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Manifolds with parallel differential forms and Kähler identities for G_2-manifolds

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ABSTRACT

Let M be a compact Riemannian manifold equipped with a parallel differential form ω. We prove a version of the Kähler identities in this setting. This is used to show that the de Rham algebra of M is weakly equivalent to its subquotient $(H_c^*(M), d)$, called *the pseudo-cohomology* of M. When M is compact and Kähler, and ω is its Kähler form, $(H_c^*(M), d)$ is isomorphic to the cohomology algebra of M. This gives another proof of homotopy formality for Kähler manifolds, originally shown by Deligne, Griffiths, Morgan and Sullivan. We compute $H_c^*(M)$ for a compact G_2-manifold, showing that $H_c^i(M) \cong H^i(M)$ unless $i = 3, 4$. For $i = 3, 4$, we compute $H_c^*(M)$ explicitly in terms of the first-order differential operator $*d : \Lambda^3(M) \longrightarrow \Lambda^3(M)$.

1. Introduction

1.1. Holonomy groups in Riemannian geometry

Let M be a Riemannian manifold equipped with a differential form ω. This form is called *parallel* if ω is preserved by the Levi-Civita connection: $\nabla\omega = 0$. This identity gives a powerful restriction on the holonomy group $\mathcal{H}\mathrm{ol}(M)$.

The structure of $\mathcal{H}\mathrm{ol}(M)$ and its relation to the geometry of a manifold is one of the main subjects of Riemannian geometry of the last 50 years. This group is compact, and hence reductive, and acts, in a natural way, on the tangent space TM. For when M is complete, Georges de Rham proved that unless this representation is irreducible, M has a finite covering, which is a product of Riemannian manifolds of smaller dimension [1]. Irreducible holonomies were classified by Berger [2], who gave a complete list of all irreducible holonomies which can occur on non-symmetric spaces. This list is quite short:

Holonomy	Geometry
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi–Yau manifolds
$\mathrm{Sp}(n)$ acting on $\mathbb{R}^{4n}$	Hyper-Kähler manifolds
$\mathrm{Sp}(n) \times \mathrm{Sp}(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	Quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	Parallel G_2-manifolds
$\mathrm{Spin}(7)$ acting on $\mathbb{R}^8$	$\mathrm{Spin}(7)$-manifolds

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Berger's list also included Spin(9) acting on $\mathbb{R}^{16}$, but Alekseevsky later observed that this case is impossible [3], unless M is symmetric. If an irreducible manifold M has a parallel differential form, its holonomy is restricted, as $SO(n)$ has no invariants in $\Lambda^i(TM)$, $0 < i < n$. Then M is locally a product of symmetric spaces and manifolds with holonomy $U(n)$, $SU(n)$, $\mathrm{Sp}(n)$, etc.

In Kähler geometry (holonomy $U(n)$) the parallel forms are the Kähler form and its powers. Studying the corresponding algebraic structures, the algebraic geometers amassed an amazing wealth of topological and geometric information. In this paper we try to generalize some of these results to other manifolds with a parallel form, especially the parallel G_2-manifolds. The results thus obtained can be summarized as "Kähler identities for G_2-manifolds".

1.2. G_2-manifolds in mathematics and physics

The theory of G_2-manifolds is one of the places where mathematics and physics interact most strongly. For many years after Berger's ground-breaking results, this subject was dormant; after Alekseevsky showed that Spin(9) cannot be realized in holonomy, there were doubts as to whether the other two exceptional entries in Berger's list (G_2 and Spin(7)) can be realized.

Only in the 1980s were manifolds with holonomy G_2 constructed. Bryant [4] found local examples, and then Bryant and Salamon found complete manifolds with holonomy G_2 [5]. The compact examples of holonomy G_2 and Spin(7)-manifolds were produced by Joyce [6,7], using difficult (but beautiful and quite powerful) arguments from analysis and PDE theory. Since then, the G_2-manifolds have become a central subject of study in some areas of string physics, and especially in M-theory. The mathematical study of G_2-geometry was less intensive, but still quite fruitful. Important results were obtained in gauge theory on the G_2-manifold (the study of Donaldson–Thomas bundles) [8–10]. Kovalev found many new examples of G_2-manifolds, using a refined version of Joyce's engine [11]. Hitchin constructed a geometric flow [12,13], which turned out to be extremely important in string physics (physicists call this flow *Hitchin's flow*). Hitchin's flow acts on the space of all "stable" (non-degenerate and positive) 3-forms on a 7-manifold. It is fixed precisely on the 3-forms corresponding to the connections with holonomy in G_2. In [14], a unified theory of gravity is introduced, based in part on Hitchin's flow. From the special cases of topological M-theory one can deduce four-dimensional loop gravity, and six-dimensional A- and B-models in string theory.

In string theory, G_2-manifolds are expected to play the same role as Calabi–Yau manifolds in the usual A- and B-models of type-II string theories. These two forms of string theory both use Calabi–Yau manifolds, but in different fashions. The duality between these theories leads to duality between Calabi–Yau manifolds, and then to far-reaching consequences, which were studied in mathematics and physics, under the name of mirror symmetry. During the last 20 years, mirror symmetry has become one of the central topics of modern algebraic geometry.

There are two important ingredients in mirror symmetry (in Strominger–Yau–Zaslow form)—one counts holomorphic curves on one Calabi–Yau manifold, and the special Lagrangian cycles on its mirror dual. Using G_2-geometry, these two kinds of objects (holomorphic curves and special Lagrangian cycles) are transformed into the same kinds of objects, called *associative cycles* on a G_2-manifold. This is done as follows.

A G_2-structure on a 7-manifold is given by a 3-form (see Section 3.1). Consider a Calabi–Yau manifold X, $\dim M = 3$, with non-degenerate holomorphic 3-form Ω, and Kähler form ω. Let $M := X \times S^1$, and let dt denote the unit cotangent form of S^1 lifted to M. Consider a 3-form $\omega \wedge dt + \mathrm{Re}\,\Omega$ on M. This form is obviously closed. It is easy to check that it defines a parallel G_2-structure on M. In this way one can convert problems from Calabi–Yau geometry to problems in G_2-geometry.

A 3-form φ on a manifold M gives an anti-symmetric map

$$\varphi^\sharp : TM \otimes TM \longrightarrow \Lambda^1(M),$$

$x, y \longrightarrow \varphi(x, y, \cdot)$. Using the Riemannian structure, we identify TM and $\Lambda^1(M)$. Then $\varphi^\sharp$ leads to a skew-symmetric vector product $V : TM \otimes TM \longrightarrow TM$. An *associative cycle* on a G_2-manifold is a three-dimensional submanifold Z such that TZ is closed under this vector product. Associative submanifolds are studied within the general framework of calibrated geometries (see [15]).

Given a Calabi–Yau 3-fold X, consider $M = X \times S^1$ with a G_2-structure defined above. Let $Z \subset X$ be a three-dimensional submanifold. It is easy to check that Z is special Lagrangian if and only if $Z \times \{t\}$ is associative in M. Also, given a 2-cycle C on X, $C \times S^1$ is associative in M if and only if C is a holomorphic curve. In this way, the instanton objects in mirror dual theories (holomorphic curves and SpLag cycles) can be studied uniformly after passing to a G_2-manifold. It was suggested that this correspondence indicates some form of string duality [16,17].

However, the main physical motivation for the study of G_2-manifolds comes from M-theory; we direct the reader to the excellent survey [18] for details and further reading. M-theory is a theory which is expected (if developed) to produce a unification of GUT (the grand unified theory of strong, weak and electromagnetic forces) with gravity, via supersymmetry. In this approach, string theories arise as approximations of M-theory. In most applications related to M-theory, a G_2-manifold is deformed to a compact G_2-variety with isolated singularities. One local construction of conical singularities of this type is based on Bryant–Salamon examples of complete G_2-manifolds (see [5]). In this approach, the study of conical singularities is essentially reduced to that of four-dimensional geometry.

An explicit mathematical study of these singular examples and their connection to physics and theory of Einstein manifolds is found in [19]. Also, Hitchin's flow can be used to produce many such examples in a uniform way (see [20]).

1.3. Structure operators on manifolds with parallel differential form

Much study in Kähler geometry is based on the interplay between the de Rham differential and the twisted de Rham differential $d^c := -I \circ d \circ I$. We construct a similar operator d_c for any manifold with a parallel differential form. This operator no longer satisfies $d_c^2 = 0$; however, it satisfies many properties expected from the twisted de Rham differential in Kähler geometry. Most importantly, a version of the dd_c-lemma is true in this setting (Proposition 1.1).

Just as in the usual case, this may lead to results in rational homotopy theory (see Section 4.1 in the present introduction).

To simplify the exposition, we restrict ourselves at present to Riemannian manifolds (M, ω) with a parallel 3-form. These include Riemannian 3-manifolds, Calabi–Yau 3-folds and G_2-manifolds. Just as in the three-dimensional case, such a 3-form defines a skew-symmetric cross-product on $\Lambda^1(M)$:

$$x, y \xrightarrow{\Psi} \omega(x^\sharp, y^\sharp, \cdot)$$

($(\cdot)^\sharp$ denotes taking the dual with respect to the metric). Consider the operator on differential forms

$$\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k} \longrightarrow \sum_{1 \leqslant a < b \leqslant k} (-1)^{a+b-1} \Psi(\xi_{i_a}, \xi_{i_b}) \wedge \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \hat{\xi}_{i_a} \wedge \cdots \wedge \hat{\xi}_{i_b} \wedge \cdots \wedge \xi_{i_k}$$

where ξ_i is an orthonormal frame in $\Lambda^1(M)$. Denote by

$$C : \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$$

the dual operator (to identify $\Lambda^i(M)$ with its dual, we use the natural metric on $\Lambda^i(M)$ induced from the Riemannian structure on M). Then C is called *the structure operator on* (M, ω).

In Section 2 we give another definition of C, which works for an arbitrary parallel i-form ω. It is not difficult to check that this definition is compatible with the one given above. When (M, ω) is Kähler, C becomes the complex structure operator on M, and the identities that we prove in the general case become the usual Kähler identities.

Denote by d_c the anticommutator $\{C, d\} = dC + Cd$. We show that d_c commutes with d, d^*, and satisfies the following version of the dd_c-lemma.

Proposition 1.1. *Consider a compact Riemannian manifold equipped with a parallel differential form. Let η be a differential k-form satisfying $d\eta = d_c\eta = 0$. Assume, moreover, that η is d_c-exact: $\eta = d_c\xi$. Then $\eta = dd_c\xi'$, for some differential form ξ.*

Proof. It follows immediately from Proposition 2.20 (see Remark 2.21). □

Remark 1.2. The operator d_c satisfies the Leibniz identity:

$$d_c(a \wedge b) = d_c(a) \wedge b + (-1)^{\tilde{a}\tilde{d_c}} a \wedge d_c(b),$$

where $\tilde{a}, \tilde{b}$ denotes parity of a form. However, $d_c^2 \neq 0$. Also, the dd_c-lemma is less strong than the usual dd^c-lemma: given a d-exact, d, d_c-closed form η, we cannot show that $\eta = dd_c\xi'$ (though this could be true in the case of G_2-manifolds).

1.4. The localization functor and rational homotopy

The homotopy formality for the Kähler manifold, observed by Deligne et al. [21], is one of the deepest and most powerful results of Kähler geometry. Since [21] appeared, there has been a whole cornucopia of research dedicated to this theme. Formality was used to study the deformations and moduli spaces (see e.g. [22–24]), in mirror symmetry and topology. The reason for all these equations lies in the so-called *master equation* (also known as *the Maurer–Cartan equation*)

$$d\gamma = -\frac{1}{2}[\gamma, \gamma]$$

in a differential graded (DG-)Lie algebra, which is responsible for deformation theory for most objects in algebraic geometry. Solutions of this equation (up to a relevant equivalence) are homotopy invariants of the DG-Lie algebra [23].

This equation can be solved recursively, if the relevant Massey products vanish (in fact, Massey products can be defined as obstructions to finding solutions of the Maurer–Cartan equation—see e.g. [25]). The homotopy formality implies the vanishing of Massey products, providing a way to solve the Maurer–Cartan equation in various contexts.

In the proof of homotopy formality for Kähler manifolds [21], the key argument hinges on the dd^c-lemma; one should expect the G_2-version of the dd^c-lemma (Proposition 1.1) to give us information about the rational homotopy of G_2-manifolds.

The topological utility of rational homotopy is based on the Quillen–Sullivan localization construction, [26,27]. The $\mathbb{Q}$-localization functor in the homotopy category maps a simply connected cellular space X to a space $X_{\mathbb{Q}} = \mathrm{Loc}_{\mathbb{Q}}(X)$ with $H^i(X_{\mathbb{Q}}, \mathbb{Z}) \cong H^i(X, \mathbb{Z}) \otimes \mathbb{Q}$ and $\pi_i(X_{\mathbb{Q}}) \cong \pi_i(X) \otimes \mathbb{Q}$. The spaces which are homotopy equivalent to their localization are called $\mathbb{Q}$-*local*. We have $\mathrm{Loc}_{\mathbb{Q}}(X) \cong \mathrm{Loc}_{\mathbb{Q}}(\mathrm{Loc}_{\mathbb{Q}}(X))$; in other words, all spaces of form $\mathrm{Loc}_{\mathbb{Q}}(X)$ are $\mathbb{Q}$-local.

Given a cellular space, one could construct its de Rham complex, using piecewise smooth differential forms. This construction maps homotopy equivalent spaces to weakly equivalent differential graded (DG-)algebras (see Definition 2.22). We obtain a functor $DR : \text{Hot} \longrightarrow \text{DG-Alg}$ for the corresponding categories. Moreover, this functor commutes with localization, and gives an equivalence of homotopy category of $\mathbb{Q}$-local simply connected spaces and the category DG-Alg of DG-algebras. This reduces the study of rational homotopies (homotopies of $\mathbb{Q}$-local spaces) to the study of DG-algebras.

The localization construction (which is defined in many other contexts; see [28]) is one of the key ideas of modern algebraic topology. Sullivan needed localization in order to prove the Adams conjecture, and Quillen used localization to give the definition of algebraic K-theory. Since then, many other uses of the same construction were found, including Voevodsky's celebrated motivic homotopy theory.

Two DG-algebras are called *quasi-isomorphic* if there exists a quasi-isomorphism (or morphism, or inducing isomorphism on the cohomology) from one to another. The equivalence relation generated by the quasi-isomorphism is called *weak equivalence* of DG-algebras (Definition 2.22).

Rational homotopy is a study of DG-algebras, up to weak equivalence.

A DG-algebra (A^*, d) is called *homotopy formal* if it is weakly equivalent to its cohomology algebra $(H^*(A), 0)$. A simply connected topological space is called *formal* if its de Rham algebra is formal. The rational homotopies of formal spaces (in particular, all rational homotopy groups) are determined by the algebraic structure on the cohomology.

Not all DG-algebras are formal; the best known obstruction to formality is called *the Massey product* (see e.g. [25]). However, there are more obstructions to formality that just the Massey product. Halperin and Stasheff [29] constructed explicitly a complete set of obstructions

$$\{O_n, n = 1, 2, 3, \dots\}$$

to homotopy formality; O_n defined if all $O_i,\ i < n$, vanish.

Since homotopy formality of Kähler manifolds has been established, many people have studied the influence of differential geometric structures on rational homotopy. Much of this work was focused on the study of rational homotopy of compact symplectic manifolds (there is a book [30] dedicated especially to this subject). Using Deligne–Griffiths–Morgan–Sullivan formality theorem, one obtains all kinds of symplectic manifolds admitting no Kähler structures.

2. Riemannian manifolds with a parallel differential form

2.1. The structure operator and the twisted differential

Let M be a C^∞-manifold. We denote the smooth forms on M by $\Lambda^*(M)$. Given an odd or even form $\alpha \in \Lambda^*(M)$, we denote by $\tilde{\alpha}$ its parity, which is equal to 0 for even forms, and 1 for odd forms. An operator $f \in \operatorname{End}(\Lambda^*(M))$ preserving parity is called *even*, and one exchanging odd and even forms is *odd*; $\tilde{f}$ is equal to 0 for even forms and 1 for odd ones.

Given a C^∞-linear map $\Lambda^1(M) \xrightarrow{p} \Lambda^{\text{odd}}(M)$ or $\Lambda^1(M) \xrightarrow{p} \Lambda^{\text{even}}(M)$, p can be uniquely extended to a C^∞-linear derivation ρ on $\Lambda^*(M)$, using the rule

$$\rho|_{\Lambda^1(M)} = p, \qquad \rho|_{\Lambda^0(M)} = 0, \qquad \rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \beta + (-1)^{\tilde{\rho}\tilde{\alpha}} \alpha \wedge \rho(\beta).$$

Then, ρ is an even (or odd) differentiation of the graded commutative algebra $\Lambda^*(M)$.

Definition 2.1. Let M be a Riemannian manifold, and $\omega \in \Lambda^k(M)$ a differential form. Consider an operator $\underline{C} : \Lambda^1(M) \longrightarrow \Lambda^{k-1}(M)$ mapping $\nu \in \Lambda^1(M)$ to $\omega \lrcorner \nu^\sharp$, where $\nu^\sharp$ is the vector field dual to ν. Alternatively, $\underline{C}(\nu)$ can be written as $\underline{C}(\nu) = *(*\omega \wedge \nu)$. The corresponding differentiation

$$C : \Lambda^*(M) \longrightarrow \Lambda^{*+k-2}(M)$$

is called *the structure operator of* (M, ω). The parity of C is equal to that of ω.

Remark 2.2. When (M, I, g) is a Kähler manifold and ω is its Kähler form, $\underline{C}(\nu) = I(\nu)$, and C is the standard Kähler–Weil operator, acting on (p, q)-forms as a multiplication by $(p - q)\sqrt{-1}$.

Definition 2.3. Let M be a Riemannian manifold, and $\omega \in \Lambda^k(M)$ a differential form, which is parallel with respect to the Levi-Civita connection. Denote by d_c the supercommutator

$$\{d, C\} := dC - (-1)^{\tilde{C}} Cd.$$

This operator is called *the twisted de Rham operator of* (M, ω). Being a graded commutator of two graded differentiations, d_c is also a graded differentiation of $\Lambda^*(M)$.

Remark 2.4. When (M, I, g) is a Kähler manifold and ω is its Kähler form, d_c is equal to the well-known twisted differential $d^c = I^{-1} \circ d \circ I$, $d^c = \frac{\partial - \bar{\partial}}{\sqrt{-1}}$. Of course, for a general form ω, d_c^2 can be non-zero.

Proposition 2.5. *Let (M, ω) be a Riemannian manifold equipped with a parallel form ω, and L_ω the operator $\eta \longrightarrow \eta \wedge \omega$. Then*

$$d_c = \{L_\omega, d^*\},$$

where $\{\cdot, \cdot\}$ denotes the supercommutator,

$$\{L_\omega, d^*\} = L_\omega d^* - (-1)^{\tilde{\omega}} d^* L_\omega,$$

and $d^* = - * d *$ is the adjoint to d.

Proof. Denote by ∇ the Levi-Civita connection

$$\nabla : \Lambda^*(M) \longrightarrow \Lambda^*(M) \otimes \Lambda^1(M).$$

Let $\eta \in \Lambda^i(M)$. Clearly, $d^*\eta$ is obtained from $\nabla\eta \in \Lambda^i(M) \otimes \Lambda^1(M)$ by applying the isomorphism

$$\Lambda^i(M) \otimes \Lambda^1(M) \cong \Lambda^i(M) \otimes TM$$

induced by the Riemannian structure and then plugging the TM-part into $\Lambda^i(M)$:

$$d^*\eta = \lrcorner(\nabla\eta). \tag{2.1}$$

Since $\nabla\omega = 0$, $\{L_\omega, d^*\}$ is equal to the composition

$$\begin{aligned} \Lambda^i(M) &\xrightarrow{\nabla} \Lambda^i(M) \otimes \Lambda^1(M) \\ &\xrightarrow{C\otimes \mathrm{Id}} \Lambda^{i+k-2}(M) \otimes \Lambda^1(M) \xrightarrow{\wedge} \Lambda^{i+k-1}(M) \end{aligned}$$

(the last arrow is exterior multiplication). Indeed, L_ω commutes with ∇, and therefore, by (2.1), $\{L_\omega, d^*\}$ is written as a composition of ∇ and a commutator of C^∞-linear maps L_ω and $\lrcorner$, where

$$\lrcorner : \Lambda^{i+k}(M) \otimes \Lambda^1(M) \longrightarrow \Lambda^{i+k-1}(M)$$

maps $\eta \otimes \nu$ to $\eta \lrcorner \nu^\sharp$. However, by definition,

$$\{L_\omega, \lrcorner\}(\eta \otimes \nu) = C(\eta) \wedge \nu.$$

This gives

$$\{L_\omega, d^*\}(\eta) = [L_\omega, \lrcorner](\nabla\eta). \tag{2.2}$$

Similarly, $[\nabla, C] = 0$, and hence d_c is written as a composition of ∇ and a C^∞-linear map

$$C \otimes \mathrm{Id} \circ \wedge - \wedge \circ C : \Lambda^i(M) \otimes \Lambda^1(M) \longrightarrow \Lambda^{i+k-1}(M), \tag{2.3}$$

where $\wedge : \Lambda^*(M)\Lambda^1(M) \longrightarrow \Lambda^{*+1}(M)$ denotes the exterior product. Since C is a differentiation, the operator (2.3) is equal to

$$\mathrm{Id} \otimes C \circ \wedge : \Lambda^i(M) \otimes \Lambda^1(M) \longrightarrow \Lambda^{i+k-1}(M).$$

This gives

$$\{d, C\}(\eta) = \mathrm{Id} \otimes C \circ \wedge(\nabla\eta). \tag{2.4}$$

However, by definition of C, we have $[L_\omega, \lrcorner](\eta \otimes \nu) = \eta \wedge C(\nu)$, and hence the right hand sides of (2.4) and (2.2) are equal. This proves Proposition 2.5. $\square$

Remark 2.6. In the Kähler case, Proposition 2.5 becomes the following well-known Kähler identity: $[L_\omega, d^*] = d^c$.

2.2. Generalized Kähler identities and the twisted Laplacian

Proposition 2.7. *Let M be a Riemannian manifold equipped with a parallel differential k-form ω, d_c the twisted de Rham operator constructed above, and d_c^* its Hermitian adjoint. Then:*

(i) *The following supercommutators vanish:*

$$\{d, d_c\} = 0, \qquad \{d, d_c^*\} = 0, \qquad \{d^*, d_c\} = 0, \qquad \{d^*, d_c^*\} = 0.$$

(ii) *The Laplacian $\Delta = \{d, d^*\}$ commutes with $L_\omega : \eta \longrightarrow \omega \wedge \eta$ and its Hermitian adjoint operator, denoted as $\Lambda_\omega : \Lambda^i(M) \longrightarrow \Lambda^{i-k}(M)$.*

(iii) *Denote the supercommutator of* d_c, d_c^* *by* Δ_c. *By definition,* $\Delta_c = d_c d_c^* + d_c^* d_c$ *when k is even, and* $\Delta_c = d_c d_c^* - d_c^* d_c$ *when k is odd. Then*

$$\Delta_c = (-1)^{\tilde{\omega}}\{d^*, [H_\omega, d]\},$$

where $H_\omega = \{L_\omega, \Lambda_\omega\}$.

Proof. We use the following basic lemma.

Basic Lemma. Let δ be an odd element in a graded Lie superalgebra A satisfying $\{\delta, \delta\} = 0$. Then $\{\delta, \{\delta, x\}\} = 0$ for all $x \in A$, assuming that the base field is not of characteristic 2.

Proof. Using the graded Jacobi identity, we obtain

$$\{\delta, \{\delta, x\}\} = -\{\delta, \{\delta, x\}\} + \{\{\delta, \delta\}, x\}.$$

This gives $2\{\delta, \{\delta, x\}\} = 0$. □

Now, $\{d, d_c\} = \{d, \{d, C\}\} = 0$ (by the basic lemma), and $\{d^*, d_c\} = \{d^*, \{d^*, L_\omega\}\} = 0$ (by the basic lemma and Proposition 2.5). Taking Hermitian adjoints of these identities, we obtain the other two equations of Proposition 2.7(i). Proposition 2.7(i) is proven.

Now, the graded Jacobi identity implies

$$[L_\omega, \Delta] = \{L_\omega, \{d, d^*\}\} = (-1)^{\tilde{\omega}}\{d, \{L_\omega, d^*\}\} \tag{2.5}$$

(we use $\{L_\omega, d\} = 0$ as ω is closed). This gives

$$[L_\omega, \Delta] = (-1)^{\tilde{\omega}}\{d, d_c\} = 0,$$

as Proposition 2.7(i) implies. Taking the Hermitian adjoint, we also obtain $[\Lambda_\omega, \Delta] = 0$. We have proved Proposition 2.7(ii).

Finally, Proposition 2.7(iii) is proven as follows:

$$\{\{L_\omega, d^*\}, \{\Lambda_\omega, d\}\} = \{\{L_\omega, \{d^*, d_c^*\}\} + (-1)^{\tilde{\omega}}\{d^*, \{L_\omega, \{\Lambda_\omega, d\}\}\}\} \tag{2.6}$$

by the graded Jacobi identity. Also,

$$\{L_\omega, \{\Lambda_\omega, d\}\} = \{H_\omega, d\} + (-1)^{\tilde{\omega}}\{\Lambda_\omega, \{L_\omega, d\}\}. \tag{2.7}$$

However, $\{L_\omega, d\} = 0$ as ω is closed. Comparing (2.7) and (2.6), we obtain

$$\Delta_c = (-1)^{\tilde{\omega}}\{d^*, \{H_\omega, d\}\}.$$

We have proved Proposition 2.7(iii). □

Remark 2.8. When (M, ω) is a Kähler manifold, Proposition 2.7(i) gives the standard commutation relations between $d, d^c, d^*, (d^c)^*$, Proposition 2.7(ii) is well known, and Proposition 2.7(iii) gives

$$\{d^c, (d^c)^*\} = \Delta_c = \{d^*, [H, d]\} = \Delta,$$

because $[H, d] = d$ as the Lefschetz theorem implies.

Corollary 2.9. *Let* (M, ω) *be a Riemannian manifold equipped with a parallel differential form, and* η *a harmonic form on* M. *Then* $\omega \wedge \eta$ *is harmonic.*

Proof. It follows from Proposition 2.7(ii).[1] □

Further on, we shall need the following trivial lemma.

Lemma 2.10. *Let* (M, ω) *be a compact Riemannian manifold equipped with a parallel differential form, and* η *a harmonic form on* M. *Consider the twisted de Rham operator* d_c *constructed above. Then* $d_c(\eta) = 0$.

Proof. Since M is compact, $d^*\eta = 0$. Then $d_c\eta = d^*L_\omega\eta$. On the other hand, $L_\omega\eta$ is harmonic, by Corollary 2.9, and hence satisfies $d^*L_\omega\eta = 0$. □

[1] This statement is due to S.-S. Chern.

2.3. The differential graded algebra $(\ker d_c, d)$

Let (M, ω) be a Riemannian manifold equipped with a parallel form, and d_c the twisted de Rham operator constructed above. By construction, d_c is a differentiation of $\Lambda^*(M)$. Therefore, $\ker d_c \subset \Lambda^*(M)$ is a subalgebra. Since d and d_c supercommute, d acts on $\ker d_c$. We consider $(\ker d_c, d)$ as a differential graded algebra (a DG-algebra).

Recall that a homomorphism of DG-algebras is called *a quasi-isomorphism* if it induces an isomorphism on the cohomology.

Proposition 2.11. *Let (M, ω) be a compact Riemannian manifold equipped with a parallel form. Consider the natural embedding*

$$(\ker d_c, d) \hookrightarrow (\Lambda^*(M), d). \tag{2.8}$$

Then this map is a quasi-isomorphism.

Proof. Let $\Lambda^*(M)_\alpha$ be the eigenspace of Δ, corresponding to the eigenvalue α. Since Δ is a self-adjoint operator with a discrete spectrum, we have a decomposition $\Lambda^*(M) \cong \oplus_\alpha \Lambda^*(M)_\alpha$. Consider the subcomplex

$$\cdots \xrightarrow{d} \Lambda^*(M)_\alpha \xrightarrow{d} \Lambda^{*+1}(M)_\alpha \xrightarrow{d} \cdots \tag{2.9}$$

corresponding to an eigenvalue α. Clearly, for $\alpha \neq 0$, the complex (2.9) is exact.

Let

$$\cdots \xrightarrow{d} (\ker d_c)_\alpha \xrightarrow{d} (\ker d_c)_\alpha \xrightarrow{d} \cdots \tag{2.10}$$

be the action of d on the α-eigenspace of Δ on $(\ker d_c)$ (Δ commutes with d_c as Proposition 2.7 implies).

For $\alpha = 0$, $(\ker d_c)_\alpha = \Lambda^*(M)_\alpha = \mathcal{H}^*(M)$ as Lemma 2.10 implies. To prove Proposition 2.11 we only need to show that (2.10) has zero cohomology for $\alpha > 0$. However, for any closed form $\eta \in (\ker d_c)_\alpha$, we have

$$\eta = \frac{1}{\alpha}(dd^* + d^*d)\eta = \frac{1}{\alpha}dd^*\eta$$

and $d^*\eta$ lies inside $(\ker d_c)_\alpha$ as d_c and d^* commute (Proposition 2.7). Therefore, η is exact. This proves Proposition 2.11. $\square$

The following claim is clear, as Δ_c and Δ commute, and $\{d_c, d_c^*\}^* = \{d_c^*, d_c\} = (-1)^{1-\tilde{d}_c}\{d_c, d_c^*\}$.

Claim 2.12. *Let (M, ω) be a compact Riemannian manifold equipped with a parallel form, and $\Delta_c = \{d_c, d_c^*\}$ the operator constructed above. Let $\Lambda^*(M)_\alpha$ be the eigenspace of the Laplacian of eigenvalue α. Then Δ_c preserves $\Lambda^*(M)_\alpha$ and acts on $\Lambda^*(M)_\alpha$ as a self-adjoint or anti-self-adjoint operator. In particular, Δ_c is diagonalizable, on some dense subspace of $\Lambda^*(M) \otimes_{\mathbb{R}} \mathbb{C}$.* $\square$

Remark 2.13. Notice that Δ_c is not a priori elliptic, and hence it has no spectral decomposition. However, it preserves the finite-dimensional eigenspaces of the Laplacian, and is diagonalizable on these eigenspaces.

2.4. Pseudo-cohomology of the operator d_c

Lemma 2.14. *Let (M, ω) be a compact Riemannian manifold equipped with a parallel form, and $(\ker d_c, d)$ the differential graded algebra constructed above. Consider the subspace*

$$V = (\ker d_c) \cap d_c(\Lambda^*(M)) \subset (\ker d_c). \tag{2.11}$$

Then V is a differential ideal in the differential graded algebra $(\ker d_c, d)$. In other words, $\ker d_c \cdot V \subset V$ and $dV \subset V$.

Proof. Given $x \in \ker d_c$, $y \in V$, $y = d_c z$, we write

$$d_c(x \wedge z) = (-1)^{\tilde{d}_c \tilde{x}} x \wedge d_c z.$$

Therefore, V is an ideal. To prove that $dV \subset V$, we write $v \in V$ as $d_c(w)$, and then $dv = (-1)^{\tilde{d}_c} d_c dw$. $\square$

Definition 2.15. The quotient $\frac{(\ker d_c)}{(\ker d_c) \cap (\operatorname{im} d_c)}$ is called a *the pseudo-cohomology* of d_c. As Lemma 2.14 implies, a pseudo-cohomology is a differential graded algebra. We denote it by $(H_c^*(M), d)$.

Remark 2.16. We do not call $H_c^*(M)$ a *cohomology* of d_c, because d_c^2 is not necessarily zero. In the literature, the pseudo-cohomology of an operator is known under the name of the *twisted cohomology* (see e.g. in [31]).

Definition 2.17. Let $\eta \in \Lambda^*(M)$ be a form which satisfies $d_c\eta = d_c^*\eta = 0$. Then η is called *pseudo-harmonic*. The space of all pseudo-harmonic forms is denoted by $\mathcal{H}_c^*(M)$. By Proposition 2.7(i), the de Rham differential preserves $\mathcal{H}_c^*(M)$.

Remark 2.18. From Lemma 2.10 it follows immediately that all harmonic forms are pseudo-harmonic: $\mathcal{H}^*(M) \subset \mathcal{H}_c^*(M)$.

Proposition 2.19. *Let (M, ω) be a compact Riemannian manifold equipped with a parallel form, and*

$$\mathcal{H}_c^*(M) \xrightarrow{i} H_c^*(M) \tag{2.12}$$

the natural projection map. Then i is an isomorphism, compatible with the de Rham differential.

Proof. We represent $\Lambda^*(M)$ as a (completion of) a direct sum of eigenvalues of the Laplacian. Using d_c, d_c^*-invariance of these eigenspaces, we may work with the associated decompositions within these eigenspaces. Abusing the language, we approach $\Lambda^*(M)$ as if it were finite dimensional, but in fact we work with these eigenspaces, which are finite dimensional.

From

$$(\Delta_c\alpha, \alpha) = (d_c\alpha, d_c\alpha) + (d_c^*\alpha, d_c^*\alpha)$$

we obtain that $\ker \Delta_c = \ker d_c \cap \ker d_c^*$. From $(d_c\alpha, \beta) = (\alpha, d_c^*\beta)$, we find that $\ker d_c = (\operatorname{im} d_c^*)^\perp$, $\ker d_c^* = (\operatorname{im} d_c)^\perp$, where $(\cdots)^\perp$ denotes the orthogonal complement. Therefore,

$$\ker \Delta_c = (\operatorname{im} d_c)^\perp \cap (\operatorname{im} d_c^*)^\perp = (\operatorname{im} d_c + \operatorname{im} d_c^*)^\perp.$$

Given $\alpha \in \Lambda^*(M)$, let $\Pi\alpha$ denote the orthogonal projection of α to $\ker \Delta_c$. Then $\alpha - \Pi\alpha$ is orthogonal to $\ker \Delta_c$, and hence

$$\alpha - \Pi\alpha \in \left(\operatorname{im} d_c + \operatorname{im} d_c^*\right). \tag{2.13}$$

Now assume that $\alpha \in \ker d_c$. The form $\Pi(\alpha)$ also lies in $\ker d_c$, because $\ker \Delta_c \subset \ker d_c$. Therefore, $\alpha - \Pi\alpha$ lies in $\ker d_c$, and hence, is orthogonal to $\operatorname{im} d_c^*$. Using (2.13), we obtain that $\alpha - \Pi\alpha \in \operatorname{im} d_c$.

Therefore,

$$\ker d_c = (\ker d_c) \cap (\operatorname{im} d_c) \oplus \mathcal{H}_c^*(M). \tag{2.14}$$

From (2.14), Proposition 2.19 follows directly. □

Proposition 2.20. *Let (M, ω) be a compact Riemannian manifold equipped with a parallel form, and*

$$(\ker d_c, d) \xrightarrow{\pi} (H_c^*(M), d) \tag{2.15}$$

the homomorphism of differential graded algebras constructed above. Then π is a quasi-isomorphism.

Proof. By definition, (2.15) is surjective. To show that it is a quasi-isomorphism, we need to prove that any d-closed $\eta \in \ker \pi$ is d-exact. However, $\ker \pi \subset (\ker d_c) \cap (\operatorname{im} d_c)$, and by (2.14) this space is orthogonal to $\mathcal{H}_c^*(M)$. Using Remark 2.18 we obtain that any $\eta \in \ker \pi$ is orthogonal to the space of harmonic forms. Using the spectral decomposition, we obtain that $\eta = \sum \eta_{\alpha_i}$, where $\Delta\eta_{\alpha_i} = \alpha_i\eta_{\alpha_i}$, and $\{\alpha_i\}$ are positive real numbers. Since Δ commutes with d_c and d_c^*, the components η_{α_i} also belong to $\ker \pi$. This gives $\eta_{\alpha_i} = \frac{1}{\alpha_i} dd^*\eta_{\alpha_i}$, and hence all the components η_{α_i} are d-exact. We obtain that η is d-exact. Proposition 2.20 is proven. □

Remark 2.21. The standard (and completely formal) argument is used to produce the dd_c-lemma from Proposition 2.20. Let η be a d_c-exact, d-, d_c-closed form on M. We need to show that $\eta = dd_c\xi$. By definition, η represents 0 in $H_c^*(M)$. Since $(\ker d_c, d)$ is quasi-isomorphic to $(H_c^*(M), d)$, η represents zero in the cohomology of $(\ker d_c, d)$. Therefore, $\eta = d\nu$, for some $\nu \in \ker d_c$. Now, the class $[\nu]$ of ν in $H_c^*(M)$ satisfies $d[\nu] = 0$. Using Proposition 2.20 again, we find that $[\nu] - [\nu'] = 0$, for some d-closed form $\nu' \in \ker d_c$. Therefore, $\nu - \nu' = d_c\xi$. Since $d\nu' = 0$, this gives $dd_c\xi = d\nu = \eta$.

Definition 2.22. Let (A^*, d), (B^*, d) be graded commutative differential graded algebras (DG-algebras, for short). If (A^*, d) and (B^*, d) can be connected by a sequence of quasi-isomorphisms

$$(A^*, d) \longrightarrow (A_1^*, d), (A_2^*, d_2) \longrightarrow (A_1^*, d), \dots, (A_n^*, d_n) \longrightarrow (B^*, d),$$

the DG-algebras (A^*, d) and (B^*, d) are called *weakly equivalent*. A DG-algebra is called *formal* if it is weakly equivalent to a DG-algebra with $d = 0$.

Corollary 2.23. *Let (M, ω) be a compact Riemannian manifold equipped with a parallel form, and $(H_c^*(M), d)$ its pseudo-cohomology DG-algebra. Then $(\Lambda^*(M), d)$ is weakly equivalent to $(H_c^*(M), d)$. Moreover, if every pseudo-harmonic form is harmonic, then $(\Lambda^*(M), d)$ is formal.*

Proof. By Proposition 2.11, the DG-algebra $(\Lambda^*(M), d)$ is quasi-isomorphic to $(\ker d_c, d)$. By Proposition 2.20, the DG-algebra $(\ker d_c, d)$ is quasi-isomorphic to $(H_c^*(M), d)$. Finally, if all pseudo-harmonic forms are harmonic, the differential d vanishes on $\mathcal{H}_c^*(M)$, and Proposition 2.19 implies that $d = 0$ on $(H_c^*(M), d)$. □

Remark 2.24. When (M, ω) is a compact Kähler manifold, $\Delta = \Delta_c$ as the Kähler identities imply. In this situation, pseudo-harmonic forms are the same as harmonic ones. This implies the celebrated result of [21]: for any compact Kähler manifold, its de Rham DG-algebra is formal.

3. The structure operator for holonomy G_2-manifolds

3.1. G_2-manifolds

We base our exposition on [12].

Claim 3.1. *Consider the natural action of $GL(7, \mathbb{R})$ on the space $\Lambda^3(V^*)$ of 3-forms on V, where $V = \mathbb{R}^7$. Then $GL(7, \mathbb{R})$ acts on $\Lambda^3(V^*)$ with two open orbits.*

Proof. It is well known (see e.g. [32]). $\square$

Definition 3.2. A 3-form ω on $V = \mathbb{R}^7$ is called *non-degenerate* if it lies in an open orbit.

The group $GL(7, \mathbb{R})$ is 49-dimensional, and the dimension of $\Lambda^3(V^*)$ is 35. Therefore, a stabilizer of a non-degenerate 3-form has dimension 14. This stabilizer is a Lie group, of dimension 14, called G_2. For one orbit it is a compact form of G_2, for another orbit a non-compact real form. We call a non-degenerate 3-form ω on $V = \mathbb{R}^7$ *positive* if its stabilizer is a compact form of G_2.

Given a 3-form $\omega \in \Lambda^3(V^*)$, consider a $\Lambda^7(V^*)$-valued scalar product $V \times V \longrightarrow \Lambda^7(V^*)$,

$$x, y \xrightarrow{\tilde{g}} \frac{1}{6}(\omega \lrcorner x) \wedge (\omega \lrcorner y) \wedge \omega.$$

It is easy to check that $\tilde{g}$ is non-degenerate when ω is non-degenerate, and sign-definite when ω is positive. Consider $\tilde{g}$ as a section of $V^* \otimes V^* \otimes \Lambda^7(V^*)$, and denote by K its determinant, $K \in \Lambda^7(V^*)^9$. Since 9 is odd, K gives an orientation on V. Let $k := \sqrt[9]{K}$ be the corresponding section of $\Lambda^7(V^*)$, and $g := k^{-1}\tilde{g}$ the $\mathbb{R}$-valued bilinear symmetric form associated with $\tilde{g}$. Assume that ω is positive. A direct calculation implies that g is positive definite, and in some orthonormal basis $e_1, \dots, e_7 \in V^*$, ω is written as

$$\omega = (e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5 + (e_1 \wedge e_3 - e_2 \wedge e_4) \wedge e_6 + (e_1 \wedge e_4 - e_2 \wedge e_3) \wedge e_7 + e_5 \wedge e_6 \wedge e_7. \tag{3.1}$$

Definition 3.3. Let M be a seven-dimensional smooth manifold, and $\omega \in \Lambda^3(M)$ a 3-form. (M, ω) is called *a G_2-manifold* if ω is non-degenerate and positive everywhere on M. We consider M as a Riemannian manifold, with the Riemannian structure determined by ω as above. The manifold (M, g, ω) is called *a holonomy G_2-manifold*, or *parallel G_2-manifold* if ω is parallel with respect to the Levi-Civita connection associated with g. Further on, we shall consider only holonomy G_2 manifolds, and sometimes (abusing the language) we omit the word "holonomy".

Remark 3.4. Holonomy G_2-manifolds have long and distinguished history. They appear in Berger's list of irreducible holonomies [2]. Local examples of holonomy G_2-manifolds were unknown until Bryant's work of the mid-1980s [4]. Then Bryant and Salamon constructed complete examples of holonomy G_2-manifolds [5], and Joyce [6] constructed and studied compact holonomy G_2-manifolds at great length. For details of Joyce's construction, see [7]. Since then, the G_2-manifolds have become crucially important in many areas of string physics, especially in M-theory.

Under the G_2-action, the space $\Lambda^*(M)$ splits into irreducible representations, as follows:

$$\begin{aligned} &\Lambda^2(M) \cong \Lambda^2_7(M) \oplus \Lambda^2_{14}(M), \\ &\Lambda^3(M) \cong \Lambda^3_1(M) \oplus \Lambda^3_7(M) \oplus \Lambda^3_{27}(M) \end{aligned} \tag{3.2}$$

where $\Lambda^i_j(M)$ is an irreducible G_2-representation of dimension j. Clearly, there is a natural isomorphism $\Lambda^*(M) \cong \Lambda^{7-*}(M)$ compatible with G_2-action, and the spaces $\Lambda^4(M)$, $\Lambda^5(M)$ split in a similar fashion. The spaces Λ^0, Λ^1 are irreducible.

The spaces $\Lambda^i_j(M)$ are defined explicitly, in the following way. $\Lambda^2_7(M)$ is $\Lambda_{*\omega}(\Lambda^6(M))$, where $\Lambda_{*\omega}$ is the Hermitian adjoint to $L_{*\omega}(\eta) = *\omega \wedge \eta$ (see Section 2). The space $\Lambda^2_{14}(M)$ is identified with $\mathfrak{g}_2 \subset \mathfrak{so}(TM)$ under the standard identification $\Lambda^2(M) = \mathfrak{so}(TM)$. The space $\Lambda^3_1(M)$ is generated by ω, and $\Lambda^3_7(M)$ is equal to $\Lambda_\omega(\Lambda^6(M))$, where Λ_ω is the Hermitian adjoint of $L_\omega(\eta) = \omega \wedge \eta$ (see Section 2). Finally, $\Lambda^3_{27}(M)$ is identified with $(\ker L_\omega) \cap (\ker \Lambda_\omega) \subset \Lambda^3(M)$.

Remark 3.5. Notice that the operators C, L_ω, Λ_ω from Section 2 are clearly G_2-invariant.

From the construction, it is clear that the splitting (3.2) can be obtained via the operators $L_\omega, \Lambda_\omega, L_{*\omega}, \Lambda_{*\omega}$. By Proposition 2.7 these operators commute with the Laplacian. Therefore, the harmonic forms also split:

$$\begin{aligned} &\mathcal{H}^2(M) \cong \mathcal{H}^2_7(M) \oplus \mathcal{H}^2_{14}(M),\\ &\mathcal{H}^3(M) \cong \mathcal{H}^3_1(M) \oplus \mathcal{H}^3_7(M) \oplus \mathcal{H}^3_{27}(M) \end{aligned} \tag{3.3}$$

and similar splitting occurs on $\mathcal{H}^4(M)$ and $\mathcal{H}^5(M)$.

The following result is well known and is implied by a Bochner–Lichnerowicz-type argument using the Ricci-flatness of holonomy G_2-manifolds.

Claim 3.6. *Let M be a compact parallel G_2-manifold, and $\eta \in \mathcal{H}^i_7(M)$ a harmonic form. Then η is parallel. Moreover, if $H^1(M) = 0$, then $\mathcal{H}^i_7(M) = 0$ ($i = 1, 2, 3, 4, 5, 6$).*

Proof. See [7]. □

Remark 3.7. A parallel G_2-manifold is Ricci-flat, as shown by Bonan [33]. Then $\pi_1(M)$ is finite, unless M has a finite covering which is isometric to $T \times M'$, where M' is a manifold with special holonomy, and T a torus. When $\pi_1(M)$ is finite, $\mathcal{H}^i_7(M) = 0$ as Claim 3.6 implies.

We shall also need the following linear-algebraic result, which is well known. Let M be a parallel G_2-manifold, and

$$\Lambda^2(M) \cong \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$$

the decomposition defined above. Consider the operator

$$* \circ L_\omega : \Lambda^2(M) \longrightarrow \Lambda^2(M).$$

This operator is G_2-invariant, and hence by Schur's lemma acts on $\Lambda^2_7(M)$ and $\Lambda^2_{14}(M)$ as scalars. These scalars are computed as follows

Claim 3.8. *For any $\alpha \in \Lambda^2_7(M)$, we have $*L_\omega\alpha = 2\alpha$. For $\alpha \in \Lambda^2_{14}(M)$, we have $*L_\omega\alpha = -\alpha$.*

Proof. See e.g. [34, (2.32)]. □

3.2. The structure operator for G_2-manifolds

Let (M, ω) be a parallel G_2-manifold. We have two parallel forms on M: ω and $*\omega$, and the results of Section 2 can be applied to ω and $*\omega$ as well.

We denote by $C, C_{*\omega}$ the corresponding structure operators, and by d_c the operator $\{C, d\}$.

This part of the paper is pure linear algebra. We never use the holonomy property: throughout this subsection, there is no need to assume that our G_2-manifold has holonomy in G_2.

Consider the operator $C^2 = \frac{1}{2}\{C, C\}$. Being a supercommutator of two differentiations, this operator is a differentiation.

Claim 3.9. *Under these assumptions,*

$$C^2 = 3C_{*\omega}. \tag{3.4}$$

Proof. Both sides of (3.4) are differentiations, and vanish on $\Lambda^0(M)$. Therefore, to prove (3.4) it suffices to check that $C^2 = 3C_{*\omega}$ on $\Lambda^1(M)$. Both C^2 and $C_{*\omega}$ define G_2-invariant maps from $\Lambda^1(M)$ to $\Lambda^3_7(M)$. By Schur's lemma, these operators are proportional. To show that the coefficient of proportionality is 3, we compute C^2 and $C_{*\omega}$ on e_1, using (3.1). □

A similar argument gives the following claim:

Claim 3.10. *Under the above assumptions, we have*

$$\{L_\omega, C^*\} = -3C_{*\omega}. \tag{3.5}$$

Proof. The operator C^* takes the form

$$C^*(e_{i_1} \wedge e_{i_2} \wedge \cdots) = \sum_{k_1<k_2} (-1)^{(i_{k_1}-1)i_{k_2}} C^*(e_{i_{k_1}} \wedge e_{i_{k_2}}) \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \check{e}_{i_{k_1}} \wedge \cdots \wedge \check{e}_{i_{k_2}} \wedge \cdots \tag{3.6}$$

where $C^*(e_{i_{k_1}} \wedge e_{i_{k_2}})$ is the usual crossed product of vectors $e_{i_{k_1}}, e_{i_{k_2}}$ on the space equipped with a 3-form and a non-degenerate bilinear symmetric form. From (3.6) it is clear that C^* is a second-order differential operator on the algebra $\Lambda^*(M)$ (differential operators on a graded commutative algebra are understood in the sense of Grothendieck—see e.g. [35]). Then $\{L_\omega, C^*\}$ is a first-order differential operator. An elementary calculation gives $C^*\omega = 0$. Therefore, $\{L_\omega, C^*\}$ is a differentiation. To compare $\{L_\omega, C^*\}$ with $-3C_{*\omega}$, we need to check that $\{L_\omega, C^*\} = -3C_{*\omega}$ on $\Lambda^1(M)$. Both of these operators are G_2-invariant, and Schur's lemma implies that they are proportional on $\Lambda^1(M)$. To compute the coefficient of proportionality, it suffices to compute $\{\{L_\omega, C^*\}, C_{*\omega}\}$on some vector, e.g. e_1. □

Claim 3.11. *Under the above assumptions, $C : \Lambda^3(M) \longrightarrow \Lambda^4(M)$ is an isomorphism. Moreover, $C\omega = 2 * \omega$.*

Proof. Clearly, C preserves the decomposition of $\Lambda^*(M)$ into G_2-invariant summands as in (3.2). We write ω in an orthonormal basis as in (3.1). The equation $C\omega = 2 * \omega$ is given by a direct calculation. Given a 3-form $\theta \in \Lambda^3_7(M)$ and applying (3.5), we obtain $\Lambda^*(C\theta) = -3(C^*_{*\omega})\theta$. However, $C^*_{*\omega} : \Lambda^3_7(M) \longrightarrow \Lambda^1(M)$ is an isomorphism, because $C_{*\omega} : \Lambda^1(M) \longrightarrow \Lambda^3_7(M)$ is non-zero. To prove Claim 3.11, it remains to show that C is an isomorphism on $\Lambda^3_{27}(M)$. By Schur's lemma, for this it suffices to show that $C|_{\Lambda^3_{27}(M)}$ is non-zero.

Consider the form $\eta = e_5 \wedge (e_1 \wedge e_2 - e_3 \wedge e_4)$. Clearly, $\Lambda_\omega \eta = 0$ and $L_\omega \eta = 0$. Therefore, $\eta \in \Lambda^3_{27}(M)$.

From (3.6) we find that $C^*(e_1 \wedge e_2 - e_3 \wedge e_4) = 0$, and hence $e_1 \wedge e_2 - e_3 \wedge e_4$ lies in $\Lambda^2_{14}(M)$. This gives

$$\begin{aligned} C(\eta) &= C(e_5 \wedge (e_1 \wedge e_2 - e_3 \wedge e_4)) \\ &= C(e_5) \wedge (e_1 \wedge e_2 - e_3 \wedge e_4) \\ &= (e_1 \wedge e_2 + e_3 \wedge e_4 + e_6 \wedge e_7) \wedge (e_1 \wedge e_2 - e_3 \wedge e_4) \\ &= e_6 \wedge e_7 \wedge (e_1 \wedge e_2 - e_3 \wedge e_4). \end{aligned} \tag{3.7}$$

We obtain that $C(\eta) \neq 0$. Claim 3.11 is proven. □

Remark 3.12. The calculation (3.7) gives

$$C(\eta) = - * \eta \tag{3.8}$$

and by Schur's lemma this equation holds for all $\eta \in \Lambda^3_{27}(M)$.

Proposition 3.13. *Let (M, ω) be a parallel G_2-manifold, and C its structure operator. Then C induces isomorphisms*

$$\Lambda^i_7(M) \xrightarrow{C} \Lambda^{i+1}_7(M), \quad (i = 1, 2, 3, 4, 5). \tag{3.9}$$

Proof. By Schur's lemma, (3.9) is either an isomorphism or zero. For $i = 1$, $i = 2$, (3.9) is non-zero as follows from Claim 3.9. For $i = 3$, (3.9) is non-zero by Claim 3.11. Using

$$C(\varphi \wedge \psi) = C(\varphi) \wedge \psi + (-1)^{\tilde{\varphi}} \varphi \wedge C(\psi),$$

we find that $*C*$ is Hermitian adjoint to C. On the other hand, (3.9) is an isomorphism if and only if

$$\Lambda^{i+1}_7(M) \xrightarrow{C^*} \Lambda^i_7(M),$$

is an isomorphism. Using $C^* = *C*$, we obtain that Proposition 3.13 for $i = k$ is implied by Proposition 3.13 for $i = 6 - k$. Therefore, the already proven assertions of Proposition 3.13 for $i = 1, 2, 3$ imply Proposition 3.13 for $i = 4, 5$. □

4. Pseudo-cohomology for G_2-manifolds

4.1. Formality for G_2-manifolds

Homotopy formality for parallel G_2-manifolds was studied by Gil Cavalcanti in his thesis (see [36]). The G_2-structure gives certain constraints on the cohomology ring of a manifold: the multiplication by the standard 3-form ω gives an isomorphism

$$H^2(M) \xrightarrow{\wedge\omega} H^5(M)$$

and the 2-form

$$\eta \longrightarrow \int_M \eta \wedge \eta \wedge \omega$$

on $H^2(M, \mathbb{R})$ must be positive definite. Also, $H^1(M) = 0$. Cavalcanti constructed examples of non-formal 7-manifolds satisfying these constraints. He also showed that for $\dim H^2(M) \leqslant 2$, these constraints do indeed imply formality.

The G_2-version of the dd_c-lemma (Proposition 1.1) should give information about rational homotopy, in the same way as the usual dd^c-lemma leads to formality of Kähler manifolds. Indeed, $(\ker d_c)$ is a subalgebra of $\Lambda^*(M)$ which is weakly equivalent to the de Rham algebra of M (Proposition 2.11), and the quotient algebra

$$(H^*_c(M), d) \cong \frac{\ker d_c}{(\ker d_c) \cap (\operatorname{im} d_c)}$$

is also weakly equivalent to $\Lambda^*(M)$. We call $(H^*_c(M), d)$ *the pseudo-cohomology* of M (Definition 2.15). We do not call it the cohomology, because $d_c^2 \neq 0$.

A form $\eta \in \Lambda^*(M)$ is called *pseudo-harmonic* if $\eta \in (\ker d_c) \cap (\ker^* d_c)$, where d_c^* is a Hermitian adjoint to d_c. Just as happens for usual cohomology, the space of pseudo-harmonic forms $\mathcal{H}_c^*(M)$ is isomorphic to the pseudo-cohomology:

$$(H_c^*(M), d) \cong (\mathcal{H}_c^*(M), d)$$

(Proposition 2.19). All harmonic forms are also pseudo-harmonic. We consider an orthogonal decomposition

$$\mathcal{H}_c^*(M) \cong \mathcal{H}^*(M) \oplus \mathcal{H}_c^*(M)_{>0},$$

where $\mathcal{H}_c^*(M)_{>0}$ is the sum of all positive eigenspaces of the Laplacian acting on $\mathcal{H}_c^*(M)$. From the arguments given above, we immediately obtain the following theorem.

Theorem 4.1. *Let M be a compact parallel G_2-manifold, and $\mathcal{H}_c^*(M)_{>0}$ the sum of all positive eigenspaces of the Laplacian acting on $\mathcal{H}_c^*(M)$. Assume that $\mathcal{H}_c^*(M)_{>0} = 0$. Then M is formal.*

Proof. This is Corollary 2.23. $\square$

We were unable to show that $\mathcal{H}_c^*(M)_{>0} = 0$ for all parallel G_2-manifolds. However, this space was computed fairly explicitly, in terms of G_2-action on differential forms.

Proposition 4.2. *Let M be a compact parallel G_2-manifold, and $\mathcal{H}_c^i(M)_{>0} = 0$ the vector space defined above. Then $\mathcal{H}_c^i(M)_{>0} = 0$ unless $i = 3$ or 4. The space $\mathcal{H}_c^3(M)_{>0}$ is generated (over $\mathbb{C}$) by the solutions of the following equation:*

$$d\alpha = \mu * \alpha, \quad \alpha \in \Lambda_{27}^3(M), \tag{4.1}$$

where $\mu \in \mathbb{C}$ is a non-zero number, and $\Lambda_{27}^3(M)$ is the 27-dimensional irreducible component of $\Lambda^3(M)$ under the G_2-action (see (3.2)). Similarly, $\mathcal{H}_c^4(M)_{>0}$ is generated by the solutions of equation $d * \eta = \mu\eta,\ \eta \in \Lambda_{27}^4(M)$.

Proof. See Theorem 4.5. $\square$

Corollary 4.3. *Let M be a compact parallel G_2-manifold, and $(H_c^*(M), d)$ its pseudo-cohomology DG-algebra. Then $(H_c^*(M), d)$ is weakly equivalent to the de Rham algebra of M, and, moreover, $d|_{H_c^i(M)} = 0$ unless $i = 3$.* $\square$

This result can be used to study the obstructions O_n to formality of the DG-algebra $(H_c^*(M), d)$, defined in [29] (see Section 1.4). It turns out that only the first obstruction O_1 is relevant for rational homotopy, and if it vanishes, the $O_i,\ i > 0$, also vanish, and the DG-algebra $(H_c^*(M), d)$ and $(\Lambda^*(M), d)$ is formal. However, the same result can be obtained from Gil Cavalcanti's work, for all simply connected 7-manifolds.

In the 1970s, Miller showed that all simply connected orientable compact manifolds of dimensions up to 6 are formal [37]. Moreover, Miller has shown that all $(k-1)$-connected orientable compact manifolds of dimension up to $2k+2$ are formal. His arguments were simplified and generalized by Fernandez and Munoz [38], who defined a notion of a *k-formal manifold*, and showed that any orientable k-formal compact manifold of dimension up to $2k+2$ is formal. They applied this theorem to obtain results about formality of compact symplectic manifolds.

Cavalcanti [36] studied 7-manifolds using the same conceptual framework, obtaining that obstructions to 3-formality for simply connected 7-manifolds can be reduced to the vanishing of the first obstruction of Halperin and Stasheff.

4.2. The de Rham differential on $\Lambda_7^*(M)$

To study the pseudo-cohomology, we use the following well-known lemma (appearing in a different form in [39,40]).

Lemma 4.4. *Let $\eta \in \Lambda_7^k(M)$ be a differential form on a holonomy G_2-manifold (not necessarily compact), where $0 < k < 5$ is an integer. Fix parallel G_2-invariant isomorphisms*

$$\Lambda_7^k(M) \xrightarrow{\tau_{i,k}} \Lambda_7^i(M), \tag{4.2}$$

for all $i = 1, 2, 3, 4, 5$ (by Schur's lemma, these isomorphisms are well defined, up to a constant).[2] *Denote by $d_7 : \Lambda_7^i(M) \longrightarrow \Lambda_7^{i+1}(M)$ the Λ_7^*-part of the de Rham differential. Then $d_7(\eta) = 0$ if and only if $d_7(\tau_{k,i}\eta) = 0$ for any $i = 1, 2, 3, 4$.*

Proof. Consider the Levi-Civita connection

$$\nabla : \Lambda_7^i(M) \longrightarrow \Lambda_7^i(M) \otimes \Lambda_7^1(M). \tag{4.3}$$

The operator d_7 is obtained as a composition of (4.3) and a G_2-invariant pairing $\Lambda_7^i(M) \otimes \Lambda_7^1(M) \longrightarrow \Lambda_7^{i+1}(M)$. Using an irreducible decomposition of $\Lambda_7^1(M) \otimes \Lambda_7^i(M)$ (see e.g. [34]), we find that $\Lambda_7^1(M) \otimes \Lambda_7^i(M)$ contains a unique irreducible

[2] Using Proposition 3.13, we could use the powers of C to define the isomorphisms (4.2).

summand isomorphic to $\Lambda^*_7(M)$ as a G_2-representation. It is clear that $d_7 : \Lambda^i_7(M) \longrightarrow \Lambda^{i+1}_7(M)$ is obtained as a composition of (4.3) and the projection to this $\Lambda^*_7(M)$-summand. Therefore, the following diagram is commutative, up to a constant multiplier:

$$\begin{array}{ccc} \Lambda^k_7(M) & \xrightarrow{d_7} & \Lambda^{k+1}_7(M) \\ \tau_{i,k}\downarrow & & \tau_{i+1,k+1}\downarrow \\ \Lambda^i_7(M) & \xrightarrow{d_7} & \Lambda^{i+1}_7(M). \end{array} \tag{4.4}$$

We obtain that $\tau_{i+1,k+1}d_7(\eta) = 0$ if and only if $d_7(\tau_{i,k}\eta) = 0$. This proves Lemma 4.4. $\square$

4.3. Computations of pseudo-cohomology

Theorem 4.5. *Let (M, ω) be a compact parallel G_2-manifold, $\mathcal{H}^*(M)$ the space of harmonic forms, and $\mathcal{H}^*_c(M) \supset \mathcal{H}^*(M)$ the space of pseudo-harmonic forms. Then:*

(i) *$\mathcal{H}^i_c(M) = \mathcal{H}^i(M)$ for all $i \neq 3, 4$.*
(ii) *The orthogonal complement[3] $\mathcal{H}^i_c(M)_{>0}$ to $\mathcal{H}^i(M)$ in $\mathcal{H}^i_c(M)$ lies in $\Lambda^i_{27}(M)$.*
(iii) *$*(\mathcal{H}^3_c(M)_{>0}) = \mathcal{H}^4_c(M)_{>0}$. Moreover, $\mathcal{H}^3_c(M)_{>0}$ is generated by all solutions of the equation $d\eta = \mu * \eta$, for all $\mu \in \mathbb{C}, \mu \neq 0, \eta \in \Lambda^3_{27}(M)$.*

Proof. Consider the orthogonal decomposition

$$\mathcal{H}^*_c(M) = \mathcal{H}^*(M) \oplus \mathcal{H}^*(M)_{>0}.$$

Since Δ preserves $\mathcal{H}^*_c(M)$, Δ acts diagonally on $\mathcal{H}^*_c(M)$, and $\mathcal{H}^*(M)_{>0}$ is generated by all eigenvectors of Δ with non-zero eigenvalue. Therefore, d preserves $\mathcal{H}^*(M)_{>0}$.

By Corollary 2.23, $\mathcal{H}^*_c(M)$ is quasi-isomorphic to $\mathcal{H}^*(M)$. Therefore, the cohomology of d on $\mathcal{H}^*(M)_{>0}$ is zero. Now, Theorem 4.5(i) is implied by the following claim:

Claim 4.6. *Let (M, ω) be a compact parallel G_2-manifold, and $\eta \in \mathcal{H}^i_c(M)$ a non-zero exact pseudo-harmonic form. Then $i = 4$.*

Proof. To prove Theorem 4.5(i) it suffices to prove Claim 4.6 for $i \leqslant 4$. Indeed, this will imply that $\mathcal{H}^i(M)_{>0} = 0$ for $i < 3$, but the Hodge $*$-operator preserves $\mathcal{H}^*_c(M)$, and exchanges $\mathcal{H}^i(M)_{>0}$ and $\mathcal{H}^{7-i}(M)_{>0}$; hence $\mathcal{H}^i(M)_{>0} = 0$ for $i = 1, 2$ implies $\mathcal{H}^i(M)_{>0} = 0$ for $i = 5, 6$.

Now, Theorem 4.5(i) is equivalent to Claim 4.6 as we have shown above. The same argument shows that Claim 4.6 for $i \leqslant 4$. This implies Theorem 4.5(i) and the full statement of Claim 4.6.

Let $\eta = d\alpha$ be a d-exact 1-form in $\mathcal{H}^1_c(M)$, $\alpha \in \mathcal{H}^2_c(M)$. Then $C\eta = Cd\alpha = -dC\alpha = 0$ (the middle equation is implied by $d_c\alpha = 0$). Therefore, $C\eta = 0$. However, C is clearly injective on $\Lambda^1(M)$. This proves Claim 4.6 for $i = 1$.

Let now $\eta = d\alpha$ be a d-exact 2-form in $\mathcal{H}^2_c(M)$, $\alpha \in \mathcal{H}^1_c(M)$. Using $d_c\eta = 0$, we obtain

$$0 = \{d, c\}\alpha = C\eta + dC\alpha. \tag{4.5}$$

Write the decomposition $\eta = \eta_7 + \eta_{14}$ induced by $\Lambda^2(M) \cong \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$. Then (4.5) gives $dC\eta = dC\eta_7 = 0$. From Lemma 4.4 we infer that $d_7\eta_7 = 0$. Consider the top degree forms

$$\eta_7 \wedge d\alpha \wedge \omega = \eta_7 \wedge \eta_7 \wedge \omega \tag{4.6}$$

(the equality holds by Schur's lemma as η_7 is the $\Lambda^2_7(M)$-part of $\eta = d\alpha$). Since $\Lambda^2_7(M)$ is an irreducible representation of G_2, by Schur's lemma the 2-form $\eta_7 \longrightarrow \int \eta_7 \wedge \eta_7 \wedge \omega$ is sign-definite (negative definite, as Claim 3.8 implies). Then $\int \eta_7 \wedge \eta_7 \wedge \omega < 0$ unless $\eta_7 = 0$. However, by (4.6),

$$\int \eta_7 \wedge \eta_7 \wedge \omega = \int \eta_7 \wedge d\alpha \wedge \omega = -\int d\eta_7 \wedge \alpha \wedge \omega = \int d_7\eta_7 \wedge \alpha \wedge \omega = 0$$

as $d_7\eta_7 = 0$. We obtain that $\eta \in \Lambda^2_{14}(M)$. Using Claim 3.8 again, we obtain that $\int \eta \wedge \eta \wedge \omega > 0$ unless $\eta = 0$. However, η is exact, and hence this integral vanishes, yielding $\eta = 0$. We proved Claim 4.6 for $i = 2$.

Now, let $\eta = d\alpha$ be a d-exact 3-form in $\mathcal{H}^3_c(M)$, $\alpha \in \mathcal{H}^2_c(M)$. To finish the proof of Claim 4.6, we need to show that $\eta = 0$.

Since d^* commutes with d_c, d_c^*, we have $d^*\alpha \in \mathcal{H}^1_c(M)$. As we have shown above, $\mathcal{H}^1_c(M) = \mathcal{H}^1(M)$, and therefore $d^*\alpha$ is harmonic. A d^*-exact harmonic form vanishes. Therefore, $d^*\alpha = 0$.

[3] This notation has the following meaning: $\mathcal{H}^i_c(M)_{>0}$ is a sum of all positive eigenspaces of Laplacian acting on $\mathcal{H}^i_c(M)$.

Then $0 = d_c\alpha = d^*L_\omega\alpha$. Similarly, $0 = d_c^*\alpha = \Lambda d\alpha$. Using

$$\Lambda^2(M) \cong \Lambda^2_7(M) \oplus \Lambda^2_{14}(M),$$

write the decomposition $\alpha = \alpha_7+\alpha_{14}$. Then $L_\omega\alpha = 2*\alpha_7 - *\alpha_{14}$ as follows from Claim 3.8. Therefore, $d^*L_\omega\alpha = *d(2\alpha_7-\alpha_{14})$. We obtain that α satisfies the following:

$$d^*\alpha = 0, \qquad d(2\alpha_7 - \alpha_{14}) = 0, \qquad \Lambda d\alpha = 0. \tag{4.7}$$

Clearly, $d^*\alpha = 0$ is equivalent to $d * \alpha = 0$. Also, $\{L_\omega, d\} = 0$ (ω is closed). Using $*\alpha_7 = \frac{1}{2}\alpha_7 \wedge \omega$, $*\alpha_{14} = -\alpha_{14} \wedge \omega$ (Claim 3.8), we rewrite $d * \alpha = 0$ as $L_\omega(d\alpha_{14} - \frac{1}{2}d\alpha_7) = 0$. From (4.7) we obtain $L_\omega(d\alpha_{14} - 2d\alpha_7) = 0$. Comparing these equations, we find

$$L_\omega(d\alpha_{14}) = 0, \qquad L_\omega(d\alpha_7) = 0. \tag{4.8}$$

Using Claim 3.8 again, we find that (4.8) implies $d^*\alpha_{14} = d^*\alpha_7 = 0$.

Now, $C^*\Lambda^2_{14}(M) = 0$ because C^* is G_2-invariant. Using $d^*\alpha = d^*\alpha_7 = 0$, we obtain

$$0 = d_c^*\alpha = \{d^*, C^*\}\alpha = d^*C^*(\alpha_{14} + \alpha_7) = d^*C^*\alpha_7 = d_c^*\alpha_7.$$

This implies

$$d_c^*\alpha_7 = d_c^*\alpha_{14} = 0. \tag{4.9}$$

Applying $d_c^* = \{d, \Lambda_\omega\}$, we find that (4.9) yields

$$\Lambda_\omega d\alpha_7 = \Lambda_\omega d\alpha_{14} = 0. \tag{4.10}$$

Comparing (4.10) and (4.8), we find that

$$d\alpha_7, d\alpha_{14} \in \Lambda^3_{27}(M). \tag{4.11}$$

This gives $\eta = d\alpha \in \Lambda^3_{27}(M)$. Since $d_c\eta = 0$, we have $dCd\alpha = 0$, and the form $C\eta = Cd\alpha$ is closed. Therefore,

$$\int \eta \wedge C\eta = \int d\alpha \wedge Cd\alpha = 0. \tag{4.12}$$

However, on $\Lambda^3_{27}(M)$, the form $\eta \longrightarrow \int \eta \wedge C\eta$ is non-zero (Claim 3.11), and hence, by Schur's lemma, sign-definite.[4] Therefore, (4.12) implies that $\eta = 0$. This proves Claim 4.6 for $i = 3$. We have finished the proof of Claim 4.6. The proof of Theorem 4.5(i) is also finished. □

Let us have $\alpha \in \Lambda^3(M)$, and $\alpha = \alpha_1 + \alpha_7 + \alpha_{27}$ its decomposition induced by (3.2). To prove Theorem 4.5(ii), we use the following trivial observation:

$$\alpha_1 = \frac{1}{7}L_\omega\Lambda_\omega\alpha, \qquad \alpha_7 = \frac{1}{4}\Lambda_\omega L_\omega\alpha. \tag{4.13}$$

Similarly, for $\eta \in \Lambda^4(M)$, $\eta = \eta_1 + \eta_7 + \eta_{27}$, we have

$$\eta_1 = \frac{1}{7}\Lambda_\omega L_\omega\eta, \qquad \eta_7 = \frac{1}{4}L_\omega\Lambda_\omega\eta. \tag{4.14}$$

Assume now that $\alpha \in \mathcal{H}^3_c(M)_{>0}$. Then $d^*\alpha = 0$ as Theorem 4.5(i) implies. Therefore

$$0 = d_c\alpha = \{L_\omega, d^*\}\alpha = d^*L_\omega\alpha.$$

From (4.13), we obtain

$$d^*\alpha_7 = \frac{1}{4}d^*\Lambda_\omega L_\omega\alpha = -\Lambda_\omega d_c\alpha = 0. \tag{4.15}$$

This implies

$$d_c^*\alpha_7 = \{d^*, C^*\}\alpha_7 = d^*C^*\alpha \tag{4.16}$$

(the last equation holds because

$$\ker C^*|_{\Lambda^3(M)} = \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$$

[4] From Remark 3.12 it follows that this form is negative definite.

as the G_2-decomposition implies). However,

$$d^*C^*\alpha = d_c^*\alpha = 0$$

since $d^*\alpha = 0$. Then (4.16) gives $d_c^*\alpha_7 = 0$. Similarly,

$$d_c\alpha_7 = d^*L_\omega\alpha_7 = d^*L_\omega\alpha \tag{4.17}$$

(here we use

$$\ker L_\omega|_{\Lambda^3(M)} = \Lambda_1^3(M) \oplus \Lambda_{27}^3(M)$$

which is also implied by the G_2-decomposition). Using

$$0 = d_c\alpha = \{d^*, L_\omega\}\alpha = d^*L_\omega\alpha,$$

we infer from (4.17) that $d_c\alpha_7 = 0$. This gives $\alpha_7 \in \mathcal{H}_c^3(M)_{>0}$.

Now, by (4.13),

$$0 = dL_\omega\Lambda_\omega\alpha_7 = L_\omega\Lambda_\omega d\alpha_7 + L_\omega d_c^*\alpha_7 = L_\omega\Lambda_\omega d\alpha_7. \tag{4.18}$$

Using (4.14), we obtain that (4.18) gives $d\alpha_7 \in \Lambda_1^4(M)$. This means that $d\alpha_7 = f * \omega$, where $f \in C^\infty(M)$ is a function. Therefore, $0 = d^2\alpha_7 = df \wedge *\omega$. This leads to $df = 0$, as the map

$$\Lambda^1(M) \xrightarrow{L_{*\omega}} \Lambda^5(M)$$

is clearly injective. Therefore, α_7 is harmonic, and hence $\alpha_7 = 0$.

We have shown that

$$\mathcal{H}_c^3(M)_{>0} \subset \Lambda_1^3(M) \oplus \Lambda_{27}^3(M).$$

Taking the adjoint, we also obtain that

$$\mathcal{H}_c^4(M)_{>0} \subset \Lambda_1^4(M) \oplus \Lambda_{27}^4(M). \tag{4.19}$$

Take an arbitrary $\alpha \in \mathcal{H}_c^3(M)_{>0}$. Then $d\alpha \in \mathcal{H}_c^4(M)_{>0}$. Using (4.19) and (4.14), we obtain that $\Lambda d\alpha = 0$. Then

$$0 = d_c^*\alpha = \{\Lambda_\omega, d\}\alpha = d\Lambda_\omega\alpha. \tag{4.20}$$

Since $\Lambda_\omega\alpha$ is a function, (4.20) gives $\alpha_1 = 0$. Then $\alpha \in \Lambda_{27}^3(M)$. We have proved Theorem 4.5(ii).

Now, every $\alpha \in \Lambda_{27}^3(M)$ satisfying $d\alpha = \mu * \alpha$ clearly belongs to $\mathcal{H}_c^3(M)$. Indeed, in this case

$$L_\omega\alpha = C^*d\alpha = C^*\alpha = 0$$

because the operators C^*, L_ω are G_2-invariant, and

$$d^*\alpha = *d*\alpha = *\mu^{-1}d^2\alpha = 0$$

because $d^2 = 0$. Taking commutators of d^* with L_ω and d^* with C^*, we find that $d_c\alpha = d_c^*\alpha = 0$. To see that such an α generates $\mathcal{H}_c^3(M)$, we use the following lemma, which finishes the proof of Theorem 4.5(iii).

Lemma 4.7. *Under the assumptions of Theorem 4.5, $\mathcal{H}_c^3(M)_{>0}$ is generated by all $\alpha \in \mathcal{H}_c^3(M)_{>0}$ which satisfy $d\alpha = \mu * \alpha$, $\mu \neq 0$.*

Proof. Since d_c, d_c^* commute with the Laplacian, $\mathcal{H}_c^3(M)_{>0}$ is generated by the eigenspaces $\mathcal{H}_c^3(M)_\lambda$ of $\Delta|_{\mathcal{H}_c^3(M)_{>0}}$, which are finite dimensional. Moreover, $*d : \Lambda^3(M) \longrightarrow \Lambda^3(M)$ also commutes with the Laplacian, and hence it acts on the finite-dimensional spaces $\mathcal{H}_c^3(M)_\lambda$. Since

$$(*d\alpha, \alpha') = \int_M d\alpha \wedge \alpha' = -\int_M \alpha \wedge d\alpha' = -\overline{(*d\alpha', \alpha)},$$

the operator $*d$ is skew-Hermitian, and hence semisimple. Therefore, $\mathcal{H}_c^3(M)_\lambda$ is generated by its eigenspaces. By Theorem 4.5(i), d^* vanishes on $\mathcal{H}_c^3(M)$, and hence $d\alpha \neq 0$ unless α is harmonic. Therefore, $*d$ acts on $\mathcal{H}_c^3(M)_\lambda$ with non-zero eigenvalues μ_i.[5] We have proved Lemma 4.7. The proof of Theorem 4.5 is finished. □

[5] In fact, $\lambda = |\mu_i|^2$, as follows from $\Delta|_{\mathcal{H}_c^3(M)_{>0}} = d^*d$.

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