{\delta \widetilde{H}}{\delta \boldsymbol{u}}$ is an adjoint symmetry [32, 33]. Therefore, the Hamiltonian structure links conservative function with adjoint symmetry and further symmetry. The existence of adjoint symmetry is necessary to permit conservation laws for systems of completely nondegenerate differential equations, and a pair of symmetry and adjoint symmetry lead to the conservation laws of any system of differential equations [34, 35]. When the underlying matrix loop algebra in zero curvature formulation is simple, the typical integrable hierarchies will be produced [3, 35]; when it's semisimple, a collection of different integrable hierarchies will be generated; and when it's non-semisimple, hierarchies of integrable couplings will be gained [36].

2.2. CQNLS hierarchies with multiple potentials

In this subsection, we will discuss the multi-component CQNLS integrable hierarchies and their bi-Hamiltonian structures, which will be use to bulid Riemann-Hilbert problems.

First, we consider an $n+1$ order matrix spectral problem:

$$\varphi_x = Y\varphi = Y(\boldsymbol{u},\lambda)\varphi, \quad Y = \begin{bmatrix} -i\lambda + \frac{1}{2}i\rho_1|\boldsymbol{q}|^2 & \boldsymbol{q} \\ -\boldsymbol{q}^* & (i\lambda - \frac{1}{2}i\rho_1|\boldsymbol{q}|^2)I_n \end{bmatrix}, \tag{10}$$

where λ is a spectral parameter and $\boldsymbol{u}$ is a $2n$-dimensional potential vector

$$\boldsymbol{u} = (\boldsymbol{q}, \boldsymbol{q}^{*\top})^\top, \quad \boldsymbol{q} = (q_1, q_2, \ldots, q_n), \quad \boldsymbol{q}^* = (q_1^*, q_2^*, \ldots, q_n^*)^\top, \tag{11}$$

where the superscript $*$ denotes the conjugate and $\top$ denotes the transpose.

When $q_j = q_j^* = 0, 3 \leq j \leq n$, Eq.(10) transforms into the CQNLS matrix spectral problem [37]. Therefore, we call Eq.(10) as a multi-component CQNLS matrix spectral problem and its associated hierarchy as a multi-component CQNLS integrable hierarchy.

Second, assume a solution T:

$$T = \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & d \end{bmatrix}, \tag{12}$$

where a is a scalar, $\boldsymbol{b}^\top$ and $\boldsymbol{c}$ are n-dimensional columns, and d is an $n \times n$ matrix. In order to obtain the multi-component CQNLS integrable hierarchies, we use the stationary zero curvature equation corresponding to Eq.(10). By direct calculation, we gain

$$\begin{cases} a_x = \boldsymbol{q}\boldsymbol{c} + \boldsymbol{b}\boldsymbol{q}^*, \\ \boldsymbol{b}_x = \boldsymbol{q}d - a\boldsymbol{q} - 2\boldsymbol{b}(i\lambda - \frac{1}{2}\rho_1|\boldsymbol{q}|^2), \\ \boldsymbol{c}_x = d\boldsymbol{q}^* - \boldsymbol{q}^* a + 2\boldsymbol{c}(i\lambda - \frac{1}{2}\rho_1|\boldsymbol{q}|^2), \\ d_x = -\boldsymbol{q}^*\boldsymbol{b} - \boldsymbol{c}\boldsymbol{q}, \end{cases} \tag{13}$$

We expand T as a formal series:

$$T = \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & d \end{bmatrix} = \sum_{m=0}^{\infty} T_m \lambda^{-m}, \quad T_m = T_m(u) = \begin{bmatrix} a^{[m]} & \boldsymbol{b}^{[m]} \\ \boldsymbol{c}^{[m]} & d^{[m]} \end{bmatrix}, \quad m \geq 0, \tag{14}$$

where $\boldsymbol{b}^{[m]}$, $\boldsymbol{c}^{[m]}$ and $d^{[m]}$ are expressed as

$$\boldsymbol{b}^{[m]} = (b_1^{[m]}, b_2^{[m]}, \ldots, b_n^{[m]}), \;\; \boldsymbol{c}^{[m]} = (c_1^{[m]}, c_2^{[m]}, \ldots, c_n^{[m]})^\top, \;\; d^{[m]} = (d_{ij}^{[m]})_{n\times n}, \quad m \geq 0. \tag{15}$$

Then, the Eqs.(13) derives the recursion relations as follows:

$$\boldsymbol{b}^{[0]} = \boldsymbol{0}, \quad \boldsymbol{c}^{[0]} = \boldsymbol{0}, \quad a_x^{[0]} = 0, \quad d_x^{[0]} = 0, \tag{16a}$$

$$\boldsymbol{b}^{[m+1]} = \frac{i}{2}(a^{[m]}\boldsymbol{q} - \boldsymbol{q}d^{[m]} - \rho_1|\boldsymbol{q}|^2\boldsymbol{b}^{[m]} + \boldsymbol{b}_x^{[m]}), \tag{16b}$$

$$\boldsymbol{c}^{[m+1]} = \frac{i}{2}(d^{[m]}\boldsymbol{q}^* - \boldsymbol{q}^* a^{[m]} - \rho_1|\boldsymbol{q}|^2\boldsymbol{c}^{[m]} - \boldsymbol{c}_x^{[m]}), \tag{16c}$$

$$a_x^{[m]} = \boldsymbol{q}\boldsymbol{c}^{[m]} + \boldsymbol{b}^{[m]}\boldsymbol{q}^*, \quad d_x^{[m]} = -\boldsymbol{c}^{[m]}\boldsymbol{q} - \boldsymbol{q}^*\boldsymbol{b}^{[m]}, \quad m \geq 1. \tag{16d}$$

If we take the initial values

$$a^{[0]} = \alpha_1, \quad d^{[0]} = \alpha_2 I_n, \tag{17}$$

where α_1, α_2 are arbitrary real constants, and by this means, all matrices $T_m, m \geq 1$, are uniquely determined according to the recursion relation (16). The coefficients present that

$$
\begin{aligned}
& b_l^{[1]} = \frac{i}{2}\alpha q_l,\ c_j^{[1]} = -\frac{i}{2}\alpha q_j^*,\ a^{[1]} = 0,\ d_{jl}^{[0]} = 0; && \text{(18a)}\\
& b_l^{[2]} = \frac{\alpha}{4}(\rho_1|\boldsymbol{q}|q_l - q_{lx}),\ c_l^{[2]} = -\frac{\alpha}{4}(\rho_1|\boldsymbol{q}|q_j^* + q_{jx}^*),\ a_x^{[2]} = -\frac{\alpha}{4}|\boldsymbol{q}|^2,\ d_{jlx}^{[2]} = \frac{\alpha}{4}q_j^* q_l; && \text{(18b)}\\
& b_l^{[3]} = \frac{i\alpha}{8}(-2|\boldsymbol{q}|^2 q_l - \rho_1^2|\boldsymbol{q}|^4 q_l + 2\rho_1|\boldsymbol{q}|^2 q_{lx} + \rho_1 \boldsymbol{q}_x \boldsymbol{q}^* q_l + \rho_1 \boldsymbol{q}\boldsymbol{q}_x^* q_l - q_{lxx}), && \text{(18c)}\\
& c_l^{[3]} = \frac{i\alpha}{8}(2|\boldsymbol{q}|^2 q_j^* + \rho_1^2|\boldsymbol{q}|^4 q_j^* + 2\rho_1|\boldsymbol{q}|^2 q_{lx}^* + \rho_1 \boldsymbol{q}^* \boldsymbol{q}_x q_j^* + \rho_1 \boldsymbol{q}_x^* \boldsymbol{q} q_j^* + q_{lxx}^*), && \text{(18d)}\\
& a^{[3]} = \frac{i\alpha}{8}(2\rho_1|\boldsymbol{q}|^4 + \boldsymbol{q}\boldsymbol{q}_x^* - \boldsymbol{q}_x\boldsymbol{q}^*),\ d_{jl}^{[3]} = -\frac{i\alpha}{8}(2\rho_1|\boldsymbol{q}|^2 q_l q_j^* + q_{jx}^* q_l - q_j^* q_{lx}), && \text{(18e)}
\end{aligned}
$$

where $\alpha = \alpha_1 - \alpha_2$ and $1 \leq j, l \leq n$. From Eqs.(16), we get a recursion relation for $\boldsymbol{b}^{[m]}$ and $c^{[m]}$:

$$
\begin{bmatrix} \boldsymbol{c}^{[m+1]} \\ \boldsymbol{b}^{[m+1]\top} \end{bmatrix} = \Psi \begin{bmatrix} \boldsymbol{c}^{[m]} \\ \boldsymbol{b}^{[m]\top} \end{bmatrix},\quad m \geq 1, \tag{19}
$$

where Ψ is a $2n \times 2n$ matrix integro-differential operator

$$
\Psi = \frac{i}{2}\begin{bmatrix} -(\partial + \rho_1|\boldsymbol{q}|^2 + \sum\limits_{j=1}^{n} q_j^*\partial^{-1}q_j)I_n - \boldsymbol{q}^*\partial^{-1}\boldsymbol{q} & -\boldsymbol{q}^*\partial^{-1}\boldsymbol{q}^{*\top} - (\boldsymbol{q}^*\partial^{-1}\boldsymbol{q}^{*\top})^\top \\ \boldsymbol{q}^\top\partial^{-1}\boldsymbol{q} - (\boldsymbol{q}^\top\partial^{-1}\boldsymbol{q})^\top & (\partial - \rho_1|\boldsymbol{q}|^2 + \sum\limits_{j=1}^{n} q_j\partial^{-1}q_j^*)I_n - \boldsymbol{q}^\top\partial^{-1}\boldsymbol{q}^{*\top} \end{bmatrix} \tag{20}
$$

Then, in order to obtain the multi-component CQNLS integrable hierarchies, we take the Lax matrix

$$
T^{[n]} = T^{[n]}(\boldsymbol{u}, \lambda) = (\lambda^n T)_+ = \sum_{m=0}^{n} T_m \lambda^{n-m},\quad n \geq 1, \tag{21}
$$

where the modification term $\triangle_n$ is equal to zero and T_m is defined in Eq.(14). Then by zero curvature equation (5), we can gain the multi-component CQNLS integrable hierarchies:

$$
\boldsymbol{u}_{t_n} = \begin{bmatrix} \boldsymbol{q}^\top \\ \boldsymbol{q}^* \end{bmatrix} = K_n = 2i\begin{bmatrix} \boldsymbol{b}^{[m+1]\top} \\ \boldsymbol{c}^{[m+1]} \end{bmatrix},\quad m \geq 0. \tag{22}
$$

When $m = 1$, we can gain a nonlinear integrable system from the above hierarchies (22) as follow:

$$
\begin{aligned}
q_{j,t_n} &= \frac{\alpha i}{2}\rho_1|\boldsymbol{q}|^2 q_j - \frac{\alpha i}{2}q_{jx},\quad 1 \leq j \leq n, && \text{(23a)}\\
q_{j,t_n}^* &= -\frac{\alpha i}{2}\rho_1|\boldsymbol{q}|^2 q_j^* + \frac{\alpha i}{2}q_{jx}^*,\quad 1 \leq j \leq n, && \text{(23b)}
\end{aligned}
$$

When $m = 2$ and $n = 2$, the multi-component CQNLS systems (24) can be reduced to the coupled cubic-quintic nonlinear Schrödinger equations [37].

$$
\begin{aligned}
q_{j,t_2} &= \frac{\alpha}{2}|\boldsymbol{q}|^2 q_j + \frac{\alpha}{4}\rho^2|\boldsymbol{q}|^4 q_j + \frac{\alpha}{4}q_{jxx} - \frac{\alpha}{2}\rho_1(|\boldsymbol{q}|^2 q_j)_x + \frac{\alpha}{4}\rho_1(|\boldsymbol{q}|^2)_x q_j,\quad 1 \leq j \leq n, && \text{(24a)}\\
q_{j,t_2}^* &= -\frac{\alpha}{2}|\boldsymbol{q}|^2 q_j^* - \frac{\alpha}{4}\rho^2|\boldsymbol{q}|^4 q_j^* - \frac{\alpha}{4}q_{jxx}^* - \frac{\alpha}{2}\rho_1(|\boldsymbol{q}|^2 q_j)_x + \frac{\alpha}{4}\rho_1(|\boldsymbol{q}|^2)_x q_j,\quad 1 \leq j \leq n. && \text{(24b)}
\end{aligned}
$$

Finally, we analyze the Liouville integrability of the multi-component CQNLS integrable hierarchies (22) through bi-Hamiltonian structures [32], which can be presented through applying the trace identity [25] or the variational identity [27].

Depending on the matrix Y, we have

$$\mathrm{tr}(T\frac{\partial Y}{\partial \lambda}) = a - \mathrm{tr}(d) = \sum_{m=0}^{\infty}(a^{[m]} - \sum_{j=1}^{n} d_{jj}^{[m]})\lambda^{-m}, \tag{25}$$

and

$$\mathrm{tr}(T\frac{\partial Y}{\partial \boldsymbol{u}}) = \begin{bmatrix} \boldsymbol{c} \\ -\boldsymbol{b}^{\top} \end{bmatrix} = \sum_{m\geq 0} G_{m-1}\lambda^{-m}. \tag{26}$$

Plugging these into the trace identity and checking the case of $m = 2$ tells $\gamma = 0$ in the trace identity, and thus, we have

$$\frac{\delta \tilde{H}_m}{\delta u} = iG_{m-1}, \quad \tilde{H}_m = -\frac{i}{m}\int (a^{[m+1]} - \sum_{j=1}^{n} d_{jj}^{[m+1]})dx, \quad G_{m-1} = \begin{bmatrix} \boldsymbol{c}^{[m]} \\ -\boldsymbol{b}^{[m]\top} \end{bmatrix}, \quad m \geq 1. \tag{27}$$

The bi-Hamiltonian structure for the multicomponent CQNLS systems is presented as following:

$$u_{t_n} = K_n = J_1\frac{\delta \tilde{H}_{m+1}}{\delta \boldsymbol{u}} = J_2\frac{\delta \tilde{H}_m}{\delta \boldsymbol{u}}, \quad m \geq 1, \tag{28}$$

where the Hamiltonian pairs $(J_1, J_2 = J_1\Psi)$ are given as follows:

$$J_1 = \begin{bmatrix} 0 & -2I_n \\ 2I_n & 0 \end{bmatrix}, \tag{29a}$$

$$J_2 = i\begin{bmatrix} -\boldsymbol{q}^{\top}\partial^{-1}\boldsymbol{q} + (\boldsymbol{q}^{\top}\partial^{-1}\boldsymbol{q})^{\top} & -(\partial - \rho_1|\boldsymbol{q}|^2 + \sum\limits_{j=1}^{n} q_j\partial^{-1}q_j^*)I_n + \boldsymbol{q}^{\top}\partial^{-1}\boldsymbol{q}^{*\top} \\ -(\partial + \rho_1|\boldsymbol{q}|^2 + \sum\limits_{j=1}^{n} q_j^*\partial^{-1}q_j)I_n - \boldsymbol{q}^*\partial^{-1}\boldsymbol{q} & -\boldsymbol{q}^*\partial^{-1}\boldsymbol{q}^{*\top} - (\boldsymbol{q}^*\partial^{-1}\boldsymbol{q}^{*\top})^{\top} \end{bmatrix}. \tag{29b}$$

Thus, each of the operators $\Phi = \Psi^+ = J_2J_1^{-1}$ presents a recursion operator [38] for every hierarchy with a fixed integer $n \geq 1$ in Eq.(23). Adjoint symmetry constraints (or equivalently symmetry constraints) decompose each multicomponent CQNLS system into two commuting finite-dimensional Liouville integrable Hamiltonian systems [32].

3. Riemann-Hilbert problem

In this section, the n-th multi-component CQNLS system (22) is the compatibility condition of the following matrix spectral problems:

$$\varphi_x = Y(\boldsymbol{u}, \lambda)\varphi, \tag{30}$$

$$\varphi_{t_n} = T^{[n]}(\boldsymbol{u}, \lambda)\varphi, \tag{31}$$

where

$$Y = i\lambda\Lambda + U(\boldsymbol{u}, \lambda), \quad T^{[n]} = i\lambda^n\Omega + V^{[n]}(\boldsymbol{u}, \lambda), \quad n \geq 0,$$

with $\Lambda = \mathrm{diag}(-1, I_n)$, $\Omega = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, and

$$U = \begin{bmatrix} \frac{i}{2}\rho_1|\boldsymbol{q}|^2 & \boldsymbol{q} \\ -\boldsymbol{q}^* & -\frac{i}{2}\rho_1|\boldsymbol{q}|^2 I_n \end{bmatrix}, \quad V^{[n]} = \sum_{m=1}^{n} V_m\lambda^{-n-m} = \sum_{m=1}^{n} \begin{bmatrix} a^{[m]} & \boldsymbol{b}^{[m]} \\ \boldsymbol{c}^{[m]} & d^{[m]} \end{bmatrix}\lambda^{n-m}. \tag{32}$$

Here $\boldsymbol{q}, \boldsymbol{q}^*$ are determined in Eq.(11), and $a^{[m]}, \boldsymbol{b}^{[m]}, \boldsymbol{c}^{[m]}, d^{[m]}, 1 \le m \le n$, are defined by Eq.(15). We can easy find that the $\mathrm{tr}(U) \neq 0$, when $n \ge 2$. So We need an appropriate transformation to make $\mathrm{tr}(U) = 0$.

Let

$$\begin{aligned}
&\tilde{\varphi} = e^{\frac{i}{2}\rho_1 \int_{-\infty}^{x} |\boldsymbol{q}|^2 d\xi \Lambda}\varphi, \tilde{\boldsymbol{q}} = \boldsymbol{q} e^{i\rho_1 \int_{-\infty}^{x} |\boldsymbol{q}|^2 d\xi},\\
&\tilde{a}^{[m]} = a^{[m]}, \tilde{\boldsymbol{b}}^{[m]} = \boldsymbol{b}^{[m]} e^{i\rho_1 \int_{-\infty}^{x} |\boldsymbol{q}|^2 d\xi},\\
&\tilde{\boldsymbol{c}}^{[m]} = \boldsymbol{c}^{[m]} e^{-i\rho_1 \int_{-\infty}^{x} |\boldsymbol{q}|^2 d\xi}, \tilde{d}^{[m]} = d^{[m]},
\end{aligned} \tag{33}$$

then substitute into Eqs.(30)–(31), we can get

$$\tilde{\varphi}_x = (i\lambda\Lambda + \tilde{U}(\boldsymbol{u}, \lambda))\tilde{\varphi} \tag{34}$$

$$\tilde{\varphi}_{t_n} = (i\lambda^n\Omega + \tilde{V}^{[n]}(\boldsymbol{u}, \lambda))\tilde{\varphi} \tag{35}$$

where

$$\tilde{U} = \begin{bmatrix} 0 & \tilde{\boldsymbol{q}} \\ -\tilde{\boldsymbol{q}}^* & 0 \end{bmatrix}, \quad \tilde{V}^{[n]} = \sum_{m=1}^{n} \begin{bmatrix} \tilde{a}^{[m]} & \tilde{\boldsymbol{b}}^{[m]} \\ \tilde{\boldsymbol{c}}^{[m]} & \tilde{d}^{[m]} \end{bmatrix} \lambda^{n-m}. \tag{36}$$

Then, we use the Riemann-Hilbert method to discuss the scattering and inverse scattering of the modified multicomponent CQNLS system [2, 10].

Assume that all the potentials rapidly vanish when $x \to \pm\infty$ and satisfy the integrable conditions:

$$\int_{-\infty}^{\infty} |x|^m \sum_{j=1}^{n} (|\tilde{q}_i| + |\tilde{q}_j^*|) dx \le \infty, \quad m = 0, 1. \tag{37}$$

Without loss of generality, we let

$$\alpha = \alpha_1 - \alpha_2 \le 0. \tag{38}$$

As $x \to \pm\infty$, we have the asymptotic behavior from Eqs.(34)–(35): $\tilde{\varphi} \sim e^{i\lambda\Lambda x + i\lambda^n \Omega t_n}$. Hence it will be convenient to express $\tilde{\varphi}$ as

$$\tilde{\varphi} = \psi e^{i\lambda\Lambda x + i\lambda^n \Omega t_n}, \tag{39}$$

where the canonical normalization $\psi \to I_{n+1}$, when $x \to \pm\infty$. Substituting Eq.(39) into Eqs.(34)–(35), we can gain

$$\psi_x = i\lambda[\Lambda,\ \psi] + \tilde{U}\psi \tag{40}$$

$$\psi_{t_n} = i\lambda^{[n]}[\Omega,\ \psi] + V^{\widetilde{[n]}}\psi \tag{41}$$

Applying a generalized Liouville's formula [39], we can gain

$$\det\ \psi = 1, \tag{42}$$

due to $\mathrm{tr}(\tilde{U}) = \mathrm{tr}(\tilde{V}) = 0$.

Next, we discuss the Riemann-Hilbert problem of the modified multicomponent CQNLS system (34). In this consideration, the time t_n is fixed and is a dummy variable, and thus it will be suppressed. In the scattering problem, we first introduce two matrix Jost solutions $\psi^{\pm}(x,\lambda)$ of Eq.(40) with the asymptotic conditions

$$\psi^{\pm} \to I_{n+1}, \quad \text{when } x \to \pm\infty, \tag{43}$$

respectively. From Eq.(42), we see that det $\psi^{\pm} = 1$ for all x. Since

$$\tilde{\varphi}^{\pm} = \psi^{\pm} E, \quad E = e^{i\lambda\Lambda x}, \tag{44}$$

are both solutions of Eq.(34), they are linearly related by a matrix $S(\lambda)$:

$$\psi^{-} E = \psi^{+} E S(\lambda), \quad \lambda \in \mathbb{R}, \tag{45}$$

where $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is the scattering matrix. Note that det $S(\lambda) = 1$ because of det $\psi^{\pm} = 1$.

Then applying the method of variation of parameters as well as the boundary conditions (43), we can transform Eq.(40) into the following Volterra integral equations for $\psi^{\pm}$ [2]:

$$\psi^{-}(\lambda, x) = I_{n+1} + \int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)} \tilde{U}(y) \psi^{-}(\lambda, y) e^{i\lambda\Lambda(y-x)} dy, \tag{46}$$

$$\psi^{+}(\lambda, x) = I_{n+1} - \int_{x}^{\infty} e^{i\lambda\Lambda(x-y)} \tilde{U}(y) \psi^{+}(\lambda, y) e^{i\lambda\Lambda(y-x)} dy, \tag{47}$$

Thus, $\psi^{\pm}$ allows analytical continuations off the real axis $\lambda \in \mathbb{R}$ as long as the integrals on the right sides converge. We can see that the integral equation for the first column of ψ^{-} involves only the exponential factor $e^{2i\lambda(x-y)}$, which decays because of $y < x$ in the integral, when λ is in the closed upper half-plane $\mathbb{C}_0^{+}$, and the integral equation for the last n columns of ψ^{+} involves only the exponential factor $e^{-2i\lambda(x-y)}$, which also decays because of $y > x$ in the integral, when λ is in the closed upper half-plane $\mathbb{C}_0^{+}$. Therefore, those $n+1$ columns can be analytically continued to the closed upper half-plane $\mathbb{C}_0^{+}$. Similarly, we can see that the last n columns of ψ^{-} and the first column of ψ^{+} can be analytically continued to the closed lower half-plane $\mathbb{C}_0^{-}$.

Below we will determine two matrix eigenfunctions $J^{\pm}(x,\lambda)$, which are analytically continued to the upper and lower half-planes, respectively. First, if we express $\psi^{\pm}$ as a collection of columns,

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \cdots, \psi_{n+1}^{\pm}), \tag{48}$$

then the matrix Jost solution

$$J^{+} = J^{+}(x,\lambda) = (\psi_1^{-}, \psi_2^{+}, \cdots, \psi_{n+1}^{+}) = \psi^{-} H_1 + \psi^{+} H_2 \tag{49}$$

is analytic in $\lambda \in \mathbb{C}^{+}$ and continuous in $\lambda \in \mathbb{C}_0^{+}$, and the matrix solution

$$(\psi_1^{+}, \psi_2^{-}, \cdots, \psi_{n+1}^{-}) = \psi^{+} H_1 + \psi^{-} H_2 \tag{50}$$

is analytic in $\lambda \in \mathbb{C}^{-}$ and continuous in $\lambda \in \mathbb{C}_0^{-}$, where H_1 and H_2 are defined by

$$H_1 = \mathrm{diag}(1, \underbrace{0, \cdots, 0}_{n}), \quad H_2 = \mathrm{diag}(0, \underbrace{1, \cdots, 1}_{n}). \tag{51}$$

In addition, from Eq.(46)–(47), we see that

$$J^+(x,\lambda) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}_0^+ \to \infty, \tag{52}$$

and

$$(\psi_1^+, \psi_2^-, \cdots, \psi_{n+1}^-) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}_0^- \to \infty. \tag{53}$$

Secondly, in order to construct the analytic counterpart of J^+ in the lower half-plane $\mathbb{C}^-$, we discuss the adjoint matrix spectral problems of Eq.(40):

$$\tilde{\psi}_x = -i\lambda[\tilde{\psi},\ \Lambda] - \tilde{\psi}\tilde{U} \tag{54}$$

The inverse matrices $\tilde{\psi}^\pm = (\psi^\pm)^{-1}$ satisfy the above adjoint equation. If we express $(\psi^\pm)^{-1}$ as a collection of rows:

$$\tilde{\psi}^\pm = (\tilde{\psi}_1^\pm, \tilde{\psi}_2^\pm, \cdots, \tilde{\psi}_{n+1}^\pm)^\top, \tag{55}$$

Then, we can show that the adjoint matrix Jost solutions

$$J^- = (\tilde{\psi}_1^-, \tilde{\psi}_2^+, \cdots, \tilde{\psi}_{n+1}^+)^\top = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1} \tag{56}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}_0^-$, and the other matrix solution

$$(\tilde{\psi}_1^+, \tilde{\psi}_2^-, \cdots, \tilde{\psi}_{n+1}^-)^\top = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1} \tag{57}$$

is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}_0^+$. Similarly, we can determine that

$$J^-(x,\lambda) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}_0^- \to \infty, \tag{58}$$

and

$$(\tilde{\psi}_1^+, \tilde{\psi}_2^-, \cdots, \tilde{\psi}_{n+1}^-) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}_0^+ \to \infty. \tag{59}$$

Now we have constructed the two matrix functions $J^+(x,\lambda)$ and $J^-(x,\lambda)$ which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, and continuous in $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$, respectively. On the real line, we can easily gain from Eqs.(45), (49) and (56):

$$J^+(x,\lambda) = J^-(x,\lambda)G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{60}$$

where

$$G(x,\lambda) = E(H_1 + H_2S(\lambda))(H_1 + S^{-1}(\lambda)H_2)E^{-1} = E\begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \cdots & 0 \\ s_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n+1,1} & 0 & 0 & \cdots & 1 \end{bmatrix} E^{-1} \tag{61}$$

with $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. The Eq.(60) presents a matrix Riemann-Hilbert problem we would like to build for the modified multicomponent CQNLS systems. The asymptotic properties

$$J^\pm(x,\lambda) \to I_{n+1}, \quad \text{when } \lambda \in \mathbb{C}_0^\pm \to \infty, \tag{62}$$

generate the canonical normalization conditions for the above presented Riemann-Hilbert problems.

To complete the direct scattering transform, let us evaluate the derivative of Eq.(44) with time t_n and use the vanishing conditions of the potentials at infinity of t_n. Clearly, we can prove that the scattering matrix S has to satisfy

$$S_{t_n} = i\lambda^n[\Omega,\ S], \tag{63}$$

which tells that the time evolution of the time-dependent scattering coefficients are given by

$$s_{1,j} = s_{1,j}(\lambda, 0)e^{i\beta\lambda^n t_n}, \quad s_{j,1} = s_{j,1}(\lambda, 0)e^{-i\beta\lambda^n t_n}, \quad 2 \le j \le n+1, \tag{64}$$

and all other scattering coefficients are independent of the time variable t_r.

4. Soliton solutions by the Riemann-Hilbert problem

In this section, we mainly discuss the soliton solution of the Riemann-Hilbert problem with $\det J^+ = 0$ and $\det J^- = 0$. The uniqueness of solutions to each associated Riemann-Hilbert problem (60) does not hold unless the zeros of det $J^\pm$ in the upper and lower half-planes are specified and the structures of ker $J^\pm$ at those zeros are determined in Ref.[40, 41, 42].

Recalling the definitions of $J^\pm$ in Eq.(49) and Eq.(56) as well as the scattering relation in Eq.(45), we see that

$$\det J^+(x,\lambda) = s_{11}(\lambda), \quad \det J^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{65}$$

where

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & \cdots & s_{2,n+1} \\ s_{32} & s_{33} & \cdots & s_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,2} & s_{n+1,3} & \cdots & s_{n+1,n+1} \end{vmatrix}. \tag{66}$$

Assume that the function s_{11} has N zeros $\{\lambda_k \in \mathbb{C}^+, 1 \le k \le N\}$, and the function $\hat{s}_{11}$ has N zeros $\{\hat{\lambda}_k \in \mathbb{C}^-, 1 \le k \le N\}$, where N is the number of these zeros. In order to compute N-soliton solutions simply, we assume that all zeros, λ_k and $\hat{\lambda}_k, 1 \le k \le N$, are simple. In this case, each of ker $J^+(x,\lambda_k), 1 \le k \le N$, contains only a single basis column vector $\boldsymbol{v}_k, 1 \le k \le N$; and each of ker $J^-(x,\hat{\lambda}_k), 1 \le k \le N$, a single basis row vector $\hat{\boldsymbol{v}}_k, 1 \le k \le N$. So, we have

$$J^+(x,\lambda_k)\boldsymbol{v}_k = 0, \quad \hat{\boldsymbol{v}}_k J^-(x,\hat{\lambda}_k) = 0, \quad 1 \le k \le N. \tag{67}$$

It is known that the Riemann-Hilbert problems (60) with the canonical normalization conditions in Eq.(62) and the zero structures in Eq.(67) can be solved explicitly in Ref.[2, 43]. To compute N-soliton solutions, we take $G = I_{n+1}$ in each above Riemann-Hilbert problem. This can be realized if we take that $s_{j,1} = \hat{s}_{1,j} = 0,\ 2 \le j \le n+1$, which equivalently requires that no reflection exists in the scattering problem. This resulting special Riemann-Hilbert problem has the solutions:

$$J^+(x,\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{\boldsymbol{v}_k (M^{-1})_{kl} \hat{\boldsymbol{v}}_l}{\lambda - \hat{\lambda}_l}, \quad J^-(x,\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{\boldsymbol{v}_k (M^{-1})_{kl} \hat{\boldsymbol{v}}_l}{\lambda - \lambda_l}, \tag{68}$$

where M is a square matrix with entries being defined by

$$M = (m_{kl})_{N\times N},\ m_{kl} = \frac{\hat{\boldsymbol{v}}_k \boldsymbol{v}_l}{\lambda_l - \hat{\lambda}_k},\ 1 \le k,l \le N. \tag{69}$$

Because the zeros λ_k and $\hat{\lambda}_k$ are constants, i.e., space and time independent, we can easily determine the spatial and temporal evolutions for the vectors, $\boldsymbol{v}_k(x,t_n)$ and $\hat{\boldsymbol{v}}_k(x,t_n), 1 \le k \le N$, in the kernels ker $J^{\pm}$. For example, let us compute the x-derivative of both sides of the first set of equations in Eq.(67). By Eq.(40) and the first set of equations in Eq.(67), we can derive

$$J^+(x,\lambda_k)(\frac{d\boldsymbol{v}_k}{dx} - i\lambda_k\Lambda\boldsymbol{v}_k) = 0, \quad 1 \le k \le N. \tag{70}$$

It then follows that for each $1 \le k \le N$, the vector $\dfrac{d\boldsymbol{v}_k}{dx} - i\lambda_k\Lambda\boldsymbol{v}_k$ must be in the kernel of $J^+(x,\lambda_k)$ and so a constant multiple of the vector $\boldsymbol{v}_k$. Without loss of generality, we consider the simplest case and assume

$$\frac{d\boldsymbol{v}_k}{dx} = i\lambda_k\Lambda\boldsymbol{v}_k, \quad 1 \le k \le N. \tag{71}$$

The time dependence of $\boldsymbol{v}_k$:

$$\frac{d\boldsymbol{v}_k}{dt_n} = i\lambda_k^n\Omega\boldsymbol{v}_k, \quad 1 \le k \le N, \tag{72}$$

can be worked out similarly by applying the temporal matrix spectral problem (41). To sum up, we can have

$$\boldsymbol{v}_k(x,t_n) = e^{i\lambda_k\Lambda x + i\lambda_k^n\Omega t_n}\boldsymbol{v}_{k_0}, \quad 1 \le k \le N, \tag{73}$$

$$\hat{\boldsymbol{v}}_k(x,t_n) = \hat{\boldsymbol{v}}_{k_0}e^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^n\Omega t_n}, \quad 1 \le k \le N, \tag{74}$$

where $\boldsymbol{v}_{k_0}$ and $\hat{\boldsymbol{v}}_{k_0}, 1 \le k \le N$, are arbitrary constant column and row vectors, respectively.

Finally, we expand J^+ at large λ as

$$J^+(x,\lambda) = I_{n+1} + \frac{1}{\lambda}J_1^+(x) + \mathrm{O}(\frac{1}{\lambda^2}), \quad \lambda \to \infty, \tag{75}$$

plugging this series expansion into Eq.(40) and balancing O(1) terms generate

$$\tilde{U} = -i[\Lambda, J_1^+] = \begin{vmatrix} 0 & 2(J_1^+)_{12} & 2(J_1^+)_{13} & \cdots & 2(J_1^+)_{1,n+1} \\ -2(J_1^+)_{21} & 0 & 0 & \cdots & 0 \\ -2(J_1^+)_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ -2(J_1^+)_{n+1,1} & 0 & 0 & \cdots & 0 \end{vmatrix}, \tag{76}$$

where $J_1^+ = ((J_1^+)_{jl})_{(n+1)\times(n+1)}$. In other words, the $2n$ potentials $\tilde{q}_j$ and $\tilde{q}_j^*, 1 \le j \le n$, can be evaluated as follows:

$$\tilde{q}_j = 2i(J_1^+)_{1,j+1}, \quad \tilde{q}_j^* = -2i(J_1^+)_{j+1,1}, \quad 1 \le j \le n. \tag{77}$$

Now from the λ-dependence of the solutions in Eq.(68), we have

$$J_1^+ = -\sum_{k,j=1}^{N} \boldsymbol{v}_k(M^{-1})_{kj}\hat{\boldsymbol{v}}_j, \tag{78}$$

and thus further through the solution expressions in Eq.(77), obtain the N-soliton solution to the modified multi-component CQNLS system:

$$\tilde{q}_j = -2i\sum_{k,l=1}^{N} \boldsymbol{v}_{k,1}(M^{-1})_{kl}\hat{\boldsymbol{v}}_{l,j+1}, \quad \tilde{q}_j^* = 2i\sum_{k,l=1}^{N} \boldsymbol{v}_{k,j+1}(M^{-1})_{kl}\hat{\boldsymbol{v}}_{l,1}, \quad 1 \le j \le n, \tag{79}$$

where the matrix M is defined by Eq.(69), and $\boldsymbol{v}_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n+1})^\top$ and $\hat{\boldsymbol{v}}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \cdots, \hat{v}_{k,n+1})$, $1 \le k \le N$, respectively.

Then, we prove that the two components in the presented solutions in Eq.(79) are conjugate to each other[18]. First, let

$$P = i\tilde{U}, \quad \tilde{U} = \begin{bmatrix} 0 & \tilde{q} \\ -\tilde{q}^* & 0 \end{bmatrix}, \tag{80}$$

. Then we easy gain that

$$P^\dagger = P, \tag{81}$$

where $\dagger$ stands for the Hermitian transpose of a matrix.

If $\psi(\lambda)$ is a matrix eigenfunction of the spectral problem in Eq.(40), then in addition to a known matrix adjoint eigenfunction $\psi^{-1}(\lambda^*)$, we have another matrix adjoint eigenfunction

$$\tilde{\psi}(\lambda^*) = \psi^\dagger(\lambda^*), \tag{82}$$

associated with an eigenvalue λ^*, ie, $\psi^\dagger(\lambda^*)$ solves the adjoint spectral problem in Eq.(54) with λ replaced with λ^*. Therefore, upon observing the asymptotic properties for $\psi^\pm$ at infinity of λ, the uniqueness of solutions tells

$$\psi^\dagger(\lambda^*) = \psi^{-1}(\lambda^*), \quad \psi = \psi^\pm. \tag{83}$$

Further from the definitions of $J^\pm$ in Eqs.(49) and (56), we see that the two matrix solutions $J^\pm$ satisfy the involution relation

$$(J^+)^\dagger(\lambda^*) = J^-(\lambda^*), \tag{84}$$

and from the definition of the scattering matrix $S(\lambda)$ in Eq.(45), we see that the scattering matrix $S(\lambda)$ has the involution property

$$S^\dagger(\lambda^*) = S^{-1}(\lambda^*). \tag{85}$$

According to Eq.(84), we have $s_{11}(\lambda^*) = \hat{s}_{11}(\lambda)$ through Eq.(65), and thus, the zeros of det $J^\pm$ satisfy the involution relation:

$$\hat{\lambda}_k = \lambda_k^*, \quad 1 \le k \le N. \tag{86}$$

To obtain involution eigenvectors v_k and $\hat{v}_k$, we check the Hermitian transpose of the first set of equations in Eq.(67):

$$0 = v_k^\dagger (J^+(\lambda_k))^\dagger = v_k^\dagger J^-(\hat{\lambda}_k), \quad 1 \le k \le N, \tag{87}$$

where we have used Eqs.(84) and (86). Therefore, we can take

$$\hat{v}_k = v_k^\dagger, \quad 1 \le k \le N, \tag{88}$$

as the solutions to the second set of equations in Eq.(67). It follows that we have the following involution eigenvectors:

$$v_k(x,t_r) = e^{i\lambda_k \Lambda x + i\lambda_k^r \Omega t_r} v_{k_0}, \quad 1 \le k \le N, \quad \hat{v}_k(x,t_r) = v_{k_0}^\dagger e^{-i\lambda_k^* \Lambda x - i\lambda_k^{*r} \Omega t_r}, \quad 1 \le k \le N, \tag{89}$$

where v_{k_0} are arbitrary constant column vectors as before. Then, we can gain that the two components in the presented solutions in Eq.(79) are conjugate to each other.

Then we can gain the N-soliton solution to the multicomponent CQNLS system (22) by the transformation $q_j = \tilde{q}_j e^{-i\rho_1 \int_{-\infty}^{x} |\boldsymbol{q}|^2 d\xi}$, and easy find $|q_j| = |\tilde{q}_j|$.

5. N-Soliton Solution

When $G = I$, the continuous scattering data $(\hat{s}_{1,j}, s_{j,1}),\ 2 \leq j \leq n+1$ is zero, and the corresponding scattering problem is called reflectionless. The corresponding solution $\tilde{\boldsymbol{q}}$ is called a reflectionless potential. This solution can be written out explicitly. Thus, from Eq.(76) we have

$$J_1^+(x, t_n) = -\sum_{k,j=1}^{N} \boldsymbol{v}_k (M^{-1})_{kj} \hat{\boldsymbol{v}}_j, \tag{90}$$

and

$$\tilde{q}_j = -2i \sum_{k,l=1}^{N} \boldsymbol{v}_{k,1} (M^{-1})_{kl} \hat{\boldsymbol{v}}_{l,j+1}, \quad 1 \leq j \leq n, \tag{91}$$

Here vectors $\boldsymbol{v}_k$ are given by Eq.(73), $\hat{\boldsymbol{v}}_j = \boldsymbol{v}_j^\dagger$, and matrix M is given by Eq.(69). Without loss of generality, we let $\boldsymbol{v}_{k_0} = (c_{k,1}, c_{k,2}, \cdots, c_{k,n+1})^\top$. In addition, we introduce the notation

$$\theta_k = i\lambda_k x + i\lambda_k^n t_n, \text{ and } \alpha_1 = -1, \alpha_2 = 1. \tag{92}$$

Then the above solution $\tilde{\boldsymbol{q}}$ can be written out explicitly as

$$\tilde{q}_j = -2i \sum_{k,l=1}^{N} c_{k,1} c_{l,j+1}^* e^{\theta_l^* - \theta_k} (M^{-1})_{kl}, \quad 1 \leq j \leq n, \tag{93}$$

where the elements of the $N \times N$ matrix M are given by

$$M_{kl} = \frac{1}{\lambda_l - \hat{\lambda}_k} \left(\sum_{m=1}^{n} (c_{k,m+1}^* c_{l,m+1}) e^{\theta_k^* + \theta_l} + c_{k,1}^* c_{l,1} e^{-(\theta_k^* + \theta_l)} \right). \tag{94}$$

Notice that M^{-1} can be expressed as the transpose of M's cofactor matrix divided by det$M$. We examine properties of these N-soliton solutions next.

5.1. Single-soliton solutions

When $N = 1$, the solution (93) is

$$\tilde{q}_j = -2i(\lambda_1 - \lambda_1^*) \frac{c_{11} c_{1,j+1}^* e^{\theta_1^* - \theta_1}}{|c_{11}|^2 e^{-(\theta_1^* + \theta_1)} + \sum\limits_{m=1}^{n} |c_{1,m+1}|^2 e^{\theta_1^* + \theta_1}}, \quad 1 \leq j \leq n. \tag{95}$$

Letting

$$\lambda_1 = \xi + i\eta, \quad c_{1,1} = e^{-2\eta x_0 + i\sigma_0}, \quad c_{1,m+1} = 1, \quad 1 \leq m \leq n, \tag{96}$$

where ξ, η are the real and imaginary parts of ξ_0, x_0 and σ_0 are real parameters, then the above solution can be rewritten as

$$\tilde{q}_j = \frac{2}{\sqrt{n}} \eta \operatorname{sech}[2(-\eta x + \sum_{0 \leq m_1 \leq n} (-1)^{\frac{m_1+1}{2}} C_n^{m_1} \eta^{m_1} \xi^{n-m_1} t_n + \eta x_0) + n_0] \exp\{-2i\xi x + 2i \sum_{0 \leq m_2 \leq n} (-1)^{\frac{m_2+2}{2}} C_n^{m_2} \eta^{m_2} \xi^{n-m_2} t_n + i\sigma_0\}, \tag{97}$$

where $e^{n_0} = \sqrt{n}$, m_1 is odd number and m_2 is even number.

This solution is a solitary wave in the modified multi-component CQNLS system (34). Its amplitude function $|\tilde{q}_j|$ has the shape of a hyperbolic secant with peak amplitude $\frac{2}{\sqrt{n}}\eta$, and its velocity is -2$\sum_{m_1}^{n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta^{m_1}\xi^{n-m_1}$. The phase of this solution depends linearly on both space x and time t_n. The spatial gradient of the phase is proportional to the speed of the wave. Parameters x_0 and σ_0 are the initial location and phase of this solitary wave. This solution is called a single-soliton solution in the modified multicomponent CQNLS system (34).

5.2. Two-soliton solutions

When $N = 2$, the solution (93) can also be written out explicitly. Here instead of giving that complicated expression, we show two typical solution behaviors, one for the case of $\xi_1 \neq \xi_2$, and the other one for the case of $\xi_1 = \xi_2$. Here

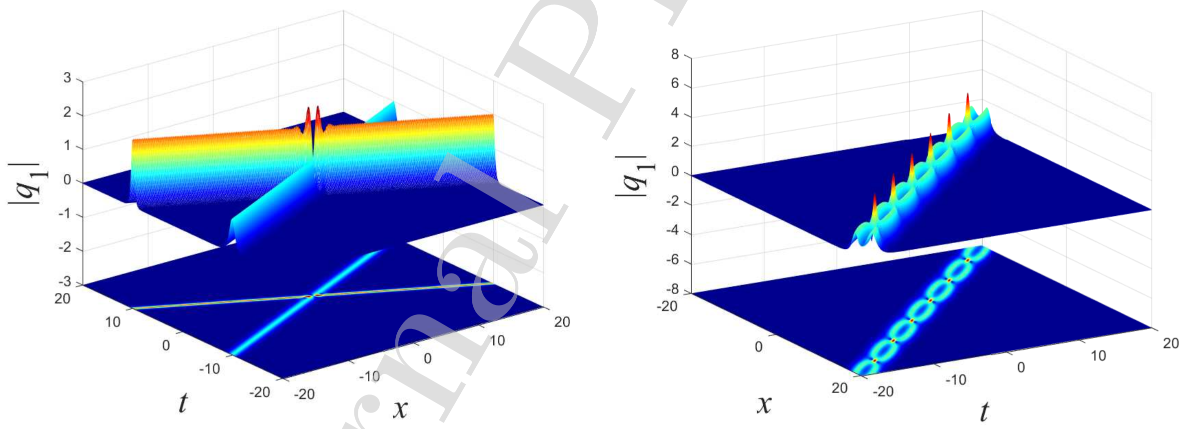
$$\lambda_k = \xi_k + i\eta_k, \quad k = 1, 2, \tag{98}$$

i.e., ξ_k and η_k are the real and imaginary parts of λ_k. Without loss of generality, we only give the figures fo $|q_1|$ and choose the solution parameters in Eqs.(93)–(94) as follows

$$\lambda_1 = 1 + i, \quad \lambda_2 = -1 - 0.5i, \quad c_{1,1} = c_{2,1} = c_{1,2} = c_{2,2} = c_{1,3} = c_{2,3} = 1, \quad j = 1 \tag{99}$$

while in Fig.1(b), the solution parameters are

$$\lambda_1 = 0.5 + i, \quad \lambda_2 = 0.5 - 0.7i, \quad c_{1,1} = c_{2,1} = c_{1,2} = c_{2,2} = c_{1,3} = c_{2,3} = 1, \quad j = 1 \tag{100}$$



(a) (b)

Figure 1: Two-soliton solutions $|q_1|$ in the modified multicomponent CQNLS system: (a) collision, where $\mathrm{Re}(\lambda_1) \neq \mathrm{Re}(\lambda_2)$; (b) bound stste, were $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$.

In the first case ($\xi_1 \neq \xi_2$), we see from Fig.1(a) that as $t_n \to \infty$, the solution consists of two single-solitons which are far apart and moving toward each other. When they collide, they interact strongly. But when $t_n \to \infty$, these solitons re-emerge out of interactions without any change of shape and velocity, and there is no energy radiation emitted to the far field. Thus the interaction of these solitons is elastic. This elastic interaction is a remarkable property which signals that the modified multi-component CQNLS system (34) is integrable. There is still some trace of the interaction however. Indeed, after the interaction, each soliton acquires a position shift and a phase shift. The position of each soliton is always shifted forward, as if the soliton accelerates during interactions.

Fig.1(a) is typical of all two-soliton solutions with $\xi_1 \neq \xi_2$. To show this fact, we analyze the asymptotic states of the solution (91) as $t_n \to \pm\infty$. Without loss of generality, let us assume that $\xi_1 > \xi_2$. This means that at $t_n = -\infty$, soliton-1 is on the right side of soliton-2 and moves slower. Note also that $\eta_k > 0$ since $\lambda_k \in \mathbb{C}_+$.

$$\mathrm{Re}(\theta_1) = -\eta_1(x + \sum_{0\le m_1\le n} C_n^{m_1}\xi_1^{n-m_1}(i\eta_1)^{m_1-1}t_n) = O(1). \tag{101}$$

Since

$$\begin{aligned}\mathrm{Re}(\theta_2) &= -\eta_2(x + \sum_{0\le m_1\le n} C_n^{m_1}\xi_2^{n-m_1}(i\eta_2)^{m_1-1}t_n)\\ &= -\eta_2(x + \sum_{0\le m_1\le n} C_n^{m_1}\xi_1^{n-m_1}(i\eta_1)^{m_1-1}t_n) - \eta_2(\sum_{0\le m_1\le n} C_n^{m_1}\xi_2^{n-m_1}(i\eta_2)^{m_1-1} - \sum_{0\le m_1\le n} C_n^{m_1}\xi_1^{n-m_1}(i\eta_1)^{m_1-1})t_n,\end{aligned} \tag{102}$$

where m_1 is odd number.

In the limit $t_n \to -\infty$, we have

$$\mathrm{Re}(\theta_1) \gg \mathrm{Re}(\theta_2). \tag{103}$$

In this case, we have to consider separately the two regions $(1-)$ and $(2-)$ defined below. It is easily seen that $\tilde{q}_j \simeq 0$ $(1 \le j \le n)$ in all other regions.

$(1-)$ finite $\mathrm{Re}(\theta_1)$, $\mathrm{Re}(\theta_2) \to -\infty$

In this case, the dominant terms are those which contain the factor $e^{-(\theta_2^*+\theta_2)}$. Then, the asymptotic state of the solution (93) is

$$\begin{aligned}\tilde{q}_j(x,t) &\simeq -2i(\lambda_1-\lambda_1^*)\frac{c_{1,1}c_{1,j+1}^*c_1^- e^{\theta_1^*-\theta_1}}{|c_{1,1}|^2e^{-(\theta_1^*+\theta_1)} + |c_1^-|^2(\sum_{m=1}^{n}|c_{1,m+1}|^2)e^{\theta_1^*+\theta_1}},\\ &= \frac{2}{\sqrt{n}}\eta_1\mathrm{sech}[2(-\eta_1 x + \sum_{0\le m_1\le n}(-1)^{\frac{m_1+1}{2}}C_n^{m_1}\eta_1^{m_1}\xi_1^{n-m_1}t_n + \eta_1 x_0) + n_1]\exp\{-2i\xi_1 x + 2i\sum_{0\le m_2\le n}(-1)^{\frac{m_2+2}{2}}C_n^{m_2}\eta_1^{m_2}\xi_1^{n-m_2}t_n + i\sigma_0\},\end{aligned} \tag{104}$$

where $c_{1,1} = e^{-2\eta x_0 + i\sigma_0}$, $c_{1,m+1} = 1$, $1 \le m \le n$, $\sqrt{n}c_1^- = e^{n_1}$, $c_1^- = \dfrac{\lambda_2-\lambda_1^*}{\lambda_2^*-\lambda_1^*}$, m_1 is odd number and m_2 is even number. Comparing this expression with Eq.(95), we see that this asymptotic solution is a single-soliton solution with peak amplitude $\frac{2}{\sqrt{n}}\eta_1$, and its velocity is -2 $\sum_{m_1}^{n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_1^{m_1}\xi_1^{n-m_1}$.

$(2-)$ finite $\mathrm{Re}(\theta_2)$, $\mathrm{Re}(\theta_1) \to +\infty$

In this case, the dominant terms are those which contain the factor $e^{(\theta_1^*+\theta_1)}$. Then, the asymptotic state of the solution (93) is

$$\begin{aligned}\tilde{q}_j(x,t) &\simeq -2i(\lambda_2-\lambda_2^*)\frac{(c_{2,2}c_{2,j+1}^*\sum_{m=1}^{n}|c_{1,m+1}|^2 - c_{2,1}c_{1,j+1}^*\frac{\lambda_1-\lambda_1^*}{\lambda_1-\lambda_2^*}\sum_{m=1}^{n}c_{2,m+1}^*c_{1,m+1})e^{\theta_2^*-\theta_2}}{|c_2^-|^2\sum_{m=1}^{n}|c_{1,m+1}|^2|c_{2,1}|^2e^{-(\theta_2^*+\theta_2)} + (\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}|c_{1,m_1+1}|^2|c_{2,m_2+1}|^2)e^{\theta_2^*+\theta_2}}\\ &= \frac{2}{\sqrt{n}}\eta_2\mathrm{sech}[2(-\eta_2 x + \sum_{0\le m_1\le n}C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_2^{m_1}\xi_2^{n-m_1}t_n) + n_2]\exp\{-2i\xi_2 x + 2i\sum_{0\le m_2\le n}C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_2^{m_2}\xi_2^{n-m_2}t_n\},\end{aligned} \tag{105}$$

where $c_{2,1} = c_{1,m+1} = c_{2,m+1} = 1$, $1 \le m \le n$, $\dfrac{\sqrt{n}}{c_2^-} = e^{n_2}$, $c_2^- = \dfrac{\lambda_1-\lambda_2^*}{\lambda_1^*-\lambda_2^*}$, m_1 is odd number and m_2 is even number.

In the limit $t_n \to +\infty$, we have

$$\mathrm{Re}(\theta_1) \ll \mathrm{Re}(\theta_2). \tag{106}$$

In this case, we have to consider separately the two regions (1+) and (2+) defined below. It is easily seen that $\tilde{q}_j \simeq 0$ $(1 \le j \le n)$ in all other regions.

(1+) finite $\mathrm{Re}(\theta_1)$, $\mathrm{Re}(\theta_2) \to +\infty$.

In this case, the dominant terms are those which contain the factor $e^{(\theta_2^* + \theta_2)}$. Then, the asymptotic state of the solution (93) is

$$\begin{aligned}\tilde{q}_j(x,t) &\simeq -2i(\lambda_1 - \lambda_1^*)\frac{c_{1,1}(c_{1,j+1}^*\sum\limits_{m=1}^{n}|c_{2,m+1}|^2 - c_{2,j+1}^*\frac{\lambda_2-\lambda_2^*}{\lambda_2-\lambda_1^*}\sum\limits_{m=1}^{n}c_{1,m+1}^*c_{2,m+1})e^{\theta_1^*-\theta_1}}{\sum\limits_{m=1}^{n}|c_{2,m+1}|^2|c_{1,1}|^2e^{-(\theta_1^*+\theta_1)} + |c^+|^2(\sum\limits_{m_1=1}^{n}\sum\limits_{m_2=1}^{n}|c_{1,m_1+1}|^2|c_{2,m_2+1}|^2)e^{\theta_1^*+\theta_1}}\\ &= \frac{2}{\sqrt{n}}\eta_1\mathrm{sech}[2(-\eta_1 x + \sum_{0\le m_1\le n}C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_1^{m_1}\xi_1^{n-m_1}t_n + \eta_1 x_0) + n_3]\exp\{-2i\xi_1 x + 2i\sum_{0\le m_2\le n}C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_1^{m_2}\xi_1^{n-m_2}t_n + i\sigma_0\}\end{aligned} \tag{107}$$

where $c_{1,1} = e^{-2\eta x_0 + i\sigma_0}$, $c_{1,m+1} = c_{2,m+1} = 1$, $1 \le m \le n$, $\sqrt{n}c_1^+ = e^{n_3}$, $c_1^+ = \dfrac{\lambda_2^* - \lambda_1^*}{\lambda_2 - \lambda_1^*}$, m_1 is odd number and m_2 is even number. This is also a single-soliton solution with velocity -2 $\sum\limits_{m_1}^{n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_1^{m_1}\xi_1^{n-m_1}$ and peak amplitude $\dfrac{2}{\sqrt{n}}\eta_1$.

(2+) finite $\mathrm{Re}(\theta_2)$, $\mathrm{Re}(\theta_1) \to -\infty$

In this case, the dominant terms are those which contain the factor $e^{-(\theta_1^*+\theta_1)}$. Then, the asymptotic state of the solution (93) is

$$\begin{aligned}\tilde{q}_j(x,t) &\simeq -2i(\lambda_2 - \lambda_2^*)\frac{c_2^+ c_{2,1}c_{2,j+1}^* e^{\theta_2^*-\theta_2}}{\sum\limits_{m=1}^{n}|c_{2,m+1}|^2e^{\theta_2^*+\theta_2} + |c_2^+|^2|c_{2,1}|^2e^{-(\theta_2^*+\theta_2)}}\\ &= \frac{2}{\sqrt{n}}\eta_2\mathrm{sech}[2(-\eta_2 x + \sum_{0\le m_1\le n}C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_2^{m_1}\xi_2^{n-m_1}t_n) + n_4]\exp\{-2i\xi_2 x + 2i\sum_{0\le m_2\le n}C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_2^{m_2}\xi_2^{n-m_2}t_n\},\end{aligned} \tag{108}$$

where $c_{2,1} = c_{2,m+1} = 1$, $1 \le m \le n$, $\dfrac{\sqrt{n}}{c_2^+} = e^{n_4}$, $c_2^+ = \dfrac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1^*}$, m_1 is odd number and m_2 is even number.

Taking the sum of Eq.(104) and Eq.(105), or Eq.(107) and Eq.(108), we arrive at the following theorem 1.

Theorem 1 *The asymptotic forms of the two-soliton solution of the n-th multi-component CQNLS system are as follows (see also Fig. 2):*
as $t_n \to -\infty$,

$$\tilde{q}_j(x,t_n) \simeq \frac{2}{\sqrt{n}}\eta_1\mathrm{sech}[2(\tau_1 + \eta_1 x_0) + n_1]\exp\{\varpi_1 + i\sigma_0\} + \frac{2}{\sqrt{n}}\eta_2\mathrm{sech}(2\tau_2 + n_2)\exp\{\varpi_2\}, \tag{109}$$

as $t_n \to +\infty$,

$$\tilde{q}_j(x,t_n) \simeq \frac{2}{\sqrt{n}}\eta_1\mathrm{sech}[2(\tau_1 + \eta_1 x_0) + n_3]\exp\{\varpi_1 + i\sigma_0\} + \frac{2}{\sqrt{n}}\eta_2\mathrm{sech}(2\tau_2 + n_4)\exp\{\varpi_2\}, \tag{110}$$

where

$$\tau_1 = -\eta_1 x + \sum_{0\le m_1\le n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_1^{m_1}\xi_1^{n-m_1}t_n,$$
$$\tau_2 = -\eta_2 x + \sum_{0\le m_1\le n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_2^{m_1}\xi_2^{n-m_1}t_n,$$
$$\varpi_1 = -2i\xi_1 x + 2i\sum_{0\le m_2\le n} C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_1^{m_2}\xi_1^{n-m_2}t_n,$$
$$\varpi_2 = -2i\xi_2 x + 2i\sum_{0\le m_2\le n} C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_2^{m_2}\xi_2^{n-m_2}t_n.$$

with $c_{1,1} = e^{-2\eta x_0 + i\sigma_0},\ c_{2,1} = c_{1,m+1} = c_{2,m+1} = 1,\ 1\le m\le n, \sqrt{n}c_1^- = e^{n_1},\ \frac{\sqrt{n}}{c_2^-} = e^{n_2},\ \sqrt{n}c_1^+ = e^{n_3},\ \frac{\sqrt{n}}{c_2^+} = e^{n_4},\ c_1^- = \frac{\lambda_2-\lambda_1^*}{\lambda_2^*-\lambda_1^*},\ c_2^- = \frac{\lambda_1-\lambda_2^*}{\lambda_1^*-\lambda_2^*},\ c_1^+ = \frac{\lambda_2^*-\lambda_1^*}{\lambda_2-\lambda_1^*},\ c_2^+ = \frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_1^*}$, m_1 *is odd number and* m_2 *is even number.*

Theorem 1 defines the collision laws of two solitons in the n-th multi-component CQNLS system.

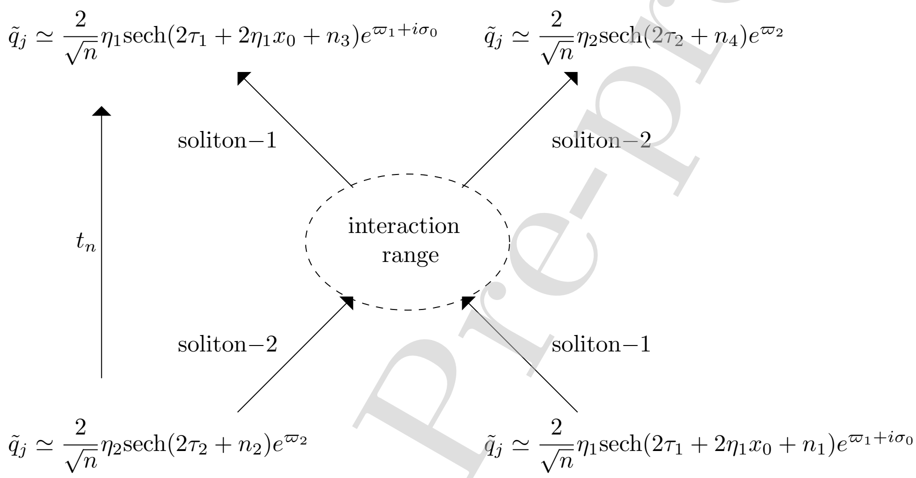


Figure 2: Two-soliton collision in the n-th multicomponent CQNLS system

This indicates that this soliton does not change its shape and velocity after collision. Its position and phase have shifted, however, as Fig.1(a) has shown. The position shift is

$$\triangle x_{01} = -\frac{\sqrt{n}}{2\eta_1}(\ln|c^+| - \ln|c^-|) = \frac{\sqrt{n}}{\eta_1}\ln|\frac{\lambda_2-\lambda_1^*}{\lambda_1^*-\lambda_2^*}|, \tag{111}$$

and the phase shift is

$$\triangle\sigma_{01} = \arg(c^+) - \arg(c^-) = -2\arg(\frac{\lambda_2-\lambda_1^*}{\lambda_1^*-\lambda_2^*}). \tag{112}$$

Notice that $\triangle x_{01} < 0$ since $\lambda_k \in \mathbb{C}_+$, and thus the (slower) soliton-1 acquires a negative position shift.

5.3. N-soliton solutions

In this subsection, we compute the asymptotic forms of the N-soliton solution in the limits $t \to \mp\infty$ and simplify them as much as possible.

We first rewrite the N-soliton solution (93) before considering the limits $t \to \mp\infty$.

$$
\begin{aligned}
\tilde{q}_j &= -2i \sum_{k,l=1}^{N} c_{k,1} c_{l,j+1}^{*} e^{\theta_l^* - \theta_k} (M^{-1})_{kl} \\
&= -2i \frac{1}{\det M} \sum_{k,l=1}^{N} \tilde{M}_{kl} c_{k,1} c_{l,j+1}^{*} e^{\theta_l^* - \theta_k} \\
&= -2i \frac{1}{\det M} c_{1,j+1}^{*} e^{\theta_1^*} \begin{vmatrix} c_{11}e^{-\theta_1} & c_{21}e^{-\theta_2} & \cdots & c_{N1}e^{-\theta_N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{vmatrix} - 2i \frac{1}{\det M} c_{2,j+1}^{*} e^{\theta_1^*} \begin{vmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ c_{11}e^{-\theta_1} & c_{21}e^{-\theta_2} & \cdots & c_{N1}e^{-\theta_N} \\ \vdots & \vdots & \vdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{vmatrix} \\
&\quad \cdots - 2i \frac{1}{\det M} c_{N,j+1}^{*} e^{\theta_N^*} \begin{vmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ M_{N-1,1} & M_{N-1,2} & \cdots & M_{N-1,N} \\ c_{11}e^{-\theta_1} & c_{21}e^{-\theta_2} & \cdots & c_{N1}e^{-\theta_N} \end{vmatrix}
\end{aligned} \tag{113}
$$

when $\xi_1 \neq \xi_2 \neq \cdots \neq \xi_N$, and investigate the asymptotic behavior of $\tilde{q}_j(x,t)$ in the limit $t \to \mp\infty$. Without loss of generality, let us assume that $\xi_1 > \xi_2 > \cdots > \xi_N$. In this case, when $t \to -\infty$, we have

$$
\mathrm{Re}(\theta_1) \gg \mathrm{Re}(\theta_2) \gg \cdots \gg \mathrm{Re}(\theta_{\mathrm{N}}), \tag{114}
$$

and we can consider separately N region (1^-)–(N^-) with the following definition

(1^-) finite $\mathrm{Re}(\theta_1), \mathrm{Re}(\theta_2), \mathrm{Re}(\theta_3), \cdots, \mathrm{Re}(\theta_N) \to -\infty$, under this choice, the dominant terms are those which contain the factor $e^{-(\theta_2+\theta_2^*+\theta_3+\theta_3^*+\cdots+\theta_N+\theta_N^*)}$. Then the numerator of $\tilde{q}_j(x,t)$ with the factor $e^{-(\theta_2+\theta_2^*+\theta_3+\theta_3^*+\cdots+\theta_N+\theta_N^*)}$ is

$$
-2ic_{1,j+1}^{*} c_{11} \prod_{k=2}^{N} |c_{k,1}|^2 e^{\theta_1^* - \theta_1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\lambda_2 - \lambda_1^*} & \frac{1}{\lambda_2 - \lambda_2^*} & \cdots & \frac{1}{\lambda_2 - \lambda_N^*} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_N - \lambda_1^*} & \frac{1}{\lambda_N - \lambda_2^*} & \cdots & \frac{1}{\lambda_N - \lambda_N^*} \end{vmatrix} \tag{115}
$$

and the denominator of $\tilde{q}_j(x,t)$ with the factor $e^{-(\theta_2+\theta_2^*+\theta_3+\theta_3^*+\cdots+\theta_N+\theta_N^*)}$ is

$$
\begin{aligned}
&\frac{1}{\lambda_1 - \lambda_1^*} \left(\sum_{m=1}^{n} |c_{1,m+1}|^2 e^{\theta_1 + \theta_1^*} \right) \prod_{k=2}^{N} |c_{k,1}|^2 \begin{vmatrix} \frac{1}{\lambda_2 - \lambda_2^*} & \frac{1}{\lambda_3 - \lambda_2^*} & \cdots & \frac{1}{\lambda_N - \lambda_2^*} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_2 - \lambda_N^*} & \frac{1}{\lambda_3 - \lambda_N^*} & \cdots & \frac{1}{\lambda_N - \lambda_N^*} \end{vmatrix} \\
&+ \prod_{k=1}^{N} |c_{k,1}|^2 e^{-\theta_1 - \theta_1^*} \begin{vmatrix} \frac{1}{\lambda_1 - \lambda_1^*} & \frac{1}{\lambda_2 - \lambda_1^*} & \cdots & \frac{1}{\lambda_N - \lambda_1^*} \\ \frac{1}{\lambda_1 - \lambda_2^*} & \frac{1}{\lambda_2 - \lambda_2^*} & \cdots & \frac{1}{\lambda_N - \lambda_2^*} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_1 - \lambda_N^*} & \frac{1}{\lambda_2 - \lambda_N^*} & \cdots & \frac{1}{\lambda_N - \lambda_N^*} \end{vmatrix}
\end{aligned} \tag{116}
$$

then the asymptotic state of the solution (93) is

$$
\begin{aligned}
\tilde{q}_j(x,t) &\simeq -2i(\lambda_1-\lambda_1^*)\frac{c_{1,1}c_{1,j+1}^*c_1^- e^{\theta_1^*-\theta_1}}{|c_{1,1}|^2 e^{-\theta_1-\theta_1^*}+|c_1^-|^2\left(\sum\limits_{m=1}^{n}|c_{1,m+1}|^2\right)e^{\theta_1^*+\theta_1}}\\
&=\frac{2}{\sqrt{n}}\eta_1 \mathrm{sech}[2(-\eta_1 x+\sum_{0\le m_1\le n}(-1)^{\frac{m_1+1}{2}}C_n^{m_1}\eta_1^{m_1}\xi_1^{n-m_1}t_n)+n_1^-]\exp\{-2i\xi_1 x+2i\sum_{0\le m_2\le n}(-1)^{\frac{m_2+2}{2}}C_n^{m_2}\eta_1^{m_2}\xi_1^{n-m_2}t_n\},
\end{aligned}
\tag{117}
$$

where $c_{1,1}=c_{1,m+1}=1,\ 1\le m\le n, \sqrt{n}c_1^-=e^{n_1^-},\ c_1^-=\prod\limits_{k\neq 1}^{N}\frac{\lambda_1-\lambda_k^*}{\lambda_1-\lambda_k}$, m_1 is odd number and m_2 is even number.

(L^-) $\mathrm{Re}(\theta_1),\cdots,\mathrm{Re}(\theta_{L-1})\to+\infty,$ finite $\mathrm{Re}(\theta_L),\mathrm{Re}(\theta_{L+1}),\cdots,\mathrm{Re}(\theta_N)\to-\infty, L=2,\cdots,N-1$, under this choice, the dominant terms are those which contain the factor$e^{\theta_1+\theta_1^*+\cdots+\theta_{L-1}+\theta_{L-1}^*-(\theta_{L+1}+\theta_{L+1}^*+\cdots+\theta_N+\theta_N^*)}$. Then, the asymptotic state of the solution (93) is

$$
\begin{aligned}
\tilde{q}_j(x,t) &\simeq -2i(\lambda_L-\lambda_L^*)\frac{\prod\limits_{k\neq L}^{N}(c_{L,2}c_{L,j+1}^*\prod\limits_{k_1\neq L}^{N}\sum\limits_{m=1}^{n}|c_{k_1,m+1}|^2-c_{L,1}\dfrac{\lambda_k-\lambda_k^*}{\lambda_k-\lambda_L^*}\prod\limits_{k_1\neq L}^{N}c_{k_1,j+1}^*\sum\limits_{m=1}^{n}c_{L,m+1}^*c_{k_1,m+1})e^{\theta_L^*-\theta_L}}{|c_L^-|^2\prod\limits_{k_1\neq L}^{N}\sum\limits_{m=1}^{n}|c_{k_1,m+1}|^2|c_{L,1}|^2e^{-(\theta_L^*+\theta_L)}+\prod\limits_{k_1\neq L}^{N}(\sum\limits_{m_1=1}^{n}\sum\limits_{m_2=1}^{n}|c_{k_1,m_1+1}|^2|c_{L,m_2+1}|^2)e^{\theta_L^*+\theta_L}}\\
&=\frac{2}{\sqrt{n}}\eta_L \mathrm{sech}[2(-\eta_L x+\sum_{0\le m_1\le n}C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_L^{m_1}\xi_L^{n-m_1}t_n)+n_L^-]\exp\{-2i\xi_L x+2i\sum_{0\le m_2\le n}C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_L^{m_2}\xi_L^{n-m_2}t_n\},
\end{aligned}
\tag{118}
$$

where $c_{L,1}=c_{k_1,m+1}=c_{L,m+1}=1,\ 1\le m\le n,\ \dfrac{\sqrt{n}}{c_L^-}=e^{n_L^-},\ c_L^-=\prod\limits_{k\neq L}^{N}\dfrac{\lambda_k-\lambda_L^*}{\lambda_k^*-\lambda_L^*}$, m_1 is odd number and m_2 is even number.

(N^-) $\mathrm{Re}(\theta_1),\cdots,\mathrm{Re}(\theta_{N-1})\to+\infty,$ finite $\mathrm{Re}(\theta_N)$, under this choice, the dominant terms are those which contain the factor$e^{\theta_1+\theta_1^*+\cdots+\theta_{N-1}+\theta_{N-1}^*}$. With calculations similar to those in the case (L^-), we obtain the asymptotic form of $\tilde{q}$ given by Eq.(118) with $L=N$.

In this case, when $t\to+\infty$, we have

$$
\mathrm{Re}(\theta_1)\ll\mathrm{Re}(\theta_2)\ll\cdots\ll\mathrm{Re}(\theta_\mathrm{N}),
\tag{119}
$$

and we can consider separately N region (1^+)–(N^+) with the following definition

(1^+) finite $\mathrm{Re}(\theta_1),\mathrm{Re}(\theta_2),\mathrm{Re}(\theta_3),\cdots,\mathrm{Re}(\theta_N)\to+\infty$, under this choice, the dominant terms are those which contain the factor $e^{\theta_2+\theta_2^*+\theta_3+\theta_3^*+\cdots+\theta_N+\theta_N^*}$. Then, the asymptotic state of the solution (93) is

$$
\begin{aligned}
\tilde{q}_j(x,t) &\simeq -2i(\lambda_1-\lambda_1^*)\frac{\prod\limits_{k\neq 1}^{N}c_{1,1}(c_{1,j+1}^*\prod\limits_{k_1\neq 1}^{N}\sum\limits_{m=1}^{n}|c_{k_1,m+1}|^2-\dfrac{\lambda_k-\lambda_k^*}{\lambda_k-\lambda_1^*}\prod\limits_{k_1\neq 1}^{N}c_{k_1,j+1}^*\sum\limits_{m=1}^{n}c_{1,m+1}^*c_{k_1,m+1})e^{\theta_1^*-\theta_1}}{\prod\limits_{k\neq 1}^{N}\sum\limits_{m=1}^{n}|c_{k_1,m+1}|^2|c_{1,1}|^2e^{-(\theta_1^*+\theta_1)}+|c_1^+|^2\prod\limits_{k_1\neq 1}^{N}(\sum\limits_{m_1=1}^{n}\sum\limits_{m_2=1}^{n}|c_{1,m_1+1}|^2|c_{k_1,m_2+1}|^2)e^{\theta_1^*+\theta_1}}\\
&=\frac{2}{\sqrt{n}}\eta_1 \mathrm{sech}[2(-\eta_1 x+\sum_{0\le m_1\le n}C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_1^{m_1}\xi_1^{n-m_1}t_n)+n_1^+]\exp\{-2i\xi_1 x+2i\sum_{0\le m_2\le n}C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_1^{m_2}\xi_1^{n-m_2}t_n\},
\end{aligned}
\tag{120}
$$

where $c_{1,1}=c_{1,m+1}=c_{k_1,m+1}=1,\ 1\le m\le n,\ \sqrt{n}c_1^+=e^{n_1^+},\ c_1^+=\prod\limits_{k\neq 1}^{N}\dfrac{\lambda_k^*-\lambda_1^*}{\lambda_k-\lambda_1^*}$, m_1 is odd number and m_2 is even number.

(L^+) $\mathrm{Re}(\theta_1),\cdots,\mathrm{Re}(\theta_{L-1})\to-\infty,$ finite $\mathrm{Re}(\theta_L),\mathrm{Re}(\theta_{L+1}),\cdots,\mathrm{Re}(\theta_N)\to+\infty, L=2,\cdots,N-1$, under this choice, the dominant terms are those which contain the factor $e^{-(\theta_1+\theta_1^*+\cdots+\theta_{L-1}+\theta_{L-1}^*)+\theta_{L+1}+\theta_{L+1}^*+\cdots+\theta_N+\theta_N^*}$. Then, the asymptotic state of the solution (93) is

$$
\begin{aligned}
\tilde{q}_j(x,t) &\simeq -2i(\lambda_L-\lambda_L^*)\frac{c_L^+ c_{L,1} c_{L,j+1}^* e^{\theta_L^*-\theta_L}}{\sum\limits_{m=1}^{n}|c_{L,m+1}|^2 e^{\theta_L^*+\theta_L}+|c_L^+|^2|c_{L,1}|^2 e^{-(\theta_L^*+\theta_L)}}\\
&=\frac{2}{\sqrt{n}}\eta_L \operatorname{sech}[2(-\eta_L x+\sum_{0\le m_1\le n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_L^{m_1}\xi_L^{n-m_1}t_n)+n_L^+]\exp\{-2i\xi_L x+2i\sum_{0\le m_2\le n} C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_L^{m_2}\xi_L^{n-m_2}t_n\},
\end{aligned}
\tag{121}
$$

where $c_{L,1}=c_{L,m+1}=1,\ 1\le m\le n,\ \dfrac{\sqrt{n}}{c_L^+}=e^{n_L^+},\ c_L^+=\prod\limits_{k\neq L}^{N}\dfrac{\lambda_L-\lambda_k}{\lambda_L-\lambda_k^*}$, m_1 is odd number and m_2 is even number.

(N^+) $\mathrm{Re}(\theta_1),\cdots,\mathrm{Re}(\theta_{N-1})\to-\infty$, finite $\mathrm{Re}(\theta_N)$, under this choice, the dominant terms are those which contain the factor$e^{-(\theta_1+\theta_1^*+\cdots+\theta_{N-1}+\theta_{N-1}^*)}$. With calculations similar to those in the case (L^+), we obtain the asymptotic form of $\tilde{q}$ given by Eq.(121) with $L=N$.

Taking the sum of Eq.(117) and Eq.(118), or Eq.(120) and Eq.(121), we arrive at the following theorem 2.

Theorem 2 *The asymptotic forms of the N-soliton solution of the n-th multi-component CQNLS system are as follows (see also Fig. 3):*

as $t_n\to-\infty$,

$$
\tilde{q}_j(x,t_n)\simeq\frac{2}{\sqrt{n}}\sum_{l=1}^{N}\eta_l \operatorname{sech}(2\tau_l+n_l^-)\exp\{\varpi_l\},
\tag{122}
$$

as $t_n\to+\infty$,

$$
\tilde{q}_j(x,t_n)\simeq\frac{2}{\sqrt{n}}\sum_{l=1}^{N}\eta_l \operatorname{sech}(2\tau_l+n_l^+)\exp\{\varpi_l\},
\tag{123}
$$

where

$$
\begin{aligned}
\tau_l&=-\eta_l x+\sum_{0\le m_1\le n} C_n^{m_1}(-1)^{\frac{m_1+1}{2}}\eta_l^{m_1}\xi_l^{n-m_1}t_n,\\
\varpi_l&=-2i\xi_l x+2i\sum_{0\le m_2\le n} C_n^{m_2}(-1)^{\frac{m_2+2}{2}}\eta_l^{m_2}\xi_l^{n-m_2}t_n,
\end{aligned}
$$

with $c_{k,m+1}=1,\ 0\le m\le n,\ 1\le k\le N, \sqrt{n}c_1^-=e^{n_1^-},\ c_1^-=\prod\limits_{k\neq1}^{N}\frac{\lambda_1-\lambda_k^*}{\lambda_1-\lambda_k},\ \dfrac{\sqrt{n}}{c_L^-}=e^{n_L^-},\ c_L^-=\prod\limits_{k\neq L}^{N}\dfrac{\lambda_k-\lambda_L^*}{\lambda_k^*-\lambda_L^*},\ \sqrt{n}c_1^+=e^{n_1^+},\ c_1^+=\prod\limits_{k\neq1}^{N}\dfrac{\lambda_k^*-\lambda_1^*}{\lambda_k-\lambda_1^*},\ \dfrac{\sqrt{n}}{c_L^+}=e^{n_L^+},\ c_L^+=\prod\limits_{k\neq L}^{N}\dfrac{\lambda_L-\lambda_k}{\lambda_L-\lambda_k^*}$, *$m_1$ is odd number and m_2 is even number.*

Theorem 2 defines the collision laws of N-solitons in the n-th multi-component CQNLS system.

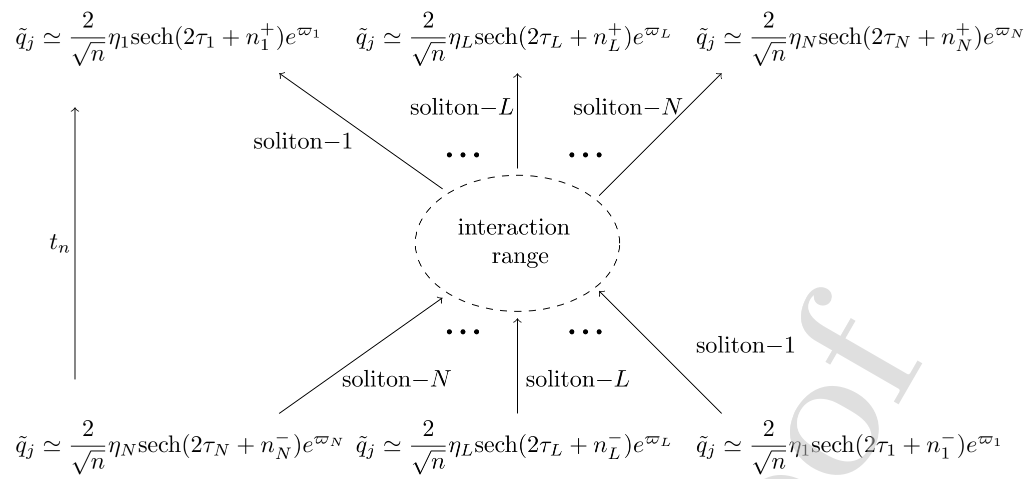


Figure 3: N-soliton collision in the n-th multicomponent CQNLS system

6. Conclusion

In this paper, we mainly study the Riemann-Hilbert problems associated with matrix spectral problems to gain soliton solutions of integrable hierarchies. We considered an arbitrary order matrix spectral problem which can reduce to the coupled cubic-quintic nonlinear Schrödinger equations and generated the corresponding integrable hierarchies possessing bi-Hamiltonian structures. For all multi-component CQNLS systems, under an appropriate transformation, we built their Riemann-Hilbert problems and presented the jump matrix in the resulting Riemann-Hilbert problems. When the jump matrix is an identity matrix, we gained soliton solutions to the multi-component CQNLS systems. Finally, we discussed the specific examples of one-soliton, two-soliton and N-soliton solutions, presented the explicit formulas, and analyzed their asymptotic behavior.

The Riemann-Hilbert method is very powerful in constructing soliton solutions. The approach has been recently generalized to solve initial-boundary value problems of integrable equations on the half-line and the finite interval [44, 45, 46, 47]. We also remake that we can get the jump matrix at the non-zero boundary branch cut and use the Riemann-Hilbert problems to gain the rogue wave. Therefore, another important questions for further study are how to formulate Riemann-Hilbert problems for solving those generalized integrable counterparts. It is hoped that our results from the perspective of the Riemann-Hilbert technique can be of great help to the study the exact solutions of to integrable equations.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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