

Kirillov's character formula, the holomorphic Peter–Weyl theorem, and the Blattner–Kostant–Sternberg pairing

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Abstract

By means of the orbit method we show that, for a compact Lie group, the Blattner–Kostant–Sternberg pairing map, with the constants being appropriately fixed, is unitary. Along the way we establish a holomorphic Peter–Weyl theorem for the complexification of a compact Lie group. Among our crucial tools is Kirillov's character formula. The basic observation is that the Weyl vector is lurking behind the Kirillov character formula, as well as behind the requisite half-form correction on which the Blattner–Kostant–Sternberg-pairing for the compact Lie group relies, and thus produces the appropriate shift which, in turn, controls the unitarity of the BKS-pairing map. Our methods are independent of heat kernel harmonic analysis, which is used by B. C. Hall to obtain a number of these results [B.C. Hall, The Segal–Bargmann Coherent State Transform for compact Lie groups, *J. Funct. Anal.* 122 (1994) 103–151; B.C. Hall, Geometric quantization and the generalized Segal–Bargmann transform for Lie groups of compact type, *Comm. Math. Phys.* 226 (2002) 233–268, quant.ph/0012015].

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0. Introduction

In this paper we will use the orbit method to show that, for a compact Lie group, the Blattner–Kostant–Sternberg (BKS) pairing map, multiplied by a suitable constant determined by the Gaussian volume in the imaginary directions of the complexified group, is unitary. Thus, let K be a compact Lie group and let $K^{\mathbb{C}}$ be its complexification. We shall first establish a Peter–Weyl theorem for the holomorphic functions on $K^{\mathbb{C}}$ relative to the inner product coming from half-form quantization, and we shall then show that the BKS-pairing map between the resulting Hilbert space of holomorphic functions on $K^{\mathbb{C}}$ and the ordinary Hilbert space of L^2 -functions on K , multiplied by the appropriate constant, is unitary. Our basic tool is Kirillov's character formula. For the special case where K is abelian, the measure on $K^{\mathbb{C}}$ is simply Haar measure on K times the appropriate Gaussian in the imaginary directions and, after decomposition of the corresponding Hilbert spaces into irreducible (1-dimensional) constituents, inspection

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of characters immediately establishes the unitarity of the pairing map, up to an overall constant determined by the Gaussian volume in the imaginary directions of the complexified group. Our argument for general not necessarily abelian K is the same, with the familiar shift given by the half-sum of the positive roots – the *Weyl vector* (sum of the fundamental highest weights when K is simply connected) – incorporated, and with those points in the dual of the abelian Lie algebra that correspond to characters being replaced with entire integral coadjoint orbits. The Weyl vector enters explicitly into the expression of the Weyl character formula and into that of the Kirillov character formula; it is, in precisely the same way, also behind the requisite half-form correction over $K^{\mathbb{C}}$ on which the BKS-pairing relies, and the unitarity of the pairing is then a formal consequence of the character formula. Here the crucial identity is (1.4) below; indeed, independently of geometric quantization, this identity itself suggests the requisite half-form correction term over $K^{\mathbb{C}}$. The coincidence of the left- and right-hand side of the quoted identity is, of course, not accidental since both sides arise, up to multiplication of the argument by the factor i , as the square root of the Jacobian of the corresponding exponential map. Moreover, in view of the *Borel–Weil* theorem, a highest weight λ determines a holomorphic line bundle β_λ over the *shifted* coadjoint orbit $\mathcal{O}_{\lambda+\rho}$ whose space of holomorphic sections $\Gamma(\beta_\lambda)$ underlies the irreducible representation having highest weight λ ; the orbit $\mathcal{O}_{\lambda+\rho}$ is necessarily of maximal dimension and hence complex algebraically a full flag manifold and, though we shall not use this fact in the paper, it is worthwhile noting that the Weyl vector induces a description of the holomorphic sections of β_λ as half-forms whence, in particular, the Weyl vector yields a *canonical* invariant inner product on $\Gamma(\beta_\lambda)$. In a sense, the holomorphic Peter–Weyl theorem assembles all these representations into a single one. The fact that the Weyl vector yields a canonical invariant inner product on each irreducible representation via the Borel–Weil theorem is presumably folk-lore; it was known to J. Wolf in the sixties of the last century (private communication), long before the advent of half-forms.

We now relate the present paper to what we already know to be in the literature. Since the measure $e^{-\kappa/t}\eta_\varepsilon$ involves a Gaussian constituent, the square-integrability of the representative functions relative to the corresponding measure can also be established directly, and Lemma 10 in [4] entails that the representative functions are dense in $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta_\varepsilon)$. The fact that the BKS-pairing map between the two Hilbert spaces, multiplied by a suitable constant, is a unitary isomorphism, has been established by Hall [7]. In that paper, the pairing map is shown to coincide, up to multiplication by a constant, with a version of the Segal–Bargmann coherent state transform developed, in turn, over Lie groups admitting a bi-invariant Riemannian metric, in a sequence of preceding papers [4–7]. See also Remark 6.8 below. Suffice it to mention at this stage that the main technique in those papers is heat kernel harmonic analysis and, in fact, in [7], Hall derives the unitarity of the pairing map by identifying the measure on $K^{\mathbb{C}}$ coming from the half-form bundle with an appropriate heat kernel measure which, in turn, he has shown in the preceding papers to furnish a unitary transform. This approach in terms of the Segal–Bargmann transform, combined with the ordinary Peter–Weyl theorem, also entails the statement of the holomorphic Peter–Weyl theorem. Though we do not spell it out, a suitable holomorphic Plancherel Theorem is an immediate consequence of our approach. A version of such a holomorphic Plancherel Theorem can be found in [20] as well as in Lemmata 9 and 10 of [4]. In [4] (Section 8), the completeness of the representative functions is established by analytical considerations. A number of results in Section 10 of [4] are actually independent of heat kernel methods in the sense that they are valid for more general measures than that coming from heat kernel analysis and, when these results are applied to the heat kernel measure, evaluation of coefficients is possible in terms of the eigenvalues of the Laplacian. The uniform convergence, on compact sets of a group of the kind $K^{\mathbb{C}}$, of the holomorphic Fourier series which, under the present circumstances, arises in an obvious manner, though not spelled out in the present paper, can already be found in Proposition 12 of [2], where this series is referred to as a Fourier–Laurent series.

Our approach provides the new insight that the unitarity of the BKS-pairing map is independent of heat kernel methods and, in particular, resolves the mystery of the unitarity of the BKS-pairing map, since the unitarity thus appears as an incarnation of the character formula or, equivalently, as a consequence of the geometry of coadjoint orbits. Moreover, the spectral decompositions of the Laplace operator refine to the Peter–Weyl decompositions, and these decompositions, in turn, explain as well why the BKS-pairing map comes out in terms of a heat equation. In this sense, the last fact is likewise seen as a consequence of the geometry of coadjoint orbits.

This paper was written during a stay at the Institute for Theoretical Physics at the University of Leipzig. This stay was made possible by the German Research Council (Deutsche Forschungsgemeinschaft) in the framework of a Mercator visiting professorship, and I wish to express my gratitude to this organization. It is a pleasure to acknowledge the stimulus of conversation with G. Rudolph and M. Schmidt at Leipzig. The paper is part of a research program aimed at exploring quantization on classical phase spaces with singularities [9–17], in particular on classical lattice

gauge theory phase spaces. Details, including quantum mechanical insight into the significance of the singular strata, are worked out in [17] for the special case of a single spatial plaquette where $K = \mathrm{SU}(2)$. The precise information needed for this research program is the equivalence of the two Hilbert spaces spelled out in [Theorem 5.3](#) below.

I am indebted to R. Szöke, B. Hall and M. Lassalle for discussion and to the referee for having prodded me to explain *why* the approach is geometric, *why* the pairing map comes out to be unitary, and *why* the pairing map is related to the heat equation.

1. The Peter–Weyl decomposition of the half-form Hilbert space

Let K be a compact Lie group and $K^{\mathbb{C}}$ its complexification, and let $\mathfrak{k}$ and $\mathfrak{k}^{\mathbb{C}}$ be the Lie algebras of K and $K^{\mathbb{C}}$, respectively. Choose an invariant inner product $\cdot : \mathfrak{k} \otimes \mathfrak{k} \rightarrow \mathbb{R}$ on $\mathfrak{k}$, and endow K with the corresponding bi-invariant Riemannian metric. Using the metric, we identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ and the total space TK of the tangent bundle with the total space T^*K of the cotangent bundle, and we will denote by $|\cdot|$ the resulting norms on $\mathfrak{k}$ and on $\mathfrak{k}^*$.

Consider the *polar decomposition* map

$$K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}}, \quad (x, Y) \mapsto x \cdot \exp(iY), \quad (x, Y) \in K \times \mathfrak{k}. \quad (1.1)$$

The composite of the inverse of left trivialization with (1.1) identifies T^*K with $K^{\mathbb{C}}$ in a $(K \times K)$ -equivariant fashion. Then the induced complex structure on T^*K combines with the symplectic structure to a K -bi-invariant (positive) Kähler structure. Indeed, the *real analytic* function

$$\kappa : K^{\mathbb{C}} \rightarrow \mathbb{R}, \quad \kappa(x \cdot \exp(iY)) = |Y|^2, \quad (x, Y) \in K \times \mathfrak{k}, \quad (1.2)$$

on $K^{\mathbb{C}}$ which is twice the *kinetic energy* associated with the Riemannian metric, is a (globally defined) K -bi-invariant *Kähler potential*, that is, the tautological cotangent bundle symplectic structure on $\mathrm{T}^*K \cong K^{\mathbb{C}}$ is given by $i \partial \bar{\partial} \kappa$.

We now introduce an additional real parameter $t > 0$; in the physical interpretation, this parameter amounts to *Planck's* constant $\hbar$. For ease of comparison with the heat kernel measure, cf. the identity (7.3) below, we prefer the notation t rather than $\hbar$. The half-form Kähler quantization, cf. e.g. [26] (chap. 10), applied to $K^{\mathbb{C}}$ relative to the tautological cotangent bundle symplectic structure on $K^{\mathbb{C}}$, multiplied by $1/t$, is accomplished by means of a certain Hilbert space of holomorphic functions on $K^{\mathbb{C}}$ which we now recall, for ease of exposition; see [7] for details. For the sake of brevity, we do not spell out the half-forms explicitly.

Let ε be the symplectic (or Liouville) volume form on $\mathrm{T}^*K \cong K^{\mathbb{C}}$; this form induces the Liouville volume measure, and we will refer to ε as *Liouville (volume) measure* as well. Further, let dx denote the volume form on K yielding *Haar* measure, *normalized* so that it coincides with the *Riemannian volume measure* on K , and let dY be the volume form inducing *Lebesgue* measure on $\mathfrak{k}$, *normalized* by the *inner product* on $\mathfrak{k}$. In terms of the polar decomposition (1.1), we then have the identity $\varepsilon = dx dY$. We prefer *not* to normalize the inner product on $\mathfrak{k}$ since this inner product yields the kinetic energy.

Define the function $\eta : K^{\mathbb{C}} \rightarrow \mathbb{R}$ by

$$\eta(x, Y) = \left(\det \left(\frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)} \right) \right)^{\frac{1}{2}}, \quad x \in K, Y \in \mathfrak{k}; \quad (1.3)$$

this yields a non-negative real analytic function on $K^{\mathbb{C}}$ which depends only on the variable $Y \in \mathfrak{k}$ and, for $x \in K$ and $Y \in \mathfrak{k}$, we will also write $\eta(Y)$ instead of $\eta(x, Y)$. The function η^2 is the density of Haar measure relative to the Liouville volume measure on $K^{\mathbb{C}}$, cf. [5] (Lemma 5). Both measures are K -bi-invariant; in particular, as a function on $\mathfrak{k}$, η is $\mathrm{Ad}(K)$ -invariant. For later reference we point out that, with the notation

$$j(Y) = \det \left(\frac{\sinh(\mathrm{ad}(Y/2))}{\mathrm{ad}(Y/2)} \right)^{\frac{1}{2}}, \quad Y \in \mathfrak{g},$$

where $\mathfrak{g}$ is a general Lie algebra, for $Y \in \mathfrak{k}$, we have the identity

$$j(iY) = \eta(Y/2). \quad (1.4)$$

The notation j is due to [19] ((2.3.6) p. 459). The function j^2 the density of Haar measure relative to canonical coordinates on K , cf. [18] (p. 143), and is the crucial correction term in Kirillov's character formula which we shall

heavily exploit later in the paper. In fact, the identity (1.4) is the reason why the BKS-pairing map comes out to be unitary. On the space of holomorphic functions on $K^{\mathbb{C}}$, we will denote by $\langle \cdot, \cdot \rangle_{t, K^{\mathbb{C}}}$ the normalized inner product given by

$$\langle \Phi, \Psi \rangle_{t, K^{\mathbb{C}}} = \frac{1}{\text{vol}(K)} \int_{K^{\mathbb{C}}} \overline{\Phi} \Psi e^{-\kappa/t} \eta \varepsilon, \quad (1.5)$$

and we denote by $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/t} \eta \varepsilon)$ the resulting Hilbert space of holomorphic functions that are square integrable with respect to the measure $e^{-\kappa/t} \eta \varepsilon$. This Hilbert space is intrinsically a Hilbert space of holomorphic half-forms on $K^{\mathbb{C}}$, cf. [7,22,26]. It is, furthermore, a unitary $(K \times K)$ -representation in an obvious fashion.

Given two holomorphic functions Φ and Ψ on $K^{\mathbb{C}}$, we define their convolution $\Phi * \Psi$ by

$$(\Phi * \Psi)(q) = \frac{1}{\text{vol}(K)} \int_K \Phi(x) \Psi(x^{-1}q) dh_K(x), \quad q \in K^{\mathbb{C}}; \quad (1.6)$$

since K is compact, the convolution $\Phi * \Psi$ is a holomorphic function on $K^{\mathbb{C}}$, indeed the unique extension to a holomorphic function on $K^{\mathbb{C}}$ of the convolution $(\Phi|_K) * (\Psi|_K)$ of the restrictions to K . Since restriction to K yields an isomorphism from $\mathbb{C}[K^{\mathbb{C}}]$ onto the space $R(K)$ of representative functions on K and since the operation of convolution turns $R(K)$ into an algebra, indeed, a topological algebra relative to the inner product determined by Haar measure on K , the operation of convolution turns the vector space $\mathbb{C}[K^{\mathbb{C}}]$ of representative functions on $K^{\mathbb{C}}$ into an algebra. We will refer to the vector space $\mathbb{C}[K^{\mathbb{C}}]$ of representative functions on $K^{\mathbb{C}}$, turned into an algebra via the convolution product, as the *convolution algebra of representative functions* on $K^{\mathbb{C}}$.

Let T be a maximal torus in K , $\mathfrak{t}$ its Lie algebra, $T^{\mathbb{C}} \subseteq K^{\mathbb{C}}$ the complexification of T , $\mathfrak{t}^{\mathbb{C}}$ the complexification of $\mathfrak{t}$, and let W denote the Weyl group. Choose a dominant Weyl chamber C^+ , and let R^+ be the corresponding system of positive *real* roots. Here and below the convention is that, given $Z \in \mathfrak{t}$ and an element A of the root space $\mathfrak{k}_{\alpha}$ associated with the root α , the bracket $[Z, A]$ is given by $[Z, A] = i\alpha(Z)A$ so that, in particular, α is a real valued linear form on $\mathfrak{t}$. In [1] (V.1.3 on p. 185) these α 's are called *infinitesimal* roots. Relative to the chosen dominant Weyl chamber, let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, the *Weyl vector*.

We will denote by $\widehat{K^{\mathbb{C}}}$ the set of isomorphism classes of irreducible rational representations of $K^{\mathbb{C}}$. As usual, we identify $\widehat{K^{\mathbb{C}}}$ with the space of highest weights relative to the chosen dominant Weyl chamber. For a highest weight λ , we denote by $T_{\lambda}: K^{\mathbb{C}} \rightarrow \text{End}(V_{\lambda})$ a representation in the class of λ and by d_{λ} the dimension of V_{λ} .

Let λ be a highest weight. For $\psi \in V_{\lambda}^*$ and $w \in V_{\lambda}$, the function $\Phi_{\psi, w}$ given by

$$\Phi_{\psi, w}(q) = \psi(qw), \quad q \in K^{\mathbb{C}}, \quad (1.7)$$

is a representative function on $K^{\mathbb{C}}$, and the assignment to $\psi \otimes w \in V_{\lambda}^* \otimes V_{\lambda}$ of the representative function $\Phi_{\psi, w}$ yields a morphism

$$\iota_{\lambda}: V_{\lambda}^* \otimes V_{\lambda} \longrightarrow \mathbb{C}[K^{\mathbb{C}}] \quad (1.8)$$

of $(K^{\mathbb{C}} \times K^{\mathbb{C}})$ -representations, necessarily injective since $V_{\lambda}^* \otimes V_{\lambda}$ is an irreducible $(K^{\mathbb{C}} \times K^{\mathbb{C}})$ -representation. We will write

$$V_{\lambda}^* \odot V_{\lambda} = \iota_{\lambda}(V_{\lambda}^* \otimes V_{\lambda}) \subset \mathbb{C}[K^{\mathbb{C}}]. \quad (1.9)$$

Given an L^2 -function f on K and the irreducible representation $T_{\lambda}: K \rightarrow \text{End}(W_{\lambda})$ of K associated with λ , following one of the possible conventions, we define the *Fourier coefficient* $\widehat{f}_{\lambda} \in \text{End}(W_{\lambda})$ of f relative to λ by

$$\widehat{f}_{\lambda} = \frac{1}{\text{vol}(K)} \int_K f(x) T_{\lambda}(x^{-1}) dx. \quad (1.10)$$

Given a holomorphic function Φ on $K^{\mathbb{C}}$ and the irreducible rational representation $T_{\lambda}: K^{\mathbb{C}} \rightarrow \text{End}(V_{\lambda})$ of $K^{\mathbb{C}}$ associated with λ , we define the *Fourier coefficient* $\widehat{\Phi}_{\lambda} \in \text{End}(V_{\lambda})$ of Φ relative to λ to be the Fourier coefficient of the restriction of Φ to K . The notational distinction between V_{λ} and W_{λ} will be justified in Section 5.

Let

$$C_{t,\lambda} = (t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2}. \quad (1.11)$$

The precise significance of the real constant $C_{t,\lambda}$ will be explained in Lemma 3.3 and Remark 3.7. On $\text{End}(V_\lambda)$, we take the standard inner product $\langle \cdot, \cdot \rangle_\lambda$ given by

$$\langle A, B \rangle_\lambda = \text{tr}(A^* B), \quad A, B \in \text{End}(V_\lambda), \quad (1.12)$$

the adjoint A^* of A being computed as usual with respect to a K -invariant inner product on V_λ . We endow $\bigoplus_{\lambda \in K^\mathbb{C}} \text{End}(V_\lambda)$ with the inner product which, on the summand $\text{End}(V_\lambda)$, is given by

$$\frac{d_\lambda}{C_{t,\lambda}} \langle \cdot, \cdot \rangle_\lambda; \quad (1.13)$$

then $\widehat{\bigoplus_{\lambda \in K^\mathbb{C}} \text{End}(V_\lambda)}$ refers to the completion relative to this inner product. Thus, up to a constant, the resulting norm on each $\text{End}(V_\lambda)$ is the familiar *Hilbert–Schmidt* norm.

Theorem 1.14 (*Holomorphic Peter–Weyl Theorem*). (i) The Hilbert space $\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon)$ contains the vector space $\mathbb{C}[K^\mathbb{C}]$ of representative functions on $K^\mathbb{C}$ as a dense subspace and, as a unitary $(K \times K)$ -representation, $\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon)$ decomposes as the direct sum

$$\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon) = \widehat{\bigoplus_{\lambda \in K^\mathbb{C}} V_\lambda^* \odot V_\lambda} \quad (1.14.1)$$

into $(K \times K)$ -isotypical summands.

(ii) The operation of convolution induces a convolution product $*$ on $\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon)$.

(ii) Relative to the convolution product, as λ ranges over the irreducible rational representations of $K^\mathbb{C}$, the assignment to a holomorphic function Φ on $K^\mathbb{C}$ of its Fourier coefficients $\widehat{\Phi}_\lambda \in \text{End}(V_\lambda)$ yields an isomorphism

$$\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon) \longrightarrow \widehat{\bigoplus_{\lambda \in K^\mathbb{C}} \text{End}(V_\lambda)} \quad (1.14.2)$$

of Hilbert algebras, where each summand $\text{End}(V_\lambda)$ is endowed with its obvious algebra structure.

The decomposition (1.14.1) of $\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon)$ is the *Peter–Weyl* decomposition of this Hilbert space alluded to earlier.

2. The convolution algebra of representative functions

For ease of exposition we recall the familiar *Cartan–Weyl* decomposition into minimal two-sided ideals of the convolution algebra of representative functions on $K^\mathbb{C}$.

The operation

$$L_x: \mathbb{C}[K^\mathbb{C}] \longrightarrow \mathbb{C}[K^\mathbb{C}], \quad (L_x(\Phi))(q) = \Phi(x^{-1}q), \quad x, q \in K^\mathbb{C}, \Phi \in \mathbb{C}[K^\mathbb{C}],$$

of left translation on $K^\mathbb{C}$ and the operation

$$R_y: \mathbb{C}[K^\mathbb{C}] \longrightarrow \mathbb{C}[K^\mathbb{C}], \quad (R_y(\Phi))(q) = \Phi(qy), \quad y, q \in K^\mathbb{C}, \Phi \in \mathbb{C}[K^\mathbb{C}],$$

of right translation on $K^\mathbb{C}$ are well known to turn $\mathbb{C}[K^\mathbb{C}]$ into an algebraic $(K^\mathbb{C} \times K^\mathbb{C})$ -representation in such a way that the operations of left and right translation commute. Furthermore, the assignment to the two representative functions f and h on $K^\mathbb{C}$ of $\langle f, h \rangle = (f * h)(e)$ yields a complex symmetric $K^\mathbb{C}$ -invariant bilinear form

$$\langle \cdot, \cdot \rangle: \mathbb{C}[K^\mathbb{C}] \otimes \mathbb{C}[K^\mathbb{C}] \longrightarrow \mathbb{C} \quad (2.1)$$

on $\mathbb{C}[K^\mathbb{C}]$, cf. e.g. [23].

Let λ be a highest weight. We endow $V_\lambda^* \odot V_\lambda \cong V_\lambda^* \otimes V_\lambda$ with the obvious complex symmetric bilinear form coming from the evaluation mapping. By construction, this form coincides with the restriction of the complex symmetric bilinear form (2.1) to $V_\lambda^* \odot V_\lambda$ whence this restriction is non-degenerate, that is, a complex inner product. The operation

of convolution is defined on $\mathbb{C}[K^{\mathbb{C}}]$ and, relative to the convolution product on $\mathbb{C}[K^{\mathbb{C}}]$, the assignment to $\Phi \in \mathbb{C}[K^{\mathbb{C}}]$ of its Fourier coefficient $\widehat{\Phi}_{\lambda} \in \text{End}(V_{\lambda})$ induces a surjective morphism of algebras

$$F_{\lambda}: \mathbb{C}[K^{\mathbb{C}}] \longrightarrow \text{End}(V_{\lambda}), \quad (2.2)$$

where $\text{End}(V_{\lambda})$ carries its obvious algebra structure, and this morphism has the property that, for every $x, y \in K^{\mathbb{C}}$ and every $w \in V_{\lambda}$,

$$(\mathbf{L}_x \mathbf{L}_y(\Phi))(w) = T_{\lambda}(x)(\Phi((T_{\lambda}(y^{-1})w))), \quad \Phi \in \mathbb{C}[K^{\mathbb{C}}]. \quad (2.3)$$

Furthermore, the composite

$$F_{\lambda} \circ \iota_{\lambda}: V_{\lambda}^* \otimes V_{\lambda} \longrightarrow \text{End}(V_{\lambda}) \quad (2.4)$$

is the canonical isomorphism.

We will use the notation $\alpha = (\alpha_{\lambda}) \in \oplus \text{End}(V_{\lambda})$ and $T = (T_{\lambda}: K^{\mathbb{C}} \rightarrow \oplus \text{End}(V_{\lambda}))$, as λ ranges over the highest weights. In terms of this notation, the obvious action of $K^{\mathbb{C}} \times K^{\mathbb{C}}$ on $\oplus \text{End}(V_{\lambda})$ is given by the association

$$(x, y, \alpha) \longmapsto T(x) \circ \alpha \circ T(y^{-1}), \quad x, y \in K^{\mathbb{C}}.$$

We now recall the algebraic analogue of the Peter–Weyl theorem; see e.g. Section 5 of [23] for details.

Proposition 2.5. (i) *The complex vector space $\mathbb{C}[K^{\mathbb{C}}]$ of representative functions decomposes as the direct sum*

$$\mathbb{C}[K^{\mathbb{C}}] = \oplus_{\lambda} V_{\lambda}^* \odot V_{\lambda} \quad (2.5.1)$$

of $(K^{\mathbb{C}} \times K^{\mathbb{C}})$ -representations and, relative to the complex symmetric bilinear form (2.1), the decomposition is orthogonal.

(ii) *For each $\lambda \in \widehat{K^{\mathbb{C}}}$, the summand $V_{\lambda}^* \odot V_{\lambda}$ is the isotypical summand of $\mathbb{C}[K^{\mathbb{C}}]$ determined by λ , and the restriction of the complex symmetric bilinear form (2.1) to this summand is non-degenerate.*

(iii) *Relative to the convolution product on $\mathbb{C}[K^{\mathbb{C}}]$, the induced morphism*

$$(F_{\lambda}): \mathbb{C}[K^{\mathbb{C}}] \longrightarrow \oplus_{\lambda} \text{End}(V_{\lambda}) \quad (2.5.2)$$

of algebras is an isomorphism of $(K^{\mathbb{C}} \times K^{\mathbb{C}})$ -representations and yields the decomposition of the convolution algebra $\mathbb{C}[K^{\mathbb{C}}]$ into minimal two-sided ideals. $\square$

3. The square integrability of the representative functions

The aim of the present section is to establish the square-integrability of the representative functions on $K^{\mathbb{C}}$ and to reduce the calculation of the requisite integrals over $K^{\mathbb{C}}$ to integrals over K .

Lemma 3.1. *Each representative function on $K^{\mathbb{C}}$ is square integrable relative to the measure $e^{-\kappa/t} \eta_{\epsilon}$.*

We shall exploit the following integration formula

$$\int_{\mathfrak{k}} f(Y) dY = \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_{K/T} f(\text{Ad}_Y(Y)) d(yT) \right\} dY, \quad (3.2)$$

valid for any integrable continuous function f on $\mathfrak{k}$. For the special case where $\mathfrak{k} = \mathfrak{su}(2)$, the formula essentially comes down to integration on $\mathbb{R}^3$ in ordinary spherical polar coordinates. For the general case, see e.g. [8] (Theorem I.5.17, p. 195) or [3] ((3.14.2) on p. 185 combined with (3.14.4) on p. 187).

Let λ be a highest weight. We will use the notation $\varphi^{\mathbb{C}}$ etc. for representative functions on $K^{\mathbb{C}}$ in the isotypical summand $V_{\lambda}^* \odot V_{\lambda}$ of $\mathbb{C}[K^{\mathbb{C}}]$ associated with λ and, accordingly, we will denote the restriction of $\varphi^{\mathbb{C}}$ to K by φ ; then φ is necessarily a representative function on K which lies in the isotypical summand of $L^2(K, dx)$ associated with λ by virtue of the ordinary Peter–Weyl theorem. Lemma 3.1 is implied by the following.

Lemma 3.3. Given the representative function $\varphi^{\mathbb{C}}$ on $K^{\mathbb{C}}$ in the isotypical summand $V_{\lambda}^* \odot V_{\lambda}$ of $\mathbb{C}[K^{\mathbb{C}}]$ associated with the highest weight $\lambda \in \widehat{K^{\mathbb{C}}}$,

$$\int_{K^{\mathbb{C}}} \overline{\varphi^{\mathbb{C}}} \varphi^{\mathbb{C}} e^{-\kappa/t} \eta_{\varepsilon} = C_{t,\lambda} \int_K \overline{\varphi} \varphi dx, \quad C_{t,\lambda} = (t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2}.$$

The operation $\lambda \rightarrow \lambda + \rho$, which appears here for the first time in the paper, is the *shift* referred to in the introduction. To prepare for the proof of the lemma, we will denote by $\chi_{\lambda}^{\mathbb{C}}$ the holomorphic character of $K^{\mathbb{C}}$ associated with the highest weight λ and, accordingly, we denote by χ_{λ} the restriction of $\chi_{\lambda}^{\mathbb{C}}$ to K ; this is plainly the irreducible character of K associated with λ .

Lemma 3.4. The character $\chi_{\lambda}^{\mathbb{C}}$ of the irreducible representation $T_{\lambda}: K^{\mathbb{C}} \rightarrow \text{End}(V_{\lambda})$ of $K^{\mathbb{C}}$ associated with the highest weight λ satisfies the identity

$$\int_{K^{\mathbb{C}}} \|\chi_{\lambda}^{\mathbb{C}}\|^2 e^{-\kappa} \eta_{\varepsilon} = \frac{1}{d_{\lambda}} \int_{K^{\mathbb{C}}} \|T_{\lambda}\|^2 e^{-\kappa} \eta_{\varepsilon}$$

where, as before, $d_{\lambda} = \dim(V_{\lambda})$.

To prepare for the proof of this Lemma recall that, given an L^2 -function f on K , the appropriate version of the *Plancherel* theorem says that a function f on K satisfying suitable hypotheses, e.g. ‘ f smooth’ suffices, admits the *Fourier* decomposition

$$f(x) = \sum_{\theta} d_{\theta} \text{tr}(\widehat{f}_{\theta} T_{\theta}(x)), \quad x \in K, \quad (3.5)$$

where θ ranges over the highest weights; see e.g. [19] (2.3.10). Furthermore, one version of the *Plancherel* formula takes the form

$$\frac{1}{\text{vol}(K)} \int_K |f(x)|^2 dx = \sum_{\theta} d_{\theta} \|\widehat{f}_{\theta}\|^2; \quad (3.6)$$

see e.g. [19] (2.3.11).

For $f = \chi_{\lambda}$, the only non-zero Fourier coefficient equals $\widehat{f}_{\lambda} = \frac{1}{d_{\lambda}} \text{Id}_{V_{\lambda}}$, and the Fourier decomposition of the character χ_{λ} takes the form

$$\chi_{\lambda}(x) = d_{\lambda} \text{tr}(\widehat{f}_{\lambda} T_{\lambda}(x)), \quad x \in K.$$

Proof of Lemma 3.4. Because the measure $\xi = e^{-\kappa} \eta_{\varepsilon}$ is K -bi-invariant it is in particular invariant under right translation by elements of K . Hence, for every function f on $K^{\mathbb{C}}$ which is square integrable relative to this measure, for each $x \in K$,

$$\int_{K^{\mathbb{C}}} \|f(y)\|^2 d\xi(y) = \int_{K^{\mathbb{C}}} \|f(yx)\|^2 d\xi(y).$$

Integrating this identity over K yields

$$\text{vol}(K) \int_{K^{\mathbb{C}}} \|f(y)\|^2 d\xi(y) = \int_{K^{\mathbb{C}}} \int_K \|f(yx)\|^2 dx d\xi(y).$$

Given $y \in K^{\mathbb{C}}$, the Fourier coefficient $\widehat{f}_{\lambda}^y$ of the function f^y on K defined by

$$f^y(x) = \chi_{\lambda}^{\mathbb{C}}(yx) = \text{tr}(T_{\lambda}(y) T_{\lambda}(x)), \quad x \in K,$$

is given by

$$\widehat{f}_{\lambda}^y = T_{\lambda}(y) \widehat{f}_{\lambda},$$

and this is the only non-zero coefficient. Hence, given $y \in K^{\mathbb{C}}$, applying the Plancherel formula (3.6) on K to the function f^y , we find

$$\frac{1}{\text{vol}(K)} \int_K \|f^y(x)\|^2 dx = d_\lambda \|T_\lambda \widehat{f_\lambda}\|^2 = d_\lambda \left\| \frac{1}{d_\lambda} T_\lambda(y) \right\|^2 = \frac{1}{d_\lambda} \|T_\lambda(y)\|^2.$$

Consequently

$$\int_{K^{\mathbb{C}}} \|\chi_\lambda^{\mathbb{C}}(y)\|^2 d\xi(y) = \frac{1}{\text{vol}(K)} \int_K \int_{K^{\mathbb{C}}} \|f^y(x)\|^2 dx d\xi(y) = \frac{1}{d_\lambda} \int_{K^{\mathbb{C}}} \|T_\lambda(y)\|^2 d\xi(y)$$

as asserted. $\square$

Proof of Lemma 3.3. We establish the statement of the Lemma for the special case where $t = 1$. The general case is reduced to the special case by a change of variables.

As a $(K \times K)$ -representation, the isotypical summand $V_\lambda^* \odot V_\lambda$ is generated by the character $\chi_\lambda^{\mathbb{C}}$. Hence it suffices to establish the assertion for $\varphi^{\mathbb{C}} = \chi_\lambda^{\mathbb{C}}$. By Lemma 3.4, it suffices to compute the integral $\int_{K^{\mathbb{C}}} \|T_\lambda\|^2 e^{-\kappa} \eta \varepsilon$. To compute this integral, let $y = x \exp(iY)$ where as before $x \in K$ and $Y \in \mathfrak{k}$. Let $T'_\lambda: \mathfrak{k}^{\mathbb{C}} \rightarrow \text{End}(V_\lambda)$ denote the corresponding Lie algebra representation and let $A(Y) \in \text{End}(V_\lambda)$ be given by $A(Y) = iT'_\lambda(Y)$. Then

$$T_\lambda(y) = T_\lambda(x)T_\lambda(\exp(iY)) = T_\lambda(x)\exp(iT'_\lambda(Y)) = T_\lambda(x)e^{A(Y)}$$

and

$$T_\lambda^*(y)T_\lambda(y) = \left(e^{A(Y)}\right)^* \left(e^{A(Y)}\right).$$

Since the endomorphism $T'_\lambda(Y)$ is skew-hermitian, the endomorphism $A(Y)$ is hermitian, that is, $A(Y)^* = A(Y)$ whence

$$\left(e^{A(Y)}\right)^* \left(e^{A(Y)}\right) = e^{A(Y)^*} e^{A(Y)} = e^{2A(Y)}$$

and thence

$$\|e^{A(Y)}\|^2 = \text{tr}(e^{2A(Y)}) = \text{tr}(e^{A(2Y)}) = \text{tr}(e^{iT'_\lambda(2Y)}) = \chi_\lambda^{\mathbb{C}}(\exp(2iY)).$$

View $\lambda + \rho$ as a point of $\mathfrak{k}^*$ via the orthogonal decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{q}^+$ where $\mathfrak{q}^+$ is the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$, and let $\Omega_{\lambda+\rho}$ be the coadjoint orbit generated by $\lambda + \rho$. Given $Y \in \mathfrak{k}$, Kirillov's character formula, evaluated at the point $\exp(2iY)$, yields the identity

$$\text{vol}(\Omega_\rho) j(2iY) \chi_\lambda^{\mathbb{C}}(\exp(2iY)) = \text{vol}(\Omega_\rho) \eta(Y) \chi_\lambda^{\mathbb{C}}(\exp(2iY)) = \int_{\Omega_{\lambda+\rho}} e^{-2\vartheta(Y)} d\sigma(\vartheta),$$

cf. [18,19]. Here ϑ refers to the variable on $\Omega_{\lambda+\rho}$ and $d\sigma$ denotes the symplectic volume form on $\Omega_{\lambda+\rho}$.

We pause for the moment and justify briefly how Kirillov's character formula applies: This formula has been shown in [18] to be valid for compact groups (and not for the complexification of a compact group). However, since K is Zariski-dense in $K^{\mathbb{C}}$, given $\vartheta: \mathfrak{k}^{\mathbb{C}} \rightarrow \mathbb{C}$, the function f_ϑ on $\mathfrak{k}^{\mathbb{C}}$ given by $f_\vartheta(Y) = \int_K \vartheta(\text{Ad}_x Y) dx$ is holomorphic. This justifies our evaluation of Kirillov's character formula at a general point of $\mathfrak{k}^{\mathbb{C}}$.

Using the diffeomorphism from K/T onto $\Omega_{\lambda+\rho}$ which sends yT ($y \in K$) to $(\text{Ad}_y^*)^{-1}(\lambda + \rho)$, we now rewrite the integral in the form

$$\begin{aligned} \int_{\Omega_{\lambda+\rho}} e^{-\vartheta(2Y)} d\sigma(\vartheta) &= \frac{\text{vol}(\Omega_{\lambda+\rho})}{\text{vol}(K/T)} \int_{K/T} e^{-(\text{Ad}_y^*)^{-1}(\lambda+\rho)(2Y)} d(yT) \\ &= \frac{d_\lambda \text{vol}(\Omega_\rho)}{\text{vol}(K/T)} \int_{K/T} e^{-2(\lambda+\rho)(\text{Ad}_y(Y))} d(yT). \end{aligned}$$

Hence

$$\eta(Y) \|e^{A(Y)}\|^2 = \eta(Y) \text{tr}(e^{iT'_\lambda(2Y)}) = \frac{d_\lambda}{\text{vol}(K/T)} \int_{K/T} e^{-2(\lambda+\rho)(\text{Ad}_y(Y))} d(yT)$$

whence, in view of the integration formula (3.2),

$$\begin{aligned} \int_{\mathfrak{k}} \|T_{\lambda}(x \exp(iY))\|^2 e^{-|Y|^2} \eta(Y) dY &= \int_{\mathfrak{k}} \|e^{A(Y)}\|^2 e^{-|Y|^2} \eta(Y) dY \\ &= d_{\lambda} \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_{K/T} e^{-2(\lambda+\rho)(\text{Ad}_Y(Y))} d(yT) \right\} e^{-|Y|^2} dY \\ &= d_{\lambda} \int_{\mathfrak{k}} e^{-2(\lambda+\rho)(Y)-|Y|^2} dY = d_{\lambda} \pi^{\dim(K)/2} e^{|\lambda+\rho|^2}. \end{aligned}$$

Consequently

$$\int_{K^{\mathbb{C}}} \|\chi_{\lambda}^{\mathbb{C}}\|^2 e^{-\kappa} \eta_{\varepsilon} = \frac{1}{d_{\lambda}} \int_{K^{\mathbb{C}}} \|T_{\lambda}\|^2 e^{-\kappa} \eta_{\varepsilon} = \pi^{\dim(K)/2} e^{|\lambda+\rho|^2} \text{vol}(K)$$

as asserted. In particular, $C_{1,\lambda} = \pi^{\dim(K)/2} e^{|\lambda+\rho|^2}$. $\square$

Remark 3.7. The number $(t\pi)^{\dim(K)/2}$ is the Gaussian volume of $\mathfrak{k}$ relative to the measure $e^{-\kappa/t} dY$ on $\mathfrak{k}$, and the constant $C_{t,0}$ is the volume of $\mathfrak{k}$ relative to the measure $e^{-\kappa/t} \eta dY$. We remind the reader that dY refers to Lebesgue measure on $\mathfrak{k}$, normalized by the inner product on $\mathfrak{k}$.

4. The constituents given by integral forms

Left and right translation turn $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/t} \eta_{\varepsilon})$ into a unitary $(K \times K)$ -representation. To establish the statement (i) of the holomorphic Peter–Weyl theorem, it remains to show that the decomposition (2.5.1) of the vector space of representative functions on $K^{\mathbb{C}}$ into isotypical summands determines the decomposition of $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/t} \eta_{\varepsilon})$ into isotypical summands, that is to say:

Proposition 4.1. *There is no irreducible $(K \times K)$ -summand in $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/t} \eta_{\varepsilon})$ beyond those which come from the decomposition (2.5.1).*

We shall establish this fact via a geometric argument which is guided by the principle that quantization commutes with reduction.

The choice of the dominant Weyl chamber in $\mathfrak{t}$ made earlier determines Borel subgroups B^+ and B^- of $K^{\mathbb{C}}$ such that the following hold: Any highest weight λ determines algebraic characters $\vartheta_{\lambda}^{\pm}: B^{\pm} \rightarrow \mathbb{C}^*$ such that the algebraic and hence holomorphic line bundles

$$\beta_{\lambda}^{\pm}: K^{\mathbb{C}} \times_{\vartheta_{\lambda}^{\pm}} \mathbb{C} \rightarrow K^{\mathbb{C}}/B^{\pm}$$

on the flag manifolds $K^{\mathbb{C}}/B^{\pm}$ realize the $K^{\mathbb{C}}$ -representations V_{λ} and V_{λ}^* ; more precisely, the complex symmetric $K^{\mathbb{C}}$ -invariant bilinear form (2.1) on $\mathbb{C}[K^{\mathbb{C}}]$ induces an isomorphism

$$\Gamma(\beta_{\lambda}^-) \longrightarrow (\Gamma(\beta_{\lambda}^+))^*$$

of algebraic and hence holomorphic $K^{\mathbb{C}}$ -representations and, when the data are suitably adjusted, the representation $\Gamma(\beta_{\lambda}^+)$ comes down to V_{λ} .

In view of the $(K \times K)$ -equivariant diffeomorphism between T^*K and $K^{\mathbb{C}}$ spelled out in Section 1 above, we consider $K^{\mathbb{C}}$, endowed with the induced symplectic structure, as a Hamiltonian $(K \times K)$ -space. Denote the symplectic structure on $K^{\mathbb{C}}$ by σ_K and the corresponding momentum mapping on $K^{\mathbb{C}}$ by

$$\mu^{K \times K}: K^{\mathbb{C}} \rightarrow \mathfrak{k}^* \times \mathfrak{k}^*.$$

Let λ be a highest weight for $K^{\mathbb{C}}$, endow $K^{\mathbb{C}}/B^+$ and $K^{\mathbb{C}}/B^-$ with the appropriate Kähler structures having Chern classes λ and $-\lambda$, respectively, let $\mu^{\pm}: K^{\mathbb{C}}/B^{\pm} \rightarrow \mathfrak{k}^*$ be the corresponding K -equivariant momentum mappings, and consider the product manifold

$$N^{\times} = K^{\mathbb{C}} \times K^{\mathbb{C}}/B^+ \times K^{\mathbb{C}}/B^-$$

endowed with the product Kähler structure. Let $\sigma^\times$ be the resulting product symplectic structure which underlies the product Kähler structure. The group $K^\mathbb{C} \times K^\mathbb{C}$ acts on $N^\times$ in the obvious way and, by construction, the symplectic structure $\sigma^\times$ is $(K \times K)$ -invariant. Let $\mu^\times: N^\times \rightarrow \mathfrak{k}^* \times \mathfrak{k}^*$ be the $(K \times K)$ -momentum mapping for the $(K \times K)$ -action on $N^\times$ relative to the symplectic structure $\sigma^\times$. This momentum mapping is essentially the sum of the momentum mapping $\mu^{K \times K}$ and the momentum mappings of $K^\mathbb{C}/B^+$ and $K^\mathbb{C}/B^-$ into the first and second copy of $\mathfrak{k}^*$ in the product $\mathfrak{k}^* \times \mathfrak{k}^*$.

Left and right translation on $K^\mathbb{C}$ turn the complex vector space $\mathcal{H}(K^\mathbb{C})$ of holomorphic functions on $K^\mathbb{C}$ into a holomorphic $(K^\mathbb{C} \times K^\mathbb{C})$ -representation. For topological reasons, the trivial complex line bundle β on $K^\mathbb{C}$ serves as prequantum bundle on $K^\mathbb{C}$, and the space of holomorphic sections of β comes down to the vector space $\mathcal{H}(K^\mathbb{C})$. By construction, the product line bundle $\beta^\times = \beta \times \beta_\lambda^+ \times \beta_\lambda^-$ is a holomorphic $(K \times K)$ -equivariant prequantum bundle on the Kähler manifold $N^\times$. Since the complex vector spaces V_λ and V_λ^* are finite-dimensional, as a $(K \times K)$ -representation, the space of holomorphic sections of $\beta^\times$ amounts to the complex vector space

$$\mathrm{Hom}_\mathbb{C}(V_\lambda^* \otimes V_\lambda, \mathcal{H}(K^\mathbb{C})).$$

The $(K \times K)$ -reduced space $(\mu^\times)^{-1}(0, 0)/(K \times K)$ at the point zero of $\mathfrak{k}^* \times \mathfrak{k}^*$ boils down to a single point. Consequently the space of $(K \times K)$ -invariant holomorphic sections of the product line bundle $\beta^\times$ is at most 1-dimensional, that is, the space

$$\mathrm{Hom}_\mathbb{C}(V_\lambda^* \otimes V_\lambda, \mathcal{H}(K^\mathbb{C}))^{K \times K}$$

is at most 1-dimensional. Since we already know that, in the decomposition (2.5.1) of the vector space $\mathbb{C}[K^\mathbb{C}]$ of representative functions on $K^\mathbb{C}$, $V_\lambda^* \odot V_\lambda$ is the isotypical summand corresponding to λ , and since, by virtue of Lemma 3.1, $V_\lambda^* \odot V_\lambda$ is actually a subspace of the Hilbert space $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon)$, we conclude that the vector space

$$\mathrm{Hom}_\mathbb{C}(V_\lambda^* \otimes V_\lambda, \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon))^{K \times K}$$

of $(K \times K)$ -invariants is 1-dimensional. However, this vector space is that of morphisms of $(K \times K)$ -representations from $V_\lambda^* \otimes V_\lambda$ to $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon)$. Since this space is 1-dimensional, it is generated by a single such morphism, and this morphism picks out the $(K \times K)$ -irreducible constituent $V_\lambda^* \odot V_\lambda$ from $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon)$. In other words, $V_\lambda^* \odot V_\lambda$ is the isotypical summand in $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon)$ determined by λ .

These observations imply that the vector space $\mathbb{C}[K^\mathbb{C}]$ of representative functions on $K^\mathbb{C}$ is dense in the Hilbert space $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon)$. This proves Proposition 4.1 and hence establishes statement (i) of the holomorphic Peter–Weyl theorem.

5. The abstract identification with the vertically polarized Hilbert space

The vertically polarized Hilbert space arising from geometric quantization on T^*K is a Hilbert space of half forms. Haar measure dx on K then yields a concrete realization of this Hilbert space as $L^2(K, dx)$. In this section we will compare the Hilbert space $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon)$ with the vertically polarized Hilbert space. This will, in particular, provide a proof of statements (ii) and (iii) of the holomorphic Peter–Weyl theorem.

Let λ be a highest weight. Let W_λ denote the space of complex representative functions on K which arise by restriction to K of the holomorphic functions in V_λ . Since a holomorphic function on $K^\mathbb{C}$ is determined by its values on K , this restriction mapping is the identity mapping of complex vector spaces, in fact, of K -representations. To justify the distinction in notation, we note that the embedding ι_λ given as (1.8) above yields an embedding

$$\iota_\lambda: W_\lambda^* \otimes W_\lambda \longrightarrow R(K) = \mathbb{C}[K^\mathbb{C}]$$

and, maintaining the notation $\odot$ introduced in Section 2, we write

$$W_\lambda^* \odot W_\lambda = \iota_\lambda(W_\lambda^* \otimes W_\lambda) \subseteq R(K).$$

The K -representation $W_\lambda^* \otimes W_\lambda$ inherits a K -invariant inner product from the embedding into $L^2(K, dx)$. On the other hand, $V_\lambda^* \otimes V_\lambda$ acquires an inner product from its embedding into the Hilbert space $\mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon)$ induced by (1.8) which turns $V_\lambda^* \otimes V_\lambda$ into a unitary K -representation, but the relationship between the inner products on $V_\lambda^* \otimes V_\lambda$ and $W_\lambda^* \otimes W_\lambda$ is not, a priori, clear. We therefore distinguish the resulting unitary K -representations W_λ and V_λ in notation as indicated.

Let $\langle \cdot, \cdot \rangle_K$ denote the normalized inner product on $L^2(K, dx)$ given by

$$\langle f, h \rangle_K = \frac{1}{\text{vol}(K)} \int_K \bar{f} h dx. \quad (5.1)$$

As usual, we endow $\bigoplus_{\lambda \in \widehat{K^\mathbb{C}}} \text{End}(W_\lambda)$ with the inner product which, on the summand $\text{End}(W_\lambda)$, is given by $d_\lambda \langle \cdot, \cdot \rangle_\lambda$. This inner product differs from the inner product (1.13); see the completion of the proof of the holomorphic Peter–Weyl theorem given below for an explanation. Then $\widehat{\bigoplus_{\lambda \in \widehat{K^\mathbb{C}}} \text{End}(W_\lambda)}$ refers to the completion relative to this inner product. As in the situation of the inner product (1.13), up to a constant, the resulting norm on each $\text{End}(W_\lambda)$ coincides with the *Hilbert–Schmidt* norm. For ease of exposition, we spell out the ordinary *Peter–Weyl* theorem in the following form.

Proposition 5.2. (i) *The space $R(K)$ of representative functions on K is dense in $L^2(K, dx)$ and, as a unitary $(K \times K)$ -representation, $L^2(K, dx)$ decomposes as the direct sum*

$$L^2(K, dx) = \widehat{\bigoplus_\lambda (W_\lambda^* \odot W_\lambda)} \cong \widehat{\bigoplus_\lambda \text{End}(W_\lambda)} \quad (5.2.1)$$

into $(K \times K)$ -isotypical summands as λ ranges over the highest weights.

(ii) *Relative to the convolution product $*$ on $L^2(K, dx)$, as λ ranges over the highest weights, the assignment to an L^2 -function f on K of its Fourier coefficients $\widehat{f}_\lambda \in \text{End}(W_\lambda)$ yields an isomorphism*

$$L^2(K, dx) \longrightarrow \widehat{\bigoplus_\lambda \text{End}(W_\lambda)} \quad (5.2.2)$$

of Hilbert algebras where $L^2(K, dx)$ is endowed with the normalized inner product $\langle \cdot, \cdot \rangle_K$.

The following is an immediate consequence of the ordinary and the holomorphic Peter–Weyl theorem, combined with the explicit determination of the constants $C_{t,\lambda}$ for the highest weights λ given in Lemma 3.3, viz. $C_{t,\lambda} = (t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2}$.

Theorem 5.3. *The association*

$$V_\lambda^* \odot V_\lambda \ni \varphi^\mathbb{C} \longmapsto C_{t,\lambda}^{1/2} \varphi = (t\pi)^{\dim(K)/4} e^{t|\lambda+\rho|^2/2} \varphi \in W_\lambda^* \odot W_\lambda,$$

as λ ranges over the highest weights, induces a unitary isomorphism

$$H_t: \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon) \longrightarrow L^2(K, dx) \quad (5.3.1)$$

of unitary $(K \times K)$ -representations. $\square$

Completion of the proof of the holomorphic Peter–Weyl theorem. Let

$$H_t^{\text{End}}: \widehat{\bigoplus_{\lambda \in \widehat{K^\mathbb{C}}} \text{End}(V_\lambda)} \rightarrow \widehat{\bigoplus_{\lambda \in \widehat{K^\mathbb{C}}} \text{End}(W_\lambda)}$$

be the obvious unitary isomorphism of $(K \times K)$ -representations which, restricted to the summand $\text{End}(V_\lambda)$, is given by multiplication by $C_{t,\lambda}^{1/2}$, as λ ranges over the highest weights. By construction, the diagram

$$\begin{array}{ccc} \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta \varepsilon) & \xrightarrow{H_t} & L^2(K, dx) \\ \downarrow & & \downarrow \\ \widehat{\bigoplus_{\lambda \in \widehat{K^\mathbb{C}}} \text{End}(V_\lambda)} & \xrightarrow{H_t^{\text{End}}} & \widehat{\bigoplus_{\lambda \in \widehat{K^\mathbb{C}}} \text{End}(W_\lambda)} \end{array}$$

is commutative where the unlabelled vertical arrows are given by the assignment to a function of its Fourier coefficients. Moreover, in view of [Theorem 5.3](#), the upper horizontal arrow is an isomorphism of unitary $(K \times K)$ -representations, the lower horizontal arrow is such an isomorphism as just pointed out and, by virtue of the ordinary Peter–Weyl theorem, the right-hand vertical arrow is an isomorphism of Hilbert algebras. In view of the algebraic version of the Peter–Weyl theorem, [Proposition 2.5](#) above, we conclude that the convolution product on the algebra $\mathbb{C}[K^{\mathbb{C}}]$ extends to a convolution product on $\mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta_{\varepsilon})$ and that the left-hand vertical arrow is an isomorphism of Hilbert algebras as asserted. $\square$

6. The Blattner–Kostant–Sternberg pairing

In this section we will show that the isomorphism [\(5.3.1\)](#) is realized by the B(lattner)K(ostant)S(ternberg)-pairing, multiplied by a global constant; see e.g. [\[22\]](#), [\[26\]](#) for details on the BKS-pairing. We maintain the notation $\langle \cdot, \cdot \rangle_{t, K^{\mathbb{C}}}$ for the normalized inner product on $\mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta_{\varepsilon})$ induced by the measure $e^{-\kappa/t}\eta_{\varepsilon}$.

Let Φ be a holomorphic function on $K^{\mathbb{C}}$ which is square integrable relative to $e^{-\kappa/t}\eta_{\varepsilon}$ and let F be a square integrable function on K ; the ordinary BKS-pairing $\langle \cdot, \cdot \rangle_{\text{BKS}}$ between the two half-form Hilbert spaces $\mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta_{\varepsilon})$ and $L^2(K, dx)$ assigns the integral

$$\langle \Phi, F \rangle_{\text{BKS}} = \frac{1}{\text{vol}(K)} \int_K \int_{\mathfrak{k}} \overline{\Phi(x \exp(iY))} F(x) e^{-\frac{|Y|^2}{2t}} \eta(Y/2) dY dx \quad (6.1.1)$$

to Φ and F provided this integral exists. The requisite calculation which yields the explicit form [\(6.1.1\)](#) of the BKS-pairing under the present circumstances is given in the appendix of [\[7\]](#), where the notation $\zeta(Y) = \eta(Y/2)$ is used ($Y \in \mathfrak{k}$).

We will now show that [\(6.1.1\)](#) extends to a pairing which is defined everywhere, that is, to a pairing of the kind

$$\langle \cdot, \cdot \rangle_{\text{BKS}}: \mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta_{\varepsilon}) \otimes L^2(K, dx) \longrightarrow \mathbb{C}. \quad (6.1.2)$$

We do not assert that the integral is absolutely convergent for every Φ and F , though. To begin with we note that it is manifest that, given a holomorphic function Φ on $K^{\mathbb{C}}$ which is square integrable relative to $e^{-\kappa/t}\eta_{\varepsilon}$, when the complex function F_{Φ} on K given by the expression

$$(F_{\Phi})(x) = \int_{\mathfrak{k}} \Phi(x \exp(iY)) e^{-\frac{|Y|^2}{2t}} \eta(Y/2) dY, \quad x \in K, \quad (6.1.3)$$

is well defined, that is, when the integral exists for every $x \in K$,

$$\langle \Phi, F \rangle_{\text{BKS}} = \langle F_{\Phi}, F \rangle_K. \quad (6.2)$$

Theorem 6.3. *Let $\varphi^{\mathbb{C}}$ be a representative function on $K^{\mathbb{C}}$, and let φ denote its restriction to K . Then the integral [\(6.1.3\)](#) is well defined for $\Phi = \varphi^{\mathbb{C}}$, and*

$$\int_{K^{\mathbb{C}}} \overline{\varphi^{\mathbb{C}}} \varphi^{\mathbb{C}} e^{-\kappa/t} \eta_{\varepsilon} = (4t\pi)^{-\dim(K)/2} \int_K \overline{F_{\varphi^{\mathbb{C}}}} F_{\varphi^{\mathbb{C}}} dx. \quad (6.3.1)$$

As for the constant which shows up in [\(6.3.1\)](#), we note that $(2t\pi)^{\dim(K)/2}$ is the Gaussian volume of $\mathfrak{k}$ relative to the measure $e^{-\kappa/2t} dY$ where dY is the normalized Lebesgue measure on $\mathfrak{k}$.

Lemma 6.4. *Let λ be a highest weight, let $\varphi^{\mathbb{C}}$ be a representative function on $K^{\mathbb{C}}$ in the isotypical summand $V_{\lambda}^* \odot V_{\lambda}$ of $\mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta_{\varepsilon})$ associated with λ and, as before, let φ denote the restriction of $\varphi^{\mathbb{C}}$ to K , necessarily a representative function on K which lies in the isotypical summand $W_{\lambda}^* \odot W_{\lambda}$ of $L^2(K, dx)$. Then, for $\Phi = \varphi^{\mathbb{C}}$, the integral [\(6.1.3\)](#) exists for every $x \in K$, and the resulting function $F_{\varphi^{\mathbb{C}}}$ on K is given by*

$$F_{\varphi^{\mathbb{C}}} = D_{t,\lambda} \varphi, \quad D_{t,\lambda} = (2t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2/2}.$$

Proof. We establish the statement of the Lemma for the special case where $t = 1$. The general case is reduced to the special case by a change of variables.

As a $(K \times K)$ -representation, the isotypical summand $V_\lambda^* \odot V_\lambda$ associated with λ is spanned by the character $\chi_\lambda^\mathbb{C}$ of $K^\mathbb{C}$ associated with the highest weight λ . Thus it suffices to establish the claim for $\varphi^\mathbb{C} = \chi_\lambda^\mathbb{C}$, and we will now do so:

In view of the integration formula (3.2), given $x \in K$,

$$\begin{aligned} F_{\chi_\lambda^\mathbb{C}}(x) &= \int_{\mathfrak{k}} \chi_\lambda^\mathbb{C}(x \exp(iY)) e^{-|Y|^2/2} \eta(Y/2) dY \\ &= \frac{1}{\text{vol}(T)} \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_K \chi_\lambda^\mathbb{C}(x \exp(\text{Ad}_y(iY))) dy \right\} e^{-|Y|^2/2} \eta(Y/2) dY \\ &= \frac{1}{\text{vol}(T)} \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_K \chi_\lambda^\mathbb{C}(y^{-1}xy \exp(iY)) dy \right\} e^{-|Y|^2/2} \eta(Y/2) dY. \end{aligned}$$

Let $x \in K$ and $Y \in \mathfrak{k}$; using the formula

$$\int_K \chi_\lambda^\mathbb{C}(y^{-1}xy \exp(iY)) dy = \frac{\text{vol}(K)}{d_\lambda} \chi_\lambda(x) \chi_\lambda^\mathbb{C}(\exp(iY))$$

where, as before, d_λ denotes the dimension of the irreducible representation associated with λ , we conclude

$$F_{\chi_\lambda^\mathbb{C}}(x) = \frac{\text{vol}(K/T)}{d_\lambda} \chi_\lambda(x) \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \chi_\lambda^\mathbb{C}(\exp(iY)) e^{-|Y|^2/2} \eta(Y/2) dY.$$

Given $Y \in \mathfrak{k}$, Kirillov's character formula, cf. [18,19], evaluated at the point $\exp(iY)$, yields the identity

$$\text{vol}(\Omega_\rho) j(iY) \chi_\lambda^\mathbb{C}(\exp(iY)) = \text{vol}(\Omega_\rho) \eta(Y/2) \chi_\lambda^\mathbb{C}(\exp(iY)) = \int_{\Omega_{\lambda+\rho}} e^{-\vartheta(Y)} d\sigma(\vartheta).$$

Now, as in the proof of Lemma 3.3, using the diffeomorphism from K/T onto $\Omega_{\lambda+\rho}$ which sends yT ($y \in K$) to $(\text{Ad}_y^*)^{-1}(\lambda + \rho)$, we rewrite the integral as an integral over K/T and obtain the identity

$$\eta(Y/2) \chi_\lambda^\mathbb{C}(\exp(iY)) = \frac{d_\lambda}{\text{vol}(K/T)} \int_{K/T} e^{-(\lambda+\rho)(\text{Ad}_y(Y))} d(yT).$$

Hence

$$\begin{aligned} F_{\chi_\lambda^\mathbb{C}}(x) &= \frac{\text{vol}(K/T)}{d_\lambda} \chi_\lambda(x) \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \chi_\lambda^\mathbb{C}(\exp(iY)) e^{-|Y|^2/2} \eta(Y/2) dY \\ &= \chi_\lambda(x) \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_{K/T} e^{-(\lambda+\rho)(\text{Ad}_y(Y))} d(yT) \right\} e^{-|Y|^2/2} dY \\ &= \chi_\lambda(x) \int_{\mathfrak{k}} e^{-(\lambda+\rho)(Y) - |Y|^2/2} dY = (2\pi)^{\dim(K)/2} e^{|\lambda+\rho|^2/2} \chi_\lambda(x) \end{aligned}$$

whence, in particular, $D_{1,\lambda} = (2\pi)^{\dim(K)/2} e^{|\lambda+\rho|^2/2}$ as asserted. $\square$

Theorem 6.5. The BKS-pairing (6.1.1) extends to a (non-degenerate) $(K \times K)$ -invariant pairing of the kind (6.1.2). Furthermore, the assignment to a representative function Φ on $K^\mathbb{C}$ of the function F_Φ on K induces a bounded $(K \times K)$ -equivariant operator

$$\Theta_t: \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t} \eta_\varepsilon) \longrightarrow L^2(K, dx) \quad (6.5.1)$$

such that

$$\langle \Phi, F \rangle_{\text{BKS}} = \langle \Theta_t(\Phi), F \rangle_K \quad (6.5.2)$$

and such that, when $\varphi^{\mathbb{C}}$ is a member of the isotypical summand $V_{\lambda}^* \odot V_{\lambda}$,

$$\Theta_t(\varphi^{\mathbb{C}}) = F_{\varphi^{\mathbb{C}}} = D_{t,\lambda}\varphi, \quad (6.5.3)$$

where as before φ refers to the restriction of $\varphi^{\mathbb{C}}$ to K . Finally, the operator

$$(4t\pi)^{-\dim(K)/4} \Theta_t: \mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta\varepsilon) \longrightarrow L^2(K, dx) \quad (6.5.4)$$

sends a representative function $\varphi^{\mathbb{C}} \in V_{\lambda}^* \odot V_{\lambda}$ to $C_{t,\lambda}^{1/2}\varphi = (t\pi)^{\dim(K)/4} e^{t|\lambda+\rho|^2/2}\varphi$ and thus coincides with the unitary isomorphism (5.3.1) of $(K \times K)$ -representations.

Lemma 6.4 and Theorem 6.5 imply in particular the statement of Theorem 6.3.

Proof. Let λ_1 and λ_2 be two highest weights, let $\varphi^{\mathbb{C}}$ be a representative function on $K^{\mathbb{C}}$ which is a member of the isotypical summand $V_{\lambda_1}^* \odot V_{\lambda_1}$ associated with the highest weight λ_1 , and let ψ be a representative function on K which is a member of the isotypical summand $W_{\lambda_2}^* \odot W_{\lambda_2}$ associated with the highest weight λ_2 . In view of the identity (6.2) and Lemma 6.4,

$$\langle \varphi^{\mathbb{C}}, \psi \rangle_{\text{BKS}} = \langle F_{\varphi^{\mathbb{C}}}, \psi \rangle_K = D_{t,\lambda_1} \langle \varphi, \psi \rangle_K.$$

Hence, by virtue of the ordinary Peter–Weyl theorem and of the holomorphic Peter–Weyl theorem, the BKS-pairing (6.1.2) is everywhere defined. By construction, the pairing is K -bi-invariant.

Let $\varphi^{\mathbb{C}}$ be a representative function on $K^{\mathbb{C}}$ which is a member of the isotypical summand $V_{\lambda}^* \odot V_{\lambda}$ associated with the highest weight λ . Since

$$\frac{C_{t,\lambda}}{D_{t,\lambda}^2} = \frac{(t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2}}{(2t\pi)^{\dim(K)} e^{t|\lambda+\rho|^2}} = ((4t\pi)^{-\dim(K)/4})^2,$$

by virtue of Lemmas 3.3 and 6.4,

$$\int_{K^{\mathbb{C}}} \overline{\varphi^{\mathbb{C}}} \varphi^{\mathbb{C}} e^{-\kappa/t}\eta\varepsilon = \frac{C_{t,\lambda}}{D_{t,\lambda}^2} \int_K \overline{F_{\varphi^{\mathbb{C}}}} F_{\varphi^{\mathbb{C}}} dx = ((4t\pi)^{-\dim(K)/4})^2 \int_K \overline{F_{\varphi^{\mathbb{C}}}} F_{\varphi^{\mathbb{C}}} dx.$$

In view of the ordinary Peter–Weyl theorem and of the holomorphic Peter–Weyl theorem, this identity implies the remaining assertions of Theorem 6.5. $\square$

Let $\Theta_t^*: L^2(K, dx) \rightarrow \mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta\varepsilon)$ be the adjoint of Θ_t . Let λ be a highest weight, let $\varphi^{\mathbb{C}} \in V_{\lambda}^* \odot V_{\lambda}$ and let $\varphi \in W_{\lambda}^* \odot W_{\lambda}$ be the restriction of $\varphi^{\mathbb{C}}$ to K . Define the number $A_{t,\lambda}$ by $\Theta_t^*(\varphi) = A_{t,\lambda}\varphi^{\mathbb{C}}$. Then

$$D_{t,\lambda} \langle \varphi, \varphi \rangle_K = \langle \Theta_t(\varphi^{\mathbb{C}}), \varphi \rangle_K = \langle \varphi^{\mathbb{C}}, \Theta_t^* \varphi \rangle_{t,K^{\mathbb{C}}} = A_{t,\lambda} \langle \varphi^{\mathbb{C}}, \varphi^{\mathbb{C}} \rangle_{t,K^{\mathbb{C}}} = A_{t,\lambda} C_{t,\lambda} \langle \varphi, \varphi \rangle_K$$

whence $A_{t,\lambda} = \frac{D_{t,\lambda}}{C_{t,\lambda}} = 2^{\dim(K)/2} e^{-t|\lambda+\rho|^2/2}$. Hence

$$\Theta_t^*(\varphi) = 2^{\dim(K)/2} e^{-t|\lambda+\rho|^2/2} \varphi^{\mathbb{C}}. \quad (6.6)$$

Corollary 6.7. *The resulting operator*

$$(4t\pi)^{-\dim(K)/4} \Theta_t^*: L^2(K, dx) \longrightarrow \mathcal{HL}^2(K^{\mathbb{C}}, e^{-\kappa/t}\eta\varepsilon) \quad (6.7.1)$$

is unitary and coincides with the inverse of the isomorphism (5.3.1).

Remark 6.8. As hinted at in the introduction, the unitarity of the BKS-pairing map, multiplied by a suitable constant, has been established in [7] by means of the heat kernel techniques developed in [4]. To this end, in [7], the BKS-pairing map has been shown to coincide with the Segal–Bargmann transform of [4] by appealing to the inversion formula in [6].

7. The spectral decomposition of the energy operator

Let Δ_K denote the *Casimir* operator on K associated with the bi-invariant Riemannian metric on K . When $X_1, \dots, X_m$ is an orthonormal basis of $\mathfrak{k}$,

$$\Delta_K = X_1^2 + \dots + X_m^2$$

in the universal algebra $U(\mathfrak{k})$ of $\mathfrak{k}$, cf. e.g. [21] (p. 591). The Casimir operator depends only on the Riemannian metric, though. Since the metric on K is bi-invariant, so is the operator Δ_K ; hence, by Schur's lemma, each isotypical summand $W_\lambda^* \odot W_\lambda \subseteq L^2(K, dx)$ is an eigenspace, whence the representative functions are eigenfunctions for Δ_K . The eigenvalue of Δ_K corresponding to the highest weight λ is known to be given explicitly by $-\varepsilon_\lambda$ where

$$\varepsilon_\lambda = (|\lambda + \rho|^2 - |\rho|^2),$$

cf. e.g. [8] (Ch. V.1 (16) p. 502). The present sign is dictated by the interpretation in terms of the energy given below. Thus Δ_K acts on each isotypical summand $W_\lambda^* \odot W_\lambda$ as scalar multiplication by $-\varepsilon_\lambda$. The Casimir operator is known to coincide with the nonpositive Laplace–Beltrami operator associated with the (bi-invariant) Riemannian metric on K , see e.g. [25] (A 1.2). In the Schrödinger picture (vertical quantization on T^*K), the unique extension $\widehat{E}_K$ of the operator $-\frac{1}{2}\Delta_K$ to an unbounded self-adjoint operator on $L^2(K, dx)$ is the quantum mechanical *energy* operator associated with the Riemannian metric, whence the spectral decomposition of this operator refines in the standard manner to the Peter–Weyl decomposition of $L^2(K, dx)$ into isotypical $(K \times K)$ -summands.

Via the embedding of $\mathfrak{k}$ into $\mathfrak{k}^\mathbb{C}$, the operator Δ_K is a differential operator on $K^\mathbb{C}$. In view of the holomorphic Peter–Weyl theorem, the unitary transform (5.3.1) (or, equivalently, (6.7.1)), is compatible with the operator Δ_K . Consequently, in the holomorphic quantization on $T^*K \cong K^\mathbb{C}$, via the transform (5.3.1) (or, equivalently, via the BKS-pairing map (6.5.1) multiplied by $(4\pi)^{-\dim(K)/4}$), the operator $\widehat{E}_{K^\mathbb{C}}$ which arises as the unique extension of the operator $-\frac{1}{2}\Delta_K$ on $\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$ to an unbounded self-adjoint operator is the quantum mechanical *energy* operator associated with the Riemannian metric, and the spectral decomposition of this operator refines to the holomorphic Peter–Weyl decomposition of $\mathcal{HL}^2(K^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$ into isotypical $(K \times K)$ -summands.

8. The heat equation

In view of the Peter–Weyl decomposition of $L^2(K, dx)$, the series

$$p_t(x) = p(t, x) = \sum_{\lambda} d_{\lambda} e^{-\frac{\varepsilon_{\lambda}}{2}t} \chi_{\lambda}(x)$$

is manifestly a formal solution of the heat equation

$$\frac{dp}{dt} = \frac{1}{2}\Delta_K p$$

on the compact group K . According to [24] (p. 38), p is a C^∞ -function, indeed the fundamental solution subject to the initial condition that p_0 be the Dirac distribution supported at the identity of K , and p extends to an analytic function on $K^\mathbb{C}$.

In terms of the Casimir operator Δ_K , the identity (6.6) may plainly be written in the form

$$\Theta_t^*(\varphi) = 2^{\dim(K)/2} e^{-t|\rho|^2/2} e^{-t\Delta_K/2} \varphi^\mathbb{C}, \quad (8.1)$$

where φ is any representative function on K . In this description of the operator Θ_t^* , the highest weights, present in the description (6.6), no longer appear explicitly. Consequently, for any smooth function f on K , $\Theta_t^*(f)$ is the unique holomorphic function on $K^\mathbb{C}$ whose restriction to K is given by

$$\Theta_t^*(f)|_K = 2^{\dim(K)/2} e^{-t|\rho|^2/2} e^{t\Delta_K/2} f. \quad (8.2)$$

In Theorem 2.6(1) of [7], this operator Θ_t^* is written as Π_h , where the parameter $\hbar$ corresponds to the present notation t .

On the other hand, the value $e^{t\Delta_K/2} f$ is given by

$$(e^{t\Delta_K/2} f)(y) = \int_K p_t(yx^{-1}) f(x) dx = (p_t * f)(y), \quad y \in K, \quad (8.3)$$

cf. [21] (Section 8). This establishes the relation between the BKS-pairing and the heat equation observed in [7].

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