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Laplacian solitons and symmetry in G_2-geometry

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ABSTRACT

In this paper, it is shown that (with no additional assumptions) on a compact 7-dimensional manifold which admits a G_2-structure, soliton solutions to the Laplacian flow of R. Bryant can only be shrinking or steady. We also show that the space of symmetries (vector fields that annihilate via the Lie derivative) of a torsion-free G_2-structure on a compact 7-manifold is canonically isomorphic to $H^1(M, \mathbb{R})$ by a direct method. Some comparisons with Ricci solitons are also discussed and emphasized, along with some future directions of exploration.

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1. Introduction

Let M be a 7-dimensional manifold that admits a G_2-structure given by a non-degenerate 3-form φ. A natural geometric flow when M is compact is the Laplacian flow first suggested by R. Bryant in [1]:

$$\frac{\partial \varphi}{\partial t} = -\Delta_\varphi \varphi \tag{1}$$

for a family $\varphi = \varphi(t)$ of G_2-structures, where Δ_φ denotes the Hodge Laplacian (hence nonnegative-definite) with respect to the metric g_φ induced by $\varphi(t)$. The original intention of Eq. (1) is to flow φ to a torsion-free G_2-structure, since φ being torsion-free is equivalent to being harmonic with respect to g_φ. Although it is not clear what it means in this case, when M is not compact the flow (1) still makes sense. In fact, when M is not compact we suspect that a more general flow is needed to prove convergence to a torsion-free G_2-structure.[1] The short-time existence and uniqueness of (1) for a closed initial G_2-structure when M is compact have been established in [3,4].[2]

As in the Ricci flow, let us consider solutions of the form

$$\varphi(t) = \tau(t) f_t^* \varphi \tag{2}$$

for a fixed G_2-structure φ, pulled back by a smooth family f_t of diffeomorphisms of M, and where $|\tau(t)| > 0$ is a scaling factor. Such special solutions are called *solitons*, and in the present context of G_2 geometry it has already appeared in [5] by Karigiannis, McKay, and Tsui. We will call solutions of the form (2) Laplacian solitons, or just solitons if no confusion

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1 See [2] for some general results in this direction.

2 We have chosen the negative sign in (1) so that it is more of a heat-like flow, but the flow originally introduced by Bryant has the plus sign. As mentioned in [5], when φ is closed $\Delta\varphi$ and $-\Delta\varphi$ turn out to be the same, up to lower order terms. If the plus sign was chosen, the results in this paper will still hold simply by cosmetic changes in sign.

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arises. Just as in the Ricci flow, we seek a static description of a soliton by substituting (2) into (1), and we will arrive at the equation[3]

$$\rho\varphi + L_X\varphi = -\Delta\varphi \tag{3}$$

for some constant ρ and vector field X. Thus equivalently, we can define a (Laplacian) soliton to be a G_2-structure φ that satisfies (3).

Analogous to Ricci solitons, we can define the following.

Definition 1. Let φ be a G_2-structure, and X a vector field on M. We say that (φ, X) is a Laplacian soliton if Eq. (3) is satisfied for some constant $\rho \in \mathbb{R}$. We say (φ, X) is an expanding soliton if $\rho > 0$, a steady soliton if $\rho = 0$, and a shrinking soliton if $\rho < 0$.

We would like to point out that a torsion-free G_2-structure is steady, and is the most trivial example of a soliton. Also, just to distinguish between the dynamic and static versions of the soliton concept, we will refer to (2) as a *soliton solution* whereas the terminology of *soliton* will be reserved for the definition above.

Work on a dual equation (*the Laplacian coflow*) to (1) and the corresponding soliton equation analogous to (3) have appeared in [5]. The original intention in [5] was to study (3), but instead they focused on the coflow version because there is available a special cohomogeneity-1 ansatz. In this paper we focus on a more detailed examination of the fundamental equation (3). In addition, we found out very recently in [6] that Weiss and Witt had also studied soliton solutions to the L^2-gradient flow of a Dirichlet-type functional they proposed in an earlier paper [7]. In [6] similar results to the ones in this paper and in [5] appeared, but note that the L^2-gradient flow of their energy functional is a different equation from the Laplacian flow (or coflow).

One of the main results of this paper is Corollary 1, which says that there are no compact expanding solitons, and no compact steady solitons except torsion-free G_2-structures. A similar result was proved in [5], where the soliton was assumed to be *coclosed* (and *closed* for the original flow (1)). Our result does not depend on the soliton φ being closed, and the proof follows directly from a fundamental identity established in Lemma 2 in the same section. Although the short-time existence of the Laplacian flow has only been established for closed G_2-structures, Corollary 1 is still valuable because we think (3) is still an interesting equation in its own right.

The other main result of this paper is Corollary 3, which says that for a torsion-free G_2-structure φ on a compact 7-manifold, the space of all vector fields X such that $L_X\varphi = 0$ is isomorphic to $H^1(M, \mathbb{R})$. We will refer to such a vector field as a *symmetry* of the G_2-structure φ, for brevity. Although Corollary 3 is not a new result—as it appeared implicitly in the book [8] by D. Joyce, we present a more direct proof partially as an application of the fundamental identity in Lemma 2. Our original goal was to prove an analogous isomorphism theorem for compact shrinking solitons as well, but at the present we are unable to do so. We also want to view Corollary 3 as a kind of rigidity result, because for a fixed φ the soliton equation (3) is invariant only by adding such vector fields. Another reason why we want to present Corollary 3 here is because we want to compare symmetries of G_2-structures to Killing vector fields for Riemannian metrics, in the context of Ricci solitons—this will be done in the concluding section of this paper.

2. G_2-structures and torsion forms

We give a brief review of the background relating to G_2-structures here. The standard reference for this is the book [8] by Dominic D. Joyce, although the papers [1,2] are also good sources.

The group G_2 is a compact, connected, simply-connected Lie group sitting in $SO(7)$. Algebraically, it can be defined as the automorphism group of the octonions. It appears as one of the exceptional holonomy groups in the classification by Berger et al. The working definition of G_2 that we will adopt is the following. Consider the differential 3-form

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356} \tag{4}$$

in $\mathbb{R}^7$, where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The group G_2 can be defined as the subgroup in $GL(7, \mathbb{R})$ that preserves φ_0, which means

$$\varphi_0\big(g(v_1), g(v_2), g(v_3)\big) = \varphi_0\big(v_1, v_2, v_3\big)$$

for all vectors $v_1, v_2, v_3 \in \mathbb{R}^7$ and every $g \in G_2$. The peculiar form that (4) takes reflects the combinatorial nature of permuting the octonions.

From a principal bundle point of view, a G_2-structure on a 7-dimensional manifold M is just a sub-bundle with structure group G_2, of the $GL(7, \mathbb{R})$-frame bundle over M. In other words, one can find local frames of the tangent bundle TM such that all the transition functions value in the group G_2. Because $G_2 \subset SO(7)$, a G_2-structure induces an orientation and a unique metric (which we will write g_φ) on M. Take any G_2-adapted frame $\{e_i\}_{i=1}^7$ in a G_2-structure, then the 3-form

$$\varphi = \omega_{123} + \omega_{145} + \omega_{167} + \omega_{246} - \omega_{257} - \omega_{347} - \omega_{356} \tag{5}$$

[3] From this point on, it is implicit that the Laplacian depends on the metric g_φ, so we will omit the subscript in the Laplacian.

is well-defined over all of M, where ω_i is the local dual 1-form to e_i. Conversely, if a 3-form φ on M can be locally represented as (5) with respect to a frame, then the transition functions of such frames value in G_2 and we have a G_2-structure. As a result, a G_2-structure is equivalent to a 3-form φ locally represented as in (5), and this is what we will refer to as a G_2-structure from this point on. It is well-known that a G_2-structure exists if and only if the 7-manifold is orientable and spin.

A G_2 structure φ induces a point-wise orthogonal decomposition (with respect to g_φ) of p-forms on M:

$$\begin{aligned}
\Omega^2_7 &= \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} = \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = -2\beta\}\\
\Omega^2_{14} &= \{\beta \in \Omega^2 \mid \beta \wedge *\varphi = 0\} = \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = \beta\}\\
\Omega^3_1 &= \{f\varphi \mid f \in C^\infty(M)\}\\
\Omega^3_7 &= \{X \lrcorner * \varphi \mid X \in \Gamma(TM)\}\\
\Omega^3_{27} &= \left\{ h_{ij} g^{jl} dx_i \wedge \left(\frac{\partial}{\partial x^l} \lrcorner \varphi \right) \mid h \in Sym^2(T^*M),\, Tr_g(h) = 0 \right\},
\end{aligned}$$

where

$$\begin{aligned}
\Omega^2 &= \Omega^2_7 \oplus \Omega^2_{14}\\
\Omega^3 &= \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}.
\end{aligned}$$

Then we can write

$$\begin{aligned}
d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3\\
d * \varphi &= 4\tau_1 \wedge *\varphi + *\tau_2,
\end{aligned}$$

where $\tau_0 \in \Omega^0_1$, $\tau_1 \in \Omega^1_7$, $\tau_2 \in \Omega^2_{14}$, and $\tau_3 \in \Omega^3_{27}$ are called the *torsion forms*. The fact that the same 1-form τ_1 appears in the decompositions of $d\varphi$ and $d * \varphi$ is non-trivial, but can be shown via some computations (see [2]). The terminology of torsion forms comes from the following. Having a G_2-structure φ does not mean the holonomy group of g_φ is contained in G_2. The additional condition that is needed is the so-called *torsion-free* condition. We say that a G_2-structure is torsion-free if φ solves the nonlinear system of partial differential equations $\nabla\varphi = 0$, where ∇ is the covariant derivative induced by g_φ. It was shown in [9] that a G_2-structure is torsion-free if and only if it is closed and co-closed (with respect to the Hodge star induced by g_φ).[4] Thus when M is compact, φ being torsion-free is equivalent to it being harmonic with respect to g_φ. In view of the torsion forms defined above, we see that φ is torsion-free if and only if all four torsion forms vanish on M. A 7-manifold M that admits a torsion-free G_2-structure has its Riemannian holonomy (with respect to g_φ) a subgroup of G_2, and such manifolds are simply known as *G_2-manifolds*.

3. The soliton equation

From direct computations, we see that a soliton solution (2) to the Laplacian flow (1) satisfies

$$\dot{\tau(t)} f_t^* \varphi + \tau(t) f_t^* \left(L_{X(t)} \varphi \right) = -\tau(t)^{1/3} f_t^* \left(\Delta\varphi \right),$$

which is exactly (3) and where the vector field $X(t)$ is the infinitesimal generator of the diffeomorphism f_t. We have also used the following fact:

Lemma 1. *If φ is a G_2-structure, then any non-zero constant multiple $c\varphi$ is also a G_2-structure, and $g_{c\varphi} = c^{2/3} g_\varphi$.*[5]

Note that we dropped the subscript on the Laplace operator since φ is now fixed. Then we see that there is a soliton solution (2) to the flow (1) only if

$$\rho\varphi + L_X \varphi = -\Delta\varphi$$

for some vector field X on M, where we have frozen at a time t with the constant $\rho = \dot{\tau}/\tau^{1/3}$ and the factor of $\tau^{2/3}$ absorbed into the vector field X above. As in the case of Ricci solitons, one can show that given a vector field X and a G_2 structure φ that satisfy (3) one can generate a solution of the form (2) to (1).

We would also like to point out that the soliton equation (3) is scale-invariant in the following sense. Note that given any G_2-structure φ satisfying (3), then for any $c\varphi$, $c \neq 0$, Eq. (3) is again satisfied for $\tilde{\rho} = c^{-2/3}\rho$ and $\tilde{X} = c^{-2/3}X$.

[4] This is an entirely local property, it is independent of whether or not M is compact.

[5] $c > 0$ preserves the orientation given by φ, $c < 0$ reverses it.

4. Compact solitons

In this section we establish the following fundamental result.

Lemma 2. *Let M be a compact 7-manifold. For any G_2-structure φ on M, vector field X, and $f \in C^\infty(M)$, we have*

$$\int_M L_X\varphi \wedge *f\varphi = -3\int_M df \wedge *X^\flat.$$

Proof. We have

$$L_X\varphi = X \lrcorner d\varphi + d(X\lrcorner\varphi).$$

From the decomposition of $d\varphi$ we see that

$$\begin{aligned}(X\lrcorner d\varphi)\wedge *f\varphi &= \tau_0 f(X\lrcorner *\varphi)\wedge *\varphi + 3f\big(X\lrcorner(\tau_1\wedge\varphi)\big)\wedge *\varphi + f(X\lrcorner *\tau_3)\wedge *\varphi\\ &= 3f\big(X\lrcorner(\tau_1\wedge\varphi)\big)\wedge *\varphi + f(X\lrcorner *\tau_3)\wedge *\varphi\\ &= -3f(\tau_1\wedge\varphi)\wedge(X\lrcorner *\varphi) - f*\tau_3\wedge(X\lrcorner *\varphi)\\ &= -3f\tau_1\wedge\varphi\wedge(X\lrcorner *\varphi)\\ &= -3f\tau_1\wedge(-4*X^\flat)\\ &= 12f\tau_1\wedge *X^\flat, \end{aligned} \tag{6}$$

where we have used the identity $\varphi\wedge(X\lrcorner *\varphi) = -4*X^\flat$ (see Appendix A in [2]) in the fifth equality and also the point-wise orthogonality of the G_2-decomposition of differential forms in the second and fourth equalities above. On the other hand, from the decomposition of $d*\varphi$ we have

$$\begin{aligned}\int_M d(X\lrcorner\varphi)\wedge *f\varphi &= \int_M (X\lrcorner\varphi)\wedge *\delta f\varphi\\ &= -\int_M (X\lrcorner\varphi)\wedge d*f\varphi\\ &= -\int_M (X\lrcorner\varphi)\wedge(df\wedge *\varphi + f\,d*\varphi)\\ &= -\int_M (X\lrcorner\varphi)\wedge df\wedge *\varphi - \int_M f(X\lrcorner\varphi)\wedge(4\tau_1\wedge *\varphi + *\tau_2)\\ &= -\int_M df\wedge *\varphi\wedge(X\lrcorner\varphi) - 4\int_M f(X\lrcorner\varphi)\wedge\tau_1\wedge *\varphi\\ &= -\int_M df\wedge *\varphi\wedge(X\lrcorner\varphi) - 4\int_M f\tau_1\wedge *\varphi\wedge(X\lrcorner\varphi)\\ &= -\int_M df\wedge 3*X^\flat - 4\int_M f\tau_1\wedge 3*X^\flat\\ &= -3\int_M df\wedge *X^\flat - 12\int_M f\tau_1\wedge *X^\flat, \end{aligned} \tag{7}$$

where we have also used the identity $*\varphi\wedge(X\lrcorner\varphi) = 3*X^\flat$ (see Appendix A in [2]). Integrating (6) and adding to (7), the lemma now follows. □

Corollary 1. *There are no compact expanding solitons, and there are no compact steady solitons except torsion-free G_2-structures.*

Proof. Wedging both sides of (3) by $*\varphi$ and integrating, we have

$$\begin{aligned}\rho\int_M \varphi\wedge *\varphi &= -\int_M \Delta\varphi\wedge *\varphi\\ &= -\int_M d\varphi\wedge *d\varphi - \int_M \delta\varphi\wedge *\delta\varphi, \end{aligned} \tag{8}$$

where we have used Lemma 2 with $f \equiv 1$. Then since $\int_M \varphi\wedge *\varphi$ is (7 times) the volume and hence non-zero, $\rho \le 0$ necessarily because the right-hand side of (8) is non-positive. It also follows from (8) that $\rho = 0$ is equivalent to $d\varphi = \delta\varphi = 0$, hence torsion-free. □

As an offshoot to the proof of Corollary 1, we also have the following observation.

Corollary 2. *For any compact soliton φ satisfying (3), the constant ρ has the Rayleigh quotient expression:*

$$\rho = \frac{-\int_M \Delta\varphi \wedge *\varphi}{\int_M \varphi \wedge *\varphi}. \tag{9}$$

Eq. (9) means that for a compact soliton, ρ is completely determined by the G_2-structure. This simple observation will play a role in Section 6. Also recall from Section 3 that $\rho = \dot{\tau}/\tau^{1/3}$ for any time t within solution (2)'s existence. Then (9) implies that $\dot{\tau}$ will always have the same sign at any time t within the soliton solution's existence, i.e. this means that a compact soliton solution is either "always" shrinking or "always" steady.

5. Eigenforms as shrinking solitons

The defining Eq. (3) for a soliton shows that it is a kind of generalized eigenvalue equation for φ (with respect to g_φ). In particular, when $\rho < 0$ an *eigenform*: $-\Delta\varphi = \rho\varphi$ solves the soliton equation with $X = 0$ ($f_t = \mathrm{Id}\ \forall t$). Eigenforms as Laplacian solitons are analogous to Einstein metrics as trivial examples of Ricci solitons. Nevertheless, it is enlightening to write out the exact solution for an eigenform in terms of (2) in the shrinking case.

Proposition 1. *Suppose φ is an eigenform then it is a shrinking soliton with a solution in the form of (2) as*

$$\varphi(t) = \left(1 + \frac{2}{3}\rho t\right)^{3/2} \varphi. \tag{10}$$

Proof. We assume the solution is of the form $\varphi(t) = R(t)\varphi$ for some real-valued function $R(t)$, and we set $R(0) = 1$. Then we see that

$$R'(t)\varphi = \frac{\partial\varphi(t)}{\partial t} = -\Delta_{\varphi(t)}\varphi(t) = -R(t)^{1/3}\Delta_\varphi\varphi = \rho R(t)^{1/3}\varphi.$$

From this we must have $R'(t) = \rho R(t)^{1/3}$, which by dividing both sides by $R(t)^{1/3}$ we can rewrite as

$$\frac{3}{2}\frac{d}{dt}\left(R(t)^{2/3}\right) = \rho. \tag{11}$$

By integrating both sides of (11) and using the initial condition, we get exactly the desired solution (10). □

From (10) we see that for an eigenform the singularity time is $t = -3/2\rho > 0$. There are no eigenforms for the expanding case, thus if we want to find expanding solitons (of course, only when M is noncompact) we must solve (3) with some nontrivial vector field X. A natural question is when is a closed G_2-structure an eigenform. Noting the characterization of Ω^3_1, we in fact have the following result.

Proposition 2. *Let φ be a closed G_2-structure. Then φ is an eigenform if and only if $\Delta\varphi \in \Omega^3_1$.*

Proof. The only if part is trivial. For the other direction, note that if $\Delta\varphi = f\varphi$ and $d\varphi = 0$, we must have $d(f\varphi) = 0$ since the exterior differential d commutes with the Laplacian. Thus we have

$$df \wedge \varphi = 0.$$

Since it is well-known that wedge product with the G_2-structure φ is injective on 1-forms, the equation above implies that $df = 0$, or f is constant on M. Hence φ must be an eigenform. □

Although we have the proposition above, it is not straightforward to find closed eigenforms. However, there are plenty of eigenforms that are not closed. We recall the following

Definition 2. A G_2-structure is called nearly parallel if its only nonzero torsion form is τ_0. We say that a 7-manifold is a nearly G_2 manifold if it admits a nearly parallel G_2-structure.

Thus (M, φ) is a nearly G_2 manifold if and only if $\delta\varphi = 0$ and $d\varphi = \tau_0 * \varphi$. We want to point out that in this case, τ_0 is necessarily a constant because $0 = d(d\varphi) = d\tau_0 \wedge *\varphi$ implies so. The squashed 7-sphere is an example of a nearly G_2 manifold. By straightforward computation, we immediately see that:

Proposition 3. *If (M, φ) is a nearly G_2 manifold, then φ is an eigenform satisfying*

$$\Delta\varphi = \tau_0^2\varphi. \tag{12}$$

In particular, φ is a shrinking soliton.

Thus we see that any nearly G_2 manifold admits a shrinking soliton.

We will end this section with some remarks on the case when a G_2-structure φ is closed. For a smooth family f_t of diffeomorphisms of M, if φ is closed then the elementary calculation

$$\begin{aligned}\frac{d}{dt}f_t^*\varphi &= f_t^*\left(L_{X_t}\varphi\right)\\ &= f_t^* d(X_t \lrcorner \varphi) = d f_t^*(X_t \lrcorner \varphi)\end{aligned} \tag{13}$$

shows that $f_t^*\varphi$ stays within the cohomology class of $f_0^*\varphi$ (φ is closed, hence $f_0^*\varphi$ is too)[6] for all t. Therefore, a closed[7] steady soliton solution remains in the original cohomology class of $f_0^*\varphi$ (normalizing the scaling factor to be 1). On the other hand, for closed expanding and shrinking solitons, we see that

$$\rho\varphi = -d(X \lrcorner \varphi + \delta\varphi). \tag{14}$$

In other words, the G_2-structure φ must be exact.[8]

6. Rigidity of Laplacian solitons

On a compact 7-manifold, if a G_2-structure φ satisfies the soliton equation (3) for some X and ρ, we may ask whether there are other vector fields X' and constants ρ' with which φ is also a soliton. We already saw at the end of Section 4 that we must have $\rho' = \rho$. On the other hand, if $-\Delta\varphi = L_{X'}\varphi + \rho\varphi$ for some other vector field X', then subtracting it from the original soliton equation gives

$$L_{X-X'}\varphi = 0.$$

In other words, for any compact soliton φ, the only symmetries of its defining Eq. (3) are $X \longrightarrow X + Y$ for vector fields such that $L_Y\varphi = 0$. With the G_2-structure fixed, this is the only change to a soliton equation that leaves it invariant. In general, a vector field X such that $L_X\varphi = 0$ is simply called a *symmetry* of the G_2-structure φ. One can go further in revealing the properties of these vector fields, in fact we have the following result.

Proposition 4. *On a compact 7-manifold M admitting a G_2-structure φ, any symmetry X of φ must satisfy* $\operatorname{div}(X) = 0$.

Proof. Again by Lemma 2, if $L_X\varphi = 0$, then

$$\begin{aligned}0 &= \int_M L_X\varphi \wedge *f\varphi = -3\int_M df \wedge *X^\flat\\ &= -3\int_M f * \delta X^\flat\end{aligned} \tag{15}$$

for all $f \in C^\infty(M)$. This implies that $\delta X^\flat = 0$, or $\operatorname{div}(X) = 0$. □

Although Proposition 4 is a nice application of Lemma 2 on compact 7-manifolds, one could also argue that it has to be true even if the 7-manifold is not compact, since the metric g_φ is completely determined by φ and thus we must also have $2\operatorname{div}(X) = L_X g_\varphi = 0$. However, a direct computational proof of this more general result is not simple, as g_φ depends on φ in a very complicated way.[9]

We want to understand the full structure of the space of symmetries for solitons in general. However, for now we will prove a partial result below, but which has immediate significance.

Lemma 3. *If a G_2-structure φ is closed, then $X \lrcorner \varphi$ is harmonic for any symmetry X of φ. Moreover, if M is compact and φ is torsion-free then $X^\flat$ is a harmonic 1-form, and in particular if g_φ has full G_2 holonomy then $X = 0$.*

Proof. If $d\varphi = 0$, then we have

$$L_X\varphi = d(X \lrcorner \varphi) = 0.$$

---

[6] Clearly, a soliton solution remains closed for all t if φ is closed. On the other hand, in general one can show from (1) that it preserves the closedness condition of the initial 3-form $\varphi(0)$.

[7] This is not to be confused with *compact* solitons, which refers to the topology of the underlying manifold.

[8] If M is compact, this immediately precludes a closed shrinking soliton from being torsion-free. See the next section for a more general derivation of this fact.

[9] See for example, [2].

Then using the fact that $X \lrcorner \varphi \in \Omega^2_7$,

$$\begin{aligned} d * (X \lrcorner \varphi) &= -\frac{1}{2} d\big[\varphi \wedge (X \lrcorner \varphi)\big) \\ &= -\frac{1}{2}\big(d\varphi \wedge (X \lrcorner \varphi) - \varphi \wedge d(X \lrcorner \varphi)\big] \\ &= 0. \end{aligned} \tag{16}$$

Thus $X \lrcorner \varphi \in \Omega^2_7$ is harmonic.[10]

Now, if φ is torsion-free then we further have

$$\begin{aligned} 0 &= d * (X \lrcorner \varphi) \\ &= d(X^\flat \wedge *\varphi) \\ &= dX^\flat \wedge *\varphi. \end{aligned}$$

This shows that $dX^\flat \in \Omega^2_{14}$. Then using the other characterization of Ω^2_{14} we see that

$$\begin{aligned} d\big(X^\flat \wedge \varphi \wedge dX^\flat\big) &= dX^\flat \wedge \varphi \wedge dX^\flat \\ &= dX^\flat \wedge *dX^\flat. \end{aligned}$$

Integrating both sides above and using Stoke's theorem, we see that $dX^\flat = 0$ necessarily. By Proposition 4, we conclude that $X^\flat$ is harmonic. If in addition we have full G_2 holonomy, then we know that $H^1(M, \mathbb{R}) = 0$[11] and so $X^\flat = 0$. □

Corollary 3. *Let φ be a torsion-free G_2-structure on a compact 7-manifold M. Then a vector field X is a symmetry of φ if and only if $X^\flat$ is a harmonic 1-form. Thus the space of symmetries of φ is isomorphic to $H^1(M, \mathbb{R})$.*

Proof. The only if part is given by Lemma 3. To prove the if part, we will employ the general relation below for the Levi-Civita connection:

$$L_X\varphi(Y_1, Y_2, Y_3) = \nabla_X\varphi\,(Y_1, Y_2, Y_3) + \varphi(\nabla_{Y_1}X, Y_2, Y_3) + \varphi(Y_1, \nabla_{Y_2}X, Y_3) + \varphi(Y_1, Y_2, \nabla_{Y_3}X) \tag{17}$$

for any vector fields X, Y_1, Y_2, Y_3. The torsion-free condition is defined by $\nabla\varphi = 0$. Furthermore, since any G_2 manifold must have zero Ricci curvature everywhere, by the Bochner's Theorem we know that any harmonic 1-form must be parallel. Then we must have $\nabla X = 0$. Combining these facts into (17) we get the desired result. □

We would like to point out that Corollary 3 is actually well-known. In particular, the isomorphism in Corollary 3 also follows from the general representation theory of G-structures presented in the book [8].

Corollary 3 contains the special case stated at the end of Lemma 3. In general, we know that for a compact G_2 manifold the following holds[12]:

$$\begin{aligned} &Hol(M) = \{1\} \Longrightarrow b_1(M) = 7 \\ &Hol(M) = SU(2) \Longrightarrow b_1(M) = 3 \\ &Hol(M) = SU(3) \Longrightarrow b_1(M) = 1 \\ &Hol(M) = G_2 \Longrightarrow b_1(M) = 0. \end{aligned} \tag{18}$$

Thus Corollary 3 shows that for a torsion-free G_2-structure, increasing the holonomy will decrease its symmetries—making it more and more "rigid". In particular, the G_2-structure of a G_2-manifold with full G_2 holonomy admits no non-trivial symmetries, hence it is "rigid". This is a rigidity result in stark contrast to that for compact manifolds with positive Ricci curvature, whose $b_1(M) = 0$ but yet typically admits a lot of Killing vector fields.

For G_2-manifolds that do not have full G_2 holonomy, the list (18) shows that there are non-trivial symmetries X. In other words, such X generates a one-parameter family f_t of diffeomorphisms homotopic to the identity such that $f_t^*\varphi = \varphi$ for all t. The existence of such symmetries is most visible in the following known examples:

1. $\mathbb{T} \times Y$, with $\varphi = dx \wedge \omega + \mathrm{Re}\,\theta$, where Y is a Calabi–Yau 3-fold with Kähler form ω and holomorphic volume form θ. The holonomy group is $SU(3)$.
2. $\mathbb{T}^3 \times Y$, with $\varphi = dx_{123} + dx_1 \wedge \omega + dx_2 \wedge \mathrm{Re}\,\theta - dx_3 \wedge \mathrm{Im}\,\theta$, where Y is a Calabi–Yau 2-fold with Kähler form ω and holomorphic volume form θ. The holonomy group is $SU(2)$.
3. $\mathbb{T}^7$, inheriting the standard G_2-structure on $\mathbb{R}^7$. The holonomy group is $\{1\}$.

We would like to point-out that Corollaries 1 and 3 together say that a compact steady soliton (φ, X) consists of a torsion-free G_2-structure φ and a symmetry of φ. As mentioned in the Introduction, it would be desirable to prove an analogous result to Corollary 3 for compact shrinking solitons. However, it seems that there are some technical difficulties.

10 Note that the computations in (16) show that if φ is closed, then $(X \lrcorner \varphi)$ being closed implies $(X \lrcorner \varphi)$ is coclosed as well, for any vector field X.

11 For example, see [8].

12 See [8].

7. Concluding remarks and questions

In this paper we have discussed fundamental properties of Laplacian solitons on manifolds admitting a G_2-structure. In particular, we have investigated solitons on compact 7-manifolds in some detail. Due to the recent flurry of interest in Ricci solitons, we wish to make some comparisons to it. Recall that a Ricci soliton is a metric g along with a vector field X on a manifold M such that

$$\mathrm{Ric}_g = L_X g + \rho g \tag{19}$$

for some constant ρ. The sign on ρ dictates the terminologies of shrinking, steady, and expanding solitons in the same way as for our Laplacian solitons. Moreover, when M is compact tracing and then integrating both sides of (19) yields

$$\rho = \frac{1}{n\,\mathrm{Vol}(M)} \int_M R_g \, dv_g,$$

where R_g is the scalar curvature and n is the dimension of M. Note that this is an analogous formula to (9), in particular since $\int \varphi \wedge *\varphi$ is also 7 times the volume.

When M is compact, we know that the only steady and expanding Ricci solitons are Einstein:

$$\mathrm{Ric}_g = \rho g,$$

i.e. X must be a Killing vector field.[13] Corollary 1 can be seen as a close parallel to this result. In fact, by the works of Richard Hamilton and Thomas Ivey (see for example, [11]) we also know that compact shrinking Ricci solitons in dimensions 2 and 3 have to also be Einstein. As mentioned in Section 5, it is clear now that eigenforms $-\Delta\varphi = \rho\varphi$ should be viewed as an analogy to Einstein metrics in the context of solitons. It is therefore natural to question whether or not compact shrinking Laplacian solitons also have to be eigenforms. If this conjecture is true, then we would be able to finish the classification of compact Laplacian solitons if we further classify all compact eigenforms.[14] The present difficulty seems to be due to a lack of maximum principle-type techniques in G_2 geometry, since such techniques were essential in proving results in Ricci solitons (or the Ricci flow in general).

Using his entropy functionals, when M is compact G. Perelman (see [12]) showed that X can always be chosen as a gradient vector field in Eq. (19), i.e. compact Ricci solitons are always *gradient* solitons. It would be interesting to know if compact Laplacian solitons are always gradient as well, in the same sense. The entropy functionals of Perelman has a deeper implication: they turn the Ricci flow into a gradient-like[15] flow with respect to the functionals. In particular, compact Ricci solitons appear as critical points of these functionals, modulo diffeomorphism and scaling in the space of metrics. Functionals associated to the Laplacian flow have been suggested in [13,7], and it would be nice to see if they (or possibly other functionals) can also be used to characterize compact Laplacian solitons as critical points.

Next, we would like to make some conjectures for an analogous result to Corollary 3 if φ is a compact shrinking Laplacian soliton. A reasonable conjecture for this remains elusive to us at this point, but we can consider the simpler case of eigenforms $-\Delta\varphi = \rho\varphi$ ($\rho < 0$). In view of Corollary 3, it is tantalizing to conjecture that the space of symmetries of an eigenform is isomorphic to the eigenspace $\{\omega \in \Omega^1(M) \mid \Delta\omega = \lambda\omega\}$ for some constant λ.[16] On the other hand, recall that on a compact manifold M of nonpositive Ricci curvature any Killing vector field is parallel, and if the Ricci curvature is negative there are no nontrivial Killing vector fields. Combined with Bochner's Theorem, this means that on a compact Ricci-flat manifold the space of Killing vector fields is isomorphic to $H^1(M, \mathbb{R})$. Note that this is perfectly analogous to the isomorphism in Corollary 3 since we are making an analogy between eigenforms and Einstein metrics (so $-\Delta\varphi = 0 \longleftrightarrow \mathrm{Ric}_g = 0$), and if we also make the analogy between symmetries of a G_2-structure and Killing vector fields of a metric. Now, there is a slight imperfection in the analogy between eigenforms and Einstein metrics when the constant ρ is not zero, since the Einstein constant ρ can have any sign while the sign of the eigenvalue ρ for an eigenform is essentially fixed (up to the artificial choice of the sign in front of the Hodge Laplacian Δ). Based on this observation and the well-known result on Killing vector fields and Ricci curvature mentioned above, we think it is also a plausible conjecture that for a shrinking Laplacian soliton the space of symmetries is in fact just $\{0\}$.

Finally, we would like to remark that our comparisons here with Ricci solitons reflect a more fundamental tenant: the study of the nonlinear operator $\Delta_\varphi\varphi$ on G_2-structures should be seen as analogous to the study of the nonlinear operator Ric_g on Riemannian metrics. We also think identifying the symmetries of solitons will be important, because one would like to construct the associated *moduli spaces* by dividing out the symmetries.[17]

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13 See [10], which is also an excellent survey of the subject of Ricci flow.

14 A relevant question here would be: are the only compact eigenforms nearly parallel G_2-structures?

15 Coupled to other equations.

16 λ is likely to be related to the constant ρ that appears in the original equation (3).

17 A discussion on such moduli spaces have also appeared in [6].

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