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Uniqueness of the momentum map

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ABSTRACT

We give a detailed discussion of existence and uniqueness of the momentum map associated to Poisson Lie actions, which was defined by Lu. We introduce a weaker notion of momentum map, called infinitesimal momentum map, which is defined on one-forms and we analyze its integrability to the Lu's momentum map. Finally, the uniqueness of the Lu's momentum map is studied by describing, explicitly, the tangent space to the space of momentum maps.

1. Introduction

In this paper we focus on the study of properties of the momentum map associated to actions of Poisson Lie groups on Poisson manifolds.

The momentum map for actions of Lie groups (without additional structures) on Poisson manifolds provides a mathematical formalization of the notion of conserved quantity associated to symmetries of a dynamical system. The definition of momentum map in this standard setting only requires a *canonical* action (i.e. an action preserving the Poisson structure on the manifold) and its existence is guaranteed whenever the infinitesimal generators of the Lie algebra action are Hamiltonian vector fields (modulo vanishing of a certain Lie algebra cohomology class), as can be seen e.g. in [1, Chapter 10] and [2, Chapter 4]. A generalization of the momentum map to the case of actions of Poisson Lie groups on Poisson manifolds has been introduced by Lu in [3,4]. The associated reduction theory has been studied by Lu (see e.g. [3]) in the case of actions on symplectic manifolds and for actions on Poisson manifolds in [5]. It is worthwhile to mention that Poisson Lie group actions on Poisson manifolds naturally appear in the study of R-matrices (see e.g. [6]); thus, studying the properties of the associated momentum map can be useful for the comprehension of the integrable systems associated to the R-matrices.

The detailed construction of Lu's momentum map and its basic properties are recalled in the following section. Basically, given a Poisson Lie group $(\mathrm{G}, \pi_{\mathrm{G}})$ one introduces its dual $(\mathrm{G}^*, \pi_{\mathrm{G}^*})$ and, under fairly general conditions, G^* carries a Poisson action of G called dressing action (and vice versa). The Lie algebra $\mathfrak{g}$ of G is naturally identified with the space of (left)-invariant one-forms on G^*:

$$\theta : \mathfrak{g} \to \Omega^1(\mathrm{G}^*)^{\mathrm{G}^*} : \xi \mapsto \theta_\xi. \tag{1.1}$$

Given a Poisson manifold (M, π) with a Poisson action of G, a momentum map is a (smooth) Poisson map

$$J : M \to \mathrm{G}^*, \tag{1.2}$$

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such that the infinitesimal action $\mathfrak{g} \to \Gamma(TM) : \xi \mapsto \xi_M$ is induced by J via

$$\xi_M = \pi^\sharp(J^*(\theta_\xi)). \tag{1.3}$$

A standard example of a momentum map is the identity map $G^* \to G^*$, which induces the infinitesimal dressing action.

On one hand, the Poisson structure on G gives its Lie algebra a structure of a Lie bialgebra $(\mathfrak{g}, \delta)$ and hence a structure of Gerstenhaber algebra on $\wedge^\bullet \mathfrak{g}$. On the other hand, the Poisson bracket on M makes $\Omega^1(M)$ into a Lie algebra with bracket $[\cdot,\cdot]_\pi$ which also induces a structure of Gerstenhaber algebra on $\Omega^\bullet(M)$. This allows us to observe that the map α, which associates to an element ξ of $\mathfrak{g}$ a one-form $\alpha_\xi = J^*(\theta_\xi)$, can be generalized to a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\cdot,\cdot]) \longrightarrow (\Omega^\bullet(M), \mathrm{d}_{dR}, [\cdot,\cdot]_\pi). \tag{1.4}$$

We refer to α as the *infinitesimal momentum map* (cf. Section 3.1 and Proposition 3.2). The existence of the infinitesimal momentum map has been discussed in [7] and its rigidity properties have been recently studied in [8] as a generalization of the results obtained in [9]. A reduction theory associated to the infinitesimal momentum map can be found in [10].

The main subject of this paper is the study of the properties of the infinitesimal momentum map and its relation to the usual momentum map. In particular, we show under which conditions it integrates to the usual momentum map. The fact that α is map of Gerstenhaber algebras reduces to two equations

$$\alpha_{[\xi,\eta]} = [\alpha_\xi, \alpha_\eta]_\pi \tag{1.5}$$

$$\mathrm{d}\alpha_\xi = \alpha \wedge \alpha \circ \delta(\xi). \tag{1.6}$$

We observe that Eq. (1.6) is a Maurer–Cartan type equation; in fact, when $M = G^*$ it is precisely the Maurer–Cartan equation for the Lie group G^*. When $\Omega^\bullet$ is formal, it admits explicit solution modulo gauge equivalence and we get the following (cf. Theorem 3.8)

Theorem 1.1. *Suppose that M is a Kähler manifold. The set of gauge equivalence classes of $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ satisfying the equation*

$$\mathrm{d}\alpha_\xi = \alpha \wedge \alpha \circ \delta(\xi) \tag{1.7}$$

is in bijective correspondence with the set of the cohomology classes $c \in H^1(M, \mathfrak{g}^*)$[1] satisfying

$$[c, c] = 0. \tag{1.8}$$

The conditions under which an infinitesimal momentum map α integrates to the momentum map J (cf. Theorem 3.6 for the details) can be summarized in the following

Theorem 1.2. *Let (M, π) be a Poisson manifold and $\alpha : \mathfrak{g} \to \Omega^1(M)$ an infinitesimal momentum map. Suppose that M and G are simply connected and G is compact. Then $\mathscr{D} = \{\alpha_\xi - \theta_\xi,\ \xi \in \mathfrak{g}\}$ generates an involutive distribution on $M \times G^*$ and a leaf $J_{\mathscr{F}}$ of $\mathscr{D}$ is a graph of a momentum map if*

$$\pi(\alpha_\xi, \alpha_\eta) - \pi_{G^*}(\theta_\xi, \theta_\eta)|_{\mathscr{F}} = 0, \quad \xi, \eta \in \mathfrak{g}. \tag{1.9}$$

In Section 3.2 we study concrete cases of this globalization question and prove the existence and uniqueness/non-uniqueness of a momentum map associated to a given infinitesimal momentum map when the dual Poisson Lie group is abelian and, respectively, the Heisenberg group. For the second case the result is as follows (cf. Theorem 3.10).

Theorem 1.3. *Let G be a Poisson Lie group acting on a Poisson manifold M with an infinitesimal momentum map α and such that G^* is the Heisenberg group. Let ξ, η, ζ denote the basis of $\mathfrak{g}$ dual to the standard basis x, y, z of $\mathfrak{g}^*$, with z central and $[x, y] = z$. Then*

$$\pi(\alpha_\xi, \alpha_\eta) = k \tag{1.10}$$

where k is a constant on M. The form α lifts to a momentum map $J : M \to G^*$ if and only if $k = 0$. When $k = 0$ the set of momentum maps with given α is one dimensional with free transitive action of $\mathbb{R}$.

Finally, in the last section, we study the question of infinitesimal deformations of a given momentum map. The main result is Theorem 4.1, which describes explicitly the space tangent to the space of momentum maps at a given point. The main result can be formulated as a statement that the space of momentum maps has a structure of manifold (in an appropriate $\mathscr{C}^\infty$ topology). Let $H : M \to \mathfrak{g}^*$ be a smooth map. We denote by H_ξ the corresponding smooth map on M given by the pullback of H.

[1] Here c can be interpreted as an element of $\mathrm{Hom}(\mathfrak{g}^*, \mathrm{H}^\bullet_{\mathrm{dR}}(M))$, which is a differential graded Lie algebra.

Theorem 1.4. *Infinitesimal deformations of a momentum map are given by smooth maps $H : M \to \mathfrak{g}^*$ satisfying the equations*

$$\xi_M H_\eta - \eta_M H_\xi = H_{[\xi,\eta]} \tag{1.11}$$

$$\{H_\xi, \cdot\} = -\pi^\sharp(\alpha_{\mathrm{ad}^*_H \xi}) \tag{1.12}$$

for all $\xi, \eta \in \mathfrak{g}$. Here ad^* denotes the coadjoint action of G *on $\mathfrak{g}^*$.*

This theorem has the following corollary (cf. Corollary 4.2).

Corollary 1.5. *Suppose that* G *is a compact and semisimple Poisson Lie group with Poisson action on a Poisson manifold M and with a momentum map J. Any smooth deformation of J is given by integrating a Hamiltonian flow on M commuting with the action of* G.

2. Preliminaries: Poisson actions and momentum maps

In this section we recall the notion of the momentum map, introduced by Lu in [3,4], in the setting of actions of Poisson Lie groups on Poisson manifolds. A Poisson Lie group $(\mathrm{G}, \pi_\mathrm{G})$ is a Lie group G equipped with a multiplicative Poisson structure π_G. The corresponding Lie algebra $\mathfrak{g}$ inherits an extra structure $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, defined by the linearization of π_G at e. This is often called cobracket, since it provides a Lie algebra structure to the dual space $\mathfrak{g}^*$; for this reason, the pair $(\mathfrak{g}, \delta)$ is said to be the Lie bialgebra correspondent to the Poisson Lie group $(\mathrm{G}, \pi_\mathrm{G})$. Integrating the Lie algebra $\mathfrak{g}^*$ we get a Lie group G^*, which is also equipped with a multiplicative Poisson structure π_{G^*}. A one-to-one correspondence between Poisson Lie groups and Lie bialgebras has been established by Drinfel'd [11], under the condition that G is connected and simply connected (we keep these assumptions throughout the paper).

Let $(\mathrm{G}, \pi_\mathrm{G})$ be a Poisson Lie group and (M, π) a Poisson manifold.

Definition 2.1. An action $\Phi : \mathrm{G} \times M \to M$ is said to be a **Poisson action** if it satisfies

$$\{f \circ \Phi, g \circ \Phi\}_{\mathrm{G}\times M} = \{f, g\}_M \circ \Phi, \quad \forall f, g \in \mathscr{C}^\infty(M), \tag{2.1}$$

where $\{\cdot,\cdot\}_{\mathrm{G}\times M}$ and $\{\cdot,\cdot\}_M$ are the Poisson bracket induced by $\pi_\mathrm{G} \oplus \pi$ and π, respectively.

Given an action $\Phi : \mathrm{G} \times M \to M$, we denote by $\xi \mapsto \xi_M$ the Lie algebra anti-homomorphism from $\mathfrak{g}$ to $\Gamma(TM)$ which defines the infinitesimal generator of this action.

Canonical actions of Lie groups on Poisson manifolds are a class of examples of Poisson actions. Indeed, if G carries the trivial Poisson structure, the action is Poisson if and only if it preserves π, i.e. if and only if it is canonical. In this setting the definition of momentum map reads:

Definition 2.2. A **momentum map** for the Poisson action $\Phi : \mathrm{G} \times M \to M$ is a map $J : M \to \mathrm{G}^*$ such that

$$\xi_M = \pi^\sharp(J^*(\theta_\xi)) \tag{2.2}$$

where θ_ξ is the left invariant 1-form on G^* defined by Eq. (1.1) and J^* is the pullback to one-forms $\Omega^1(\mathrm{G}^*) \to \Omega^1(M)$.

In other words, the momentum map generates ξ_M by means of the following construction

$$\mathfrak{g} \xrightarrow{\theta} \Omega^1(\mathrm{G}^*) \xrightarrow{J^*} \Omega^1(M) \xrightarrow{\pi^\sharp} \Gamma(TM)$$

where, for $\xi \in \mathfrak{g}$, $\alpha_\xi = J^*(\theta_\xi)$. Notice that the map θ is a Lie algebra morphism. It is useful to recall that a Poisson structure π defines a skew-symmetric operation $[\cdot,\cdot]_\pi : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$ by

$$[\alpha, \beta]_\pi = \mathscr{L}_{\pi^\sharp(\alpha)}\beta - \mathscr{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)). \tag{2.3}$$

Furthermore, it provides $\Omega^1(M)$ with a Lie algebra structure such that $\pi^\sharp$ is a Lie algebra morphism (for more details see e.g. [12]).

In general, J^* is not a Lie algebra homomorphism; for this reason we need the concept of equivariance of the momentum map.

Definition 2.3. A momentum map for the Poisson action Φ is said to be G-**equivariant** if it intertwines Φ and the dressing action of G on G^*.

It is useful to recall that a momentum map is G-equivariant if and only if it is a Poisson map, i.e.

$$J_*\pi = \pi_{\mathrm{G}^*}. \tag{2.4}$$

The proof of this claim can be found in [3]. Finally, we can say that a **Hamiltonian action** in this context is a Poisson action induced by an G-equivariant momentum map. This definition generalizes Hamiltonian actions in the canonical setting of Lie group actions.

3. The infinitesimal momentum map

The conditions for the existence and the uniqueness of the momentum map J are quite complicated, as it has been showed by Ginzburg in [7]. For this reason the author introduces an infinitesimal version of the momentum map, in terms of one-forms and he proves its existence and uniqueness. In the following we study the conditions under which the infinitesimal momentum map determines a momentum map J. We describe the theory of reconstruction of the momentum map J from the infinitesimal one α in two explicit cases. Finally, we provide the conditions which ensure the uniqueness of the momentum map.

3.1. The structure of a momentum map

Given the Poisson Lie group G^*, we identify $\mathfrak{g}$ with the space of left invariant 1-forms on G^* via the map θ defined in Eq. (1.1); this space is closed under the bracket defined by π_{G^*} and the induced bracket on $\mathfrak{g}$, by the above identification, coincides with the original Lie bracket on $\mathfrak{g}$ (the proof can be found in [13]). For any $\xi \in \mathfrak{g}$, we denote by θ_ξ the left invariant form on G^*, whose value at identity is ξ. The basic property of θ's is the Maurer–Cartan equation for G^*:

$$d\theta_\xi = \theta \wedge \theta \circ \delta(\xi). \tag{3.1}$$

Let us denote by $\mathrm{ad}^*_x : \mathfrak{g} \to \mathfrak{g}$ the coadjoint action for a fixed $x \in \mathfrak{g}^*$.

Proposition 3.1. *Let θ_ξ, θ_η be two left invariant one-forms on G^*, such that $\theta_\xi(e) = \xi$, $\theta_\eta(e) = \eta$. Then we have*

(i) *The map $\theta : \mathfrak{g} \to \Omega^1(G^*)^{G^*}$ is a Lie algebra morphism, i.e.*

$$\theta_{[\xi,\eta]} = [\theta_\xi, \theta_\eta]_{\pi_{G^*}}; \tag{3.2}$$

(ii) *For any left-invariant vector field $X \in \Gamma(TG^*)$ and the corresponding $x \in \mathfrak{g}^*$*

$$\mathscr{L}_X \pi_{G^*}(\theta_\xi, \theta_\eta) = x([\xi, \eta]) + \pi_{G^*}(\theta_{\mathrm{ad}^*_x \xi}, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_{\mathrm{ad}^*_x \eta}). \tag{3.3}$$

Proof. (i) First, we prove that $[\theta_\xi, \theta_\eta]_{\pi_{G^*}}$ is a left-invariant 1-form. Let us consider an element $x \in \mathfrak{g}^*$ and the corresponding left-invariant vector field $X \in \Gamma(TG^*)$. We contract X with the bracket $[\theta_\xi, \theta_\eta]_{\pi_{G^*}}$ to show that we obtain a constant. Using the properties of Lie derivative we easily get

$$\begin{aligned} \iota_X d\pi_{G^*}(\theta_\xi, \theta_\eta) &= (\mathscr{L}_X \pi_{G^*})(\theta_\xi, \theta_\eta) + \pi_{G^*}(\mathscr{L}_X \theta_\xi, \theta_\eta) + \pi_{G^*}(\theta_\xi, \mathscr{L}_X \theta_\eta) \\ \iota_X \mathscr{L}_{\pi^\sharp_{G^*}(\theta_\xi)} \theta_\eta &= \mathscr{L}_X \pi_{G^*}(\theta_\eta, \theta_\xi) - \pi_{G^*}(\theta_\xi, \mathscr{L}_X \theta_\eta) \\ \iota_X \mathscr{L}_{\pi^\sharp_{G^*}(\theta_\eta)} \theta_\xi &= \mathscr{L}_X \pi_{G^*}(\theta_\eta, \theta_\xi) - \pi_{G^*}(\theta_\eta, \mathscr{L}_X \theta_\xi). \end{aligned}$$

Thus, from the formula

$$\mathscr{L}_X \pi_{G^*}(\theta_\xi, \theta_\eta) = (\mathscr{L}_X \pi_{G^*})(\theta_\xi, \theta_\eta) + \pi_{G^*}(\mathscr{L}_X \theta_\xi, \theta_\eta) + \pi_{G^*}(\theta_\xi, \mathscr{L}_X \theta_\eta) \tag{3.4}$$

we get

$$\iota_X [\theta_\xi, \theta_\eta]_{\pi_{G^*}} = (\mathscr{L}_X \pi_{G^*})(\theta_\xi, \theta_\eta).$$

Finally, since $\mathscr{L}_X \pi_{G^*}(e) = {}^t\delta(x)$ and ${}^t\delta(x)(\xi, \eta) = x([\xi, \eta])$, Eq. (3.2) is proved.

(ii) Moreover, we have

$$\begin{aligned} \mathscr{L}_X \pi_{G^*}(\theta_\xi, \theta_\eta) &= (\mathscr{L}_X \pi_{G^*})(\theta_\xi, \theta_\eta) + \pi_{G^*}(\mathscr{L}_X \theta_\xi, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_\eta) \\ &= {}^t\delta(x)(\xi, \eta) + \pi_{G^*}(\theta_{\mathrm{ad}^*_x \xi}, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_{\mathrm{ad}^*_x \eta}), \end{aligned}$$

since $\mathscr{L}_X \theta_\xi = \theta_{\mathrm{ad}^*_x \xi}$. This concludes the proof. □

As a direct consequence of the above proposition, we can prove the following

Proposition 3.2. *Given a Poisson action $\Phi : G \times M \to M$ with G-equivariant momentum map $J : M \to G^*$, the forms $\alpha_\xi = J^*(\theta_\xi)$ satisfy the following identities:*

$$\alpha_{[\xi,\eta]} = [\alpha_\xi, \alpha_\eta]_\pi \tag{3.5}$$

$$d\alpha_\xi = \alpha \wedge \alpha \circ \delta(\xi). \tag{3.6}$$

Proof. Let us prove the first relation.

$$\begin{aligned} [\alpha_\xi, \alpha_\eta]_\pi &= [J^*(\theta_\xi), J^*(\theta_\eta)]_\pi \\ &\overset{(a)}{=} J^*([\theta_\xi, \theta_\eta]_{\pi_{G^*}}) \\ &\overset{(b)}{=} J^* \theta_{[\xi,\eta]} \\ &= \alpha_{[\xi,\eta]}. \end{aligned}$$

We used the G-equivariance of J in (a) and Eq. (3.2) in (b). The second relation is quite easy, in fact we have

$$\begin{aligned} d\alpha_\xi &= dJ^*(\theta_\xi) \\ &= J^* d\theta_\xi \\ &= J^*(\theta) \wedge J^*(\theta) \circ \delta(\xi) \\ &= \alpha \wedge \alpha \circ \delta(\xi). \quad \square \end{aligned}$$

Last proposition motivates a new, weaker, definition of momentum map in terms of one-forms on M.

Definition 3.3. An **infinitesimal momentum map** for the action Φ is a map $\alpha : \mathfrak{g} \to \Omega^1(M)$ such that

(i) It induces the fundamental vector field of Φ via

$$\xi_M = \pi^\sharp(\alpha_\xi) \tag{3.7}$$

(ii) It is a Lie algebra morphism

$$\alpha_{[\xi,\eta]} = [\alpha_\xi, \alpha_\eta]_\pi \tag{3.8}$$

(iii) It satisfies the Maurer–Cartan condition

$$d\alpha_\xi = \alpha \wedge \alpha \circ \delta(\xi). \tag{3.9}$$

As already said, a similar definition was already introduced in [7] (what is called *cotangent lift*) and the problem of its existence and uniqueness has been exhaustively discussed. It is important to mention that the definition of cotangent lift does not require conditions (3.8) and (3.9).

The above definition can be generalized to a morphism of Gerstenhaber algebras, which is defined as follows:

Definition 3.4. A **Gerstenhaber algebra** is a triple $(A, \wedge, [\cdot,\cdot])$ with $A = \oplus_{i\in\mathbb{Z}} A^i$ such that $(A, \wedge)$ is a graded commutative associative algebra, $(A, [\cdot,\cdot])$ with $A^{(i)} = A^{i+1}$, is a graded Lie algebra, and for each $a \in A^{(i)}$ one has that $[a, \cdot]$ is a derivation of degree i with respect to $\wedge$.

A complete description of such structure can be found in [14].

On one hand, the Poisson structure on G gives its Lie algebra a structure of a Lie bialgebra $(\mathfrak{g}, \delta)$ and hence a structure of Gerstenhaber algebra on $\wedge^\bullet \mathfrak{g}$. On the other hand, the Poisson bracket on M induces a structure of Lie algebra on $\Omega^1(M)$ with bracket $[\cdot,\cdot]_\pi$; this induces a Gerstenhaber algebra structure on $\Omega^\bullet(M)$. Thus, the following generalization is quite natural:

Definition 3.5. An **infinitesimal momentum map** for the Poisson action Φ is a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\cdot,\cdot]) \longrightarrow (\Omega^\bullet(M), d_{dR}, [\cdot,\cdot]_\pi). \tag{3.10}$$

Notice that the induced action does not reduce to the canonical one when the Poisson structure on G is trivial. In fact, if $\delta = 0$ the Maurer–Cartan condition implies that α_X is a closed form, but in general this form is not exact. If, for example, M is simply connected, α_X is also exact and we can recover the usual definition of momentum map and Hamiltonian action.

The following theorem describes the conditions in which an infinitesimal momentum map $\alpha : \mathfrak{g} \to \Omega^1(M)$ determines a momentum map $J : M \to G^*$.

Theorem 3.6. *Let (M, π) be a Poisson manifold and $\alpha : \mathfrak{g} \to \Omega^1(M)$ an infinitesimal momentum map. Then:*

(i) *The set $\{\alpha_\xi - \theta_\xi,\ \xi \in \mathfrak{g}\}$ generate an involutive distribution $\mathscr{D}$ on $M \times G^*$.*

(ii) *If M is connected and simply connected, the leaves $\mathscr{F}$ of $\mathscr{D}$ coincide with the graphs of the maps $J_{\mathscr{F}} : M \to G^*$ satisfying $\alpha = J^*_{\mathscr{F}}(\theta)$ and G^* acts freely and transitively on the space of leaves by left multiplication on the second factor.*

(iii) *The vector fields $\pi^\sharp(\alpha_\xi)$ define a morphism from $\mathfrak{g}$ to $\Gamma(TM)$. If they integrate to the action $\Phi : G \times M \to M$, then Φ is a Poisson action and $J_{\mathscr{F}}$ is its G-equivariant momentum map if and only if the functions*

$$\varphi(\xi, \eta) := \pi(\alpha_\xi, \alpha_\eta) - \pi_{G^*}(\theta_\xi, \theta_\eta) \tag{3.11}$$

satisfy the condition

$$\varphi(\xi, \eta)|_{\mathscr{F}} = 0 \tag{3.12}$$

for all $\xi, \eta \in \mathfrak{g}$.

Proof. (i) Using the Maurer–Cartan equations for α and θ, the $\mathfrak{g}$-valued form $\alpha - \theta$ on $M \times G^*$ satisfies $d(\alpha - \theta) = (\alpha - \theta) \wedge (\alpha - \theta)$. As a consequence, from the dual formulation of the Frobenius theorem, it defines a distribution on $M \times G^*$. Let $\mathscr{F}$ be a generic leaf and let p_i, $i = 1, 2$ denote the projection onto the first (resp. second) factor in $M \times G^*$. Since the linear span of θ_ξ, $\xi \in \mathfrak{g}$ at any point $u \in G^*$ coincides with $T^*_u G^*$, the restriction of the projection $p_1 : M \times G \to M$ to $\mathscr{F}$ is an immersion. Finally, since $\dim(M) = \dim(\mathscr{F})$, p_1 is a covering map.

(ii) Under the hypothesis that M is simply connected, p_1 is a diffeomorphism and

$$J_{\mathscr{F}} = p_2 \circ p_1^{-1}$$

is a smooth map whose graph coincides with $\mathscr{F}$. It is immediate, that $\alpha = J_{\mathscr{F}}^*(\theta)$. Moreover, since θ's are left invariant it follows immediately that the action of G^* on the space of leaves by left multiplication of the second factor is free and transitive.

(iii) Suppose that Eq. (3.12) is satisfied. Then

$$\pi(\alpha_\xi, \alpha_\eta) = J_{\mathscr{F}}^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\eta))$$

and $\ker J_{\mathscr{F}*}$ coincides with the set of zero's of $\alpha_\xi,\ \xi \in \mathfrak{g}$. Hence, $J_{\mathscr{F}}$ is a Poisson map and, in particular,

$$J_{\mathscr{F}*}(\pi^\sharp(\alpha_\xi)) = \pi^\sharp_{\mathrm{G}^*}(\theta_\xi),$$

i.e. it is a G-equivariant map. □

3.1.1. The Kähler case

Let us consider the case in which M is a Kähler manifold (the compatibility between Poisson structures and Kähler forms has been studied, e.g. [15,16]). This case plays an important role in our theory, since the Maurer–Cartan equation (3.9) can be explicitly solved. More precisely, we can define an equivalent class of solutions of the Maurer–Cartan equation as follows:

Definition 3.7. Two solutions α and α' of Eq. (3.9) are said to be **gauge equivalent**, if there exists a smooth function $H : M \to \mathfrak{g}^*$ such that

$$\alpha' = \exp(\mathrm{ad}H)(\alpha) + \int_0^1 \mathrm{d}t \exp t(\mathrm{ad}H)(\mathrm{d}H). \tag{3.13}$$

The gauge equivalence classes defined above can be related to the cohomological classes $H^1(M, \mathfrak{g}^*)$, as proved in the following

Theorem 3.8. *Suppose that M is a Kähler manifold. The set of gauge equivalence classes of $\alpha \in \Omega^1(M, \mathfrak{g}^*)$, satisfying Eq. (3.9) is in bijective correspondence with the set of the cohomology classes $c \in H^1(M, \mathfrak{g}^*)$ satisfying*

$$[c, c] = 0. \tag{3.14}$$

Proof. Given a manifold M, the de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ always has a structure of commutative differential graded algebra (CDGA). If M is a Kähler manifold, it follows that this CDGA is formal (as proved in [17]) i.e. there is a map between the complex $\Omega^\bullet(M)$ and its cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$. As a consequence,

$$(\mathrm{Hom}\,(\mathfrak{g}^*, \Omega^\bullet(M)), \mathrm{d}, [\cdot, \cdot]) \tag{3.15}$$

is a formal differential graded Lie algebra and, in particular, there exists a bijection between the equivalence classes of Maurer–Cartan elements of $(\mathrm{Hom}\,(\mathfrak{g}^*, \Omega^\bullet(M)), \mathrm{d}, [\cdot, \cdot])$ and Maurer–Cartan elements of $(\mathrm{Hom}\,(\mathfrak{g}^*, \mathrm{H}^\bullet_{\mathrm{dR}}(M)), [\cdot, \cdot])$. Since a Maurer–Cartan element in $(\mathrm{Hom}\,(\mathfrak{g}^*, \mathrm{H}^\bullet_{\mathrm{dR}}(M)), [\cdot, \cdot])$ is an element $c \in H^1(M, \mathfrak{g}^*)$ satisfying

$$[c, c] = 0, \tag{3.16}$$

the claim is proved. □

3.2. The reconstruction problem

In this section we discuss the conditions under which the distribution $\mathscr{D}$ defined in Theorem 3.6 admits a leaf satisfying Eq. (3.12). In particular, we analyze the case where the structure on G^* is trivial and the Heisenberg group case.

3.2.1. The standard case

Suppose that $\mathrm{G}^* = \mathfrak{g}^*$ is abelian. Then, the forms α_ξ satisfy $\mathrm{d}\alpha_\xi = 0$ for any $\xi \in \mathfrak{g}$ and, under the hypothesis of Theorem 3.6, we have $\alpha_\xi = \mathrm{d}H_\xi$, for some $H_\xi \in \mathscr{C}^\infty(M)$.

Theorem 3.9. *Suppose that G is a Lie group with trivial Poisson structure and M is compact. Then an infinitesimal momentum map defines a map $H : \mathfrak{g} \to \mathscr{C}^\infty(M) : \xi \mapsto H_\xi$ such that*

$$\mathrm{d}\{H_\xi, H_\eta\} = \mathrm{d}H_{[\xi,\eta]}, \quad \forall \xi, \eta \in \mathfrak{g}. \tag{3.17}$$

The element $c(\xi, \eta) = \{H_\xi, H_\eta\} - H_{[\xi,\eta]}$ is a two-cocycle c on $\mathfrak{g}$ with values in $\mathbb{R}$. The infinitesimal momentum map H is generated by a momentum map J if this cocycle vanishes and, in this case, J is unique.

Proof. Let us denote by $\mathtt{ev}_\xi$ the linear functions $\mathfrak{g}^* \ni z \to z(\xi)$. Then $\theta_\xi = \mathrm{d}(\mathtt{ev}_\xi)$ and the leaves of the distribution $\mathscr{D}$ coincide with the level sets on $M \times \mathfrak{g}^*$ of the functions

$$\{H_\xi - \mathtt{ev}_\xi \mid \xi \in \mathfrak{g}\}. \tag{3.18}$$

Furthermore, we have

$$\varphi(\xi,\eta)(m,z) = \{H_\xi, H_\eta\} - z([\xi,\eta]), \quad \forall m \in M. \tag{3.19}$$

In this case, Eq. (3.8) reduces to

$$\mathrm{d}\{H_\xi, H_\eta\} = \mathrm{d}H_{[\xi,\eta]}, \tag{3.20}$$

thus

$$\{H_\xi, H_\eta\} - H_{[\xi,\eta]} = c(\xi,\eta), \tag{3.21}$$

for some constants $c(\xi,\eta)$. By the Jacobi identity, the constants $c(\xi,\eta)$ define a class $[c] \in H^2(\mathfrak{g},\mathbb{R})$. Suppose that this class vanishes (for instance, if $\mathfrak{g}$ semisimple). Then, there exists a $z_0 \in \mathfrak{g}^*$ such that $c(\xi,\eta) = z_0([\xi,\eta])$. Thus, given a leaf $\mathscr{F}$, we have

$$\varphi(\xi,\eta)|_{\mathscr{F}} = 0 \tag{3.22}$$

if and only if $\mathscr{F}$ is given by

$$H_\xi - \mathtt{ev}_\xi - z_0(\xi) = 0. \tag{3.23}$$

In other words, the space of leaves of $\mathscr{D}$ which give a momentum map coincides with the affine space modeled on $\{z \in \mathfrak{g}^* : z|_{[\mathfrak{g},\mathfrak{g}]} = 0\}$ (which vanishes when $\mathfrak{g}$ is semisimple). This concludes the proof. □

3.2.2. The Heisenberg group case

Suppose now that G^* is the Heisenberg group. Let x, y, z be a basis for $\mathfrak{g}^*$, with $[x,y] = z$. Let ξ, η, ζ be its dual basis of $\mathfrak{g}$. The cocycle δ on $\mathfrak{g}$ is given by

$$\delta(\xi) = \delta(\eta) = 0 \quad \text{and} \quad \delta(\zeta) = \xi \wedge \eta, \tag{3.24}$$

thus

$$\mathrm{d}\alpha_\xi = \mathrm{d}\alpha_\eta = 0 \quad \text{and} \quad \mathrm{d}\alpha_\zeta = \alpha_\xi \wedge \alpha_\eta. \tag{3.25}$$

The Lie bialgebra structure on $\mathfrak{g}^*$ can be either

$$[\xi,\eta] = 0, \qquad [\xi,\zeta] = \xi, \qquad [\eta,\zeta] = \eta \tag{3.26}$$

or

$$[\xi,\eta] = 0, \qquad [\xi,\zeta] = \eta, \qquad [\eta,\zeta] = -\xi. \tag{3.27}$$

The result below turns out to be independent of the choice (the computations has been done using the second choice, which corresponds to $G = \mathbb{R} \ltimes \mathbb{R}^2$, with $\mathbb{R}$ acting by rotation on $\mathbb{R}^2$).

Let us denote by

$$\delta(\xi) = \sum_i \xi_i^1 \wedge \xi_i^2. \tag{3.28}$$

Applying the Cartan formula $\mathscr{L} = [\iota, \mathrm{d}]$ and Eq. (3.8) we get

$$\sum_i \pi(\alpha_\eta, \alpha_{\xi_i^1})\alpha_{\xi_i^2} - \sum_i \pi(\alpha_\xi, \alpha_{\eta_i^1})\alpha_{\eta_i^2} = \alpha_{[\eta,\xi]} - \mathrm{d}\pi(\alpha_\eta, \alpha_\xi). \tag{3.29}$$

In our case it gives the following relations

$$\begin{aligned}
&\mathrm{d}\pi(\alpha_\xi, \alpha_\eta) = \alpha_{[\xi,\eta]}\\
&\mathrm{d}\pi(\alpha_\zeta, \alpha_\eta) = \alpha_{[\zeta,\eta]} + \pi(\alpha_\eta, \alpha_\xi)\alpha_\eta\\
&\mathrm{d}\pi(\alpha_\zeta, \alpha_\xi) = \alpha_{[\zeta,\xi]} - \pi(\alpha_\xi, \alpha_\eta)\alpha_\xi
\end{aligned}$$

which are also satisfied for θ. Let $\mathscr{I}$ denote the ideal generating our distribution $\mathscr{D}$. Then, from above,

$$\mathrm{d}\varphi(\xi,\eta) \in \mathscr{I} \tag{3.30}$$

and

$$\varphi(\xi,\eta)|_{\mathscr{F}} = 0 \Longrightarrow \mathrm{d}\varphi(\zeta,\eta)|_{\mathscr{F}} \quad \text{and} \quad \mathrm{d}\varphi(\zeta,\xi)|_{\mathscr{F}} \in \mathscr{I}. \tag{3.31}$$

Let $\mathscr{F}$ be a leaf of $\mathscr{D}$ and X, Y and Z the left-invariant vector fields on G^* corresponding to x, y, z, resp. From Eq. (3.3), we get

$$\begin{aligned}
&\mathscr{L}_Z^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\eta)) = \mathscr{L}_X^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\eta)) = \mathscr{L}_Y^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\eta)) = 0\\
&\mathscr{L}_Z^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\zeta)) = \mathscr{L}_Y^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\zeta)) = 0\\
&\mathscr{L}_Z^*(\pi_{\mathrm{G}^*}(\theta_\eta, \theta_\zeta)) = \mathscr{L}_X^*(\pi_{\mathrm{G}^*}(\theta_\eta, \theta_\zeta)) = 0\\
&\mathscr{L}_X^*(\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\zeta)) = 1\\
&\mathscr{L}_Y^*(\pi_{\mathrm{G}^*}(\theta_\eta, \theta_\zeta)) = 1.
\end{aligned}$$

In particular, $\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\eta)$ is invariant under left translations. Since π_{G^*} is zero at the identity, we get

$$\pi_{\mathrm{G}^*}(\theta_\xi, \theta_\eta) = 0. \tag{3.32}$$

Since $d\varphi(\xi, \eta) \in \mathscr{I}$, the function $\varphi(\xi, \eta)$ is leafwise constant. Using Eqs. (3.11) and (3.32) it follows that $\pi(\alpha_\xi, \alpha_\eta)$ is also leafwise constant. Thus, we have that the Poisson structure $\pi(\alpha_\xi, \alpha_\eta)$ is a constant k on M. Furthermore, if $k = 0$, from Eq. (3.30) follows that

$$\varphi(\eta, \zeta)|_{\mathscr{F}} = c_1 \quad \text{and} \quad \varphi(\xi, \zeta)|_{\mathscr{F}} = c_2 \tag{3.33}$$

for some constants c_1 and c_2. Setting $\mathscr{F}' = \mathtt{id} \times \exp(c_1 x) \exp(c_2 y)$ we get

$$\varphi(\eta, \zeta)|_{\mathscr{F}'} = \varphi(\xi, \zeta)|_{\mathscr{F}'} = \varphi(\xi, \eta)|_{\mathscr{F}'} = 0. \tag{3.34}$$

We can summarize the results obtained in the Heisenberg case in the following

Theorem 3.10. *Let* G *be a Poisson Lie group acting on a Poisson manifold* M *with an infinitesimal momentum map* α *and such that* G^* *is the Heisenberg group. Let* ξ, η, ζ *denote the basis of* $\mathfrak{g}$ *dual to the standard basis* x, y, z *of* $\mathfrak{g}^*$*, with* z *central and* $[x, y] = z$*. Then*

$$\pi(\alpha_\xi, \alpha_\eta) = k \tag{3.35}$$

where k *is a constant on* M*. The form* α *lifts to a momentum map* $J : M \to \mathrm{G}^*$ *if and only if* $k = 0$*. When* $k = 0$ *the set of momentum maps with given* α *is one dimensional with free transitive action of* $\mathbb{R}$*.*

4. Infinitesimal deformations of a momentum map

Finally, we study the infinitesimal deformations of a given momentum map J. Let us consider a Hamiltonian action of a Poisson Lie group $(\mathrm{G}, \pi_{\mathrm{G}})$ on a Poisson manifold (M, π), with momentum map $J : M \to \mathrm{G}^*$ and suppose that $[-\epsilon, \epsilon] \ni t \to J_t : M \to \mathrm{G}^*, \epsilon > 0$, is a differentiable path of G-equivariant momentum maps for this action. We can assume that $J_t(m)$ is of the form

$$J(m)\exp(tH(m) + t^2\lambda(t, m)), \tag{4.1}$$

where $\exp : \mathfrak{g}^* \to \mathrm{G}^*$ is the exponential map; we assume $H : M \to \mathfrak{g}^* : m \mapsto H(m)$ to be differentiable and λ to be a map from $]-\epsilon, \epsilon[\, \times M$ to G^*.

Theorem 4.1. *In above notation, for any* $\xi, \eta \in \mathfrak{g}$*, we have*

$$\xi_M H_\eta - \eta_M H_\xi = H_{[\xi,\eta]} \tag{4.2}$$

$$\{H_\xi, \cdot\} = -\pi^\sharp(\alpha_{\mathrm{ad}_H^* \xi}). \tag{4.3}$$

Proof. Here we can assume that $J_t(m) = J(m)\exp(tH(m))$ and we consider $\alpha_\xi^t = J_t^*(\theta_\xi) = \langle dJ_t, \theta_\xi \rangle$. Let us compute

$$\beta_\xi = \left.\frac{d}{dt}\right|_{t=0} \langle dJ_t, \theta_\xi \rangle = \left.\frac{d}{dt}\right|_{t=0} \langle d(J\exp(tH)), \theta_\xi \rangle. \tag{4.4}$$

First note that

$$d(J\exp(tH)) = (\mathsf{R}_{\exp(tH)})_* dJ + (\mathsf{L}_J)_* d\exp(tH) \tag{4.5}$$

where R and L are the right and left multiplication, respectively. Calculating the derivative $\left.\frac{d}{dt}\right|_{t=0}$ we get:

$$\left.\frac{d}{dt}\right|_{t=0} \langle (\mathsf{R}_{\exp(tH)})_* dJ, \theta_\xi \rangle = \left.\frac{d}{dt}\right|_{t=0} \langle dJ, (\mathsf{R}_{\exp(tH)})^* \theta_\xi \rangle \tag{4.6}$$

$$= \langle dJ, \theta_{\mathrm{ad}_H^* \xi} \rangle = \alpha_{\mathrm{ad}_H^* \xi} \tag{4.7}$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle(\mathrm{L}_J)_*\mathrm{d}\exp(tH),\theta_\xi\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle\mathrm{d}\exp(tH),(\mathrm{L}_J)^*\theta_\xi\rangle \tag{4.8}$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle\mathrm{d}\exp(tH),\theta_\xi\rangle. \tag{4.9}$$

The differential of the exponential map $\exp : \mathfrak{g}^* \to \mathrm{G}^*$ is a map from the cotangent bundle of $\mathfrak{g}^*$ to the cotangent bundle of G^*. It can be trivialized as $\mathrm{d}\exp : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathrm{G}^* \times \mathfrak{g}^*$. Furthermore, $(\exp^{-1}, \mathtt{id}) : \mathrm{G}^* \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^*$, thus the map $\mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^*$ is given by $tH + o(t^2)$. We obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle\mathrm{d}\exp(tH),\theta_\xi\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle\mathrm{d}(tH+o(t)),\theta_\xi\rangle$$
$$= \mathrm{d}\langle H,\xi\rangle$$

and finally

$$\beta_\xi = \alpha_{\mathrm{ad}^*_H\xi} + \mathrm{d}H_\xi. \tag{4.10}$$

Since $\pi^\sharp(\alpha^t_\xi) = \xi_M$ is independent of t, $\pi^\sharp\beta_\xi = 0$ and Eq. (4.3) is proved. Now, since J_t is a family of Poisson maps, one has

$$\pi(\alpha^t_\xi,\alpha^t_\eta)(m) = \pi_{\mathrm{G}^*}(\theta_\xi,\theta_\eta)(J_t(m)). \tag{4.11}$$

Applying $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}$ to both sides, we get

$$\pi(\beta_\xi,\alpha_\eta)(m) + \pi(\alpha_\xi,\beta_\eta)(m) = \mathscr{L}_H(\pi_{\mathrm{G}^*}(\theta_\xi,\theta_\eta))(J(m)). \tag{4.12}$$

Using Eq. (4.10) we obtain the formula

$$\mathscr{L}_H(\pi_{\mathrm{G}^*}(\theta_\xi,\theta_\eta))(J(m)) = H([\xi,\eta]) + \pi_{\mathrm{G}^*}(\theta_{\mathrm{ad}^*_H\xi},\theta_\eta) + \pi_{\mathrm{G}^*}(\theta_\xi,\theta_{\mathrm{ad}^*_H\eta}) \tag{4.13}$$

which proves Eq. (4.2). $\square$

This theorem provides an explicit description of the tangent space of the momentum map at a given point and it implies the following

Corollary 4.2. *Suppose that M is a Poisson manifold with a Poisson action of a compact semisimple Poisson Lie group G. Then any infinitesimal deformation of a momentum map $J : M \to \mathrm{G}^*$ as above is generated by a one parameter family of gauge transformations.*

Proof. Eq. (4.2) implies that $H \in H^1(\mathfrak{g}, C^\infty(M,\mathfrak{g}^*))$. Since G is compact semisimple, H is a Lie coboundary, i.e. there exists a function

$$j : M \to \mathfrak{g}^* \tag{4.14}$$

such that

$$\xi_M j = H_\xi. \tag{4.15}$$

In particular, it is easy to check that $\pi^\sharp(\alpha_{\mathrm{ad}^*_H\xi})f = \sum X_{\xi^1_i j}\,\xi^2_i(f)$, where we use the notation $\delta(\xi) = \sum \xi^1_i \otimes \xi^2_i$. Now observe that

$$\xi\{j,f\} = \xi_M\pi(\mathrm{d}j,\mathrm{d}f) = (\xi_M\pi)(\mathrm{d}j,\mathrm{d}f) + \{\xi_M j,f\} + \{j,\xi_M f\} \tag{4.16}$$

hence

$$\{H_\xi,f\} = \{\xi_M j,f\} = \xi\{j,f\} - (\xi_M\pi)(\mathrm{d}j,\mathrm{d}f) - \{j,\xi_M f\} \tag{4.17}$$
$$= \xi\{j,f\} - \delta(\xi)(j,f) - \{j,\xi_M f\} \tag{4.18}$$
$$= \xi\{j,f\} - \xi^1_M j\,\xi^2(f) - \{j,\xi_M f\}. \tag{4.19}$$

Substituting Eqs. (4.15) and (4.17) in Eq. (4.3) we get

$$\xi\{j,f\} - \{j,\xi_M f\} = 0. \tag{4.20}$$

In other words the Hamiltonian vector field associated to j commutes with the group action and is tangent to the derivative of J_t at $t = 0$ as claimed. $\square$

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