Journal of Geometry and Physics 127 (2018) 101–111

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Coulomb dynamics of three equal negative charges in field of fixed two equal positive charges

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ARTICLE INFO

Article history:
Received 4 September 2017
Received in revised form 31 January 2018
Accepted 5 February 2018
Available online 8 February 2018

Keywords:
Three equal charges
Quasi-periodic motion
Equilibrium
Elimination of node

ABSTRACT

Quasi-periodic solutions of the Coulomb equation of motion for three identical negative charges in the field of two fixed point positive charges are found. The center Lyapunov theorem is applied to the transformed equation of motion.

1. Introduction

It is the important task of mathematics to find solutions of newtonian equations of motion of manybody d-dimensional mechanical systems represented in terms of series converging on the infinite time interval. These equations for N-body systems with a potential energy U and masses $m_j, j = 1, \dots, N$ look like

$$m_j \frac{d^2 x_j}{dt^2} = -\frac{\partial U(x_{(N)})}{\partial x_j}, \qquad j = 1, \dots, N, \quad x_{(N)} = (x_1, \dots, x_N) \in \mathbb{R}^{dN}, x_j = (x_j^1, \dots, x_j^d). \tag{1.1}$$

The existence of an equilibrium of the potential energy makes it possible to find such solutions with the help of fundamental theorems among which is the Lyapunov center theorem.

The Coulomb systems of two and three negative equal charges e_0 in the field of fixed two equal positive charges e' have equilibrium configurations [1–3]. This fundamental fact allowed us to construct periodic [1,2], bounded [1,2] and quasi-periodic [3] solutions close to the equilibria of the Coulomb equation of motion for the negative charges. For the line systems we applied the Weinstein [4,5], Moser [6] and center Lyapunov [7–12] theorems which demand a knowledge of the spectra of the matrix U^0 of second derivatives of the potential energy at the equilibrium(for equal masses). The last two theorems, guaranteeing the existence of the periodic solutions in terms of convergent series, restrict the values of $\frac{e_0}{e'}$ through a non-resonance condition. The Weinstein theorem establishes the existence of the periodic solutions without this condition but can be applied only for mechanical systems with a stable equilibrium (U^0 is positive definite and the equilibrium is a minimum of the potential energy). The construction of the bounded solutions in [2] for the line and planar systems does not demand the non-resonant condition to be true and is based on the generalization of the semi-linearization Siegel technique exposed in the section Lyapunov theorem in [7]. Periodic solutions are also found in planar Coulomb systems of $n-1, n > 2$ equal negative charges and one and three positive charges [13,14]. These solutions describe positions of the negative charges in the form of regular polygons.

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https://doi.org/10.1016/j.geomphys.2018.02.006
0393-0440/

The dynamics in the Coulomb system of a negative charge in a field of many fixed positive charges can be deduced from [15–17].

Mechanical systems with an integral of motion and an equilibrium have U^0 characterized by the zero eigenvalue [7]. This does not allow one to find directly the solutions of their equations of motion on the infinite time interval with the help of the mentioned theorems.

The mentioned space systems of two and three equal negative charges possess a rotational symmetry, its angular momentum Q is an integral of motion and there is the continuum of equilibria parametrized by points of a circle centered at the origin. The matrices of second derivatives of the potential energy at the equilibria are similar to the simplest matrix U^0 which corresponds to the equilibrium determined by the three equal negative charges located at a coordinate axis and the two fixed positive charges located at an equal distance from the origin at another coordinate axis.

In this paper we find quasi-periodic solutions of the Coulomb equation of motion whose Euclidean norms are periodic functions for the system of three equal negative charges in the field of two fixed positive charges. This result is an analog of the result of [3] and a consequence of the representation of U^0 as the direct sum of three three-dimensional matrices whose eigenvalues are found explicitly.

We circumvent the obstruction of the zero eigenvalue as in [3] with the help of the procedure of the elimination of node from the celestial mechanics (section 18 in [7]). The main idea of this procedure is to produce a canonical transformation turning the integrals of motions into cyclic variables (a transformed Hamiltonian does not depend on them). Then linear part of the equation of motion for them will be zero and the canonical matrix, i.e the matrix generating the linear part of the equation of motion, for remaining variables will not contain the zero eigenvalue (see the Appendix). The procedure of the elimination of node is described in the following theorem (the proof of its first two statements are given in [7]).

Theorem 1.1. *Let $H(x,p)$ be a $2n$-dimensional Hamiltonian, Q be its time independent integral and $w(u,p)$ be a generating function of the canonical transformation such that*

$$v_k = \frac{\partial w}{\partial u_k}, \qquad x_k = \frac{\partial w}{\partial p_k}, \qquad k = 1, \dots, n, \tag{1.2}$$

$$\frac{\partial w}{\partial u_n} = Q(x,p), \qquad W_{k,j} = \frac{\partial^2 w}{\partial u_k \partial p_j}, \qquad DetW \neq 0. \tag{1.3}$$

Then the transformed Hamiltonian $H'(u,v)$ does not depend on u_n. Let also the canonical matrix of H have doubly degenerate zero eigenvalue, this canonical transformation and the Hamiltonian be holomorphic at a neighborhood of the equilibrium. Then the canonical matrix of the $2(n-1)$-dimensional Hamiltonian equation

$$\dot{u}_j = \frac{\partial H'}{\partial v_j}, \qquad \dot{v}_j = -\frac{\partial H'}{\partial u_j}, \qquad j = 1, \dots, n-1, \tag{1.4}$$

does not have the zero eigenvalue for the equilibrium value Q^0 of Q and eigenvalues of the canonical matrices of H and H' are identical.

A separation of cyclic variables, generated by integrals of motion, in a Hamiltonian equation is also described in [18].

We find w as

$$w = \sum_{j=1}^{n} g_k(u_1, \dots, u_n) p_k,$$

where $n = 9$, for a special numeration of charge coordinates and momenta $(x_j; p_j) = (x_j^\alpha; p_j^\alpha)$, $j = 1, 2, 3$, $\alpha \leq 3$. Such the representation for w, the equation for g_k and its solution is inspired by the Celestial Mechanics [7]. As a result solutions of the Coulomb equation are given by

$$x_j^\alpha(t) = \sum_{k=1}^{n-1} u_k(t)[\gamma_{k,j,\alpha} + \gamma'_{k,j,\alpha}\cos(u_n(t)) + \gamma''_{k,j,\alpha}\sin(u_n(t))], \tag{1.5}$$

where $\gamma, \gamma', \gamma''$ are constants and $u_k(t)$, $k = 1, \dots, n$ are solutions of the equation with the Hamiltonian H'. We show with the help of the center Lyapunov theorem that (1.4) for our system possesses periodic solutions.

Let $u_{(n-1)}, v_{(n-1)}$ be a periodic solution of (1.4) with a period τ. Then

$$(\frac{\partial H'}{\partial v_n})(u_{(n-1)}, v_{(n-1)}, Q^0) = H'_{n-1}(t)$$

is also a periodic function such that

$$H'_{n-1}(t) = H^0_{n-1}(t) + \xi, \qquad \int_t^{t+\tau} H^0_{n-1}(s)ds = 0,$$

where ξ is a constant. This implies that

$$u_n(t) = \int_0^t H'_{n-1}(s)ds = \int_0^t H^0_{n-1}(s)ds + \xi t = u_0(t) + \xi t,$$

where u_0 is periodic with the period τ. The last equality and (1.4) determine the quasi-periodic(doubly periodic) functions $x_j^\alpha(t)$ if $\gamma' \neq 0$ or $\gamma'' \neq 0$. We will show that their Euclidean norms are periodic functions.

Our paper is organized as follows. In the second section we find the structure of U^0 for the simplest equilibrium. In the third section we find the spectrum of U^0. In the fourth section the node transformation is produced and the main result is formulated in Theorem 4.1. In the Appendix we prove the second and third statements of Theorem 1.1.

2. Characteristics of equilibrium

For three identical negative charges in $\mathbb{R}^3$ in the field of two positive fixed charges located at the points b_1, b_2 the Coulomb potential energy is given by

$$U(x_{(3)}) = \frac{1}{2}\sum_{j\neq k=1}^{3} \frac{e_j e_k}{|x_j - x_k|} - e_0 e' \sum_{j,k=1,2} |x_j - b_k|^{-1}, \tag{2.1}$$

where $e_j = -e_0 < 0$, $e' > 0$ and

$$|x|^2 = (x_j^1)^2 + (x_j^2)^2 + (x_j^3)^2, \quad x_j = (x_j^1, x_j^2, x_j^3) \in \mathbb{R}^3,$$

$$b_j = (b_j^1, b_j^2, b_j^2) \in \mathbb{R}^3, \quad b_j^1 = b_j^3 = 0, \quad b_1^2 = -b_2^2 = b.$$

The first partial derivatives of U look like

$$\frac{\partial U(x_{(3)})}{\partial x_j^\alpha} = -e_0^2 \sum_{k=1,k\neq j}^{3} \frac{x_j^\alpha - x_k^\alpha}{|x_j - x_k|^3} + e_0 e' \sum_{k=1}^{2} \frac{x_j^\alpha - b_k^\alpha}{|x_j - b_k|^3}.$$

The equilibrium is determined by

$$x_1^{01} = -a, x_3^{01} = a, x_2^{01} = 0, x_j^{0\alpha} = 0, \qquad \alpha = 2, 3.$$

The first derivative in x_j^2 is equal zero at it since $b_1^2 = -b_2^2 = b$. The other derivatives generate the following equilibrium relation

$$\frac{5e_0^2}{4a^2} - \frac{2e' e_0 a}{(\sqrt{a^2+b^2})^3} = 0.$$

There are also the equilibria, which are characterized by this relation, determined by

$$x_1^{01} = -a\cos\xi, x_1^{03} = -a\sin\xi, x_3^{01} = a\cos\xi, x_3^{03} = a\sin\xi, x_2^{0\alpha} = 0, \quad \alpha = 1, 2, 3, \quad x_j^{02} = 0.$$

From the equilibrium relation one derives

$$\frac{e_0^2}{4a^3} = u', \qquad \frac{2e_0 e'}{(\sqrt{a^2+b^2})^3} = 5u', \qquad (\sqrt{a^2+b^2})^{-1} = \frac{1}{2a}(\frac{5e_0}{e'})^{\frac{1}{3}}, \tag{2.2}$$

$$a = (4-\eta')^{-\frac{1}{2}}\sqrt{\eta'}b, \qquad \eta' = (\frac{5e_0}{e'})^{\frac{2}{3}} < 4, \quad \frac{e_0}{e'} < \frac{8}{5}. \tag{2.3}$$

The second partial derivatives of U look like

$$\frac{\partial U(x_{(3)})}{\partial x_j^\alpha \partial x_k^\beta} = \frac{\partial U(x_{(3)})}{\partial x_k^\beta \partial x_j^\alpha} = e_0^2[\frac{\delta_{\alpha,\beta}}{|x_j - x_k|^3} - 3\frac{(x_j^\alpha - x_k^\alpha)(x_j^\beta - x_k^\beta)}{|x_j - x_k|^5}], \quad \alpha, \beta = 1, 2, 3, j \neq k,$$

$$\frac{\partial^2 U(x_{(3)})}{\partial x_j^\beta \partial x_j^\alpha} = e_0^2 \sum_{k=1,k\neq j}^{3} [-\frac{\delta_{\alpha,\beta}}{|x_j - x_k|^3} + 3\frac{(x_j^\alpha - x_k^\alpha)(x_j^\beta - x_k^\beta)}{|x_j - x_k|^5}]$$
$$+ e_0 e' \sum_{k=1}^{2} [\frac{\delta_{\alpha,\beta}}{|x_j - b_k|^3} - 3\frac{(x_j^\alpha - b_k^\alpha)(x_j^\beta - b_k^\beta)}{|x_j - b_k|^5}].$$

Let U^0 be the (main equilibrium) matrix of second derivatives at the equilibrium with $\xi = 0$ and $\delta_{j,k}$ be the Kronecker symbol. We will prove that

$$U^0_{j,\alpha;k,\beta} = \delta_{\alpha,\beta} U^0_{\alpha;j,k}, \qquad j, k, \alpha, \beta = 1, 2, 3. \tag{2.4}$$

This representation leads immediately to the following result.

Proposition 2.1. *Let the following numeration be true*

$$(1,1) = 1; \ (2,1) = 2, \quad (3,1) = 3, \quad (1,2) = 4, \quad (2,2) = 5, \quad (3,2) = 6,$$
$$(1,3) = 7, \quad (2,3) = 8, \quad (3,3) = 9, \tag{2.5}$$

where the first and second number in the pairs is the lower and upper index of a charge coordinate. Then

$$U^0 = U^0_1 \oplus U^0_2 \oplus U^0_3,$$

where U^0_α is the three dimensional square matrix with the components $U^0_{\alpha;j,k}$ determined by (2.4).

Let us prove (2.4). It is clear that the 3×3 matrix U^0_3 coincides with the contribution of the first terms in the three square brackets in the two expressions for the second derivatives of U. The contribution of the second terms in the first two square brackets also have the similar structure at the equilibrium. Hence we have to prove only that

$$T_j(\alpha, \beta) = \sum_{k=1}^{2} \frac{(x^\alpha_j - b^\alpha_k)(x^\beta_j - b^\beta_k)}{|x_j - b_k|^5}$$

has the structure of (2.4) for $j = k$ at the equilibrium. Let $T^0_j(\alpha, \beta)$ is the equilibrium value of $T_j(\alpha, \beta)$ and take into account that $|x^0_j - b_k|^2 = a^2 + b^2, j = 1, 3, |x^0_2 - b_k|^2 = b^2$. We will prove that

$$T^0_j(\alpha, \beta) = 2(a^2 + b^2)^{-\frac{5}{2}} \delta_{\alpha,\beta}(a^2\delta_{\alpha,1} + b^2\delta_{\alpha,2}), j = 1, 3; \qquad T^0_2(\alpha, \beta) = 2b^{-3}\delta_{\alpha,\beta}\delta_{\alpha,2}. \tag{2.6}$$

These equalities hold for $\alpha = 3$. Let us put

$$\tilde{T}^0_j(\alpha, \beta) = \sum_{k=1}^{2} (x^{0\alpha}_j - b^\alpha_k)(x^{0\beta}_j - b^\beta_k).$$

Then

$$\tilde{T}^0_1(1, 2) = -((a - b^1_1)b^2_1 + (a - b^1_2)b^2_2) = -(ab - ab) = 0,$$

$$\tilde{T}^0_3(1, 2) = -((-a - b^1_1)b^2_1 + (-a - b^1_2)b^2_2) = ab - ab = 0,$$

$$\tilde{T}^0_1(1, 1) = (a - b^1_1)(a - b^1_1) + (a - b^1_2)(a - b^1_2) = 2a^2,$$

$$\tilde{T}^0_3(1, 1) = (-a - b^1_1)(-a - b^1_1) + (-a - b^1_2)(-a - b^1_2) = 2a^2,$$

$$\tilde{T}^0_2(2, 2) = \tilde{T}^0_1(2, 2) = \tilde{T}^0_3(2, 2) = (b^2_1)^2 + (b^2_2)^2 = 2b^2.$$

Simple arguments prove the second equality in (2.6) since $x^{0\alpha}_2 = 0$ and b_k has non-zero only in the second component. Hence we proved (2.6) and (2.4).

Now let us calculate the elements of the matrices U^0_j.

$$U^0_{3;2,1} = U^0_{3;1,2} = U^0_{3;2,3} = U^0_{3;3,2} = \frac{e^2_0}{a^3} = 4u', \qquad U^0_{3;3,1} = U^0_{3;1,3} = \frac{u'}{2}.$$

(2.2)–(2.3) yield

$$U^0_{3;1,1} = U^0_{3;3,3} = -\frac{e^2_0}{a^3}(1 + \frac{1}{8}) + \frac{2e_0e'}{(\sqrt{a^2 + b^2})^3} = \frac{u'}{2}, \quad U^0_{3;2,2} = -8u' + \frac{2e_0e'}{b^3} = u'g',$$

where

$$g'(u) = -8 + g_* > -3, \qquad g_* = \frac{2e_0e'}{u'b^3} = 40(4-\eta')^{-\frac{3}{2}} = 5\sqrt{27}(3-u)^{-\frac{3}{2}}, \quad u = \frac{3\eta'}{4}.$$

Hence

$$U_*(g') = u'^{-1}U_3^0 = \begin{pmatrix} \frac{1}{2} & 4 & \frac{1}{2} \\ 4 & g' & 4 \\ \frac{1}{2} & 4 & \frac{1}{2} \end{pmatrix}.$$

Let

$$u_* = 5\frac{3e_0^2}{4^2a^3}\eta' = 5uu', \qquad u'' = 5(3-u).$$

Then from (2.2)–(2.3) one derives

$$u''_* = \frac{6e_0e'b^2}{(\sqrt{a^2+b^2})^5} = \frac{6e_0e'}{(2a)^5}\left(\frac{5e_0}{e'}\right)^{\frac{5}{3}}\frac{4-\eta'}{\eta'}a^2 = u''u', \qquad \frac{6e_0e'a^2}{(\sqrt{a^2+b^2})^5} = u_*. \tag{2.7}$$

We see that U_2^0 differs from U_3^0 only by the diagonal elements due to T_j^0. Hence

$$U^0_{2;1,1} = U^0_{2;3,3} = \frac{u'}{2} - u''_* = \left(\frac{1}{2} - u''\right)u', \quad U^0_{2;2,2} = -8u' - \frac{4e_0e'}{b^3} = u'g'',$$

$$g'' = -(8 + 10\sqrt{27}(3-u)^{-\frac{3}{2}}) = -8 - 2g_*.$$

That is

$$u'^{-1}U_2^0 = \begin{pmatrix} \frac{1}{2} - u'' & 4 & \frac{1}{2} \\ 4 & g'' & 4 \\ \frac{1}{2} & 4 & \frac{1}{2} - u'' \end{pmatrix} = U_{*2} - u''I,$$

where

$$U_*(g_2) = U_{*2} = \begin{pmatrix} \frac{1}{2} & 4 & \frac{1}{2} \\ 4 & g_2 & 4 \\ \frac{1}{2} & 4 & \frac{1}{2} \end{pmatrix}, \quad g_2 = g'' + u''.$$

U_1^0 differs from U_3^0 by the contributions of the second terms in the square brackets in the expressions of the second derivatives of U.

$$U^0_{1;2,1} = U^0_{1;1,2} = U^0_{1;2,3} = U^0_{1;3,2} = -\frac{2e_0^2}{a^3} = -8u', \qquad U^0_{1;3,1} = U^0_{1;1,3} = -u',$$

$$U^0_{1;2,2} = u'g, \qquad U^0_{1;1,1} = U^0_{1;3,3} = 14u' - u_* = (14-5u)u'. \qquad g = 16 + g_*.$$

That is

$$u'^{-1}U_1^0 = \begin{pmatrix} 14-5u & -8 & -1 \\ -8 & g & -8 \\ -1 & -8 & 14-5u \end{pmatrix} = -2U_{*1} + u''I,$$

$$U_*(g_1) = U_{*1} = 2^{-1}\begin{pmatrix} 1 & 8 & 1 \\ 8 & 2g_1 & 8 \\ 1 & 8 & 1 \end{pmatrix}, \qquad 2g_1 = -g + u'',$$

where I is the unit matrix.

3. Spectrum of main equilibrium matrix

$DetU_{*j} = 0$ since $U_*(q)$ has identical first and third rows. This permits to find roots of the characteristic polynomials $p_*(q)$ for $U_*(q)$ and p'_j for $U'_j = u'^{-1}U_j^0$.

$$p_*(\lambda, q) = -Det(-\lambda I + U_*(q)) = -Det \begin{pmatrix} \frac{1}{2} - \lambda & 4 & \frac{1}{2} \\ 4 & q - \lambda & 4 \\ \frac{1}{2} & 4 & \frac{1}{2} - \lambda \end{pmatrix}$$

$$= -Det \begin{pmatrix} -\lambda & 0 & \lambda \\ 4 & q-\lambda & 4 \\ \frac{1}{2} & 4 & \frac{1}{2} - \lambda \end{pmatrix} = \lambda[(q-\lambda)(\frac{1}{2} - \lambda) - 16] - \lambda[16 - \frac{1}{2}(q-\lambda)]$$

$$= [(q-\lambda)(1-\lambda) - 32]\lambda = [\lambda^2 - (q+1)\lambda + q - 32]\lambda.$$

Here we subtracted the third row from the first one of $U_*(q) - \lambda I$ and expanded the determinant in the elements of the first row which are proportional to λ.

Roots of $p_*(q)$ are given as follows

$$2\lambda = q + 1 \pm \sqrt{(q-1)^2 + 128}, \qquad \lambda = 0.$$

The roots of p'_1

$$p'_1(\lambda') = -2^3 p_*(-\frac{\lambda'}{2} + \frac{5(3-u)}{2}, g_1)$$

look like

$$\lambda' = 5(3-u) - g_1 - 1 \pm \sqrt{(g_1 - 1)^2 + 128}, \qquad \lambda' = 5(3-u) = \zeta'_1. \tag{3.1}$$

Let ζ'_2, ζ'_3 be the roots corresponding to plus and minus before the sign of the square root

$$\zeta'_2 = \frac{g + 5(3-u)}{2} - 1 + \sqrt{(\frac{g + 5(u-3)}{2} + 1)^2 + 128}, \tag{3.2}$$

$$\zeta'_3 = \frac{g + 5(3-u)}{2} - 1 - \sqrt{(\frac{g + 5(u-3)}{2} + 1)^2 + 128}. \tag{3.3}$$

Proposition 3.1. *Root ζ'_3 is positive at $0 \le u \le \frac{6}{5}$ and negative at $\frac{13}{5} < u < 3$. Moreover the inequalities*

$$\zeta'_2 > \zeta'_1 > 0, \qquad \zeta'_3 < \zeta'_1 \tag{3.4}$$

are true

Proof. (3.4) follows from $g > 21$, $0 < u < 3$ and $\frac{g-5(3-u)}{2} > 2$. Further the inequality

$$\zeta'_3 < 5(3-u) + \frac{g + 5(u-3)}{2} - 1 - \frac{g + 5(u-3)}{2} - 1 < 5(3-u) - 2$$

means $\zeta'_3 < 0$ if $u > \frac{13}{5}$.

The condition of positivity of ζ'_3 looks like

$$(\frac{g + 5(3-u)}{2} - 1)^2 > (\frac{g + 5(u-3)}{2} + 1)^2 + 128.$$

That is

$$g(5(3-u) - 2) > 128. \tag{3.5}$$

It is easy to see that $\zeta'_3 > 0$ for $0 \le u \le \frac{6}{5}$ since $g \ge 21$, $5(3-u) - 2 \ge 7$ and

$$g(5(3-u) - 2) > 147. \quad \blacksquare$$

It is easy to check that $\zeta'_3 > 0$ at $u = \frac{7}{5}, \frac{8}{5}, \frac{9}{5}$ and $\zeta'_3 < 0$ at $u = 2$. A conjecture arises that $\zeta'_3 > 0$ at $0 \le u \le \frac{9}{5}$ and $\zeta'_3 < 0$ at $2 \le u < 3$.

The neutrality condition $3e_0 = 2e'$ leads to

$$u = \frac{3}{4}(\frac{5e_0}{e'})^{\frac{2}{3}} = \frac{3}{4}(\frac{10}{3})^{\frac{2}{3}} = \frac{5}{2}(\frac{3}{10})^{\frac{1}{3}} < \frac{5}{2}(\frac{1}{3})^{\frac{1}{3}} < \frac{5}{2}(1,4)^{-1} = \frac{25}{14} < \frac{9}{5}.$$

Characteristic polynomials p_{*2} of U_{*2} and p'_2 of $U'_2 = u'^{-1}U^0_2$ look like

$$p_{*2}(\lambda) = p_*(\lambda, g_2), \qquad p'_2(\lambda) = p_{*2}(\lambda + u'', g_2).$$

Hence the roots of $p'_2(\lambda')$ are found as follows

$$2\lambda' = -2u'' + g_2 + 1 \pm \sqrt{(g_2 - 1)^2 + 128}, \qquad \lambda' = -u'' = \zeta'_4.$$

Let ζ'_5, ζ'_6 be the roots corresponding to plus and minus before the sign of the square root.

Proposition 3.2. *$\zeta'_5 < 0$ and $\zeta'_5 > 0$ at $0 < u \le 2$, $\frac{14}{5} \le u < 3$, respectively. Moreover $\zeta'_4 < 0$, $\zeta'_6 < 0$ and $|\zeta'_4| < |\zeta'_6|$, $|\zeta'_5| < |\zeta'_6|$, $\zeta'_5 > \zeta'_4$.*

Proof. We have

$$2\zeta'_5 = g'' - u'' + 1 + \sqrt{(g'' + u'' - 1)^2 + 128},$$

$$2\zeta'_6 = g'' - u'' + 1 - \sqrt{(g'' + u'' - 1)^2 + 128}.$$

Then $\zeta'_4, \zeta'_6 < 0$ since $g'' + 1\langle 0, u''\rangle 0$, ànd $\zeta'_5 > 0$ if

$$(|g''| - u'' + 1)^2 + 128 > (|g''| + u'' - 1)^2, \qquad 4(u'' - 1)|g''| < 128. \tag{3.6}$$

This is true if $u'' = 5(3 - u) \le 1$ that is $u \ge \frac{14}{5}$. Let us show that $\zeta'_5 < 0$ at the interval $[0, 2]$ that is the inverse to (3.6) holds

$$(|g''| - u'' + 1)^2 + 128 < (|g''| + u'' - 1)^2, \qquad 4(u'' - 1)|g''| > 128. \tag{3.7}$$

This inequality is true at $u = 0, 2$. This follows that at these values of u it coincides with

$$56|g''| > 128, \qquad 16|g''| > 128,$$

respectively. These inequalities are true since at these values of u

$$g_* = 5, \quad |g''| = 8 + 2g_* = 18; \qquad g_* = 15\sqrt{3}a > 25, \quad |g''| = 8 + 30\sqrt{3} > 58. \tag{3.8}$$

(3,7) is valid at the interval $[0, 2]$. This is checked substituting the minimum of $|g''|$, u'' at it into (3.7). We also have $\zeta'_5 > \zeta'_4$ since

$$2(\zeta'_5 - \zeta'_4) = g'' + u'' + 1 + \sqrt{(g'' + u'' - 1)^2 + 128},$$

implying for $g' + u'' < 0$

$$2(\zeta'_5 - \zeta'_4) > g'' + u'' + 1 + |g'' + u'' - 1| > 0. \quad \blacksquare$$

The roots of p'_3 are given by

$$2\lambda = g' + 1 \pm \sqrt{(g' - 1)^2 + 128}, \qquad \lambda = 0 = \zeta'_7.$$

Let ζ'_8, ζ'_9 be the roots corresponding to plus and minus before the sign of the square root.

Proposition 3.3. *Eigenvalues ζ_j of the nine-dimensional matrix*

$$U^0 = U^0_1 \oplus U^0_2 \oplus U^0_3$$

are determined in the previous propositions and by the following relations

$$\zeta_j = u'\zeta'_j, \quad j = 1, \dots, 9, \quad \zeta'_7 = 0, \quad \zeta_8 > 0,$$

$\zeta_9 < 0$ at $0 < u < \frac{9}{4}$, $\zeta_9 > 0$ at $\frac{9}{4} < u < 3$ and $\zeta_9 = 0$ at $u = \frac{9}{4}$.

Proof. We have

$$2\zeta'_8 = g' + 1 + \sqrt{(g' - 1)^2 + 128},$$

$$2\zeta'_9 = g' + 1 - \sqrt{(g' - 1)^2 + 128}.$$

It is obvious that $\zeta_8' > 0$ since $g' > -3$ and the square root is greater than 2. It is obvious also that $\zeta_9' < 0$ if

$$(g'-1)^2 + 128 > (g'+1)^2, \qquad 4g' < 128,$$

that is

$$g' < 32, \quad g_* < 40.$$

This is true if $0 < u < \frac{9}{4} = u_0, (5^{-1}g_*(u_0))^2 = \frac{27}{(3-u_0)^3} = 64$. $\zeta_9' > 0$ if $g_* > 40$, $\frac{9}{4} < u < 3$ and $\zeta_9' = 0$ if $g_* = 40$, $u = \frac{9}{4}$. ■

Proposition 3.4. *The inequalities $\zeta_j < \zeta_2$ for $j \neq 5$ and $\zeta_5 < \zeta_2$ are true for $0 < u < 3$ and $0 < u < 2$, respectively. Moreover (3.9) holds for $0 < u < 3$,*

$$\zeta_8' < \zeta_2', \qquad \zeta_9' < \zeta_2'. \tag{3.9}$$

$\zeta_8 < \zeta_1$, $\zeta_9 > \zeta_4 > \zeta_6$ are true for $0 < u < 1$ and $\zeta_8' > \zeta_1' = u''$ holds for $2 \le u < 3$.

Proof. We have

$$\frac{g'-1}{2} = \frac{g_*}{2} - \frac{9}{2}, \qquad \frac{g+5(u-3)}{2} + 1 > \frac{g_*}{2} + \frac{3}{2}, \qquad \frac{g+5(3-u)}{2} - 1 > \frac{g_*}{2} + 7.$$

From these relations one deduces (3.9). Besides $\zeta_8' < \zeta_1' = u'' = 5(3-u)$ and $\zeta_9' > \zeta_4' = -u''$ for $0 \le u \le 1$ since

$$2(\zeta_8' - \zeta_1') = g' - 2u'' + 1 + \sqrt{(g'-1)^2 + 128} < 0,$$

$$2(\zeta_9' - \zeta_4') = g' + 2u'' + 1 - \sqrt{(g'-1)^2 + 128} > 0,$$

$-3 \le g' \le -8+5\sqrt{\frac{27}{8}} \le 2$ and $10 \le u'' \le 15$. We derive also relying on (3.8) the inequality $\zeta_8' > \zeta_1' = u''$ for $2 \le u < 3$. ■

Let us consider

$$2(\zeta_5' - \zeta_9') = -g_* - u'' + \sqrt{(g''+u''-1)^2 + 128} + \sqrt{(g'-1)^2 + 128}.$$

For $u = 0, 1$ the sum of first two terms are greater than -20 while the sum of the square roots exceeds 22 and the right-hand side is greater than zero. The same is true for $u \ge 2$ ((3.8) and $u'' \le 5$ are taken into account). That is we expect that $\zeta_5' > \zeta_9'$. We see also that there is the point at (0.3) at which $\zeta_1 = \zeta_8$.

The result about absence of resonances between ζ_2 and other ζ_j allows one to apply the center Lyapunov theorem after the elimination of node if all $\zeta_j, j \neq 7$ are non-zero. This condition is satisfied for $0 < u \le \frac{6}{5}$. Note that we have to rely also on the well-known fact that the eigenvalues λ_j of the linear part of the Coulomb Hamiltonian vector field, i.e. canonical matrix, are given by $\lambda_j = \pm\sqrt{-\zeta_j}$ [1].

4. Elimination of node. Main result

Now we begin to apply the technique of the elimination of node. Let

$$x_1^2 = x_1, \quad x_2^2 = x_2, \quad x_3^2 = x_3, \quad x_1^1 = x_4, \quad x_2^1 = x_5, \quad x_3^1 = x_6, \quad x_1^3 = x_7, \quad x_2^3 = x_8, \quad x_3^3 = x_9 \tag{4.1}$$

and the same numeration be true for momenta. Then the angular moment (the integral of motion corresponding to the rotation around the second coordinate axis) is given by

$$Q = \sum_{j=1}^{3}(x_j^3 p_j^1 - x_j^1 p_j^3) = \sum_{j=4}^{6}(x_{3+j}p_j - p_{3+j}x_j).$$

Further the upper index will show a power. The generating function $w(u_{(9)}, p_{(9)})$ of the canonical transformation (1.2) with $x_j = u_j, j = 1, 2, 3$ is given by

$$w = \sum_{j=4}^{9} g_k(u_{(9\setminus 3)})p_k + u_1p_1 + u_2p_2 + u_3p_3,$$

where $(9 \setminus 3) = 4, \ldots, 9$. Let $Q = v_9$ then (1.3) results in

$$\sum_{j=4}^{9} \frac{\partial g_k}{\partial u_9} p_k = \sum_{j=4}^{6}(g_{j+3}p_j - g_jp_{j+3}),$$

or

$$\frac{\partial g_k}{\partial u_9} = g_{k+3}, \qquad \frac{\partial g_{k+3}}{\partial u_9} = -g_k, \qquad k = 4, 5, 6.$$

The partial solution is given by

$$g_4 = u_4 c, \quad g_7 = -u_4 s, \quad g_5 = u_5 c + u_6 s, \quad g_8 = -u_5 s + u_6 c,$$

$$g_6 = u_7 c + u_8 s, \quad g_9 = -u_7 s + u_8 c, \quad c = cosu_9, \quad s = sinu_9.$$

This solution is generated by the non-singular canonical transformation if $u_4 \neq 0$ since

$$DetW = DetG, \qquad W_{k,j} = \frac{\partial^2 w}{\partial u_k \partial p_j}, \qquad G_{j,k} = \frac{\partial g_j}{\partial u_k}.$$

It is not difficult to check that $DetG = DetG_*$, $|DetG_*| = |u_4|$, where

$$G_{*j,k} = G_{j+3,k+3}, \qquad G_* = \begin{pmatrix} c & 0 & 0 & 0 & 0 & -u_4 s \\ -s & 0 & 0 & 0 & 0 & -u_4 c \\ 0 & c & s & 0 & 0 & -u_5 s + u_6 c \\ 0 & -s & c & 0 & 0 & -u_5 c - u_6 s \\ 0 & 0 & 0 & c & s & -u_7 s + u_8 c \\ 0 & 0 & 0 & -s & c & -u_7 c - u_8 s \end{pmatrix}.$$

G_* is obtained from G by changing the order of rows through the rule: $4 \to 2, 2 \to 3, 3 \to 5, 5 \to 4$. One has to take into account that a determinant is multiplied by minus one if a row indexed by k becomes a row indexed by $k+1$ or $k-1$. $DetG_*$ is easily calculated expanding it in the elements of the first and second rows

$$DetG_* = cDet \begin{pmatrix} 0 & 0 & 0 & 0 & -u_4 c \\ c & s & 0 & 0 & -u_5 s + u_6 c \\ -s & c & 0 & 0 & -u_5 c - u_6 s \\ 0 & 0 & c & s & -u_7 s + u_8 c \\ 0 & 0 & -s & c & -u_7 c - u_8 s \end{pmatrix} + u_4 sDet \begin{pmatrix} -s & 0 & 0 & 0 & 0 \\ 0 & c & s & 0 & 0 \\ 0 & -s & c & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{pmatrix}$$

$$= -c^2 u_4 Det \begin{pmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix} - s^2 u_4 Det \begin{pmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix} = -u_4.$$

As a result

$$x_j = u_j, j = 1, 2, 3; \quad x_4 = u_4 c, \quad x_7 = -u_4 s, \quad x_5 = u_5 c + u_6 s, \quad x_8 = -u_5 s + u_6 c, \tag{4.2}$$

$$x_6 = u_7 c + u_8 s, \quad x_9 = -u_7 s + u_8 c, \quad c = cosu_9, \quad s = sinu_9. \tag{4.3}$$

From (4.2) one obtains

$$v_4 = cp_4 - sp_7, \quad v_5 = cp_5 - sp_8, \quad v_6 = sp_5 + cp_8,$$

$$v_7 = cp_6 - sp_9, \quad v_8 = sp_6 + cp_9, \quad v_j = p_j, j = 1, 2, 3.$$

Let

$$v_0 = sp_4 + cp_7$$

then

$$p_4 = cv_4 + sv_0, \quad p_7 = cv_0 - sv_4, \qquad p_5 = cv_5 + sv_6, \quad p_8 = cv_6 - sv_5,$$

$$p_6 = cv_7 + sv_8, \quad p_9 = cv_8 - sv_7$$

and

$$Q = v_9 = Q_0 - u_4 v_0, \qquad Q_0 = u_6 v_5 - u_5 v_6 + u_8 v_7 - u_7 v_8, \quad v_0 = u_4^{-1}(Q_0 - v_9).$$

From these equalities one derives

$$v_5^2 + v_6^2 = p_5^2 + p_8^2, \qquad v_0^2 + v_4^2 = p_4^2 + p_7^2, \quad v_7^2 + v_8^2 = p_6^2 + p_9^2.$$

The squared Euclidean norms of the re-numerated charge coordinates are represented by

$$x_4^2 + x_7^2 = u_4^2, \qquad x_5^2 + x_8^2 = u_5^2 + u_6^2, \qquad x_6^2 + x_9^2 = u_7^2 + u_8^2.$$

The squared Euclidean distances in re-numerated variables between three charges are given by

$$(x_4 - x_5)^2 + (x_7 - x_8)^2 = (u_5 - u_4)^2 + u_6^2,$$

$$(x_4 - x_6)^2 + (x_7 - x_9)^2 = (u_7 - u_4)^2 + u_8^2,$$

$$(x_5 - x_6)^2 + (x_8 - x_9)^2 = (u_7 - u_5)^2 + (u_8 - u_6)^2.$$

The new Hamiltonian in the new variables is given by (w does not depend on t)

$$H' = (2m)^{-1} \sum_{j=1}^{8} v_j^2 + (2mu_4^2)^{-1}(Q_0 - v_9)^2 + U'(u_{(8)}), \tag{4.4}$$

$$\begin{aligned} U'(u_{(8)}) = e_0^2[&((u_1 - u_2)^2 + (u_5 - u_4)^2 + u_6^2)^{-\frac{1}{2}} + ((u_1 - u_3)^2 + (u_7 - u_4)^2 + u_8^2)^{-\frac{1}{2}} \\ &+ ((u_3 - u_2)^2 + (u_7 - u_5)^2 + (u_6 - u_8)^2)^{-\frac{1}{2}}] \\ &- e_0 e'[(u_4^2 + (b - u_1)^2)^{-\frac{1}{2}} + (u_5^2 + u_6^2 + (b - u_2)^2)^{-\frac{1}{2}} + (u_7^2 + u_8^2 + (b - u_3)^2)^{-\frac{1}{2}}] \\ &- e_0 e'[(u_4^2 + (b + u_1)^2)^{-\frac{1}{2}} + (u_5^2 + u_6^2 + (b + u_2)^2)^{-\frac{1}{2}} + (u_7^2 + u_8^2 + (b + u_3)^2)^{-\frac{1}{2}}]. \end{aligned}$$

The equilibrium in new variables is determined by $v_s = 0, s = 1, \ldots, 9$ and

$$u_4 = -a, u_7 = a, \quad u_j = 0, j \neq 4, 7, \quad u_9 = \xi.$$

The variable u_9 is cyclic. Since $v_9 = Q = const$ we have

$$\dot{u}_9 = \frac{\partial H'}{\partial v_9} = (mu_4^2)^{-1}(Q - Q_0), \qquad \dot{v}_9 = 0. \tag{4.5}$$

The last two equations correspond to the zero eigenvalue of the canonical matrix of the Hamiltonian H and are easily integrated since the right-hand side of the equation for u_9 in (4.5) is a holomorphic function at the equilibrium.

Making the translation $u_4 \to u_4 + a$, $u_7 \to u_7 - a$ we can apply the Lyapunov center theorem to the set of Eqs. (1.4) with the Hamiltonian H' in (4.4) which is a holomorphic function in a neighborhood of the origin.

The center Lyapunov theorem, the four propositions of the previous section and Theorem 1.1 imply the following theorem.

Theorem 4.1. *Let u be such that all $\zeta_j, j \neq 7$ are non-zero. Let also $0 < u \leq 2$ or $2 < u < 3$ and $\frac{\zeta_5}{\zeta_2} \neq n^2$, where n is an integer. Then equation of motion (1.4) for $n = 9$, $v_9 = Q^0 = 0$ and H', given by (4.4), which corresponds to the Coulomb equation of motion (1.1) for $N = 3, d = 3$, the potential energy (2.1), possesses a periodic solution. It and its period $\tau(c)$ are holomorphic functions in the parameter c at the origin and $\tau(0) = 2\pi\sqrt{\frac{m}{\zeta_2}}$. Moreover this solution generates the quasi-periodic solution of (1.1) given by (1.5) with $n = 9$, where $\gamma, \gamma', \gamma''$ correspond to (4.1)–(4.3), such that the Euclidean norm $|x_j|^2$ is a periodic function.*

5. Conclusion

We formulated our main result in Theorem 4.1. In other words we established the existence of quasi-periodic solutions of the Coulomb equation of motion whose Euclidean norms are periodic functions for the considered Coulomb systems. We found the explicit form of eigenvalues of the matrix U^0 of second derivatives of the potential energy at the equilibrium. This enabled us to avoid a resonance between them and apply the center Lyapunov theorem and Theorem 1.1 for the proof of the main result.

We hope that this result will help to understand better the corresponding quantum system and find a connection of a discrete spectrum of the quantum Hamiltonian and the found periodic solutions of the classical Coulomb systems.

Appendix

Here we prove the third and second statements of Theorem 1.1. We deal with the Hamiltonian H in $\mathbb{R}^{2n}$ which is a holomorphic function at its equilibrium q^0

$$H(q) = \frac{1}{2}(h^0(q - q^0), (q - q^0)) + \cdots, \quad q = (x; p), x \in \mathbb{R}^n, p \in \mathbb{R}^n,$$

where (.,.) is the Euclidean scalar product in $\mathbb{R}^{2n}$, h^0 is a symmetric matrix and the three dots imply higher power terms in q_j in the Taylor expansion. The canonical matrix Jh^0 is found from the linear part of the equation of motion $\dot{q} = J\partial H$, where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

I is the $n \times n$ unit matrix and ∂ is the vector of first partial derivatives. The direct and inverse canonical transformations of $q_{(n)} = (x, p)_{(n)}$ into $q'_{(n)} = (x', p')_{(n)}$ are given by

$$q'_j - q'^0_j = \sum_{k=1}^{2n} M^{-1}_{j,k}(q_k - q^0_k) + \cdots, \qquad q_j - q^0_j = \sum_{k=1}^{2n} M_{j,k}(q'_k - q'^0_k) + \cdots,$$

where $M_{j,k}$ is an invertible matrix of the linear symplectic transformation in $\mathbb{R}^{2n}$ and q'^0 is the new equilibrium. If $M_{*j,k} = M_{k,j}$ then (see the sections 2 and 15 in [7])

$$H'(q') = \frac{1}{2}(h'^0(q' - q'^0), q' - q'^0) + \cdots,$$

where

$$h'^0 = M_* h^0 M, \qquad M_* JM = J, \qquad J = -J^{-1}$$

and J determines the symplectic structure in $\mathbb{R}^{2n}$. This yields

$$-Det(\lambda I - Jh^0) = Det(\lambda J + h^0) = (DetM)^{-2} Det(\lambda J + h'^0).$$

That is the characteristic polynomials of the canonical matrices of the transformed and initial Hamiltonians have the same roots.

Statement 2 follows from $\dot{v}_n = 0$, the fact that the characteristic polynomial of a canonical matrix is an even function in the spectral parameter λ (see section 15 in [7]) and the fact that the characteristic polynomial of Jh^0 is proportional to λ^2. ■

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