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The index theory on non-compact manifolds with proper group action

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ABSTRACT

We construct a regularized index of a generalized Dirac operator on a complete Riemannian manifold endowed with a proper action of a unimodular Lie group. We show that the index is preserved by a certain class of non-compact cobordisms and prove a gluing formula for the regularized index. The results of this paper generalize our previous construction of index for compact group action and the recent paper of Hochs and Mathai who studied the case of a Hamiltonian action on a symplectic manifold. As an application of the cobordism invariance of the index we give an affirmative answer to a question of Hochs and Mathai about the independence of the Hochs–Mathai quantization of the metric, connection and other choices.

1. Introduction

Paradan [1] introduces a regularized topological index of a Dirac-type operator on a non-compact manifold M endowed with an action of a compact group G. In [2] (see also [3] for a review) we constructed an analytic counterpart of this index and proved that the two indexes coincide (see also [4]). The regularized index depends on an additional data, namely an equivariant map from M to the Lie algebra of G, called the *taming map*. The regularized index was used in [5] as a method to prove a conjecture of Vergne [6]. The method of [2] was also used in [7] to construct a regularized Dolbeault cohomology of a non-compact G-manifold.

Mathai and Zhang [8] defined an index of a Dirac operator on a manifold endowed with a proper cocompact action of a non-compact group. Hochs and Mathai [9] considered a Hamiltonian action of a non-compact group G on a non-compact symplectic manifold M. They constructed a regularized index of the spin^c-Dirac operator without assuming that the action is cocompact and showed that this index has the Guillemin–Sternberg "quantization commutes with reduction" property. The regularization of the index used in [9] is very similar to the one introduced in [2], with the moment map playing the role of the taming map. In this sense, the Hochs–Mathai construction can be viewed as a combination of ideas from [8] and [2].

In this note we combine the methods of [2] and [9] to construct an analytical index of a generalized Dirac operator on a non-compact manifold endowed with a proper action of a non-compact Lie group. In the case of a Hamiltonian group action on a symplectic manifold our index coincides with the construction of [9]. We show that our index is invariant under a certain class of non-compact cobordisms. We also prove a gluing formula for this index. From the cobordism invariance of the index we immediately conclude that the index is independent of the metric, the connection and other data used in its definition. That gives an affirmative answer to a question posed by Hochs and Mathai, cf. Remark 3.8 and 6.2 of [9].

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1.1. The construction of the index

Suppose M is a complete Riemannian manifold on which a non-compact unimodular Lie group G acts by isometries. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a G-equivariant $\mathbb{Z}_2$-graded self-adjoint Clifford module over M. We refer to the pair $(\mathcal{E}, \mathbf{v})$ as a *tamed Clifford module*.

Consider a Dirac operator $D^\pm : \Gamma(M, \mathcal{E}^\pm) \to \Gamma(M, \mathcal{E}^\mp)$ associated to a Clifford connection on $\mathcal{E}$.

Following [8,9] we consider a smooth *cutoff function* $\chi : M \to [0, \infty)$, whose support intersects all G-orbits in compact sets and which satisfies $\int_G \chi(g \cdot x)^2\, dg = 1$. Since the group G is unimodular, such a function always exists by [10, Ch. VII, Section 2.4].

A section $s \in \Gamma(M, \mathcal{E})$ is called *transversally compactly supported* if the support of s is cocompact. We denote by $\Gamma^\infty_{tc}(M, \mathcal{E})$ the space of smooth transversally compactly supported sections of $\mathcal{E}$ and by $\Gamma^\infty_{tc}(M, \mathcal{E})^G$ the subspace of G-invariant elements of $\Gamma^\infty_{tc}(M, \mathcal{E})$. Consider the operator

$$D_\chi : \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G \to \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G, \qquad D_\chi(\chi s) := \chi\, Ds, \quad \text{for } s \in \Gamma^\infty_{tc}(M, \mathcal{E}).$$

It is shown in [8] that if the quotient space M/G is compact than the operator D_χ is Fredholm and its index is independent of the choice of the function χ. Thus one can define the regularized index of D by $\mathrm{ind}_G D := \mathrm{ind}\, D_\chi$ for any cutoff function χ. If M/G is not compact then D_χ is not necessarily Fredholm and a regularization is needed to define its index.

Let $\mathbf{v} : M \to \mathfrak{g} = \mathrm{Lie}\, G$ be a G-equivariant map, such that the induced vector field v on M does not vanish outside of a cocompact subset of M. We call $\mathbf{v}$ a *taming* map, and we refer to the pair $(\mathcal{E}, \mathbf{v})$ as a *tamed Clifford module*.

Let $f : M \to [0, \infty)$ be a G-invariant function which increases fast enough at infinity (see Section 2.10 for the precise condition on f). We consider the *deformed Dirac operator*

$$D_{\chi, fv} := D_\chi + \sqrt{-1}\, c(fv) : \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G \to \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G,$$

where $c : TM \simeq T^*M \to \mathrm{End}\, \mathcal{E}$ is the Clifford module structure on $\mathcal{E}$. Our principal result is Theorem 2.15, which states that the deformed Dirac operator is Fredholm and its index $\mathrm{ind}_G D_{\chi, fv}$ is independent of the choice of the functions χ, f and of the Clifford connection on $\mathcal{E}$, used in the definition of D. We denote this index by $\mathrm{ind}_G(\mathcal{E}, \mathbf{v})$ and call it the *(analytic) index of* $(\mathcal{E}, \mathbf{v})$.

1.2. The cobordism invariance

In Section 3, we introduce the notion of a cobordism between tamed Clifford modules. Roughly speaking, this is a usual cobordism, which carries a taming map. Our notion of cobordism is very close to the notion of non-compact cobordism developed by V. Ginzburg, V. Guillemin and Y. Karshon [11–13]. We prove that *the index is preserved by a cobordism*.

1.3. The gluing formula

Suppose $\Sigma \subset M$ is a cocompact G-invariant hypersurface, such that the vector field v does not vanish anywhere on Σ. We endow the open manifold $M \setminus \Sigma$ with a complete Riemannian metric and we denote by $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ the induced tamed Clifford module on $M \setminus \Sigma$. In Section 4, we prove that *the tamed Clifford modules $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ and $(\mathcal{E}, \mathbf{v})$ are cobordant. In particular, they have the same index*. We refer to this result as the *gluing formula*.

The gluing formula takes especially nice form if Σ divides M into 2 disjoint manifolds M_1 and M_2. Let $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ be the restrictions of $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ to M_1 and M_2, respectively. Then the gluing formula implies

$$\mathrm{ind}_G(\mathcal{E}, \mathbf{v}) = \mathrm{ind}_G(\mathcal{E}_1, \mathbf{v}_1) + \mathrm{ind}_G(\mathcal{E}_2, \mathbf{v}_2).$$

In other words, *the index is additive*.

1.4. $\mathrm{Spin}^{\mathbb{C}}$-reduction and a topological formula for the index

After the first version of this paper has been released, Hochs and Mathai, [14], proved a version of "quantization commutes with reduction" for $\mathrm{ind}_G(\mathcal{E}, \mathbf{v})$ in the following interesting situation.

Consider a G-equivariant $\mathrm{Spin}^{\mathbb{C}}$-structure on M. Let L be the determinant line bundle of this $\mathrm{Spin}^{\mathbb{C}}$-structure. We also assume that the taming map $\mathbf{v}$ is constructed using the $\mathrm{Spin}^{\mathbb{C}}$-connection, as in section 6.2 of [14]. Hochs and Mathai showed that for large enough integer p, the index $\mathrm{ind}_G(\mathcal{E} \otimes L^p, \mathbf{v})$ is equal to the index of the Dirac operator on the $\mathrm{Spin}^{\mathbb{C}}$*-reduction* M_0 of M, see [15] and section 3 of [14] for the definition of $\mathrm{Spin}^{\mathbb{C}}$-reduction.

Since, by our assumptions, the $\mathrm{Spin}^{\mathbb{C}}$-reduction M_0 is a compact manifold, the Atiyah–Singer index theorem gives a topological formula for the index Dirac operator on M_0. Hence, the Hochs–Mathai "quantization commutes with reduction" result gives, in particular, a nice topological formula for $\mathrm{ind}_G(\mathcal{E} \otimes L^p, \mathbf{v})$, cf. formula (1.2) of [14].

2. Equivariant index for a unimodular group action on non-compact manifolds

In this section we introduce our main objects of study: tamed non-compact manifolds, tamed Clifford modules, and the (analytic) equivariant index of such modules.

Throughout the paper (M, g^M) is a complete Riemannian manifold without boundary.

2.1. Clifford module and Dirac operator

Let $C(M)$ denote the Clifford bundle of M (cf. [16, Section 3.3]), i.e., a vector bundle, whose fiber at every point $x \in M$ is isomorphic to the Clifford algebra $C(T^*_xM)$ of the cotangent space.

A *($\mathbb{Z}_2$-graded self-adjoint) Clifford module* on M is a $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ over M endowed with a graded action

$$(a, s) \mapsto c(a)s, \quad \text{where } a \in \Gamma(M, C(M)),\ s \in \Gamma(M, \mathcal{E}),$$

of the bundle $C(M)$ such that the operator $c(v) : \mathcal{E}_x \to \mathcal{E}_x$ is skew-adjoint, for all $x \in M$ and $v \in T^*_xM$.

A *Clifford connection* on $\mathcal{E}$ is a Hermitian connection $\nabla^{\mathcal{E}}$, which preserves the subbundles $\mathcal{E}^\pm$ and

$$[\nabla^{\mathcal{E}}_X, c(a)] = c(\nabla^{\mathrm{LC}}_X a), \quad \text{for any } a \in \Gamma(M, C(M)),\ X \in \Gamma(M, TM),$$

where ∇^{LC}_X is the Levi-Civita covariant derivative on $C(M)$ associated with the Riemannian metric on M.

The *Dirac operator* $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E})$ associated to a Clifford connection $\nabla^{\mathcal{E}}$ is defined by the following composition:

$$\Gamma(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \Gamma(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} \Gamma(M, \mathcal{E}).$$

In local coordinates, this operator may be written as $D = \sum c(dx^i)\, \nabla^{\mathcal{E}}_{\partial_i}$. Note that D sends even sections to odd sections and vice versa: $D : \Gamma(M, \mathcal{E}^\pm) \to \Gamma(M, \mathcal{E}^\mp)$.

Consider the L^2-scalar product on the space of sections $\Gamma(M, \mathcal{E})$ defined by the Riemannian metric on M and the Hermitian structure on $\mathcal{E}$. By [16, Proposition 3.44], the Dirac operator associated to a Clifford connection $\nabla^{\mathcal{E}}$ is formally self-adjoint with respect to this scalar product. Moreover, it is essentially self-adjoint with the initial domain smooth, compactly supported sections, cf. [17], [18, Th. 1.17].

If W is a manifold with boundary, then by a Clifford module over it we will understand a smooth vector bundle $\mathcal{E}$ over W, whose restriction to the set of the interior points W^{int} of W has a structure of a Clifford module over W^{int}. (Usually we require some additional structure of $\mathcal{E}$ near the boundary of W, but we will formulate these requirements when we need them.)

2.2. Group action. The index.

Let G be a unimodular Lie group and suppose that there is a proper action of G on M by isometries. Assume that there is given a lift of this action to $\mathcal{E}$, which preserves the grading, the connection and the Hermitian metric on $\mathcal{E}$. Then the Dirac operator D commutes with the action of G. Hence, $\mathrm{Ker}D = \mathrm{Ker}D^+ \oplus \mathrm{Ker}D^-$ is a G-invariant subspace of $\Gamma(M, \mathcal{E})$. Let

$$(\mathrm{Ker}D^\pm)^G \subset \mathrm{Ker}D^\pm$$

denote the space of G-invariant elements of $\mathrm{Ker}D^\pm$.

If M is compact, then the spaces $\mathrm{Ker}D^\pm$ and, hence, $(\mathrm{Ker}D^\pm)^G$ are finite dimensional. In this situation we define an *equivariant index* of D by

$$\mathrm{ind}_G(D) = \dim(\mathrm{Ker}D^+)^G - \dim(\mathrm{Ker}D^-)^G. \tag{2.1}$$

The index depends only on M and the equivariant Clifford module $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ and does not depend on the choice of the connection $\nabla^{\mathcal{E}}$ and the metric $h^{\mathcal{E}}$. We set

$$\mathrm{ind}_G(\mathcal{E}) := \mathrm{ind}_G(D),$$

and refer to it as the *index* of $\mathcal{E}$.

Our goal is to define an analogue of (2.1) for a G-equivariant Clifford module over a complete *non-compact* manifold. If the group G is compact, then this was done on [2]. If the group G is not compact, but M is a symplectic manifold and the action of G on M is Hamiltonian, the index was constructed in [9]. We now combine the ideas of [2] and [9] to construct the index for a general complete G-manifold.

2.3. Cutoff function along the orbits

Let dg denote the Haar measure on G. Following [8,9] we make the following definition.

Definition 2.4. A smooth function $\chi : M \to [0, \infty)$, whose support intersects all G-orbits in compact sets and which satisfies

$$\int_G \chi(g \cdot x)^2\, dg = 1 \tag{2.2}$$

is called a *cutoff function* on M.

Note that a cutoff function always exists by [10, Ch. VII, §2.4].

Let M/G denote the space of G-orbits in M and let $q : M \to M/G$ denote the quotient map.

Definition 2.5. A subset $V \subset M$ is called *cocompact* if $q(V) \subset M/G$ is compact.

Definition 2.6. A section $s \in \Gamma(M, \mathcal{E})$ is called *transversally compactly supported* if the support of s is cocompact. We denote by $\Gamma^\infty_{tc}(M, \mathcal{E})$ the space of smooth transversally compactly supported sections of $\mathcal{E}$ and by $\Gamma^\infty_{tc}(M, \mathcal{E})^G$ the subspace of G-invariant elements in $\Gamma^\infty_{tc}(M, \mathcal{E})$.

Notice that if $s \in \Gamma^\infty_{tc}(M, \mathcal{E})$ then χs is a smooth compactly supported section of $\mathcal{E}$.
Define the operator

$$D_\chi : \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G \to \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G$$

by

$$D_\chi(\chi s) := \chi\, Ds, \quad s \in \Gamma^\infty_{tc}(M, \mathcal{E}). \tag{2.3}$$

It is shown in [8] that if the quotient space M/G is compact than the operator D_χ is Fredholm and its index is independent of the choice of the function χ. Thus one can define the regularized index of D by

$$\operatorname{ind}_G D := \operatorname{ind} D_\chi$$

for any cutoff function χ. As before, $\operatorname{ind}_G(D)$ depends only on M and the equivariant Clifford module $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ and we set $\operatorname{ind}_G(\mathcal{E}) := \operatorname{ind}_G(D)$.

2.7. A tamed non-compact manifold

When M/G is not compact to define the index we need an additional structure on M. This structure is given by an equivariant map $\mathbf{v} : M \to \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G and G acts on it by the adjoint representation. Such a map induces a vector field v on M defined by

$$v(x) := \frac{d}{dt}\Big|_{t=0} \exp(t\mathbf{v}(x)) \cdot x. \tag{2.4}$$

The following definition extends Definition 2.4 of [2].

Definition 2.8. Let M be a complete G-manifold. A *taming map* is a G-equivariant map $\mathbf{v} : M \to \mathfrak{g}$, such that the vector field v on M, defined by (2.4), does not vanish anywhere outside of a cocompact subset of M. If $\mathbf{v}$ is a taming map, we refer to the pair $(M, \mathbf{v})$ as a *tamed G-manifold.*

If, in addition, $\mathcal{E}$ is a G-equivariant $\mathbb{Z}_2$-graded self-adjoint Clifford module over M, we refer to the pair $(\mathcal{E}, \mathbf{v})$ as a *tamed Clifford module* over M.

The index we are going to define depends on the (equivalence class) of $\mathbf{v}$.

Remark 2.9. Suppose M is a symplectic manifold and that the action of G on M is Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$. Following Hochs and Mathai [9] we introduce a family of scalar products $\langle\cdot,\cdot\rangle_x$ $(x \in M)$ on $\mathfrak{g}^*$ which is G-invariant in the sense that

$$\langle \operatorname{Ad}_g(\xi), \operatorname{Ad}_g(\eta)\rangle_{g\cdot x} = \langle \xi, \eta\rangle_x, \quad \text{for all } x \in M,\ \xi, \eta \in \mathfrak{g}^*.$$

For $x \in M$ set $\mathcal{H}(x) := \langle \mu(x), \mu(x)\rangle_x$ and let $\mathbf{v}(x) \in \mathfrak{g}$ denote the dual of $\mu(x)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_x$. Then the vector field (2.4) is equal to one half of the vector field $X_1^{\mathcal{H}}$ introduced in [9, Section 2]. Thus the construction of the index below generalizes the construction of Hochs and Mathai to manifolds which are not symplectic.

2.10. A rescaling of v

Our definition of the index uses certain rescaling of the vector field v. By this we mean the product $f(x)v(x)$, where $f : M \to [0, \infty)$ is a smooth positive function. Roughly speaking, we demand that $f(x)v(x)$ tends to infinity "fast enough" when $q(x) \subset M/G$ tends to infinity. The precise conditions we impose on f are quite technical, cf. Definition 2.12. Luckily, our index turns out to be independent of the concrete choice of f. It is important, however, to know that at least one admissible function exists. This is guaranteed by Lemma 2.13 below.

We need to introduce some additional notations.

For a vector $\mathbf{u} \in \mathfrak{g}$, we denote by $\mathcal{L}^{\mathcal{E}}_{\mathbf{u}}$ the infinitesimal action of $\mathbf{u}$ on $\Gamma(M, \mathcal{E})$ induced by the action on G on $\mathcal{E}$. Let $\nabla^{\mathcal{E}}_u : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E})$ denote the covariant derivative along the vector field u induced by $\mathbf{u}$. The difference between those two operators is a bundle map, which we denote by

$$\mu^{\mathcal{E}}(\mathbf{u}) := \nabla^{\mathcal{E}}_u - \mathcal{L}^{\mathcal{E}}_{\mathbf{u}} \in \operatorname{End} \mathcal{E}. \tag{2.5}$$

We will use the same notation $|\cdot|$ for the norms on the bundles TM, T^*M, $\mathcal{E}$. Let End (TM) and End $(\mathcal{E})$ denote the bundles of endomorphisms of TM and $\mathcal{E}$, respectively. We will denote by $\|\cdot\|$ the norms on these bundles induced by $|\cdot|$. Let χ be a cutoff function as in Section 2.3. To simplify the notation, set

$$\nu = |\mathbf{v}| + \|\nabla^{\mathrm{LC}} v\| + \|\mu^{\mathcal{E}}(\mathbf{v})\| + |v| + 1. \tag{2.6}$$

Definition 2.11. We say that a function $h : M \to [0, \infty)$ tends to infinity as $q(x) \to \infty$ and write $\lim_{q(x)\to\infty} h(x) = \infty$ if for arbitrary large number $R > 0$ there exists a compact set $K \Subset M/G$ such that for all $x \notin q^{-1}(K)$ we have $h(x) > R$.

Definition 2.12. We say that a smooth G-invariant function $f : M \to [0, \infty)$ on a tamed G-manifold $(M, \mathbf{v})$ is *admissible* for $(\mathcal{E}, \mathbf{v}, \nabla^{\mathcal{E}})$ if

$$\lim_{q(x)\to\infty} \frac{f^2|v|^2}{|df||v| + f\nu + 1} = \infty. \tag{2.7}$$

Lemma 2.13. *Let $(\mathcal{E}, \mathbf{v})$ be a tamed Clifford module and let $\nabla^{\mathcal{E}}$ be a G-invariant Clifford connection on $\mathcal{E}$. Let $\chi : M \to [0, \infty)$ be a cutoff function. Then there exists an admissible function f for $(\mathcal{E}, \mathbf{v}, \nabla^{\mathcal{E}}, \chi)$.*

Proof. The proof for the case when the group G is compact given in [2, Section 8] does not use the compactness of G and works in our new situation. □

2.14. Index on non-compact manifolds

We use the Riemannian metric on M, to identify the tangent and the cotangent bundles of M. In particular, we consider v as a section of T^*M.

Let f be an admissible function and let D_χ be the operator defined in (2.3). Consider the *deformed Dirac operator*

$$D_{\chi,fv} = D_\chi + \sqrt{-1}\, c(fv) : \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G \to \chi\, \Gamma^\infty_{tc}(M, \mathcal{E})^G. \tag{2.8}$$

This operator is essentially self-adjoint, cf. the remark on page 411 of [17].

Our first result is the following analogue of Theorem 2.9 from [2].

Theorem 2.15. *Suppose f is an admissible function. Then*

1. *The kernel of the deformed Dirac operator $D_{\chi,fv}$ has finite dimension.*
2. *The index*

$$\mathrm{ind}_G D_{\chi,fv} := \dim \mathrm{Ker} D^+_{\chi,fv} - \dim \mathrm{Ker} D^-_{\chi,fv} \tag{2.9}$$

is independent of the choices of the cutoff function χ, the admissible function f, and the G-invariant Clifford connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$.

The proof of the first part of the theorem is given in Section 6. The second part of the theorem is proven in Section 3.7 as an immediate consequence of Theorem 3.6 about cobordism invariance of the index.

We refer to the pair $(D, \mathbf{v})$ as a *tamed Dirac operator*. The above theorem allows us to defined the index of a tamed Dirac operator $\mathrm{ind}_G(D, \mathbf{v}) := \mathrm{ind}_G(D_{\chi,fv})$. Since $\mathrm{ind}_G(D, \mathbf{v})$ is independent of the choice of the connection on $\mathcal{E}$, it is an invariant of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$. We set $\mathrm{ind}_G(\mathcal{E}, \mathbf{v}) := \mathrm{ind}_G(D, \mathbf{v})$ and refer to it as the *(analytic) index of a tamed Clifford module* $(\mathcal{E}, \mathbf{v})$.

3. Cobordism invariance of the index

In this section we adopt the notion of cobordism between tamed Clifford modules and tamed Dirac operators introduced in [2] to the case of a non-compact group G. We show that the index introduced in Section 2.14 is invariant under a cobordism. We use this result to prove Theorem 2.15.1.

3.1. Cobordism between tamed G-manifolds

Note, first, that for cobordism to be meaningful one must make some compactness assumption. Otherwise, every manifold is cobordant to the empty set via the noncompact cobordism $M \times [0, 1)$. Since our manifolds are non-compact themselves, we cannot demand cobordism to be compact. Instead, we demand the cobordism to carry a taming map to $\mathfrak{g}$.

Definition 3.2. A *cobordism* between tamed G-manifolds $(M_1, \mathbf{v}_1)$ and $(M_2, \mathbf{v}_2)$ is a triple $(W, \mathbf{v}, \phi)$, where

(i) W is a complete Riemannian G-manifold with boundary;
(ii) $\mathbf{v} : W \to \mathfrak{g}$ is a smooth G-invariant map, such that the corresponding vector field v does not vanish anywhere outside of a cocompact subset of W;
(iii) ϕ is a G-equivariant, metric preserving diffeomorphism between a neighborhood U of the boundary ∂W of W and the disjoint union $\big(M_1 \times [0,\varepsilon)\big) \bigsqcup \big(M_2 \times (-\varepsilon, 0]\big)$. We will refer to U as the *neck* and we will identify it with $\big(M_1 \times [0,\varepsilon)\big) \bigsqcup \big(M_2 \times (-\varepsilon, 0]\big)$.
(iv) the restriction of $\mathbf{v}\big(\phi^{-1}(x,t)\big)$ to $M_1 \times [0,\varepsilon)$ (resp. to $M_2 \times (-\varepsilon, 0]$) is equal to $\mathbf{v}_1(x)$ (resp. to $\mathbf{v}_2(x)$).

3.3. Cobordism between tamed Clifford modules

If M is a Riemannian G-manifold, then, for any interval $I \subset \mathbb{R}$, the product $M \times I$ carries natural Riemannian metric and G-action. Let $\pi : M \times I \to M$, $t : M \times I \to I$ denote the natural projections. We refer to the pull-back $\pi^*\mathcal{E}$ as a vector bundle *induced* by $\mathcal{E}$. We view t as a real valued function on M, and we denote by dt its differential.

Definition 3.4. Let $(M_1, \mathbf{v}_1)$ and $(M_2, \mathbf{v}_2)$ be tamed G-manifolds. Suppose that each M_i, $i = 1, 2$, is endowed with a G-equivariant self-adjoint Clifford module $\mathcal{E}_i = \mathcal{E}_i^+ \oplus \mathcal{E}_i^-$. A *cobordism* between the tamed Clifford modules $(\mathcal{E}_i, \mathbf{v}_i)$, $i = 1, 2$, is a cobordism $(W, \mathbf{v}, \phi)$ between $(M_i, \mathbf{v}_i)$ together with a pair $(\mathcal{E}_W, \psi)$, where

(i) $\mathcal{E}_W$ is a G-equivariant (non-graded) self-adjoint Clifford module over W;
(ii) ψ is a G-equivariant isometric isomorphism between the restriction of $\mathcal{E}_W$ to U and the Clifford module induced on the neck $\big(M_1 \times [0,\varepsilon)\big) \bigsqcup \big(M_2 \times (-\varepsilon, 0]\big)$ by $\mathcal{E}_i$.
(iii) On the neck U we have $c(dt)|_{\psi^{-1}\mathcal{E}_i^{\pm}} = \pm\sqrt{-1}$.

Remark 3.5. Let $\mathcal{E}_1^{\mathrm{op}}$ denote the Clifford module $\mathcal{E}_1$ with the opposite grading, i.e., $\mathcal{E}_1^{\mathrm{op}\pm} = \mathcal{E}_1^{\mp}$. Then, $\mathrm{ind}_G(\mathcal{E}_1, \mathbf{v}_1) = -\,\mathrm{ind}_G(\mathcal{E}_1^{\mathrm{op}}, \mathbf{v}_1)$.

Consider the Clifford module $\mathcal{E}$ over the disjoint union $M = M_1 \sqcup M_2$ induced by the Clifford modules $\mathcal{E}_1^{\mathrm{op}}$ and $\mathcal{E}_2$. Let $\mathbf{v} : M \to \mathfrak{g}$ be the map such that $\mathbf{v}|_{M_i} = \mathbf{v}_i$. A cobordism between $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ may be viewed as a cobordism between $(\mathcal{E}, \mathbf{v})$ and (the Clifford module over) the empty set.

One of the main results of this paper is the following theorem, which asserts that the index is preserved by a cobordism.

Theorem 3.6. *Suppose $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ are cobordant tamed Clifford modules. Let D_1, D_2 be Dirac operators associated to G-invariant Clifford connections on $\mathcal{E}_1$ and $\mathcal{E}_2$ and let χ_1, χ_2 be cutoff functions on M_1 and M_2. Then, for any admissible functions f_1, f_2*

$$\mathrm{ind}_G\big(D_{1,\chi_1} + \sqrt{-1}\,c(f_1 v_1)\big) = \mathrm{ind}_G\big(D_{2,\chi_2} + \sqrt{-1}\,c(f_2 v_2)\big).$$

The proof of the theorem is given in Section 7.

3.7. The definition of the analytic index of a tamed Clifford module

Theorem 3.6 implies, in particular, that, if $(\mathcal{E}, \mathbf{v})$ is a tamed Clifford module, then the index $\mathrm{ind}_G(D_{\chi, fv})$ is independent of the choice of the admissible function f, the cutoff function χ, and the Clifford connection on $\mathcal{E}$. This proves part 2 of Theorem 2.15 and (cf. Section 2.14) allows us to define the *(analytic) index* of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$

$$\mathrm{ind}_G(\mathcal{E}, \mathbf{v}) := \mathrm{ind}_G(D_{\chi, fv}), \quad f \text{ is an admissible function.}$$

Theorem 3.6 can be reformulated now as follows.

Theorem 3.8. *The indexes of cobordant tamed Clifford modules coincide.*

3.9. Index and zeros of v

As a simple corollary of Theorem 3.6, we obtain the following Lemma.

Lemma 3.10. *If the vector field $v(x) \neq 0$ for all $x \in M$, then $\mathrm{ind}_G(\mathcal{E}, \mathbf{v}) = 0$.*

Proof. Consider the product $W = M \times [0, \infty)$ and define the map $\tilde{\mathbf{v}} : W \to \mathfrak{g}$ by the formula: $\tilde{\mathbf{v}}(x,t) = \mathbf{v}(x)$. Clearly, $(W, \tilde{\mathbf{v}})$ is a cobordism between the tamed G-manifold M and the empty set. Let $\mathcal{E}_W$ be the lift of $\mathcal{E}$ to W. Define the Clifford module structure $c : T^*W \to \mathrm{End}\,\mathcal{E}_W$ by the formula

$$c(x,a)e = c(x)e \pm \sqrt{-1}\,ae, \quad (x,a) \in T^*W \simeq T^*M \oplus \mathbb{R},\ e \in \mathcal{E}_W^{\pm}.$$

Then $(\mathcal{E}_W, \tilde{\mathbf{v}})$ is a cobordism between $(\mathcal{E}, \mathbf{v})$ and the Clifford module over the empty set. $\square$

3.11. The excision property

We will now amplify the above lemma and show that the index is independent of the restriction of $(\mathcal{E}, \mathbf{v})$ to a subset, where $v \neq 0$.

Let $(M_i, \mathbf{v}_i)$ $i = 1, 2$, be tamed n-dimensional G-manifolds. Let U be an open n-dimensional G-manifold. For each $i = 1, 2$, let $\phi_i : U \to M_i$ be a smooth G-equivariant embedding. Set $U_i = \phi_i(U) \subset M_i$. Assume that the boundary $\Sigma_i = \partial U_i$ of U_i is a smooth hypersurface in M_i. Assume also that the vector field v_i induced by $\mathbf{v}_i$ on M_i does not vanish anywhere on $M_i \setminus U_i$.

Lemma 3.12. *Let $(\mathcal{E}_1, \mathbf{v}_1)$, $(\mathcal{E}_2, \mathbf{v}_2)$ be tamed Clifford modules over M_1 and M_2, respectively. Suppose that the pull-backs $\phi_i^*\mathcal{E}_i$, $i = 1, 2$ are G-equivariantly isomorphic as $\mathbb{Z}_2$-graded self-adjoint Clifford modules over U. Assume also that $\mathbf{v}_1 \circ \phi_1 \equiv \mathbf{v}_2 \circ \phi_2$. Then $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ are cobordant. In particular, $\operatorname{ind}_G(\mathcal{E}_1, \mathbf{v}_1) = \operatorname{ind}_G(\mathcal{E}_2, \mathbf{v}_2)$.*

Proof. An explicit cobordism between $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ is constructed in section 12.2 of [2]. □

The following lemma is, in a sense, opposite to Lemma 3.12.

Lemma 3.13. *Let $\mathbf{v}_1, \mathbf{v}_2 : M \to \mathfrak{g}$ be taming maps, which coincide out of a compact subset of M. Then the tamed Clifford modules $(\mathcal{E}, \mathbf{v}_1)$ and $(\mathcal{E}, \mathbf{v}_2)$ are cobordant. In particular, $\operatorname{ind}_G(\mathcal{E}, \mathbf{v}_1) = \operatorname{ind}_G(\mathcal{E}, \mathbf{v}_2)$.*

Proof. The proof is a verbatim repetition of the proof of Lemma 3.16 in [2]. □

4. The gluing formula

If we cut a tamed G-manifold along a G-invariant hypersurface Σ, we obtain a manifold with boundary. By rescaling the metric near the boundary we may convert it to a complete manifold without boundary, in fact, to a tamed G-manifold. In this section, we show that the index is invariant under this type of surgery. In particular, if Σ divides M into two pieces M_1 and M_2, we see that the index on M is equal to the sum of the indexes on M_1 and M_2. In other words, the index is *additive*.

4.1. The surgery

Let $(M, \mathbf{v})$ be a tamed G-manifold. Suppose $\Sigma \subset M$ is a smooth G-invariant hypersurface in M. For simplicity, we assume that Σ is cocompact. Assume also that the vector field v induced by $\mathbf{v}$ does not vanish anywhere on Σ. Suppose that $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a G-equivariant $\mathbb{Z}_2$-graded self-adjoint Clifford module over M. Denote by $\mathcal{E}_\Sigma$ the restriction of the $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{E}$ to $M_\Sigma := M \setminus \Sigma$.

Let g^M denote the Riemannian metric on M. In section 4.2 of [2] by a rescaling of g^M we constructed a complete Riemannian metric g^{M_Σ} on M_Σ and a Clifford action $c_\Sigma : T^*M_\Sigma \to \operatorname{End}(\mathcal{E}_\Sigma)$ compatible with this metric. It follows, from the cobordism invariance of the index (Lemma 3.12), that our index theory is essentially independent of the concrete choice of g^{M_Σ}.

Theorem 4.2. *The tamed Clifford modules $(\mathcal{E}, \mathbf{v})$ and $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ are cobordant. In particular,*

$$\operatorname{ind}_G(\mathcal{E}, \mathbf{v}) = \operatorname{ind}_G(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma).$$

We refer to Theorem 4.2 as a *gluing formula*, meaning that M is obtained from M_Σ by gluing along Σ.

Proof. To prove the theorem it is enough to construct a cobordism W between M and M_Σ.

Consider the product $M \times [0, 1]$, and the set

$$Z := \left\{ (x, t) \in M \times [0, 1] : t \leq 1/3, x \in \Sigma \right\}.$$

Set $W := (M \times [0, 1]) \setminus Z$. Then W is a G-manifold, whose boundary is diffeomorphic to the disjoint union of $M \setminus \Sigma \simeq (M \setminus \Sigma) \times \{0\}$ and $M \simeq M \times \{1\}$. To finish the proof we need to construct a complete Riemannian metric g^W on W, so that the condition (iii) of Definition 3.2 is satisfied. For the case when the group G is compact it is done in section 13 of [2]. The same construction works without any changes for a non-compact G. □

4.3. The additivity of the index

Suppose that Σ divides M into two open submanifolds M_1 and M_2, so that $M_\Sigma = M_1 \sqcup M_2$. The metric g^{M_Σ} induces complete G-invariant Riemannian metrics g^{M_1}, g^{M_2} on M_1 and M_2, respectively. Let $\mathcal{E}_i, \mathbf{v}_i$ $(i = 1, 2)$ denote the restrictions of the Clifford module $\mathcal{E}_\Sigma$ and the taming map $\mathbf{v}_\Sigma$ to M_i. Then Theorem 4.2 implies the following Corollary.

Corollary 4.4. $\operatorname{ind}_G(\mathcal{E}, \mathbf{v}) = \operatorname{ind}_G(\mathcal{E}_1, \mathbf{v}_1) + \operatorname{ind}_G(\mathcal{E}_2, \mathbf{v}_2)$.

Thus, we see that the index of non-compact manifolds is *"additive"*.

5. Functions on a cobordism

In this section we define the notions of an admissible and cutoff functions on a cobordism and prove the existence of such functions.

5.1. A cobordism

Let $(\mathcal{E}, \mathbf{v})$ be a tamed Clifford module over a complete G-manifold M. Let $(W, \mathbf{v}_W, \phi)$ be a cobordism between $(M, \mathbf{v})$ and the empty set, cf. Definition 3.2. In particular, W is a complete G-manifold with boundary and ϕ is a G-equivariant metric preserving diffeomorphism between a neighborhood U of $\partial W \simeq M$ and the product $M \times [0, \varepsilon)$.

Let $\pi : M \times [0, \varepsilon) \to M$ be the projection. A G-invariant Clifford connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ induces a connection $\nabla^{\pi^*\mathcal{E}}$ on the pull-back $\pi^*\mathcal{E}$, such that

$$\nabla^{\pi^*\mathcal{E}}_{(u,a)} := \pi^*\nabla^{\mathcal{E}}_u + a\frac{\partial}{\partial t}, \quad (u, a) \in TM \times \mathbb{R} \simeq T(M \times [0, \varepsilon)). \tag{5.1}$$

Let $(\mathcal{E}_W, \mathbf{v}_W, \psi)$ be a cobordism between $(\mathcal{E}, \mathbf{v})$ and the unique Clifford module over the empty set, cf. Definition 3.4. In particular, $\psi : \mathcal{E}_W|_U \to \pi^*\mathcal{E}$ is a G-equivariant isometry. Let $\nabla^{\mathcal{E}_W}$ be a G-invariant connection on $\mathcal{E}_W$, such that $\nabla^{\mathcal{E}_W}|_{\phi^{-1}(M\times[0,\varepsilon/2))} = \psi^{-1} \circ \nabla^{\pi^*\mathcal{E}} \circ \psi$.

5.2. An admissible function on a cobordism

Definition 5.3. A smooth G-invariant function $f : W \to [0, \infty)$ is an admissible function for $(\mathcal{E}_W, \mathbf{v}_W, \nabla^{\mathcal{E}_W}, \chi)$, if it satisfies (2.7) and there exists a function $h : M \to [0, \infty)$ such that $f\big(\phi^{-1}(y, t)\big) = h(y)$ for all $y \in M$, $t \in [0, \varepsilon/2)$.

Lemma 5.4. *Suppose h is an admissible function for $(\mathcal{E}_M, \mathbf{v}, \nabla^{\mathcal{E}})$. Then there exists an admissible function f on $(\mathcal{E}_W, \mathbf{v}_W, \nabla^{\mathcal{E}_W})$ such that the restriction $f|_M = h$.*

Proof. For the case when G is compact the proof is given in section 8 of [2]. The proof does not use the compactness of G and extends to non-compact case without any changes. □

5.5. A cutoff function on a cobordism

Definition 5.6. A smooth function $\chi : W \to [0, \infty)$ is called a *cutoff function on cobordism* if its support intersects all G-orbits in compact sets, it satisfies (2.2), and there exists a cutoff function $\eta : M \to [0, \infty)$ such that $\chi\big(\phi^{-1}(y, t)\big) = \eta(y)$ for all $y \in M$, $t \in [0, \varepsilon/2)$.

Lemma 5.7. *Suppose η is a cutoff function on M. Then there exists a cutoff function χ on W, such that $\chi|_M = \eta$.*

Proof. Recall that we are given a neighborhood U of ∂W and a diffeomorphism $\phi : U \to M \times I$. We write $\phi(x) = (y, t)$ where $y \in M$ and $t \in [0, \varepsilon)$.

Let α and β be smooth functions $[0, \infty) \to [0, 1]$ such that

$$\alpha(t) = \begin{cases} 0, & \text{for } t < \dfrac{\varepsilon}{3}, \\[2mm] 1, & \text{for } t > \dfrac{2\varepsilon}{3}, \end{cases}$$

and

$$\alpha^2 + \beta^2 \equiv 1. \tag{5.2}$$

Let χ_1 be any cutoff function on W and define a new function χ_2 on W by

$$\chi_2(x) = \begin{cases} \chi_1(x), & \text{for } x \notin U, \\ \alpha(t)\chi_1(x) + \beta(t)\eta(y), & \text{for } x = (y, t) \in U \simeq M \times I. \end{cases}$$

Then the support of χ_2 intersects the orbits of G in compact sets and

$$\chi_2^2 = \alpha^2\chi_1^2 + \beta^2\eta^2 + 2\alpha\beta\chi_1\eta = \alpha^2\chi_1^2 + \beta^2\eta^2 + \psi,$$

where we set $\psi := 2\alpha\beta\chi_1\eta$. Hence, from (5.2) and the definition of the cutoff function (2.2) we obtain

$$\int_G \chi_2^2(g \cdot x)\, dg = 1 + \int_G \psi(g \cdot x)\, dg.$$

It follows that the function

$$\chi(x) := \frac{\chi_2(x)}{\left(1+\int_G \psi(g\cdot x)dg\right)^{1/2}}$$

is a cutoff function whose restriction to ∂W is equal to η. □

6. Proof of Theorem 2.15.1

6.1. The operator D^2_{fv}

Consider the operator

$$D_{fv} := D + \sqrt{-1}c(fv).$$

It follows from (2.3) that

$$D_{\chi,fv}(\chi s) := \chi\, D_{fv}s, \quad s \in \Gamma^\infty_{tc}(M,\mathcal{E}),$$

and, hence,

$$D^2_{\chi,fv}(\chi s) := \chi\, D^2_{fv}s, \quad s \in \Gamma^\infty_{tc}(M,\mathcal{E}), \tag{6.1}$$

6.2. Calculation of $D^2_{\chi,fv}$

Let f be an admissible function and set $u = fv$. Consider the operator

$$A_u = \sum c(e_i)\, c(\nabla^{\mathrm{LC}}_{e_i}u) : \mathcal{E} \to \mathcal{E}, \tag{6.2}$$

where $\bar{e} = \{e_1\cdots e_n\}$ is an orthonormal frame of $TM \simeq T^*M$ and ∇^{LC} is the Levi-Civita connection on TM. One easily checks that A_u is independent of the choice of $\bar{e}$ (it follows, also, from equation (6.3) below).

Recall that we denote by $\mathcal{L}_{\mathbf{u}}$ the infinitesimal action of $\mathbf{u}$ on $\Gamma(M,\mathcal{E})$ induced by the action of G on $\mathcal{E}$. Then the restriction of $\mathcal{L}_{\mathbf{u}}$ to $\Gamma^\infty_{tc}(M,\mathcal{E})^G$ is equal to 0. Combining Lemma 9.2 of [2] with (2.5) we obtain

$$D^2_u = D^2 + |u|^2 + \sqrt{-1}A_u - 2\sqrt{-1}\,\mu^{\mathcal{E}}(\mathbf{u}) : \Gamma^\infty_{tc}(M,\mathcal{E})^G \to \Gamma^\infty_{tc}(M,\mathcal{E})^G. \tag{6.3}$$

Since both A_u and $\mu^{\mathcal{E}}(\mathbf{u})$ commute with multiplication by χ, we conclude from (2.3), (6.1) and (6.3) that

$$D^2_{\chi,u} = D^2_\chi + |u|^2 + \sqrt{-1}A_u - 2\sqrt{-1}\,\mu^{\mathcal{E}}(\mathbf{u}) : \chi\Gamma^\infty_{tc}(M,\mathcal{E})^G \to \chi\Gamma^\infty_{tc}(M,\mathcal{E})^G. \tag{6.4}$$

6.3. Proof of Theorem 2.15

It is shown in section 9.3 of [2] that there exists a real valued function $r(x)$ on M such that $\lim_{q(x)\to\infty} r(x) = +\infty$ and

$$|u|^2 + \sqrt{-1}A_u - 2\sqrt{-1}\,\mu^{\mathcal{E}}(\mathbf{u}) \geq r(x).$$

It follows now from (6.4) that

$$D^2_{\chi,u} \geq D^2_\chi + r(x). \tag{6.5}$$

By Proposition 3.9 of [9] the operator $D_{\chi,u}$ is Fredholm. □

7. Proof of Theorem 3.6

The proof is a minor modification of the proof of Theorem 3.7 in [2]. We only sketch it here, stressing the couple of places where it defers from [2].

By Remark 3.5, it is enough to show that, if $(\mathcal{E},\mathbf{v})$ is cobordant to (the Clifford module over) the empty set, then $\mathrm{ind}_G(D_{\chi,fv}) = 0$ for any admissible function f.

Let $(W,\mathcal{E}_W,\mathbf{v})$ be a cobordism between the empty set and $(\mathcal{E},\mathbf{v})$ (slightly abusing the notation, we denote by the same letter $\mathbf{v}$ the taming maps on W and M).

In Section 5 we showed that there exist a cutoff function and an admissible function on W whose restriction to M are equal to χ and f respectively. By a slight abuse of notation, we denote these functions by the same letters χ and f.

Let $\tilde{W}$ be the manifold obtained from W by attaching a cylinder to the boundary, i.e.,

$$\tilde{W} = W \sqcup \big(M \times (0,\infty)\big).$$

The action of G, the Riemannian metric, the map $\mathbf{v}$, the functions χ, f and the Clifford bundle $\mathcal{E}_W$ extend naturally from W to $\tilde{W}$.

Let us consider two anti-commuting actions (left and right action) of $\mathbb{R}$ on the exterior algebra $\Lambda^\bullet\mathbb{C} = \Lambda^0\mathbb{C}\oplus\Lambda^1\mathbb{C}$, given by the formulas

$$c_L(t)\,\omega = t\wedge\omega - \iota_t\omega; \qquad c_R(t)\,\omega = t\wedge\omega + \iota_t\omega. \tag{7.1}$$

Note, that $c_L(t)^2 = -t^2$, while $c_R(t)^2 = t^2$.

Set $\tilde{\mathcal{E}} = \mathcal{E}\otimes\Lambda^\bullet\mathbb{C}$ and define the grading and the Clifford action $\tilde{c} : T^*\tilde{W} \to \operatorname{End}\,\tilde{\mathcal{E}}$ on $\tilde{\mathcal{E}}$ by the formulas

$$\tilde{\mathcal{E}}^+ := \mathcal{E}_W\otimes\Lambda^0\mathbb{C}; \qquad \tilde{\mathcal{E}}^- := \mathcal{E}_W\otimes\Lambda^1\mathbb{C}; \qquad \tilde{c}(v) := \sqrt{-1}\,c(v)\otimes c_L(1) \quad (v\in T^*\tilde{W}).$$

Let $\tilde{D}$ be a Dirac operator on $\tilde{\mathcal{E}}$ and consider the operator

$$\tilde{D}_{\chi,fv} := \tilde{D}_\chi + \sqrt{-1}\,\tilde{c}(fv).$$

Note that v might vanish somewhere near infinity on the cylindrical end of $\tilde{W}$. In particular, the index of $\tilde{D}_{\chi,fv}$ is not defined in general.

Let $p : \tilde{W} \to \mathbb{R}$ be a map, whose restriction to $M\times(1,\infty)$ is the projection on the second factor, and such that $p(W) = 0$. For every $a\in\mathbb{R}$, consider the operator

$$\mathbf{D}_a := \tilde{D}_{\chi,fv} - 1\otimes c_R((p(t)-a)).$$

By Lemma 10.4 of [2] we have

$$\mathbf{D}_a^2 = \tilde{D}_{\chi,fv}^2 - B + |p(x)-a|^2, \tag{7.2}$$

where $B : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ is a uniformly bounded bundle map.

It follows now from Proposition 3.9 of [9] that the operator $\mathbf{D}_a$ is Fredholm.

Proposition 7.1. $\operatorname{ind}_G(\mathbf{D}_a) = 0$ *for all* $a\in\mathbb{R}$.

Proof. Since

$$\mathbf{D}_a - \mathbf{D}_b = c_R(b-a) :\ \chi\,\Gamma^\infty_{tc}(\tilde{W},\tilde{E}) \ \to\ \chi\,\Gamma^\infty_{tc}(\tilde{W},\tilde{E})$$

is bounded operator depending continuously on $a, b\in\mathbb{R}$, the index $\operatorname{ind}_G(\mathbf{D}_a)$ is independent of a. Therefore, it is enough to prove the proposition for one particular value of a.

Let $\|B(x)\|$, $x\in\tilde{W}$, denote the norm of the bundle map $B_x : \tilde{\mathcal{E}}_x \to \tilde{\mathcal{E}}_x$ and let $\|B\|_\infty = \sup_{x\in\tilde{W}}\|B(x)\|$. Choose $a \ll 0$ such that $a^2 > \|B\|_\infty$. Since $p(x)\ge 0$ we have $|p(x)-a|^2 \ge a^2 > \|B\|_\infty$. Thus it follows from (7.2) that $\mathbf{D}_a^2 > 0$, so that $\operatorname{Ker}\mathbf{D}_a^2 = 0$. Hence, $\operatorname{ind}_G(\mathbf{D}_a^2) = 0$. □

Theorem 3.6 follows now from Proposition 7.1 and the following Theorem.

Theorem 7.2. $\operatorname{ind}_G(\mathbf{D}_a) \ = \ \operatorname{ind}_G(D_{\chi,fv})$.

Proof. For compact group G the proof is given in section 11 of [2]. This proof works without any changes in our current situation. □

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