Journal of Geometry and Physics 101 (2016) 1–10

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Biharmonic maps on tangent and cotangent bundles

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ARTICLE INFO

Article history:
Received 6 May 2015
Accepted 17 November 2015
Available online 2 December 2015

MSC:
53C50
53B30

Keywords:
Harmonic and biharmonic map
Tangent and cotangent bundles
Complete lift metric
Riemannian extension

ABSTRACT

We examine the harmonicity of some natural maps associated to the tangent and cotangent bundles, providing some new examples of proper biharmonic maps between pseudo-Riemannian manifolds.

1. Introduction

A major flaw in Riemannian geometry (as compared to other subjects) is a shortage of suitable kind of functions from one manifold to another that will compare their geometric properties. Generalizing the local isometries, totally geodesic maps preserve geodesics and hence they are suitable for comparing the local geometries of the domain and the range manifolds. However totally geodesic maps are too rigid and, for many purposes, harmonic maps are of major interest. As a generalization of harmonic maps, the study of biharmonic maps was suggested by Eells and Sampson [1]. While harmonic maps between compact Riemannian manifolds are defined as being critical points of the energy functional, biharmonic maps are critical points of the bienergy functional (the L^2-norm of the tension field).

During the last decades there has been a growing interest in the theory of biharmonic maps which can be divided in two main directions. The analytic point of view focuses on the study of biharmonic maps as solutions of a fourth order strongly elliptic semilinear PDE. However the geometric approach has been mainly devoted to the construction of examples and classification results. The latter was also considered in the pseudo-Riemannian case and the main purpose of this work is to contribute into this direction by constructing new examples of biharmonic maps.

Clearly any harmonic map is biharmonic but the converse is not true in general. Thus the task of constructing biharmonic maps which are not harmonic is an interesting one to pursue. One such construction has consisted in chosen suitable conformal deformations so that a map becomes biharmonic. In particular, it turns out that the existence problem for proper biharmonic submanifolds in pseudo-Riemannian manifolds often appears considerably different from its Riemannian counterpart, and the classification problems are more complicated. In this paper we support the above by showing

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http://dx.doi.org/10.1016/j.geomphys.2015.11.007

many examples of proper biharmonic maps between pseudo-Riemannian manifolds. All these maps are constructed from geometric data and their domain and/or range are the tangent or cotangent bundle of a given pseudo-Riemannian manifold. It is worth to emphasize that although the kind of maps we are interested in are not harmonic, their tension field is a non-zero null vector field for the corresponding pseudo-Riemannian metrics.

2. Harmonic and biharmonic maps

2.1. Harmonic maps

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. Define the *energy density of* φ by $e(\varphi) := \frac{1}{2}\|d\varphi\|^2$. The *energy functional* (assuming either that M is compact or that φ has compact support) is given by:

$$E(\varphi) := \int_M e(\varphi) dM,$$

where dM denotes the volume element of (M, g). In this setting, φ is said to be *harmonic* if it is a critical point of the energy functional $E(\varphi)$.

Let $\varphi^{-1}(TN)$ be the pull-back bundle. Then the Levi-Civita connections on TM and TN induce a connection ∇ in the bundle of one-forms on M with values in $\varphi^{-1}(TN)$. Then $\nabla d\varphi$ is a symmetric bilinear form on TM which is called the *second fundamental form* of φ. The trace of $\nabla d\varphi$ with respect to g is called the *tension field* of φ, and is denoted by $\tau(\varphi)$.

It turns out that the Euler–Lagrange equations corresponding to the critical points of the energy functional correspond to the vanishing of the tension field of φ, and hence the map φ is said to be *harmonic* if $\tau(\varphi) = 0$ and *totally geodesic* if the second fundamental form vanishes. (See [1] and [2,3] for more details and references.)

Now, let $U \subset M$ be a domain with coordinates $(x^1, \ldots, x^m)$, $m = \dim M$ and $V \subset N$ be a domain with coordinates $(z^1, \ldots, z^n)$, $n = \dim N$, such that $\varphi(U) \subset V$ and suppose that φ is locally represented by $z^\alpha = \varphi^\alpha(x^1, \ldots, x^m)$, $\alpha = 1, \ldots, n$. Then we have:

$$(\nabla d\varphi)^\gamma_{ij} = \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - {}^g\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} + {}^h\Gamma^\gamma_{\alpha\beta}(\varphi) \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j}. \tag{1}$$

Here ${}^g\Gamma^k_{ij}$ and ${}^h\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of (M, g) and (N, h), respectively. So, φ is harmonic if and only if

$$\tau(\varphi) = \mathrm{trace}_g(\nabla d\varphi) = g^{ij}(\nabla d\varphi)^\gamma_{ij} \frac{\partial}{\partial z^\gamma} = 0. \tag{2}$$

2.2. Biharmonic maps

In analogy with the definition of harmonic maps, the *bienergy density* of a map $\varphi : (M, g) \longrightarrow (N, h)$ is defined by $e^2(\varphi) = \frac{1}{2}\|\tau(\varphi)\|^2$, and the corresponding *bienergy functional*:

$$E^2(\varphi) = \int_M e^2(\varphi) dM,$$

as a kind of L^2-norm of the tension field of φ. Now, a smooth map $\varphi : (M, g) \to (N, h)$ is called *biharmonic* if it is a critical point of the bienergy functional.

It now follows that the Euler–Lagrange equations for the critical points of the bienergy functional are equivalent to the vanishing of the *bitension field* $\tau^2(\varphi)$ given by (see for example [4–8] and references therein):

$$\tau^2(\varphi) = \mathrm{trace}_g(\nabla^2 d\varphi) = \mathrm{trace}_g\{({}^h\nabla^h\nabla - {}^h\nabla_{{}^g\nabla})\tau(\varphi) - {}^hR(\varphi_*, \tau(\varphi))\varphi_*\},$$

where the curvature tensor is taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

A coordinate expression of the bitension field is obtained in an analogous way as in the above as follows (see also [9]),

$$\begin{aligned}\tau^2(\varphi) &= \mathrm{trace}_g(\nabla^2 d\varphi) \\ &= g^{ij}\left\{\frac{\partial^2 \tau^\sigma}{\partial x^i \partial x^j} + \frac{\partial \tau^\alpha}{\partial x^j}\frac{\partial \varphi^\beta}{\partial x^i}\,{}^h\Gamma^\sigma_{\alpha\beta} + \frac{\partial}{\partial x^i}\left(\tau^\alpha \frac{\partial \varphi^\beta}{\partial x^j}\,{}^h\Gamma^\sigma_{\alpha\beta}\right)\right. \\ &\quad \left. +\ \tau^\alpha \frac{\partial \varphi^\beta}{\partial x^i}\frac{\partial \varphi^\rho}{\partial x^i}\,{}^h\Gamma^\nu_{\alpha\beta}\,{}^h\Gamma^\sigma_{\nu\rho} - {}^g\Gamma^k_{ij}\left(\frac{\partial \tau^\alpha}{\partial x^k} + \tau^\alpha \frac{\partial \varphi^\beta}{\partial x^k}\,{}^h\Gamma^\sigma_{\alpha\beta}\right) - \tau^\nu \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial \varphi^\beta}{\partial x^j}\,{}^hR^\sigma_{\beta\alpha\nu}\right\}\frac{\partial}{\partial z^\sigma},\end{aligned} \tag{3}$$

where $\tau(\varphi) = \tau^\sigma \frac{\partial}{\partial z^\sigma}$ is the coordinate expression of the tension field of φ and superindices denote the components $\varphi = (\varphi^1, \ldots, \varphi^n)$. Moreover, ${}^g\Gamma$ and ${}^h\Gamma$ denote the Christoffel symbols of the Levi-Civita connections of (M, g) and (N, h), and ${}^hR$ is the curvature tensor of (N, h), where we follow the notation ${}^hR^\sigma(\partial_{z^\alpha}, \tau)\partial_{z^\beta} = \tau^\nu\, {}^hR^\sigma_{\beta\alpha\nu}$.

3. Biharmonic maps on tangent bundles

3.1. The complete lift metric

Let TM be the tangent bundle of a manifold M and let $\pi : TM \to M$ be the canonical projection. For each pseudo-Riemannian metric on M, there are many induced metrics on the tangent bundle. The complete lift metric, which is always of neutral signature (m, m) is one of the most natural ones. We refer to [10] for more information on the geometry of the tangent bundle and the complete lift metric.

One of the basic objects behind the structure of TM is the evaluation map. Each one-form ω on M induces a function $\iota\omega : TM \to \mathbb{R}$, the *evaluation map*, defined by $\iota\omega(p, v) = \omega_p(v)$. As a special case, for each function $f \in \mathcal{C}^\infty(M)$, $f^C = \iota(df)$ is called the complete lift of the function f. The special significance of these functions on TM is that they characterize vector fields on TM, since $\tilde{X}(f^C) = \tilde{Y}(f^C)$ for all $f \in \mathcal{C}^\infty(M)$ if and only if $\tilde{X} = \tilde{Y}$ for all vector fields on TM.

For each vector field X on M, its complete lift X^C is the vector field on TM defined by $X^C(f^C) = (Xf)^C$. It is a remarkable fact that covariant tensor fields on TM are characterized by their action on complete lifts of vector fields on M. Hence if (M, g) is a pseudo-Riemannian manifold, the tangent bundle TM is naturally equipped with its *complete lift* which is characterized by:

$$g^C(X^C, Y^C) = g(X, Y)^C.$$

A coordinate description of the complete lift metric is as follows. Let $(x^1, \dots, x^m)$ be local coordinates on an open subset $U \subset M$ and consider the induced coordinates on $\pi^{-1}(U) \subset TM(x^1, \dots, x^m, x^{\bar{1}}, \dots, x^{\bar{m}})$, where $x^{\bar{1}}, \dots, x^{\bar{m}}$ are the components of vector fields on M with respect to the coordinate vector fields $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\}$. In what follows we use the notation $\bar{i} = i + m$, where $i = 1, \dots, m$.

Now if g_{ij} denote the component functions of g in the coordinates $(U, (x^1, \dots, x^m))$, then the complete lift g^C is given in matrix form by:

$$g^C = \begin{pmatrix} x^{\bar{k}} \dfrac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

where $(\pi^{-1}(U), (x^1, \dots, x^m, x^{\bar{1}}, \dots, x^{\bar{m}}))$ are the induced coordinates on TM.

In order to study the biharmonicity of certain maps on TM, we need the expressions of the Levi-Civita connection and the curvature of (TM, g^C). Let $(U, (x^1, \dots, x^m))$ and $(\pi^{-1}(U), (x^1, \dots, x^m, x^{\bar{1}}, \dots, x^{\bar{m}}))$ be local coordinates on M and TM. Then the Levi-Civita connections of g^C and g are related as follows:

$$^{g^C}\Gamma^k = \begin{pmatrix} ^g\Gamma_{ij}^k & 0 \\ 0 & 0 \end{pmatrix}, \qquad ^{g^C}\Gamma^{\bar{k}} = \begin{pmatrix} x^{\bar{l}} \dfrac{\partial}{\partial x^l} {}^g\Gamma_{ij}^k & ^g\Gamma_{ij}^k \\ ^g\Gamma_{ij}^k & 0 \end{pmatrix}. \tag{4}$$

Moreover, the components of the curvature tensors $^{g^C}R$ of (TM, g^C) and ^gR of (M, g) are also related by:

$$\begin{aligned} &^{g^C}R_{ij\bar{k}}^l = {}^{g^C}R_{\bar{i}jk}^l = {}^{g^C}R_{i\bar{j}k}^l = {}^{g^C}R_{\bar{i}\bar{j}k}^l = {}^{g^C}R_{\bar{i}j\bar{k}}^l = 0, \\ &^{g^C}R_{ijk}^l = {}^{g^C}R_{ij\bar{k}}^{\bar{l}} = {}^{g^C}R_{\bar{i}jk}^{\bar{l}} = {}^gR_{ijk}^l, \\ &^{g^C}R_{ijk}^{\bar{l}} = {}^{g^C}R_{\bar{i}j\bar{k}}^{\bar{l}} = {}^{g^C}R_{\bar{i}\bar{j}k}^{\bar{l}} = 0. \end{aligned} \tag{5}$$

Remark 1. The complete lift g^C is a pseudo-Riemannian metric of neutral signature (m, m), $m = \dim M$, whose properties have a nice correspondence with those of (M, g). For instance (see [11,10]) (M, g) is locally symmetric if and only if (TM, g^C) is so, (M, g) is a real space form if and only if (TM, g^C) is locally conformally flat, (M, g, J) is a complex space form if and only if (TM, g^C, J^C) is a Bochner-flat Kähler manifold.

Projections and sections are the most natural maps associated to a given bundle structure. It is important to emphasize that although the projection $\pi : (TM, g^C) \to (M, g)$ is a submersion, it is not a Riemannian submersion. However it behaves nicely as concerns harmonicity questions since $\pi : (TM, g^C) \to (M, g)$ is a totally geodesic submersion [12]. The evaluation map, which associated a real function on TM to each one-form on M, is not only useful to understand the structure of the tangent bundle, but it also provides a large family of biharmonic functions on (TM, g^C) as follows.

Theorem 2. *Let ω be a one-form on a pseudo-Riemannian manifold (M, g) and consider its evaluation $\iota\omega : (TM, g^C) \to \mathbb{R}$. Then*

(i) $\iota\omega : (TM, g^C) \to \mathbb{R}$ *is a harmonic function if and only if ω is co-closed (i.e., $\delta\omega = 0$).*
(ii) $\iota\omega : (TM, g^C) \to \mathbb{R}$ *is a biharmonic function.*

Proof. Let $(U,(x^1,\dots,x^m))$ and $(\pi^{-1}(U),(x^1,\dots,x^m,x^{\bar 1},\dots,x^{\bar m}))$ be local coordinates on M and TM as above. Considering the local expression of the one-form $\omega=\sum_{i=1}^m \omega_i dx^i$, the evaluation map reads in induced coordinates as:

$$\iota\omega(x^1,\dots,x^m,x^{\bar 1}\dots,x^{\bar m})=\sum_{l=1}^m \omega_l x^{\bar l}.$$

Hence, one has the following expression for the second fundamental form

$$(\nabla d\iota\omega)_{ij}=x^{\bar l}\left(\frac{\partial^2\omega_l}{\partial x^i\partial x^j}-{}^g\Gamma_{ij}^k\frac{\partial\omega_l}{\partial x^k}\right),\qquad (\nabla d\iota\omega)_{\bar i\bar j}=0,$$
$$(\nabla d\iota\omega)_{\bar i j}=({}^g\nabla_{\frac{\partial}{\partial x^j}}\omega)_i,\qquad (\nabla d\iota\omega)_{i\bar j}=({}^g\nabla_{\frac{\partial}{\partial x^i}}\omega)_j,$$

where ${}^g\nabla$ is the Levi-Civita connection of (M,g) and ${}^g\Gamma_{ij}^k$ are the corresponding Christoffel symbols.

Observe that the inverse of the matrix of g^C is given by

$$(g^C)^{-1}=\begin{pmatrix}0 & g^{ij}\\ g^{ij} & x^{\bar k}\dfrac{\partial gij}{\partial x^k}\end{pmatrix}.$$

Hence in order to compute the tension field $\tau(\iota\omega)$ we only need to consider the following components of the second fundamental form

$$(\nabla d\iota\omega)_{i\bar j},\qquad (\nabla d\iota\omega)_{\bar i j},\qquad (\nabla d\iota\omega)_{\bar i\bar j},$$

from where it follows that $\tau(\iota\omega)=\delta\omega$, where δ denotes the coderivative operator in (M,g), thus showing (i).

Next in order to analyse the biharmonicity of $\iota\omega$, observe that the bitension field is given by

$$\tau^2(\iota\omega)=\mathrm{trace}_{g^C}(\nabla^2 d\iota\omega)=g^{AB}\left\{\frac{\partial^2\tau(\iota\omega)}{\partial x^A\partial x^B}-{}^{g^C}\Gamma_{AB}^C\frac{\partial\tau(\iota\omega)}{\partial x^C}\right\}\frac{\partial}{\partial t},$$

where t denotes the coordinate in $\mathbb{R}$, ${}^{g^C}\Gamma_{AB}^C$ are the Christoffel symbols of the Levi-Civita connection of (TM,g^C) and $A,B,C=1,\dots,2m$.

Now, since

$$(\nabla^2 d\iota\omega)_{\bar i j}=\frac{\partial^2\tau(\iota\omega)}{\partial x^{\bar i}\partial x^j}-{}^{g^C}\Gamma_{\bar i j}^{\bar k}\frac{\partial\tau(\iota\omega)}{\partial x^{\bar k}}=0,$$
$$(\nabla^2 d\iota\omega)_{\bar i\bar j}=\frac{\partial^2\tau(\iota\omega)}{\partial x^{\bar i}\partial x^{\bar j}}=0,$$

it follows that $\iota\omega$ is a biharmonic function for any one-form ω, thus proving (ii). □

Vector fields on a manifold M are defined as sections of the tangent bundle and hence, it is natural to consider the induced maps $X:(M,g)\to(TM,g^C)$. Vector fields defining harmonic sections have been investigated in [13], where it is shown that the tension field of $X:(M,g)\to(TM,g^C)$ satisfies

$$\tau(X)^k=0,\qquad \tau(X)^{\bar k}=\mathrm{trace}(\mathcal{L}_X{}^g\nabla)^k,$$

where $\mathcal{L}$ denotes the Lie derivative. Now, since $\pi_*\tau(X)$, it follows that $\tau(X)$ is tangent to the fibres of TM, and hence one may consider the tension field as the vertical lift of a map $\tau(X):p\in M\mapsto\tau(X)_p\in T_{X(p)}T_pM\simeq T_pM$ and thus consider $\tau(X)$ as a map $\tau(X):(M,g)\to(TM,g^C)$. Now we have:

Theorem 3. *Let (M,g) be a pseudo-Riemannian manifold. A vector field X on M defines a biharmonic section $X:(M,g)\to(TM,g^C)$ if and only if its tension field defines a harmonic section $\tau(X):(M,g)\to(TM,g^C)$.*

Proof. Let $\tau\equiv\tau(X)$ be the tension field of $X:(M,g)\to(TM,g^C)$. A calculation from (3) using the expressions (4) and (5) shows that the bitension field of the section $X:(M,g)\to(TM,g^C)$ satisfies $\tau^2(X)=\mathrm{trace}_g(\nabla^2 dX)$, where

$$(\nabla^2 dX)_{ij}^k=0,$$
$$(\nabla^2 dX)_{ij}^{\bar k}=\frac{\partial^2\tau^{\bar k}}{\partial x^i\partial x^j}+\frac{\partial\tau^{\bar l}}{\partial x^j}{}^g\Gamma_{li}^k+\frac{\partial}{\partial x^i}(\tau^{\bar l}\,{}^g\Gamma_{lj}^k)+\tau^{\bar l}\,{}^g\Gamma_{lj}^t\,{}^g\Gamma_{ti}^k-{}^g\Gamma_{ij}^l\left(\frac{\partial\tau^{\bar k}}{\partial x^l}+\tau^{\bar t}\,{}^g\Gamma_{tl}^k\right)-\tau^{\bar l}\,{}^gR_{jil}^k. \tag{6}$$

Moreover, the tension field of $\tau(X)$ is given by $\tau(\tau(X))=\mathrm{trace}_g(\nabla d\tau(X))$, where

$$(\nabla d\tau(X))_{ij}^k=0,$$
$$(\nabla d\tau(X))_{ij}^{\bar k}=\frac{\partial^2\tau^{\bar k}}{\partial x^i\partial x^j}-\frac{\partial\tau^{\bar k}}{\partial x^l}{}^g\Gamma_{ij}^l+\frac{\partial\tau^{\bar l}}{\partial x^i}{}^g\Gamma_{lj}^k+\frac{\partial\tau^{\bar l}}{\partial x^l}{}^g\Gamma_{il}^k+\tau^{\bar l}\frac{\partial^g\Gamma_{ij}^k}{\partial x^l}. \tag{7}$$

Now it immediately follows from the curvature term in (6)

$$ {}^gR^k_{jil} = \frac{\partial {}^g\Gamma^k_{lj}}{\partial x^i} - \frac{\partial {}^g\Gamma^k_{ij}}{\partial x^l} + {}^g\Gamma^t_{lj}\,{}^g\Gamma^k_{it} - {}^g\Gamma^t_{ij}\,{}^g\Gamma^k_{lt}, $$

that both expressions (6) and (7) coincide with each other, from where the result follows. □

3.2. φ-morphisms

Let $\varphi : (M, g) \to (N, h)$ be a map between pseudo-Riemannian manifolds. Following the terminology of [14], a map $\Phi : TM \to TN$ is said to be a *φ-morphism* if its fibre restriction $\Phi_p : T_pM \to T_{\varphi(p)}N$ is linear for each $p \in M$. A coordinate description of φ-morphisms is as follows. Let $(U, (x^i))$ and $(\pi^{-1}(U), (x^i, x^{\bar{i}}))$, $i = 1, \ldots, m$ and $\bar{i} = i+m$, be local coordinates on M and TM. Set $\varphi = (\varphi^1, \ldots, \varphi^n)$ where $\varphi^\alpha = \varphi^\alpha(x^1, \ldots, x^m)$. Then Φ is a φ-morphism if its local expression becomes

$$ \Phi(x^i, x^{\bar{i}}) = \left(\Phi^\alpha(x^i, x^{\bar{i}}), \Phi^{\bar{\alpha}}(x^i, x^{\bar{i}})\right) = \left(\varphi^\alpha(x^i), x^{\bar{l}}\Phi^{\bar{\alpha}}_l(x^i)\right), \tag{8} $$

for some functions $\Phi^{\bar{\alpha}}_l(x^i)$, $\alpha = 1, \ldots, n$ and $\bar{\alpha} = \alpha + n$.

The corresponding tension field $\tau(\Phi)$ is given by

$$ \tau(\Phi)^\sigma = 0 \quad \text{and} \quad \tau(\Phi)^{\bar{\sigma}} = 2g^{ij}\left\{\frac{\partial \Phi^{\bar{\sigma}}_j}{\partial x^i} + {}^N\Gamma^\sigma_{\alpha\beta}\Phi^{\bar{\alpha}}_j\varphi^\beta_i - {}^M\Gamma^k_{ij}\Phi^{\bar{\sigma}}_k\right\}. \tag{9} $$

Now, although it is clear that not all φ-morphism is harmonic, one has

Theorem 4. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map. Then any φ-morphism $\Phi : (TM, g^C) \to (TN, h^c)$ is biharmonic.*

Proof. Since the bitension field

$$ \tau^2(\Phi)^\gamma = \operatorname{trace}_{g^C}(\nabla^2 d\Phi), $$

we only need to compute the terms $(\nabla^2 d\Phi)^\sigma_{\bar{i}\bar{j}}$, $(\nabla^2 d\Phi)^{\bar{\sigma}}_{ij}$, and $(\nabla^2 d\Phi)^{\bar{\sigma}}_{\bar{i}j}$, $(\nabla^2 d\Phi)^{\bar{\sigma}}_{i\bar{j}}$. It immediately follows from (3) using the expressions of the Christoffel symbols and the curvature of a complete lift metric together with (8) and (9), that $(\nabla^2 d\Phi)^\sigma_{\bar{i}\bar{j}} = (\nabla^2 d\Phi)^\sigma_{ij} = 0$, which shows that $\tau^2(\Phi)^\sigma = 0$.

Next we compute the terms $(\nabla^2 d\Phi)^{\bar{\sigma}}_{i\bar{j}}$, $(\nabla^2 d\Phi)^{\bar{\sigma}}_{ij}$. It follows from (3), (8) and (9) that

$$ \begin{aligned} (\nabla^2 d\Phi)^{\bar{\sigma}}_{i\bar{j}} &= \frac{\partial^2 \tau(\Phi)^{\bar{\sigma}}}{\partial x^i \partial x^{\bar{j}}} + \frac{\partial \tau(\Phi)^R}{\partial x^{\bar{j}}}\frac{\partial \Phi^S}{\partial x^i}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{RS} + \frac{\partial}{\partial x^i}\left(\tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^{\bar{j}}}\,{}^{g^C}\Gamma^{\bar{\sigma}}_{RS}\right) + \tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^{\bar{j}}}\frac{\partial \Phi^T}{\partial x^i}\,{}^{h^C}\Gamma^U_{RS}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{TU} \\ &\quad - {}^{g^C}\Gamma^A_{i\bar{j}}\left(\frac{\partial \tau(\Phi)^S}{\partial x^A} + \tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^A}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{RS}\right) - \tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^i}\frac{\partial \Phi^T}{\partial x^{\bar{j}}}\,{}^{h^C}R^{\bar{\sigma}}_{TSR} \\ &= \frac{\partial}{\partial x^i}\left(\tau(\Phi)^{\bar{\alpha}}\,\Phi^{\bar{\beta}}\,{}^{g^C}\Gamma^{\bar{\sigma}}_{\bar{\alpha}\bar{\beta}}\right) + \tau(\Phi)^{\bar{\alpha}}\,\Phi^{\bar{\beta}}_j\frac{\partial \varphi^\gamma}{\partial x^i}\,{}^{h^C}\Gamma^U_{\bar{\alpha}\bar{\beta}}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{\bar{\gamma}U} - {}^{g^C}\Gamma^A_{i\bar{j}}\left(\frac{\partial \tau(\Phi)^{\bar{\alpha}}}{\partial x^A} + \tau(\Phi)^{\bar{\beta}}\frac{\partial \Phi^S}{\partial x^A}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{\bar{\beta}S}\right) \\ &\quad - \tau(\Phi)^{\bar{\alpha}}\frac{\partial \Phi^S}{\partial x^i}\Phi^{\bar{\beta}}_j\,{}^{h^C}R^{\bar{\sigma}}_{\bar{\beta}S\bar{\alpha}}, \end{aligned} $$

and

$$ \begin{aligned} (\nabla^2 d\Phi)^{\bar{\sigma}}_{\bar{i}\bar{j}} &= \frac{\partial^2 \tau(\Phi)^{\bar{\sigma}}}{\partial x^{\bar{i}} \partial x^{\bar{j}}} + \frac{\partial \tau(\Phi)^R}{\partial x^{\bar{j}}}\frac{\partial \Phi^S}{\partial x^{\bar{i}}}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{RS} + \frac{\partial}{\partial x^{\bar{i}}}\left(\tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^{\bar{j}}}\,{}^{g^C}\Gamma^{\bar{\sigma}}_{RS}\right) + \tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^{\bar{j}}}\frac{\partial \Phi^T}{\partial x^{\bar{i}}}\,{}^{h^C}\Gamma^U_{RS}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{TU} \\ &\quad - {}^{g^C}\Gamma^A_{\bar{i}\bar{j}}\left(\frac{\partial \tau(\Phi)^S}{\partial x^A} + \tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^A}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{RS}\right) - \tau(\Phi)^R\frac{\partial \Phi^S}{\partial x^{\bar{i}}}\frac{\partial \Phi^T}{\partial x^{\bar{j}}}\,{}^{h^C}R^{\bar{\sigma}}_{TSR} \\ &= \frac{\partial}{\partial x^{\bar{i}}}\left(\tau(\Phi)^{\bar{\alpha}}\,\Phi^{\bar{\beta}}\,{}^{g^C}\Gamma^{\bar{\sigma}}_{\bar{\alpha}\bar{\beta}}\right) - {}^{g^C}\Gamma^A_{\bar{i}\bar{j}}\left(\frac{\partial \tau(\Phi)^{\bar{\alpha}}}{\partial x^A} + \tau(\Phi)^{\bar{\beta}}\frac{\partial \Phi^S}{\partial x^A}\,{}^{h^C}\Gamma^{\bar{\sigma}}_{\bar{\beta}S}\right) - \tau(\Phi)^{\bar{\alpha}}\frac{\partial \Phi^S}{\partial x^{\bar{i}}}\Phi^{\bar{\beta}}_j\,{}^{h^C}R^{\bar{\sigma}}_{\bar{\beta}S\bar{\alpha}}, \end{aligned} $$

where $i, j = 1, \ldots, m = \dim M$; $\overline{latin} = latin + m$; $A = 1, \ldots, 2m$; $\alpha, \beta, \sigma = 1, \ldots, n = \dim N$; $\overline{greek} = greek + n$ and $R, S, T, U = 1, \ldots, 2n$.

It is immediate now to recognize from (4) and (5) that $(\nabla^2 d\Phi)^{\bar{\sigma}}_{i\bar{j}} = (\nabla^2 d\Phi)^{\bar{\sigma}}_{\bar{i}\bar{j}} = 0$, from where it follows that Φ is biharmonic. □

3.3. Applications

In what follows we use Theorem 4 to construct examples of proper biharmonic maps between tangent bundles.

3.3.1. Biharmonic metrics

A metric $\tilde{g}$ on a pseudo-Riemannian manifold (M, g) is said to be harmonic if the identity map $\mathrm{id}_M : (M, g) \to (M, \tilde{g})$, whose tension field is given by $\tau(\mathrm{id}_M)^k = g^{ij}\left\{{}^{\tilde{g}}\Gamma_{ij}^k - {}^{g}\Gamma_{ij}^k\right\}$, is a harmonic map [15]. Generalizing the above, a metric $\tilde{g}$ on (M, g) is said to be biharmonic if the identity map $\mathrm{id}_M : (M, g) \to (M, \tilde{g})$ is a biharmonic map. Examples of biharmonic metrics where $\tilde{g}$ is obtained by a conformal deformation of g were constructed in [16], showing a nice relation with the theory of isoparametric functions. Now it immediately follows from Theorem 4 that

- *Let (M, g) be a pseudo-Riemannian manifold and let $\tilde{g}$ an arbitrary metric on M. Then the complete lift metric $\tilde{g}^C$ is biharmonic with respect to (TM, g^C).*

Observe that the complete lift metrics g^C and $\tilde{g}^C$ are not conformally related neither harmonically related in the general situation.

3.3.2. The tangent map

Let $\varphi : (M, g) \to (N, h)$ be a map between pseudo-Riemannian manifolds. The *tangent map* is the induced map $T_\varphi : (TM, g^C) \to (TN, h^C)$ given by $T_\varphi(\xi) = d\varphi_p(v)$ for each $\xi = (p, v) \in TM$, where $p \in M$ and $v \in T_pM$. Harmonicity of tangent maps has been investigated by the last author in [17], showing that $T_\varphi : (TM, g^C) \to (TN, h^C)$ is harmonic if and only if $\varphi : (M, g) \to (N, h)$ is so. Since tangent maps are the tautological example of φ-morphisms, Theorem 4 applies in sharp contrast with the harmonic case

- *For any smooth map $\varphi : (M, g) \to (N, h)$, its tangent map $T_\varphi : (TM, g^C) \to (TN, h^C)$ is always biharmonic.*

3.3.3. Endomorphism fields

A $(1, 1)$-tensor field T on a pseudo-Riemannian manifold (M, g) induces a map $\mathcal{T} : TM \to TM$ defined by $\mathcal{T}(p, v) = (p, T_p(v))$ for all $(p, v) \in TM$. Clearly $\mathcal{T}$ is a id_M-morphism and hence Theorem 4 gives

- *For each $(1, 1)$-tensor field T on (M, g), its induced map $\mathcal{T} : TM \to TM$ is biharmonic.*

Once again observe that the above result is in sharp contrast with the harmonicity situation previously investigated in [18], where it was shown that $\mathcal{T} : TM \to TM$ is harmonic if and only if $\delta T = g^{ij}(\nabla_{\frac{\partial}{\partial x^i}} T)_j = 0$.

4. Biharmonic maps on cotangent bundles

4.1. The Riemannian extension

Let T^*M be the cotangent bundle of M and let $\pi : T^*M \to M$ be the natural projection on M. For each vector field X on M, the *evaluation map* ιX is the function $\iota X : T^*M \to \mathbb{R}$ given by $\iota X(p, \omega) = \omega(X_p)$ for all $(p, \omega) \in T^*M$. Following a complete analogy with the previously considered case of TM, two vector fields $\tilde{X}$ and $\tilde{Y}$ on T^*M coincide with each other if and only if $\tilde{X}(\iota Z) = \tilde{Y}(\iota Z)$ for all vector fields Z en M. Hence the complete lift X^C to T^*M of a vector field X on M is defined by $X^C(\iota Z) = \iota[X, Z]$ for all vector fields Z on M (see [10] for details).

It is important to observe that $(0, s)$-tensor fields on T^*M are characterized by their action on complete lifts as follows: $\tilde{T}(X_1^C, \ldots, X_s^C) = \tilde{S}(X_1^C, \ldots, X_s^C)$ for all vector fields $X_1, \ldots, X_s$ on M if and only if $\tilde{T} = \tilde{S}$, for all tensor fields $\tilde{T}, \tilde{S}$ of type $(0, s)$ on T^*M. Then, the *Riemannian extension* of a torsion-free connection D on M is defined by

$$g_D(X^C, Y^C) = -\iota(D_X Y + D_Y X).$$

It turns out that g_D is a symmetric $(0, 2)$-tensor field on T^*M and moreover, it defines a pseudo-Riemannian metric of signature (m, m).

In order to express the Riemannian extension and its curvature in local coordinates, let $(U, (x^i))$, $i = 1 \ldots m = \dim M$ be local coordinates on M and consider the induced coordinates on T^*M given by $(\pi^{-1}(U), (x^i, x_{\hat{i}}))$, $\hat{i} = i + m$, where $x_{\hat{1}}, \ldots, x_{\hat{m}}$ are the coordinate functions of the one-forms with respect to $\{dx^1, \ldots, dx^m\}$. Then one has the local expression

$$g_D = \begin{pmatrix} -2\, x_{\hat{k}}\, {}^D\Gamma_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix},$$

where ${}^D\Gamma_{ij}^k$ are the Christoffel symbols of the affine connection D and δ_i^j is the Kronecker's delta.

The Christoffel symbols ${}^{g_D}\Gamma_{AB}^C$, $A, B, C = 1, \ldots, 2m$, of the Riemannian extension g_D are related with those of the base affine connection D by the following

$$\begin{aligned} {}^{g_D}\Gamma^k &= \begin{pmatrix} {}^D\Gamma_{ij}^k & 0 \\ 0 & 0 \end{pmatrix}, \\ {}^{g_D}\Gamma^{\hat{k}} &= \begin{pmatrix} x_{\hat{l}} \left\{ -\dfrac{\partial\, {}^D\Gamma_{jk}^l}{\partial x^i} - \dfrac{\partial\, {}^D\Gamma_{ik}^l}{\partial x^j} + \dfrac{\partial\, {}^D\Gamma_{ij}^l}{\partial x^k} + 2\, {}^D\Gamma_{kt}^l\, {}^D\Gamma_{ij}^t \right\} & -{}^D\Gamma_{ik}^j \\ -{}^D\Gamma_{ik}^j & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

Moreover, the non-zero components of the curvature tensor ${}^{g_D}R$ of (T^*M, g_D) are given by:

$$
\begin{aligned}
&{}^{g_D}R_{ijk}^{l} = {}^{D}R_{ijk}^{l},\\
&{}^{g_D}R_{ijk}^{\hat{l}} = x_{\hat{l}}\left\{D_l{}^{D}R_{ijk}^{t} - D_k{}^{D}R_{ijk}^{l} + {}^{D}\Gamma_{ls}^{t}\,{}^{D}R_{ijk}^{s} + {}^{D}\Gamma_{ls}^{t}\,{}^{D}R_{klj}^{s} + {}^{D}\Gamma_{js}^{t}\,{}^{D}R_{lki}^{s} + {}^{D}\Gamma_{ks}^{t}\,{}^{D}R_{ijl}^{s}\right\},\\
&{}^{g_D}R_{\hat{i}jk}^{\hat{l}} = -{}^{D}R_{lkj}^{i}, \qquad {}^{g_D}R_{i\hat{j}k}^{\hat{l}} = -{}^{D}R_{lki}^{j}, \qquad {}^{g_D}R_{ij\hat{k}}^{\hat{l}} = -{}^{D}R_{ijl}^{k},
\end{aligned}
\tag{11}
$$

where ${}^{D}R$ denotes the curvature tensor of the affine manifold (M, D).

Remark 5. The geometry of the Riemannian extensions has been extensively studied in the literature, and it is known that Riemannian extensions nicely relate affine and metric properties. For example (M, D) is locally symmetric if and only if (T^*M, g_D) is so, (M, D) is projectively flat if and only if (T^*M, g_D) is locally conformally flat, (T^*M, g_D) is Einstein if and only if the Ricci tensor of (M, D) is skew-symmetric.

Despite the fact that the complete lift metric g^C on TM and the Riemannian extension g_D on T^*M have some similarities, the later are much less rigid due to the greater flexibility of affine geometry. The special situation when D is the Levi-Civita connection ∇ of a pseudo-Riemannian metric g on M will be clarified by analysing the musical isomorphisms between (TM, g^C) and (T^*M, g_∇).

The canonical projection $\pi : (T^*M, g_D) \to (M, D)$ is a totally geodesic submersion. Indeed, a simple calculation shows that its second fundamental form satisfies

$$
(\nabla d\pi)_{ij}^{k} = {}^{D}\Gamma_{ij}^{k} - {}^{g_D}\Gamma_{ij}^{k}, \qquad (\nabla d\pi)_{i\hat{j}}^{k} = -{}^{g_D}\Gamma_{i\hat{j}}^{k}, \qquad (\nabla d\pi)_{\hat{i}\hat{j}}^{k} = -{}^{g_D}\Gamma_{\hat{i}\hat{j}}^{k},
$$

from where one $\nabla d\pi = 0$ as a consequence of (10).

Let X be a vector field on a pseudo-Riemannian manifold (M, g). Its evaluation is the function $\iota X : T^*M \to \mathbb{R}$ given by $\iota X(\xi) = \iota X(p, \omega) = \omega(X_p)$, where $\omega \in \wedge^1(T_pM)$. As well as for the tangent bundle, evaluations of vector fields provide a large family of proper biharmonic functions on the cotangent bundle.

Theorem 6. *Let X be a vector field on an affine manifold (M, D). Then*

(1) $\iota X : (T^*M, g_D) \to \mathbb{R}$ *is harmonic if and only if* $\operatorname{trace}(DX) = 0$.
(2) $\iota X : (T^*M, g_D) \to \mathbb{R}$ *is a biharmonic function.*

Proof. Take coordinates $(x^1, \ldots, x^m)$ on M and put $X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}$ so that the evaluation $\iota X : T^*M \to \mathbb{R}$ reads in induced coordinates on T^*M as the function

$$
\iota X(x^i, x_{\hat{i}}) = \sum_{l=1}^{m} x_{\hat{l}} X^l.
$$

A direct calculation shows that its second fundamental form is given by

$$
\begin{aligned}
&(\nabla d\iota X)_{ij} = x_{\hat{l}}\left\{\frac{\partial^2 X}{\partial x^i \partial x^j} - {}^{D}\Gamma_{ij}^{k}\frac{\partial X^l}{\partial x^k} + \frac{\partial\, {}^{D}\Gamma_{jk}^{l}}{\partial x^i} + \frac{\partial\, {}^{D}\Gamma_{ik}^{l}}{\partial x^j} - \frac{\partial\, {}^{D}\Gamma_{ij}^{l}}{\partial x^k} - 2\,{}^{D}\Gamma_{kt}^{l}\,{}^{D}\Gamma_{ij}^{t} X^k\right\},\\
&(\nabla d\iota X)_{\hat{i}j} = D_{\frac{\partial}{\partial x^i}} X, \qquad (\nabla d\iota X)_{i\hat{j}} = D_{\frac{\partial}{\partial x^j}} X, \qquad (\nabla d\iota X)_{\hat{i}\hat{j}} = 0,
\end{aligned}
$$

where ${}^{D}\Gamma_{ij}^{k}$ are the Christoffel symbols of the connection D and $i, j, k, l = 1, \ldots, m = \dim M$, $\widehat{latin} = latin + m$. Since the inverse of the matrix of g_D is of the form

$$
(g_D)^{-1} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_i^j & 2\, x_{\hat{k}}\, {}^{D}\Gamma_{ij}^{k} \end{pmatrix},
\tag{12}
$$

the tension field is given by

$$
\tau(\iota X) = 2 \sum_{i=1}^{m} D_{\frac{\partial}{\partial x^i}} X = 2 \operatorname{trace}(DX),
$$

from where (1) follows.

Next, in order to consider the bitension field

$$
\tau^2(\iota X) = \operatorname{trace}_{g_D}(\nabla^2 d\iota X) = (g_D)^{AB}\left\{\frac{\partial^2 \tau(\iota X)}{\partial x_A \partial x_B} - {}^{g_D}\Gamma_{AB}^{C}\frac{\partial \tau(\iota X)}{\partial x_C}\right\},
$$

where ${}^{g_D}\Gamma_{AB}^{C}(A, B, C = 1, \ldots, 2m = 2\dim M)$ are the Christoffel symbols of the Levi-Civita connection of (T^*M, g_D). Since the inverse of the matrix of g_D is of the form (12) one only needs to show that $(\nabla^2 d\iota X)_{\hat{i}j} = 0$ and $(\nabla^2 d\iota X)_{\hat{i}\hat{j}} = 0$.

A straightforward calculation shows that

$$(\nabla^2 d\iota X)_{\hat{i}j} = \frac{\partial^2 \tau(\iota X)}{\partial x_{\hat{i}} \partial x^j} - {}^{g_D}\Gamma^k_{\hat{i}j} \frac{\partial \tau(\iota X)}{\partial x^k} - {}^{g_D}\Gamma^{\hat{k}}_{\hat{i}j} \frac{\partial \tau(\iota X)}{\partial x_{\hat{k}}} = 0,$$

and

$$(\nabla^2 d\iota X)_{\hat{i}\hat{j}} = \frac{\partial^2 \tau(\iota X)}{\partial x_{\hat{i}} \partial x_{\hat{j}}} - {}^{g_D}\Gamma^k_{\hat{i}\hat{j}} \frac{\partial \tau(\iota X)}{\partial x^k} - {}^{g_D}\Gamma^{\hat{k}}_{\hat{i}\hat{j}} \frac{\partial \tau(\iota X)}{\partial x_{\hat{k}}} = 0,$$

from where it follows that $\tau^2(\iota X) = 0$, thus showing that ιX is a biharmonic function on (T^*M, g_D). □

4.2. Morphisms between tangent and cotangent bundles

Let $\varphi : M \to N$, following the ideas of Section 3.2, a morphism between tangent and cotangent bundles is a smooth map that is linear on the fibres, i.e., it corresponds to one of the following:

(i) $\Phi : (TM, g^C) \to (T^*N, g_{D_N})$, where (M, g) is a pseudo-Riemannian manifold and D_N an affine connection on N, which expresses locally as

$$\Phi(x^i, x^{\bar{i}}) = \left(\varphi^\alpha(x^i), x^{\bar{l}} \Phi_{l\hat{\alpha}}(x^i)\right) \tag{13}$$

with tension field

$$\tau^\gamma(\Phi) = 0, \quad \tau^{\hat{\gamma}}(\Phi) = g^{ij} \left(\frac{\partial \Phi_{j\hat{\gamma}}}{\partial x^i} - {}^g\Gamma^k_{ij} \Phi_{k\hat{\gamma}} - {}^{D_N}\Gamma^\beta_{\alpha\gamma} \frac{\partial \varphi^\alpha}{\partial x^i} \Phi_{j\hat{\beta}} \right). \tag{14}$$

(ii) $\Psi : (T^*M, g_{D_M}) \to (TN, h^C)$, where D_M is an affine connection on M and (N, h) is a pseudo-Riemannian manifold, which expresses locally as

$$\Psi(x^i, x_{\hat{i}}) = \left(\varphi^\alpha(x^i), x_{\hat{l}} \Psi^{l\bar{\alpha}}(x^i)\right) \tag{15}$$

with tension field

$$\tau^\gamma(\Psi) = 0, \quad \tau^{\bar{\gamma}}(\Psi) = 2 \sum_{i=1}^n \left(\frac{\partial \Psi^{i\hat{\gamma}}}{\partial x^i} + {}^{D_M}\Gamma^i_{ik} \Psi^{k\hat{\gamma}} + {}^h\Gamma^\gamma_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \Psi^{i\hat{\beta}} \right). \tag{16}$$

(iii) $\Upsilon : (T^*M, g_{D_M}) \to (T^*N, g_{D_N})$, where D_M and D_N are affine connections on M and N, which expresses locally as

$$\Upsilon(x^i, x_{\hat{i}}) = \left(\varphi^\alpha(x^i), x_{\hat{l}} \Upsilon^l_{\hat{\alpha}}(x^i)\right) \tag{17}$$

with tension field

$$\tau^\gamma(\Upsilon) = 0, \quad \tau^{\hat{\gamma}}(\Upsilon) = 2 \sum_{i=1}^n \left(\frac{\partial \Upsilon^i_{\hat{\gamma}}}{\partial x^i} + {}^{D_M}\Gamma^i_{ik} \Upsilon^k_{\hat{\gamma}} - {}^{D_N}\Gamma^\beta_{\alpha\gamma} \frac{\partial \varphi^\alpha}{\partial x^i} \Upsilon^j_{\hat{\beta}} \right). \tag{18}$$

A completely analogous calculation as that carried out in Section 3.2 shows that

Theorem 7. *Let (M, g) and (N, h) be pseudo-Riemannian manifolds and let D and D^* be affine connections on M and N, respectively. Then the bundle morphisms Φ, Ψ, Υ defined by (13), (15), (17) are biharmonic maps.*

An immediate application of previous results is the following

(1) *Let D, D^* be arbitrary torsion-free connections on a smooth manifold M. Then the Riemannian extensions g_D and g_{D^*} are biharmonic metrics on T^*M.*

4.3. The musical isomorphisms

Let (M, g) be a pseudo-Riemannian manifold. Since the metric g is nondegenerated, the musical isomorphisms $\flat : X \mapsto \flat(X) = g(X, \cdot)$ and $\sharp : \omega \mapsto \sharp(\omega)$, where $\sharp(\omega)$ is the vector field on M defined by $g(\sharp(\omega), Y) = \omega(Y)$, define a bundle isomorphism between the tangent and the cotangent bundle of M. As an immediate application of the previous results one has that

- *Let (M, g) be a pseudo-Riemannian manifold and D a torsion-free connection on M. Then the musical isomorphisms $\flat : (TM, g^C) \to (T^*M, g_D)$ and $\sharp : (T^*M, g_D) \to (TM, g^C)$ are biharmonic maps.*

Further observe that the musical isomorphisms are not harmonic in general. Indeed, a straightforward calculation shows that the corresponding tension fields are given by

$$\tau(\flat)^k = 0, \qquad \tau(\flat)^{\hat{k}} = 2g^{ij} \left\{ \frac{\partial g_{ki}}{\partial x^j} - {}^g\Gamma^t_{ij} g_{kt} - {}^D\Gamma^t_{kj} g_{ti} \right\},$$

and

$$\tau(\sharp)^k = 0, \qquad \tau(\sharp)^{\bar{k}} = 2\sum_i \left\{ \frac{\partial g^{ki}}{\partial x^i} + {}^D\Gamma_{it}^i g^{k\gamma} + {}^g\Gamma_{i\gamma}^k g^{it} \right\}.$$

Remark 8. A connection D on a pseudo-Riemannian manifold (M, g) is said to be g-conjugate to the Levi-Civita connection ∇ if and only if $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, D_X Z)$, for all vector fields X, Y, Z on M. Now it follows from the expressions above that, if D is g-conjugate to the Levi-Civita connection of g, then $\flat$ is harmonic.

Remark 9. A simple observation of the components of the second fundamental form of the musical isomorphisms shows that

$$(\nabla d\flat)_{ij}^k = -{}^g\Gamma_{ij}^k + {}^D\Gamma_{ij}^k, \qquad (\nabla d\sharp)_{ij}^k = {}^g\Gamma_{ij}^k - {}^D\Gamma_{ij}^k,$$

from where it immediately follows that the musical isomorphisms are totally geodesic isomorphisms if and only if the affine connection D coincides with the Levi-Civita connection of (M, g) [12]. Indeed, in this case they induce global isometries.

Theorem 10. *Let (M, g) be a pseudo-Riemannian manifold. Then the musical isomorphisms induced by g are isometries between (TM, g^C) and (T^*M, g_∇), where ∇ is the Levi-Civita connection of g.*

Proof. Let $\flat : TM \to T^*M$ given in local coordinates by

$$\flat(x^1, \dots, x^m, x^{\bar{1}}, \dots, x^{\bar{m}}) = \left(x^1, \dots, x^m, x^{\bar{l}} g_{1l}, \dots, x^{\bar{l}} g_{ml}\right).$$

Then

$$\flat_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + x^{\bar{l}} \frac{\partial g_{rl}}{\partial x^i} \frac{\partial}{\partial x_{\hat{r}}}, \quad \text{and} \quad \flat_*\left(\frac{\partial}{\partial x^{\bar{i}}}\right) = g_{ir} \frac{\partial}{\partial x_{\hat{r}}}.$$

Hence one has

$$\begin{aligned}
(\flat^* g_\nabla)\left(\frac{\partial}{\partial x^{\bar{i}}}, \frac{\partial}{\partial x^{\bar{j}}}\right) &= 0, \\
(\flat^* g_\nabla)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{j}}}\right) &= g_{js} g_\nabla\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_{\hat{s}}}\right) = g_{ij}, \\
(\flat^* g_\nabla)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g_\nabla\left(\frac{\partial}{\partial x^i} + x^{\bar{l}} \frac{\partial g_{rl}}{\partial x^i} \frac{\partial}{\partial x_{\hat{r}}}, \frac{\partial}{\partial x^j} + x^{\bar{l}} \frac{\partial g_{sl}}{\partial x^j} \frac{\partial}{\partial x_{\hat{s}}}\right) \\
&= x^{\bar{l}}\left\{-2{}^g\Gamma_{ij}^r g_{lr} + \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j}\right\} = x^{\bar{l}} \frac{\partial g_{ij}}{\partial x^l},
\end{aligned}$$

which shows that $\flat : (TM, g^C) \to (T^*M, g_\nabla)$ is an isometry.

In an analogous way, considering the expression in local coordinates of $\sharp : T^*M \to TM$ given by

$$\sharp(x^1, \dots, x^m, x_{\hat{1}}, \dots, x_{\hat{m}}) = \left(x^1, \dots, x^m, x_{\hat{l}} g^{1l}, \dots, x_{\hat{l}} g^{ml}\right),$$

one has

$$\sharp_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + x_{\hat{l}} \frac{\partial g^{rl}}{\partial x^i} \frac{\partial}{\partial x^{\bar{r}}}, \quad \text{and} \quad \sharp_*\left(\frac{\partial}{\partial x_{\hat{i}}}\right) = g^{ir} \frac{\partial}{\partial x^{\bar{r}}}.$$

Hence,

$$\begin{aligned}
(\sharp^* g^C)\left(\frac{\partial}{\partial x_{\hat{i}}}, \frac{\partial}{\partial x_{\hat{j}}}\right) &= 0 \\
(\sharp^* g^C)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_{\hat{j}}}\right) &= g^{rs} g^C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_{\hat{s}}}\right) = g^{js} g_{is} = \delta_i^j \\
(\sharp^* g^C)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g^C\left(\frac{\partial}{\partial x^i} + x_{\hat{l}} \frac{\partial g^{rl}}{\partial x^i} \frac{\partial}{\partial x^{\bar{r}}}, \frac{\partial}{\partial x^j} + x_{\hat{l}} \frac{\partial g^{sl}}{\partial x^j} \frac{\partial}{\partial x^{\bar{s}}}\right) \\
&= x_{\hat{l}}\left\{g^{rl} \frac{\partial g_{ij}}{\partial x^r} + g_{ik} \frac{\partial g^{kl}}{\partial x^j} + g_{jk} \frac{\partial g^{kl}}{\partial x^i}\right\} = -2 x_{\hat{l}} \Gamma_{ij}^l,
\end{aligned}$$

which completes the proof. □

Remark 11. Recall that the composition $F \circ f$ of a biharmonic map f with a totally geodesic one F remains biharmonic and thus, since the musical isomorphisms between (TM, g^C) and (T^*M, g_∇) are totally geodesic, a one form ω defines a biharmonic section $\omega : (M, g) \to (T^*M, g_\nabla)$ if and only if the corresponding vector field $\sharp\omega$ defines a biharmonic section $\sharp\omega : (M, g) \to (TM, g^C)$ and conversely.

Acknowledgements

Supported by projects GRC2013-045, MTM2013-41335-P and EM2014/009 with FEDER funds (Spain).

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