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Lie bialgebras of complex type and associated Poisson Lie groups

A. Andrada[a,*], M.L. Barberis[a], G. Ovando[b]

[a] CIEM, FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, (5000) Córdoba, Argentina

[b] CONICET, FCEIA, Universidad Nacional de Rosario, Av. Pellegrini 250, (2000) Rosario, Argentina

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ABSTRACT

In this work we study a particular class of Lie bialgebras arising from Hermitian structures on Lie algebras such that the metric is ad-invariant. We will refer to them as Lie bialgebras of complex type. These give rise to Poisson Lie groups G whose corresponding duals G^* are complex Lie groups. We also prove that a Hermitian structure on $\mathfrak{g}$ with ad-invariant metric induces a structure of the same type on the double Lie algebra $\mathcal{D}\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$, with respect to the canonical ad-invariant metric of neutral signature on $\mathcal{D}\mathfrak{g}$. We show how to construct a $2n$-dimensional Lie bialgebra of complex type starting with one of dimension $2(n-2)$, $n \geq 2$. This allows us to determine all solvable Lie algebras of dimension ≤ 6 admitting a Hermitian structure with ad-invariant metric. We present some examples in dimensions 4 and 6, including two one-parameter families, where we identify the Lie–Poisson structures on the associated simply connected Lie groups, obtaining also their symplectic foliations.

1. Introduction

The notion of Poisson Lie group was first introduced by Drinfeld [10] and studied by Semenov-Tyan-Shanskii [29] to understand the Hamiltonian structure of the hidden symmetry group of a completely integrable system. These groups appear in the theory of Poisson–Lie T-dual sigma models [16]. Lie bialgebras, the infinitesimal counterpart of Poisson Lie groups (see [10,31]), are in a natural one-to-one correspondence with Lie algebra structures on the vector space $\mathcal{D}\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ with some compatibility conditions. $\mathcal{D}\mathfrak{g}$ with this Lie algebra structure is called the double of the Lie algebra $\mathfrak{g}$. The classification of Lie bialgebras for complex semisimple $\mathfrak{g}$ was carried out in [7]. On the other hand, since non-semisimple Lie algebras play an important role in physical problems, there is more recent work on the classification problem of low-dimensional Lie bialgebras (see, for instance, [25] and the references therein).

The purpose of this paper is to study a particular class of Lie bialgebras arising from classical r-matrices of a special type and to present some new examples of Poisson Lie groups in this setting. We obtain Poisson Lie groups such that their duals turn out to be complex Lie groups. In fact, if J is a complex structure on a Lie algebra $\mathfrak{g}$, then a new bracket $[\cdot,\cdot]_J$ can be defined on $\mathfrak{g}$ as follows:

$$[x, y]_J = [Jx, y] + [x, Jy], \quad x, y \in \mathfrak{g},$$

and the integrability condition for J (see Section 2.2) implies that $[\cdot,\cdot]_J$ satisfies the Jacobi identity, that is, J is a classical r-matrix on $\mathfrak{g}$ (see Lemma 3). Moreover, $\mathfrak{g}_J := (\mathfrak{g}, [\cdot,\cdot]_J)$ is a complex Lie algebra. If $\mathfrak{g}$ carries also an ad-invariant metric g such that (J, g) is a Hermitian structure, then $\mathfrak{g}$ admits a Lie bialgebra structure. Such a Lie bialgebra will be called of *complex type*. We prove that the Lie algebra structure on the dual $\mathfrak{g}^*$ is isomorphic to $\mathfrak{g}_J$, and therefore $\mathfrak{g}^*$ is a complex Lie algebra. This

* Corresponding author. Fax: +54 351 4334054.
E-mail addresses: andrada@mate.uncor.edu (A. Andrada), barberis@mate.uncor.edu (M.L. Barberis), ovando@mate.uncor.edu (G. Ovando).

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implies that the simply connected Poisson Lie group G associated to the Lie bialgebra $\mathfrak{g}$ has the remarkable property that its dual Poisson Lie group G^* is a complex Lie group. Furthermore, we prove that the double Lie algebra $\mathcal{D}\mathfrak{g}$ admits a complex structure $\mathcal{J}$ (induced by J), such that $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ is a Hermitian structure on $\mathcal{D}\mathfrak{g}$, where $\langle\cdot,\cdot\rangle$ is the canonical ad-invariant metric of neutral signature on $\mathcal{D}\mathfrak{g}$. It turns out that when $\mathfrak{g}$ is a Lie bialgebra of complex type, then so is $\mathcal{D}\mathfrak{g}$.

Also, beginning with a Lie algebra $\mathfrak{g}$ equipped with a Hermitian structure such that the metric is ad-invariant, we can apply, under certain conditions, the double extension method (see [13,15,23]) in order to construct a new Lie algebra of dimension $\dim\mathfrak{g}+4$ endowed with a similar structure. This construction leads us to determine all solvable Lie algebras of dimension ≤ 6 admitting a Hermitian structure with ad-invariant metric.

The paper is organized as follows. In Section 2.1 we define Lie bialgebras, Poisson Lie groups and recall from [2] how to obtain a Lie bialgebra from a classical r-matrix. Section 2.2 is devoted to preliminary results on the complex Lie algebra $\mathfrak{g}_J$ constructed from a complex structure J on $\mathfrak{g}$ and in Section 2.3 we define metric Lie algebras and discuss some properties of Hermitian structures with ad-invariant metrics.

In Section 3 we prove that every Hermitian structure with ad-invariant metric gives rise to a Lie bialgebra of complex type. In this case, we show that the double Lie algebra $\mathcal{D}\mathfrak{g}$ also has such a structure. We obtain, as a corollary, that a complex structure on $\mathfrak{g}$ induces a Lie bialgebra structure of complex type on the double $\mathcal{D}(T^*\mathfrak{g})$ of the cotangent Lie algebra $T^*\mathfrak{g}$. We provide several examples starting with solvable Lie algebras. In case $\mathfrak{g}$ is compact semisimple, we show that our construction yields a Lie bialgebra which is not equivalent to the standard one obtained by [21].

Section 4 is devoted to the study of Lie bialgebras of complex type obtained by a double extension process. By applying this construction we can prove that every solvable metric Lie algebra of dimension at most 6 with metric of signature $(2r, 2s)$ admits a Lie bialgebra structure of complex type.

In the last section we study the simply connected Poisson Lie groups associated to some Lie bialgebras of complex type in dimension 4 and 6, obtaining also their symplectic foliations. We get a linearizable Poisson structure on $\mathbb{R}^4$ and we exhibit two one-parameter families $\{\Pi_1^\lambda\}$, $\{\Pi_2^\lambda\}$ of Lie–Poisson structures on $\mathbb{R}^6$ $(\lambda > 0)$ such that Π_1^λ is not equivalent to Π_2^λ for each $\lambda > 0$, but as $\lambda \to 0$ these families converge to the same Poisson structure Π^0. When $\lambda \to \infty$ we have an analogous situation, with both families converging to the same Poisson structure Π^∞. We point out that the structures Π^0 and Π^∞ on $\mathbb{R}^6$ are not equivalent. On the other hand, there is a unique nilpotent Lie algebra of dimension 6 admitting Lie bialgebra structures of complex type, giving rise to Lie–Poisson structures on $\mathbb{R}^6$ which cannot be obtained as deformations of the previous ones.

2. Preliminaries and basic results

2.1. Lie bialgebras and Poisson Lie groups

Let $\mathfrak{g}$ be a real Lie algebra and $\delta : \mathfrak{g} \to \bigwedge^2\mathfrak{g}$ a 1-cocycle with respect to the adjoint representation. The pair $(\mathfrak{g}, \delta)$ is called a *Lie bialgebra* if $\delta^* : \bigwedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ induces a Lie algebra structure on $\mathfrak{g}^*$ (see [10]).

A *multiplicative* Poisson structure on a Lie group G is a smooth section Π of $\bigwedge^2 TG$ such that the following conditions are satisfied:

(1) $\{f, g\} := \langle df \wedge dg, \Pi\rangle$ defines a Lie bracket on $C^\infty(G, \mathbb{R})$;
(2) the multiplication $m : G \times G \to G$ is a Poisson mapping, that is, the pull back mapping $m^* : C^\infty(G, \mathbb{R}) \to C^\infty(G \times G, \mathbb{R})$ is a homomorphism for the Poisson brackets.[1]

A Lie group equipped with a multiplicative Poisson structure is called a *Poisson Lie group*.

If G is a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, there is a one-to-one correspondence between multiplicative Poisson structures on G and Lie bialgebra structures δ on $\mathfrak{g}$ (see [11]). The correspondence is established as follows:

$$(\mathrm{d}\Pi)_e = \delta. \tag{1}$$

We will say that a linear operator $r \in \mathrm{End}(\mathfrak{g})$ is a *classical r-matrix* if the $\mathfrak{g}$-valued skew-symmetric bilinear form on $\mathfrak{g}$ given by

$$[x, y]_r = [rx, y] + [x, ry]$$

is a Lie bracket, that is, it satisfies the Jacobi identity. $\mathfrak{g}$ equipped with the Lie bracket $[\cdot,\cdot]_r$ will be denoted by $\mathfrak{g}_r$. The next result is a consequence of [2, Corollary 2.9] (see also [28]):

Proposition 1. *If r is a classical r-matrix and $\mathfrak{g}$ admits a non-degenerate symmetric ad-invariant bilinear form such that r is skew-symmetric, then r gives rise to a Lie bialgebra $(\mathfrak{g}, \delta_r)$.*

[1] We recall that the Poisson bracket on $C^\infty(G \times G, \mathbb{R})$ is defined by

$$\{f, g\}(x, y) = \{f(x,\cdot), g(x,\cdot)\}(y) + \{f(\cdot, y), g(\cdot, y)\}(x), \quad x, y \in G.$$

The 1-cocycle δ_r above is defined as follows. Let g be a non-degenerate symmetric ad-invariant bilinear form on $\mathfrak{g}$ such that r is skew-symmetric. We consider the metric g on $\mathfrak{g}$ as a linear isomorphism $g : \mathfrak{g} \to \mathfrak{g}^*$, $g(x)(y) = g(x, y)$. Via this isomorphism, the Lie bracket $[\cdot,\cdot]_r$ on $\mathfrak{g}_r$ induces a Lie bracket $[\cdot,\cdot]^*$ on $(\mathfrak{g}_r)^* = \mathfrak{g}^*$, given as follows:

$$[\alpha, \beta]^* = g([g^{-1}\alpha, g^{-1}\beta]_r), \quad \alpha, \beta \in \mathfrak{g}^*.$$

We denote by $\delta_r : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ the dual of $[\cdot,\cdot]^*$. If we look upon $R := r \circ g^{-1}$ as an element in $\bigwedge^2 \mathfrak{g}$, we observe that $(\mathfrak{g}, \delta_r)$ is an exact Lie bialgebra, since

$$\delta_r(x) = \mathrm{ad}(x)R \quad \text{for all } x \in \mathfrak{g},$$

that is, δ_r is a coboundary.

For a Lie bialgebra coming from a classical r-matrix r, the corresponding Poisson structure on G given by (1) is the bivector field $\overleftarrow{R} - \overrightarrow{R}$, where $\overleftarrow{R}$ (resp. $\overrightarrow{R}$) is the left (resp. right) invariant bivector field on G whose value at the identity e of G is $R = r \circ g^{-1}$ (see [2,8]).

In Section 3 we will apply the above proposition to the particular case when the r-matrix is a complex structure on the Lie algebra.

Remark 2. Given a bivector $R \in \bigwedge^2 \mathfrak{g}$, the coboundary δ_R defined by $\delta_R(x) = \mathrm{ad}(x)R$ for $x \in \mathfrak{g}$ induces a Lie bialgebra structure on $\mathfrak{g}$ if and only if the Schouten bracket[2] $[R, R]$ is $\mathrm{ad}(\mathfrak{g})$-invariant:

$$[R, R] \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}}.$$

When R satisfies this condition we say that R is a solution of the *modified Yang–Baxter equation* (MYBE). In particular, this is trivially satisfied when $[R, R] = 0$, which is known as the *classical Yang–Baxter equation* (CYBE).

If $\mathfrak{g}$ admits a metric as above and $r = R \circ g \in \mathrm{End}(\mathfrak{g})$, then the MYBE and the CYBE in terms of r are equivalent, respectively, to:

$$[x, B_r(y, z)] + [y, B_r(z, x)] + [z, B_r(x, y)] = 0, \quad x, y, z \in \mathfrak{g},$$
$$B_r(x, y) = 0, \quad x, y \in \mathfrak{g},$$

where

$$B_r(x, y) := [rx, ry] - r([x, y]_r).$$

2.2. Complex structures as r-matrices and associated Lie algebras

A complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism J of $\mathfrak{g}$ satisfying $J^2 = -\mathrm{id}$ and the integrability condition $N_J \equiv 0$, where

$$N_J(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, Jy] \quad \text{for } x, y \in \mathfrak{g}. \tag{2}$$

If G is a Lie group with Lie algebra $\mathfrak{g}$, by left translating the endomorphism J we obtain a complex manifold (G, J) such that left translations are holomorphic maps. We point out that (G, J) is not necessarily a complex Lie group since right translations are not in general holomorphic.

Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$, then we have a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0}$ (resp. $\mathfrak{g}^{0,1}$) is the eigenspace of J of eigenvalue i (resp. $-i$). It follows that

$$\mathfrak{g}^{1,0} = \{x - iJx : x \in \mathfrak{g}\}, \qquad \mathfrak{g}^{0,1} = \{x + iJx : x \in \mathfrak{g}\},$$

hence $\mathfrak{g}^{0,1} = \sigma(\mathfrak{g}^{1,0})$, where $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is conjugation with respect to the real form $\mathfrak{g}$, that is, $\sigma(x + iy) = x - iy$, $x, y \in \mathfrak{g}$. The integrability of J is equivalent to the fact that both, $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, are complex Lie subalgebras of $\mathfrak{g}^{\mathbb{C}}$. When $\mathfrak{g}^{1,0}$ is abelian J is called an *abelian* complex structure (see [3]) and it follows that $\mathfrak{g}^{0,1}$ is also abelian. This is equivalent to the condition $[Jx, Jy] = [x, y]$, $x, y \in \mathfrak{g}$.

When $\mathfrak{g}^{1,0}$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{g}^{0,1}$ is also an ideal and $(\mathfrak{g}, J)$ is called a complex Lie algebra. In this case, $\mathrm{ad}(x)$, $x \in \mathfrak{g}$, are complex linear maps:

$$\mathrm{ad}(x) \circ J = J \circ \mathrm{ad}(x). \tag{3}$$

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and (G, J) is the complex manifold obtained by left translating J, the above condition is equivalent to (G, J) being a complex Lie group, that is, both, right and left translations on G are holomorphic maps.

[2] For the definition and properties of the Schouten bracket see, for instance, [17].

We will say that two complex structures J_1, J_2 on $\mathfrak{g}$ are equivalent if there exists a Lie algebra automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi \circ J_1 = J_2 \circ \varphi$.

As a consequence of the integrability condition of a complex structure J we obtain the following lemma (see also [5,18]).

Lemma 3. *If J is a complex structure on $\mathfrak{g}$ then J is a classical r-matrix. Moreover, $(\mathfrak{g}_J, J)$ is a complex Lie algebra.*

Proof. Using that $N_J \equiv 0$ (see (2)), we compute

$$J[x, y]_J = J([Jx, y] + [x, Jy]) = -[x, y] + [Jx, Jy],$$
$$[x, Jy]_J = [Jx, Jy] + [x, J(Jy)] = [Jx, Jy] - [x, y],$$

therefore

$$J[x, y]_J = [x, Jy]_J. \tag{4}$$

Setting $\mathrm{ad}_J(x) = [x, \cdot]_J$, (4) says that $\mathrm{ad}_J(x)$ satisfies (3) for all $x \in \mathfrak{g}_J$, in other words, $(\mathfrak{g}_J, J)$ is a complex Lie algebra and the lemma follows. □

Lemma 4. (1) *$\mathfrak{g}_J$ is an abelian Lie algebra if and only if J is an abelian complex structure.*
(2) *$(\mathfrak{g}, J)$ is a complex Lie algebra if and only if it is isomorphic to $(\mathfrak{g}_J, J)$.*
(3) *$(\mathfrak{g}, J)$ is a complex Lie algebra if and only if $\mathrm{ad}_J(x) = 2J \circ \mathrm{ad}(x)$ for any $x \in \mathfrak{g}$.*

Proof. Assertion (1) is straightforward from the definition of $[\cdot, \cdot]_J$.

To prove (2), assume first that $(\mathfrak{g}, J)$ is a complex Lie algebra. Then

$$[x, y]_J = 2J[x, y], \quad x, y \in \mathfrak{g}.$$

It follows that $\varphi = -\frac{1}{2}J$ is a Lie algebra isomorphism from $\mathfrak{g}$ onto $\mathfrak{g}_J$ that commutes with J.

The converse follows from Lemma 3.

The only if part of (3) is straightforward. To prove the converse, assume that $\mathrm{ad}_J(x) = 2J \circ \mathrm{ad}(x)$ for any $x \in \mathfrak{g}$. Therefore,

$$\mathrm{ad}(x) = -\frac{1}{2} J \circ \mathrm{ad}_J(x),$$

and composing with J on the left we obtain

$$J \circ \mathrm{ad}(x) = \frac{1}{2}\, \mathrm{ad}_J(x) = -\frac{1}{2}\, \mathrm{ad}_J(x) \circ J^2 = -\frac{1}{2} J \circ \mathrm{ad}_J(x) \circ J = \mathrm{ad}(x) \circ J,$$

where the third equality follows from Lemma 3. Hence, $(\mathfrak{g}, J)$ is a complex Lie algebra. □

The next proposition gives a complex Lie algebra isomorphism between $(\mathfrak{g}_J, J)$ and $\mathfrak{g}^{1,0}$ (see also [18]).

Proposition 5. *If J is a complex structure on $\mathfrak{g}$, then $(\mathfrak{g}_J, J)$ and $\mathfrak{g}^{1,0}$ are isomorphic as complex Lie algebras.*

Proof. Consider $\mathfrak{g}$ as a complex vector space with multiplication by i given by the endomorphism J. Let $\varphi : \mathfrak{g}_J \to \mathfrak{g}^{1,0}$ be the complex linear map defined as follows:

$$\varphi(x) = i(x - iJx) = (J + i\,\mathrm{id})(x), \quad x \in \mathfrak{g}_J,$$

where id is the identity operator. It is easily checked that φ is a Lie algebra isomorphism, where the Lie bracket on $\mathfrak{g}^{1,0}$ is the complex bilinear extension of $[\cdot, \cdot]$. □

Remark 6. (i) Observe that in the above proposition $J - i\,\mathrm{id}$ gives an isomorphism between $\mathfrak{g}_J$ and $\mathfrak{g}^{0,1}$.
(ii) We point out that if $\mathfrak{g}$ is a solvable (resp. nilpotent) Lie algebra with a complex structure J, then $\mathfrak{g}_J$ is solvable (resp. nilpotent). The converse is not true, that is, we can obtain a solvable Lie algebra $\mathfrak{g}_J$ starting from a non-solvable Lie algebra $\mathfrak{g}$. We recall from [30] the definition of a regular complex structure on a reductive Lie algebra $\mathfrak{g}$. A complex structure J on $\mathfrak{g}$ is called *regular* if there exists a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ such that $[\mathfrak{h}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$, where σ is conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. If $\mathfrak{g}$ is a product of an abelian ideal and a semisimple ideal $\mathfrak{s}$, it was shown in [30] that when the simple factors in $\mathfrak{s}$ are compact or belong to the following list:

$$\mathfrak{sl}(2, \mathbb{R}), \qquad \mathfrak{sp}(2n, \mathbb{R}), \mathfrak{so}^*(2n), \qquad \mathfrak{e}_{6(-14)}, \qquad \mathfrak{e}_{6(2)}, \quad \text{any real form of: } \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2,$$

then any complex structure J on $\mathfrak{g}$ is regular and satisfies $\mathfrak{u} \subset \mathfrak{g}^{1,0} \subset \mathfrak{b}$ for some Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}^{\mathbb{C}}$ with unipotent radical $\mathfrak{u}$. Therefore, $\mathfrak{g}_J \cong \mathfrak{g}^{1,0}$ is always solvable.

On the other hand, if $\mathfrak{g}$ is a real non-compact semisimple Lie algebra with a compact Cartan subalgebra or a real split semisimple Lie algebra, it follows from results of [19] that for the canonical Koszul operator J on $\mathfrak{g}$, $\mathfrak{g}_J$ is also solvable.

The next result is a straightforward consequence of Proposition 5.

Corollary 7. *If J_1 and J_2 are equivalent complex structures on $\mathfrak{g}$, then $\mathfrak{g}_{J_1}$ is isomorphic to $\mathfrak{g}_{J_2}$.*

2.3. Metric Lie algebras and Hermitian structures

A *metric* on $\mathfrak{g}$ is a non-degenerate symmetric bilinear form on $\mathfrak{g}$. A Hermitian structure (J, g) on $\mathfrak{g}$ is a pair of a complex structure J and a metric g on $\mathfrak{g}$ such that J is skew-symmetric. We will say that two Lie algebras with Hermitian structures $(\mathfrak{g}_1, J_1, g_1)$ and $(\mathfrak{g}_2, J_2, g_2)$ are *equivalent* if there exists a Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ which is also an isometry between g_1 and g_2 satisfying $\varphi \circ J_1 = J_2 \circ \varphi$.

A *metric Lie algebra* is a pair $(\mathfrak{g}, g)$ of a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant metric g, that is, the following condition holds for all $x, y, z \in \mathfrak{g}$:

$$g([x, y], z) = g(x, [y, z]).$$

Equivalently, the endomorphisms $\mathrm{ad}(x)$ are skew-symmetric for all $x \in \mathfrak{g}$. Note that in this case $\mathfrak{g}$ is unimodular (i.e., $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for all $x \in \mathfrak{g}$). The Killing form on a semisimple Lie algebra is an example of an ad-invariant metric. However, for non-semisimple Lie algebras, there are obstructions for the existence of ad-invariant metrics: for instance, if such a metric exists, then $\mathfrak{z}^\perp = [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}$ denotes the centre of $\mathfrak{g}$. In particular, we have that $\dim \mathfrak{z} + \dim[\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g}$, so that a solvable Lie algebra with trivial centre cannot admit an ad-invariant metric. The metric Lie algebra $(\mathfrak{g}, g)$ is called *indecomposable* if every proper ideal of $\mathfrak{g}$ is degenerate.

We will study Hermitian structures (J, g) on $\mathfrak{g}$ where g is ad-invariant. First, we prove some results which impose restrictions on the complex structure or the metric. The next lemma gives a necessary condition on the signature of the metric. It can be shown by induction on $\dim V$ and its proof is omitted.

Lemma 8. *Let V be a real vector space with a Hermitian structure (J, g). Then the signature of g is of the form $(2r, 2s)$, $2r + 2s = \dim V$.*

Proposition 9. *Let $\mathfrak{g}$ be a Lie algebra endowed with a Hermitian structure (J, g) such that the metric g is ad-invariant. If any of the conditions* (i)– (iii) *below holds, then $\mathfrak{g}$ is abelian.*

(i) *The Kähler form ω on $\mathfrak{g}$ defined by $\omega(x, y) = g(Jx, y)$ is closed.*
(ii) *J is abelian.*
(iii) $J \circ \mathrm{ad}(x) = \mathrm{ad}(x) \circ J$ *for all $x \in \mathfrak{g}$.*

Proof. (i) Suppose that the 2-form ω is closed, so that

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Since g is ad-invariant and J skew-symmetric, from the definition of ω we obtain

$$\begin{aligned}\omega([x, y], z) &= g(J[x, y], z), \omega([y, z], x) \\ &= g(z, [y, Jx]), \omega([z, x], y) = -g(z, [x, Jy]),\end{aligned}$$

so that

$$g(J[x, y] - [Jx, y] - [x, Jy], z) = g(J[Jx, Jy], z) = 0$$

for all $x, y, z \in \mathfrak{g}$, where we have used the integrability of J. Since g is non-degenerate, we have that $[Jx, Jy] = 0$ for all $x, y \in \mathfrak{g}$, and hence $\mathfrak{g}$ is abelian.
(ii) Let us assume now that J is abelian. Then

$$g([Jx, y], z) = -g(y, [Jx, z]) \stackrel{\dagger}{=} g(y, [x, Jz]) = g(J[x, y], z),$$

where in equality (†) we have used the fact that J is abelian. Since g is non-degenerate, we have that $J[x, y] = [Jx, y]$ for all $x, y \in \mathfrak{g}$, and from this we obtain easily that $[Jx, y] = [x, Jy]$. However, $[Jx, y] = -[x, Jy]$ as J is abelian, and hence $[Jx, y] = 0$ for all x, y and $\mathfrak{g}$ is abelian.
(iii) Suppose now that $J \circ \mathrm{ad}(x) = \mathrm{ad}(x) \circ J$, or equivalently, $J[x, y] = [Jx, y]$ for all $x, y \in \mathfrak{g}$. Then we have

$$g([Jx, y], z) = -g(y, [Jx, z]) = -g(y, J[x, z]) = -g([x, Jy], z),$$

so that $[Jx, y] = -[x, Jy]$ and therefore J is abelian. From (ii) $\mathfrak{g}$ is abelian. □

3. Lie bialgebras and Hermitian structures

Let $\mathfrak{g}$ be a Lie algebra with a Hermitian structure (J, g). We consider the metric g on $\mathfrak{g}$ as a linear isomorphism $g : \mathfrak{g} \to \mathfrak{g}^*$, $g(x)(y) = g(x, y)$. Via this isomorphism, the Lie bracket $[\cdot, \cdot]_J$ on $\mathfrak{g}_J$ induces a Lie bracket $[\cdot, \cdot]^*$ on $(\mathfrak{g}_J)^* = \mathfrak{g}^*$, given as follows:

$$[\alpha, \beta]^* = g([g^{-1}\alpha, g^{-1}\beta]_J), \quad \alpha, \beta \in \mathfrak{g}^*$$

and J induces an endomorphism $J^* : \mathfrak{g}^* \to \mathfrak{g}^*$,

$$J^*\alpha = -\alpha \circ J, \quad \alpha \in \mathfrak{g}^*. \tag{5}$$

We denote by $\delta_J : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ the dual of $[\cdot, \cdot]^*$.

Theorem 10. *Let (J, g) be a Hermitian structure on $\mathfrak{g}$ such that g is ad-invariant. Then $(\mathfrak{g}, \delta_J)$ is a Lie bialgebra such that $(\mathfrak{g}^*, [\cdot, \cdot]^*, J^*)$ is a complex Lie algebra isomorphic to $(\mathfrak{g}_J, [\cdot, \cdot]_J, J)$. In particular, if G and G^* are the corresponding simply connected Poisson Lie groups, then G^* is a complex Lie group.*

Proof. The fact that $(\mathfrak{g}, \delta_J)$ is a Lie bialgebra is a consequence of Proposition 1. It follows from [2, Proposition 2.8] that $(\mathfrak{g}^*, [\cdot, \cdot]^*) \cong (\mathfrak{g}_J, [\cdot, \cdot]_J)$. Moreover, since $J^* \circ g = g \circ J$, Lemma 3 implies that

$$(\mathfrak{g}^*, [\cdot, \cdot]^*, J^*) \cong (\mathfrak{g}_J, [\cdot, \cdot]_J, J)$$

as complex Lie algebras. □

A Lie bialgebra of the form $(\mathfrak{g}, \delta_J)$ as above will be called a *Lie bialgebra of complex type* and a Poisson Lie group corresponding to it, a *Poisson Lie group of complex type*.

Example 11. The *cotangent Lie algebra* $T^*\mathfrak{h}$ of a Lie algebra $\mathfrak{h}$ is defined as the semidirect product $T^*\mathfrak{h} := \mathfrak{h} \ltimes_{\text{coad}} \mathfrak{h}^*$. If H denotes the simply connected Lie group with Lie algebra $\mathfrak{h}$, the cotangent bundle T^*H has a natural Lie group structure with corresponding Lie algebra $T^*\mathfrak{h}$. There is a canonical ad-invariant metric of neutral signature $\langle \cdot, \cdot \rangle$ on $T^*\mathfrak{h}$, defined as in (6) below. According to Theorem 10, every skew-symmetric complex structure on $T^*\mathfrak{h}$ determines a multiplicative Poisson structure on T^*H such that $(T^*H)^*$ is a complex Lie group. It should be pointed out that Hermitian structures $(J, \langle \cdot, \cdot \rangle)$ on $T^*\mathfrak{h}$ are in one-to-one correspondence with left invariant generalized complex structures on H, as shown in [1].

It is well known [10] that given a Lie bialgebra $(\mathfrak{g}, \delta)$, the vector space $\mathcal{D}\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ becomes a Lie algebra with the following bracket:

$$[(x, \alpha), (y, \beta)] = \big([x, y] + \text{coad}^*(\alpha)y - \text{coad}^*(\beta)x, [\alpha, \beta]^* + \text{coad}(x)\beta - \text{coad}(y)\alpha\big),$$

for $x, y \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*$, where coad stands for the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$, and coad^* stands for the coadjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}$. Note that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras of $\mathcal{D}\mathfrak{g}$ with this Lie bracket. There is a canonical ad-invariant metric $\langle \cdot, \cdot \rangle$ of neutral signature on $\mathcal{D}\mathfrak{g}$ given by

$$\langle (x, \alpha), (y, \beta) \rangle = \alpha(y) + \beta(x) \tag{6}$$

for $x, y \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*$. Given an almost complex structure J on $\mathfrak{g}$ we define an almost complex structure $\mathcal{J}$ on $\mathcal{D}\mathfrak{g}$ by

$$\mathcal{J}(x, \alpha) = (Jx, J^*\alpha), \tag{7}$$

where J^* is defined as in (5). Note that $\mathcal{J}$ is skew-symmetric with respect to the metric (6) and that both subalgebras, $\mathfrak{g}$ and $\mathfrak{g}^*$, are $\mathcal{J}$-invariant.

When the bialgebra structure on $\mathfrak{g}$ is trivial, that is, $\delta = 0$, the Lie algebra $\mathcal{D}\mathfrak{g}$ is the cotangent Lie algebra $T^*\mathfrak{g}$. It was proved in [4] that if J is a complex structure on $\mathfrak{g}$ and we define $\mathcal{J}$ as in (7), then $\mathcal{J}$ is a complex structure on the cotangent algebra $T^*\mathfrak{g}$. Moreover, $(\mathcal{J}, \langle \cdot, \cdot \rangle)$ is a Hermitian structure on $T^*\mathfrak{g}$, where $\langle \cdot, \cdot \rangle$ is the canonical ad-invariant neutral metric. More generally, we have:

Proposition 12. *Let $\mathfrak{g}$ be a Lie algebra with a Hermitian structure (J, g) such that g is ad-invariant and $\mathcal{J}$ the almost complex structure defined in (7). Then $\mathcal{J}$ is integrable on $\mathcal{D}\mathfrak{g}$.*

Proof. Since J is integrable on $\mathfrak{g}$, J^* is integrable on $\mathfrak{g}^*$ (see Theorem 10) and both $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras of $\mathcal{D}\mathfrak{g}$, we only have to check that

$$N_{\mathcal{J}}((x, 0), (0, \beta)) = 0, \quad \text{for all } x \in \mathfrak{g}, \beta \in \mathfrak{g}^*.$$

To calculate $N_{\mathcal{J}}((x, 0), (0, \beta))$ (see (2)), we do the following computations, using the definition of the Lie bracket:

$$\begin{aligned}
&[\mathcal{J}(x, 0), \mathcal{J}(0, \beta)] = (-\text{coad}^*(J^*\beta)(Jx), \text{coad}(Jx)(J^*\beta)),\\
&[(x, 0), (0, \beta)] = (-\text{coad}^*(\beta)x, \text{coad}(x)\beta),\\
&\mathcal{J}[\mathcal{J}(x, 0), (0, \beta)] = (-J\,\text{coad}^*(\beta)(Jx), J^*\,\text{coad}(Jx)\beta),\\
&\mathcal{J}[(x, 0), \mathcal{J}(0, \beta)] = (-J\,\text{coad}^*(J^*\beta)x, J^*\,\text{coad}(x)J^*\beta).
\end{aligned}$$

Let us begin with the first coordinate; then, for $\psi \in \mathfrak{g}^*$, we compute

$$\begin{aligned}
(\text{coad}^*(J^*\beta)(x))\psi &= (-x \circ \text{ad}^*(J^*\beta))\psi = -x([J^*\beta, \psi]^*)\\
&= -x(J^*[\beta, \psi]^*) \quad (\text{since } (\mathfrak{g}^*, J^*) \text{ is a complex Lie algebra})\\
&= -J^*[\beta, \psi]^*(x) = [\beta, \psi]^*(Jx)\\
&= (Jx)[\beta, \psi]^* = -(\text{coad}^*(\beta)(Jx))\psi,
\end{aligned}$$

and from this one easily verifies that the first coordinate of $N_{\mathcal{J}}((x,0),(0,\beta))$ vanishes. Let us compute now its second coordinate. For $v \in \mathfrak{g}$, we have

$$\begin{aligned}(\mathrm{coad}(Jx)(J^*\beta))v &= -(J^*\beta)[Jx,v] = \beta(J[Jx,v]) \\ &= \beta([Jx,Jv]-[x,v]-J[x,Jv]) \quad (\text{using that } N_J \equiv 0) \\ &= J^*(\mathrm{coad}(Jx)\beta)v + (\mathrm{coad}(x)\beta)v + (J^*\,\mathrm{coad}(x)J^*\beta)v,\end{aligned}$$

and from this we get that the second coordinate of $N_{\mathcal{J}}((x,0),(0,\beta))$ also vanishes. Therefore, $N_{\mathcal{J}} \equiv 0$. □

Proposition 12 implies that $(\mathcal{J},\langle\cdot,\cdot\rangle)$ is a Hermitian structure on $\mathcal{D}\mathfrak{g}$, therefore, Theorem 10 yields:

Corollary 13. (1) *Under the same hypotheses of the previous proposition, $(\mathcal{D}\mathfrak{g},\delta_{\mathcal{J}})$ is a Lie bialgebra of complex type.*
(2) *If J is a complex structure on an arbitrary Lie algebra $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle$ is the canonical ad-invariant neutral metric on $T^*\mathfrak{g}$, then $(\mathcal{J},\langle\cdot,\cdot\rangle)$ is a Hermitian structure on $T^*\mathfrak{g}$. In particular, $\mathcal{D}(T^*\mathfrak{g})$ is a Lie bialgebra of complex type.*

Proof. The first assertion is a consequence of Proposition 12. For (2), use the fact that $\mathcal{J}$ defined as in (7) is a complex structure on $T^*\mathfrak{g}$ (see [4]) and therefore $(\mathcal{J},\langle\cdot,\cdot\rangle)$ is a Hermitian structure on $T^*\mathfrak{g}$. The second assertion now follows by applying (1). □

Example 14. Let $\mathfrak{aff}(\mathbb{R})$ be the non-abelian real 2-dimensional Lie algebra and let $\mathfrak{g} := L_2(1,1)$ denote its cotangent Lie algebra (see Example 11), which has a basis $\{e_1,\dots,e_4\}$ such that the non-vanishing Lie bracket relations are

$$[e_1,e_2]=e_2, \qquad [e_1,e_3]=-e_3, \qquad [e_2,e_3]=e_4. \tag{8}$$

Let (J,g) be the Hermitian structure on $\mathfrak{g}$ with ad-invariant g given by

$$\begin{aligned}&Je_1=e_2, \qquad Je_3=-e_4,\\ &g(e_1,e_4)=g(e_2,e_3)=1,\end{aligned}$$

(see [24]).

The Lie bracket $[\cdot,\cdot]_J$ is given by

$$[e_1,e_3]_J=e_4, \qquad [e_1,e_4]_J=-e_3, \qquad [e_2,e_3]_J=e_3, \qquad [e_2,e_4]_J=e_4, \tag{9}$$

hence, $\mathfrak{g}_J$ is isomorphic, as a complex Lie algebra, to the complex analogue $\mathfrak{aff}(\mathbb{C})$ of $\mathfrak{aff}(\mathbb{R})$. If $\{e^1,\dots,e^4\}$ denotes the dual basis of $\mathfrak{g}^*$, from (9) and the definition of g, we have that the only non-trivial brackets on $\mathfrak{g}^*$ are

$$[e^1,e^3]^*=-e^1, \qquad [e^1,e^4]^*=e^2, \qquad [e^2,e^3]^*=-e^2, \qquad [e^2,e^4]^*=-e^1.$$

To determine the Lie algebra structure on $\mathcal{D}\mathfrak{g}$, it remains to compute the brackets of the form $[e_i,e^j] = (-\mathrm{coad}^*(e^j)e_i, \mathrm{coad}(e_i)e^j)$, which is done below:

$$\begin{array}{lll}[e_1,e^1]=-e_3, & [e_2,e^1]=e_4, & \\ [e_1,e^2]=-(e_4+e^2), & [e_2,e^2]=-e_3+e^1, & \\ [e_1,e^3]=e_1+e^3, & [e_2,e^3]=e_2, & [e_3,e^3]=-e^1, \\ [e_1,e^4]=e_2, & [e_2,e^4]=-(e_1+e^3), & [e_3,e^4]=e^2.\end{array}$$

$\mathcal{D}\mathfrak{g}$ is 3-step solvable, and observe that

$$\mathfrak{z}(\mathcal{D}\mathfrak{g}) = \mathrm{span}\left\{e_4, e_3-e^1\right\}, \qquad [\mathcal{D}\mathfrak{g},\mathcal{D}\mathfrak{g}] = \mathrm{span}\left\{e_2,e_3,e_4,e^1,e^2,e_1+e^3\right\}.$$

The ad-invariant neutral metric $\langle\cdot,\cdot\rangle$ on $\mathcal{D}\mathfrak{g}$ is given by $\langle e_j,e^j\rangle = 1$ for $j=1,\dots,4$, and the induced complex structure $\mathcal{J}$ on $\mathcal{D}\mathfrak{g}$ is as follows:

$$\mathcal{J}|_{\mathfrak{g}}=J, \qquad \mathcal{J}e^1=e^2, \qquad \mathcal{J}e^3=-e^4.$$

3.1. Compact semisimple bialgebras of complex type

We recall Samelson's construction of a complex structure on a compact semisimple even dimensional Lie algebra $\mathfrak{g}$ [27]. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Then we have the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha\in\Phi} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$

where Φ is a finite subset of $(\mathfrak{h}^{\mathbb{C}})^*$ and

$$\mathfrak{g}^{\mathbb{C}}_{\alpha} = \{x \in \mathfrak{g}^{\mathbb{C}} : [h,x] = \alpha(h)x, \forall h \in \mathfrak{h}^{\mathbb{C}}\}$$

are the one dimensional root subspaces. Since $\mathfrak{h}$ is even dimensional, one can choose a skew-symmetric endomorphism I of $\mathfrak{h}$ with respect to the Killing form such that $I^2 = -\mathrm{id}$. Samelson defines a complex structure on $\mathfrak{g}$ by considering a positive system Φ^+ of roots, which is a set $\Phi^+ \subset \Phi$ satisfying

$$\Phi^+ \cap (-\Phi^+) = \emptyset, \qquad \Phi^+ \cup (-\Phi^+) = \Phi, \quad \alpha, \beta \in \Phi^+, \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Phi^+.$$

Setting

$$\mathfrak{m} = \mathfrak{h}^{1,0} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha^{\mathbb{C}},$$

where $\mathfrak{h}^{1,0}$ is the eigenspace of I of eigenvalue i, it follows that $\mathfrak{m}$ is a (solvable) complex Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$ which induces a complex structure J on $\mathfrak{g}$ such that $\mathfrak{g}^{1,0} = \mathfrak{m}$. This complex structure is skew-symmetric with respect to the Killing form on $\mathfrak{g}$, hence we obtain a Lie bialgebra $(\mathfrak{g}, \delta_J)$ of complex type. This is not equivalent to the standard Lie bialgebra structure on $\mathfrak{g}$ constructed in [21] (see also [20]). In fact, in the standard case, the Lie algebra $\mathfrak{g}^*$ is completely solvable,[3] whereas in the complex type case, it is not.

Example 15. Let us consider $\mathfrak{g} = \mathfrak{su}(2n+1)$ for $n \geq 2$, hence $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2n+1, \mathbb{C})$. We can take

$$\mathfrak{h} = \left\{ \begin{pmatrix} it_1 & & \\ & \ddots & \\ & & it_{2n+1} \end{pmatrix} : t_j \in \mathbb{R}, t_1 + \cdots + t_{2n+1} = 0 \right\},$$

and

$$\mathfrak{h}^{1,0} = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_{2n+1} \end{pmatrix} : \begin{array}{l} z_l \in \mathbb{C}, \quad z_1 - z_{2n+1} = iz_{n+1}, \\ z_r = iz_{n+r}, \quad r = 2, \ldots, n, \sum_l z_l = 0 \end{array} \right\}.$$

The corresponding complex structure J on $\mathfrak{g}$ as above is determined by the complex subalgebra $\mathfrak{g}^{1,0}$ of $\mathfrak{sl}(2n+1, \mathbb{C})$ given by $\mathfrak{g}^{1,0} = \mathfrak{h}^{1,0} \oplus \mathfrak{n}$ where

$$\mathfrak{n} = \{\text{all } (2n+1) \times (2n+1) \text{ strictly upper triangular complex matrices}\}.$$

According to Theorem 10 and Proposition 5, the Lie algebra $\mathfrak{g}^*$ associated to $(\mathfrak{g}, \delta_J)$ is isomorphic to $\mathfrak{g}^{1,0}$.

On the other hand, we recall from [21] that if we consider the standard Lie bialgebra structure on $\mathfrak{g}$, then $\mathfrak{g}^*$ is isomorphic to the solvable Lie algebra

$$\mathfrak{sb}(2n+1, \mathbb{C}) = \left\{ \begin{array}{l} \text{all } (2n+1) \times (2n+1) \text{ traceless upper triangular} \\ \text{complex matrices with real diagonal entries} \end{array} \right\}.$$

Note that $\mathfrak{sb}(2n+1, \mathbb{C})$ is not a complex subalgebra of $\mathfrak{sl}(2n+1, \mathbb{C})$. Therefore, our construction yields a Lie bialgebra structure on $\mathfrak{su}(2n+1)$ which is not equivalent to the standard one.

We next provide families of metric Lie algebras admitting skew-symmetric complex structures where the ad-invariant metric has signature $(2, n)$.

3.2. Examples arising from Lorentzian ad-invariant metrics

We recall from [22] the classification of Lie algebras admitting ad-invariant metrics of signature $(1, n-1)$.

Theorem 16 *([22]). Each indecomposable non-simple Lie algebra with an ad-invariant Lorentzian metric is isomorphic to exactly one Lie algebra in the family*

$$\mathfrak{osc}(\underline{\lambda}) = span\{e_0, e_1, \ldots, e_{2m+1}\}$$

with $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, *where*

$$[e_{2i-1}, e_{2i}] = \lambda_i e_0, \quad i = 1, \ldots, m,$$

[3] Recall that a real solvable Lie algebra $\mathfrak{u}$ is *completely solvable* when $\mathrm{ad}(x)$ has real eigenvalues for all $x \in \mathfrak{u}$.

and the adjoint action of e_{2m+1} *on* $\operatorname{span}\{e_1, e_2, \ldots, e_{2m}\}$ *is given by*

$$A_{\underline{\lambda}} := \begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_m \\ & & & \lambda_m & 0 \end{pmatrix}.$$

The simply connected Lie group corresponding to $\mathfrak{osc}(\underline{\lambda})$ is known as the *oscillator* group. The 4-dimensional Lie algebra $\mathfrak{osc}(1)$ will be denoted simply by $\mathfrak{osc}$. Since the metric on any Lie algebra of the family $\mathfrak{osc}(\underline{\lambda})$ is Lorentzian these Lie algebras do not admit Lie bialgebra structures of complex type (Lemma 8). However, trivial central extensions of them do admit such structures.

Theorem 17. *The direct extensions* $\mathfrak{osc}(\underline{\lambda}) \times \mathbb{R}^{1,1}$ *are Lie bialgebras of complex type.*

Proof. Let $\mathfrak{osc}(\underline{\lambda}) = \operatorname{span}\{e_0, e_1, \ldots, e_{2m+1}\}$ be as above. An ad-invariant metric on $\mathfrak{osc}(\underline{\lambda})$ is defined by the following non-trivial relations

$$g(e_0, e_{2m+1}) = g(e_j, e_j) = 1, \quad \text{for } j = 1, \ldots, 2m.$$

If we consider $\mathbb{R}^{1,1} = \operatorname{span}\{e_{2m+2}, e_{2m+3}\}$ with the metric $g(e_{2m+2}, e_{2m+3}) = 1$, we obtain on $\mathfrak{osc}(\underline{\lambda}) \times \mathbb{R}^{1,1}$ an ad-invariant metric of signature $(2, 2m+2)$. The almost complex structure defined by

$$Je_0 = e_{2m+3}, \qquad Je_{2m+1} = e_{2m+2}, \qquad Je_{2i-1} = e_{2i}, \quad \text{for } i = 1, \ldots, m,$$

is integrable and skew-symmetric with respect to g, proving the assertion. □

3.3. *Examples arising from ad-invariant metrics of signature* $(2, n-2)$

We recall from [6] the classification of the Lie algebras with one dimensional centre admitting an ad-invariant metric of signature $(2, n-2)$.

Theorem 18 (*[6]*). *Let* $\mathfrak{g}$ *be an indecomposable Lie algebra with one dimensional centre endowed with an ad-invariant metric of signature* $(2, n-2)$*. Then* $\mathfrak{g}$ *is isomorphic to one of the following Lie algebras:* $L_2(1,1)$, $L_3(1,2)$, $L_{2,\underline{\lambda}}(1, n-3)$ *or* $L_{3,\underline{\lambda}}(1, n-3)$.

We describe below the Lie algebras mentioned in the above theorem: $L_2(1,1)$ was introduced in Example 14, and for the remaining cases we have

◇ $L_3(1,2) = \operatorname{span}\{e_0, e_1, e_2, e_3, e_4\}$ with $[e_1, e_2] = e_0$ and the adjoint action of e_4 on the subspace $\operatorname{span}\{e_1, e_2, e_3\}$ is given by

$$L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

◇ $L_{2,\underline{\lambda}}(1, n-3) = \operatorname{span}\{e_0, e_1, \ldots, e_{n-1}\}$ for $n = 2m+2 > 5$ even and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_1 \le \cdots \le \lambda_m$, with $[e_1, e_2] = e_0$, $[e_{2i-1}, e_{2i}] = \lambda_i e_0$, $2 \le i \le m$, and the adjoint action of e_{n-1} on $\operatorname{span}\{e_1, \ldots, e_{n-2}\}$ is given by

$$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & A_{\underline{\lambda}} \end{pmatrix},$$

with $A_{\underline{\lambda}}$ as in Theorem 16.

◇ $L_{3,\underline{\lambda}}(1, n-3) = \operatorname{span}\{e_0, e_1, \ldots, e_{n-1}\}$ for $n = 2m+3 > 5$ odd, with $[e_1, e_2] = e_0$, $[e_{2i}, e_{2i+1}] = \lambda_i e_0$, $2 \le i \le m$, and the adjoint action of e_{n-1} on $\operatorname{span}\{e_1, \ldots, e_{n-2}\}$ is given by

$$\begin{pmatrix} L_3 & 0 \\ 0 & A_{\underline{\lambda}} \end{pmatrix},$$

with $\underline{\lambda}$ and $A_{\underline{\lambda}}$ as above.

Theorem 19. *Let* $\mathfrak{g}$ *denote a Lie algebra as in the previous theorem. Then* $\mathfrak{g} \times \mathbb{R}^s$ *is a Lie bialgebra of complex type, where* $s = 0$ *(resp.* $s = 1$*) if* $\dim \mathfrak{g}$ *is even (resp. odd).*

Proof. We exhibit in each case a Hermitian structure (J, g) where g is an ad-invariant metric. The proofs of the ad-invariance property and the integrability of J follow by standard computations.
$\diamond$ $L_2(1,1)$: see Example 14.
$\diamond$ $L_3(1,2) \times \mathbb{R}e_5$:

$$g(e_0,e_4) = -g(e_1,e_3) = g(e_2,e_2) = g(e_5,e_5) = 1,$$
$$Je_0 = e_3, \qquad Je_1 = e_4, \qquad Je_2 = e_5.$$

$\diamond$ $L_{2,\underline{\lambda}}(1, n-3)$:

$$g(e_0,e_{n-1}) = -g(e_1,e_1) = g(e_i,e_i) = 1, \quad 2 \le i \le n-2,$$
$$Je_0 = \frac{1}{\sqrt{2}}(e_1+e_2), \qquad Je_{n-1} = \frac{1}{\sqrt{2}}(e_2-e_1), \qquad Je_{2i-1} = e_{2i}, \quad 2 \le i \le m.$$

$\diamond$ $L_{3,\underline{\lambda}}(1, n-3) \times \mathbb{R}e_n$:

$$g(e_0,e_{n-1}) = -g(e_1,e_3) = g(e_{2i},e_{2i}) = g(e_{2i+3},e_{2i+3}) = 1, \quad 1 \le i \le m,$$
$$Je_3 = e_0, \qquad Je_{n-1} = e_1, \qquad Je_n = e_2, \qquad Je_{2i} = e_{2i+1}, \quad 2 \le i \le m. \quad \square$$

We can obtain Lie bialgebras of complex type in arbitrarily high dimensions by applying an iterative doubling procedure. In fact, it is a consequence of (1) in Corollary 13 that if (J, g) is a Hermitian structure on $\mathfrak{g}$ such that g is ad-invariant, then $\mathcal{D}^k\mathfrak{g}$ is a Lie bialgebra of complex type of dimension $2^k m$, $m = \dim \mathfrak{g}$, $k \ge 1$, where $\mathcal{D}^k\mathfrak{g}$ is defined inductively by $\mathcal{D}^1\mathfrak{g} = \mathcal{D}\mathfrak{g}$, $\mathcal{D}^k\mathfrak{g} = \mathcal{D}\left(\mathcal{D}^{k-1}\mathfrak{g}\right)$.

In a similar way, the second assertion in Corollary 13 implies that if J is a complex structure on any Lie algebra $\mathfrak{g}$, then J induces a Hermitian structure on $T^*\mathfrak{g}$ with respect to the neutral ad-invariant metric, therefore, $\mathcal{D}^k(T^*\mathfrak{g})$ is a Lie bialgebra of complex type for $k \ge 1$.

We point out that the ad-invariant metric on either $\mathcal{D}^k\mathfrak{g}$ or $\mathcal{D}^k(T^*\mathfrak{g})$ has neutral signature for all $k \ge 1$. In the next section we will develop a method to construct a Lie bialgebra of complex type with ad-invariant metric of signature $(2p+2, 2q+2)$ starting with one whose corresponding metric has signature $(2p, 2q)$.

4. Construction of examples via double extensions

The aim of this section is to obtain new examples of Lie bialgebras of complex type starting with a Hermitian structure on a lower dimensional metric Lie algebra by a double extension process. As a consequence of our construction we show that all solvable metric Lie algebras of dimension 4 and 6 with metrics of suitable signature admit Lie bialgebra structures of complex type.

We begin by recalling the following result (see, for instance, [23]):

Theorem 20. *Let $(\mathfrak{d}, g)$ be an indecomposable metric Lie algebra. Then one of the following conditions holds:*

(1) *$\mathfrak{d}$ is simple;*
(2) *$\mathfrak{d}$ is 1-dimensional;*
(3) *$\mathfrak{d}$ is a double extension $\mathfrak{d} = \mathfrak{d}_\pi(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is 1-dimensional or simple and $\pi : \mathfrak{h} \to \mathrm{Der}_{\mathrm{skew}}(\mathfrak{g}, g)$ is a representation of $\mathfrak{h}$ on $\mathfrak{g}$ by skew-symmetric derivations.*

We will restrict ourselves to the solvable case, and therefore we will consider only double extensions of a metric Lie algebra by 1-dimensional Lie algebras. Following the notation in [6], we give the description of such extensions. Let $\mathfrak{h} = \mathbb{R}H$ and $\mathfrak{h}^* = \mathbb{R}\alpha$, with $\alpha(H) = 1$. The double extensions of the metric Lie algebra $(\mathfrak{g}, g)$ by the one dimensional Lie algebra $\mathfrak{h}$ are determined by skew-symmetric derivations $D \in \mathrm{Der}_{\mathrm{skew}}(\mathfrak{g}, g)$:

$$\mathfrak{d}_D := \mathfrak{d}_D(\mathfrak{g}, \mathbb{R}) := \mathbb{R}\alpha \oplus \mathfrak{g} \oplus \mathbb{R}H$$

with Lie bracket

$$\alpha \in \mathfrak{z}(\mathfrak{d}_D), \qquad [x,y]_{\mathfrak{d}_D} = g(Dx,y)\alpha + [x,y]_{\mathfrak{g}}, \qquad [H,x]_{\mathfrak{d}_D} = Dx, \tag{10}$$

for all $x, y \in \mathfrak{g}$. The ad-invariant metric on the double extension $\mathfrak{d}_D$ is obtained extending the ad-invariant metric on $\mathfrak{g}$ via the single relation $g(\alpha, H) = 1$.

We will show next that beginning with a metric Lie algebra $(\mathfrak{g}, g)$ equipped with a skew-symmetric complex structure and certain skew-symmetric derivations, we can produce a skew-symmetric complex structure on double extensions of $\mathfrak{g}$.

Let (J, g) be a Hermitian structure on a Lie algebra $\mathfrak{g}$ with ad-invariant metric g, and denote by $\mathbb{R}^{1,1} := \mathrm{span}\{x_0, y_0\}$ the abelian pseudo-Euclidean Lie algebra with metric $g(x_0, y_0) = 1$. Consider the product metric Lie algebra $\tilde{\mathfrak{g}} = \mathbb{R}^{1,1} \times \mathfrak{g}$, fix $c \in \mathbb{R}$, $\tilde{y} \in \mathfrak{z}(\mathfrak{g})$ and let A be a skew-symmetric derivation of $(\mathfrak{g}, g)$ commuting with J. Define the following skew-symmetric derivation $\tilde{A}$ of $\tilde{\mathfrak{g}}$:

$$\tilde{A}x_0 = cx_0, \qquad \tilde{A}y_0 = -cy_0 + \tilde{y}, \qquad \tilde{A}x = -g(x,\tilde{y})x_0 + Ax, \tag{11}$$

for all $x \in \mathfrak{g}$. We extend the complex structure J on $\mathfrak{g}$ to a skew-symmetric almost complex structure $\mathbb{J}$ on the double extension $\mathfrak{d}_{\tilde{A}} = \mathfrak{d}_{\tilde{A}}(\tilde{\mathfrak{g}}, \mathbb{R})$ (see (10)) as follows:

$$\mathbb{J}|_{\mathfrak{g}} = J, \qquad \mathbb{J}x_0 = \alpha, \qquad \mathbb{J}y_0 = H, \qquad \mathbb{J}^2 = -\mathrm{id}. \tag{12}$$

We show next that $\mathbb{J}$ is integrable, that is, $N_{\mathbb{J}} \equiv 0$. It follows from (2) that $N_{\mathbb{J}}$ is skew-symmetric and $N_{\mathbb{J}}(u, \mathbb{J}v) = -\mathbb{J}N_{\mathbb{J}}(u, v)$, for all $u, v \in \mathfrak{d}_{\tilde{A}}$, therefore we only need to verify the vanishing of $N_{\mathbb{J}}(x_0, y_0)$, $N_{\mathbb{J}}(x_0, x)$, $N_{\mathbb{J}}(y_0, x)$ and $N_{\mathbb{J}}(x, y)$ for all $x, y \in \mathfrak{g}$. In order to simplify notation we will denote $\mathfrak{d} := \mathfrak{d}_{\tilde{A}}$ and $k(x) = -g(x, \tilde{y})$, $x \in \mathfrak{g}$. For $x_0, y_0 \in \mathbb{R}^{1,1}$, $x \in \mathfrak{g}$, we compute

$$\begin{aligned}
N_{\mathbb{J}}(x_0, y_0) &= -[x_0, y_0]_{\mathfrak{d}} - \mathbb{J}[\mathbb{J}x_0, y_0]_{\mathfrak{d}} - \mathbb{J}[x_0, \mathbb{J}y_0]_{\mathfrak{d}} + [\mathbb{J}x_0, \mathbb{J}y_0]_{\mathfrak{d}} \\
&= -\left(g(\tilde{A}x_0, y_0)\alpha + [x_0, y_0]_{\tilde{\mathfrak{g}}}\right) - \mathbb{J}[\alpha, y_0]_{\mathfrak{d}} - \mathbb{J}[x_0, H]_{\mathfrak{d}} + [\alpha, H]_{\mathfrak{d}} \\
&= -g(\tilde{A}x_0, y_0)\alpha + \mathbb{J}\tilde{A}x_0 = -c\alpha + c\mathbb{J}x_0 \\
&= 0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
N_{\mathbb{J}}(y_0, x) &= -[y_0, x]_{\mathfrak{d}} - \mathbb{J}[H, x]_{\mathfrak{d}} - \mathbb{J}[y_0, Jx]_{\mathfrak{d}} + [H, Jx]_{\mathfrak{d}} \\
&= -\left(g(\tilde{A}y_0, x)\alpha + [y_0, x]_{\tilde{\mathfrak{g}}}\right) - \mathbb{J}\tilde{A}x - \mathbb{J}\left(g(\tilde{A}y_0, Jx)\alpha + [y_0, Jx]_{\tilde{\mathfrak{g}}}\right) + \tilde{A}Jx \\
&= g(y_0, \tilde{A}x)\alpha - (k(x)\alpha + JAx) - g(y_0, \tilde{A}Jx)x_0 + (k(Jx)x_0 + AJx) \\
&= k(x)\alpha - k(x)\alpha - JAx - k(Jx)x_0 + k(Jx)x_0 + AJx \\
&= -JAx + AJx \\
&= 0,
\end{aligned} \tag{14}$$

$$\begin{aligned}
N_{\mathbb{J}}(x_0, x) &= -[x_0, x]_{\mathfrak{d}} - \mathbb{J}[\alpha, x]_{\mathfrak{d}} - \mathbb{J}[x_0, \mathbb{J}x]_{\mathfrak{d}} + [\alpha, \mathbb{J}x]_{\mathfrak{d}} \\
&= -\left(g(\tilde{A}x_0, x)\alpha + [x_0, x]_{\tilde{\mathfrak{g}}}\right) - \mathbb{J}\left(g(\tilde{A}x_0, Jx)\alpha + [x_0, Jx]_{\tilde{\mathfrak{g}}}\right) \\
&= 0,
\end{aligned}$$

since $\tilde{A}x_0 = cx_0$ is orthogonal to $\mathfrak{g}$ and x_0 is central in $\tilde{\mathfrak{g}}$. Now, take $x, y \in \mathfrak{g}$ and compute

$$\begin{aligned}
N_{\mathbb{J}}(x, y) &= -[x, y]_{\mathfrak{d}} - \mathbb{J}[Jx, y]_{\mathfrak{d}} - \mathbb{J}[x, Jy]_{\mathfrak{d}} + [Jx, Jy]_{\mathfrak{d}} \\
&= -\left(g(\tilde{A}x, y)\alpha + [x, y]_{\mathfrak{g}}\right) - \mathbb{J}\left(g(\tilde{A}Jx, y)\alpha + [Jx, y]_{\mathfrak{g}}\right) - \mathbb{J}\left(g(\tilde{A}x, Jy)\alpha + [x, Jy]_{\mathfrak{g}}\right) + g(\tilde{A}Jx, Jy)\alpha + [Jx, Jy]_{\mathfrak{g}} \\
&= -g(k(x)x_0 + Ax, y)\alpha - [x, y]_{\mathfrak{g}} + g(k(Jx)x_0 + AJx, y)x_0 - J[Jx, y]_{\mathfrak{g}} + g(k(x)x_0 + Ax, Jy)x_0 - J[x, Jy]_{\mathfrak{g}} \\
&\quad + g(k(Jx)x_0 + AJx, Jy)\alpha + [Jx, Jy]_{\mathfrak{g}} \\
&= -g(Ax, y)\alpha + g(AJx, y)x_0 + g(Ax, Jy)x_0 + g(AJx, Jy)\alpha + N_J^{\mathfrak{g}}(x, y) \\
&= 0,
\end{aligned}$$

since J is a skew-symmetric integrable almost complex structure on $\mathfrak{g}$ which commutes with A. Therefore, $N_{\mathbb{J}} \equiv 0$, as asserted.

We show next that, conversely, if $\tilde{A}$ is a skew symmetric derivation of $\tilde{\mathfrak{g}}$ and $\mathbb{J}$ is the almost complex structure on $\mathfrak{d}_{\tilde{A}} = \mathfrak{d}_{\tilde{A}}(\tilde{\mathfrak{g}}, \mathbb{R})$ defined by (12), then the integrability of $\mathbb{J}$ implies that there exist $c \in \mathbb{R}$, $\tilde{y} \in \mathfrak{z}(\mathfrak{g})$ and a skew-symmetric derivation A of $(\mathfrak{g}, g)$ commuting with J such that $\tilde{A}$ takes the form (11).

Theorem 21. *Let (J, g) be a Hermitian structure on a Lie algebra $\mathfrak{g}$ with ad-invariant metric g, and consider the product metric Lie algebra $\tilde{\mathfrak{g}} = \mathbb{R}^{1,1} \times \mathfrak{g}$ as above. Let $\tilde{A}$ be a skew-symmetric derivation of $\tilde{\mathfrak{g}}$, $\mathfrak{d}_{\tilde{A}} = \mathfrak{d}_{\tilde{A}}(\tilde{\mathfrak{g}}, \mathbb{R})$ the double extension of $(\mathfrak{g}, g)$ given by (10) and $\mathbb{J}$ the skew-symmetric almost complex structure on $\mathfrak{d}_{\tilde{A}}$ defined in (12). Then $\mathbb{J}$ is integrable if and only if there exist $c \in \mathbb{R}$, $\tilde{y} \in \mathfrak{z}(\mathfrak{g})$ and a skew-symmetric derivation A of $(\mathfrak{g}, g)$ commuting with J such that $\tilde{A}$ is given by (11).*

Proof. We have shown in the above paragraphs that when $\tilde{A}$ is given by (11) for $c \in \mathbb{R}$, $\tilde{y} \in \mathfrak{z}(\mathfrak{g})$ and a skew-symmetric derivation A of $(\mathfrak{g}, g)$ commuting with J, then $\mathbb{J}$ is integrable on $\mathfrak{d}_{\tilde{A}}$.

Conversely, suppose that the almost complex structure $\mathbb{J}$ is integrable. We will determine the form of the derivation $\tilde{A}$. Recall that the integrability of $\mathbb{J}$ is equivalent to the vanishing of the Nijenhuis tensor $N_{\mathbb{J}}$. From the calculation of $N_{\mathbb{J}}(x_0, y_0)$ in (13) and the fact that $N_{\mathbb{J}}(x_0, y_0) = 0$ we get

$$0 = N_{\mathbb{J}}(x_0, y_0) = -g(\tilde{A}x_0, y_0)\alpha + \mathbb{J}\tilde{A}x_0.$$

Hence, applying $\mathbb{J}$ and setting $c := g(\tilde{A}x_0, y_0)$, we obtain that

$$\tilde{A}x_0 = cx_0.$$

Since $\tilde{A}$ is skew-symmetric, we see that $g(\tilde{A}y_0, y_0) = 0$, so that

$$\tilde{A}y_0 \in (\mathrm{span}\{y_0\})^{\perp} = \mathrm{span}\{y_0\} \oplus \mathfrak{g}$$

and therefore $\tilde{A}y_0 = \beta y_0 + \tilde{y}$ for some $\beta \in \mathbb{R}$ and $\tilde{y} \in \mathfrak{g}$. But, since $\tilde{A}$ is skew-symmetric, we have

$$c = g(\tilde{A}x_0, y_0) = -g(x_0, \tilde{A}y_0) = -g(x_0, \beta y_0 + \tilde{y}) = -\beta$$

and using now that $\tilde{A}$ is a derivation of $\tilde{\mathfrak{g}}$, we obtain for $x \in \mathfrak{g}$

$$0 = \tilde{A}[y_0, x]_{\tilde{\mathfrak{g}}} = [\tilde{A}y_0, x]_{\tilde{\mathfrak{g}}} + [y_0, \tilde{A}x_0]_{\tilde{\mathfrak{g}}} = [\beta y_0 + \tilde{y}, x]_{\tilde{\mathfrak{g}}} = [\tilde{y}, x]_{\mathfrak{g}},$$

so that $\tilde{y} \in \mathfrak{z}(\mathfrak{g})$. Thus,

$$\tilde{A}y_0 = -cy_0 + \tilde{y}, \quad \tilde{y} \in \mathfrak{z}(\mathfrak{g}).$$

Now take $x \in \mathfrak{g}$ and observe that $g(\tilde{A}x, x_0) = -g(x, \tilde{A}x_0) = -cg(x_0, x) = 0$, and hence $\tilde{A}x \in (\operatorname{span}\{x_0\})^{\perp} = \operatorname{span}\{x_0\} \oplus \mathfrak{g}$, so that there exist $k \in \mathfrak{g}^*$ and $A \in \operatorname{End}(\mathfrak{g})$ such that

$$\tilde{A}x = k(x)x_0 + Ax, \quad \text{for all } x \in \mathfrak{g}.$$

Let us determine k. For $x \in \mathfrak{g}$, we compute $g(\tilde{A}y_0, x) = g(-cy_0 + \tilde{y}, x) = g(\tilde{y}, x)$, but, on the other hand $g(\tilde{A}y_0, x) = -g(y_0, \tilde{A}x) = -g(y_0, k(x)x_0 + Ax) = -k(x)$, so that

$$k(x) = -g(x, \tilde{y}), \quad x \in \mathfrak{g}. \tag{15}$$

We observe that $k\left([\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}\right) = 0$ since the metric g on $\mathfrak{g}$ is ad-invariant and $\tilde{y} \in \mathfrak{z}(\mathfrak{g})$. It follows that $\tilde{A}$ takes the form (11).

Now let us prove the statements regarding $A \in \operatorname{End}(\mathfrak{g})$. For $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} g(Ax, y) &= g(\tilde{A}x - k(x)x_0, y) = g(\tilde{A}x, y) \\ &= -g(x, \tilde{A}y) = -g(x, k(y)x_0 + Ay) = -g(x, Ay), \end{aligned}$$

so that A is skew-symmetric. Next, we see that, for $x, y \in \mathfrak{g}$:

$$\tilde{A}[x, y]_{\tilde{\mathfrak{g}}} = \tilde{A}[x, y]_{\mathfrak{g}} = \underbrace{k([x, y]_{\mathfrak{g}})}_{=0} x_0 + A[x, y]_{\mathfrak{g}} = A[x, y]_{\mathfrak{g}}$$

$$\begin{aligned} [\tilde{A}x, y]_{\tilde{\mathfrak{g}}} + [x, \tilde{A}y]_{\tilde{\mathfrak{g}}} &= [k(x)x_0 + Ax, y]_{\tilde{\mathfrak{g}}} + [x, k(y)x_0 + Ay]_{\tilde{\mathfrak{g}}} \\ &= [Ax, y]_{\mathfrak{g}} + [x, Ay]_{\mathfrak{g}}. \end{aligned}$$

Therefore, using that $\tilde{A}$ is a derivation of $\tilde{\mathfrak{g}}$, we obtain that A is a derivation of $\mathfrak{g}$. Finally, from the calculation of $N_{\mathbb{J}}(y_0, x)$ in (14) and $N_{\mathbb{J}}(y_0, x) = 0$ we obtain that

$$0 = N_{\mathbb{J}}(y_0, x) = -JAx + AJx$$

for all $x \in \mathfrak{g}$, so that $JA = AJ$. This completes the proof. □

Let us compute next the associated Lie bracket $[\cdot, \cdot]_{\mathbb{J}}$ on $\mathfrak{d}_{\tilde{A}}$. Keeping the notation from the previous theorem, it is easily verified that the following relations hold for $x, y \in \mathfrak{g}$:

$$\begin{array}{ll} [x, y]_{\mathbb{J}} = [x, y]_J, & [x_0, y_0]_{\mathbb{J}} = -cx_0, \\ [y_0, x]_{\mathbb{J}} = k(x)x_0 - k(Jx)\alpha + Ax, & \\ [\alpha, y_0]_{\mathbb{J}} = -c\alpha, & [\alpha, H]_{\mathbb{J}} = cx_0, \\ [H, x]_{\mathbb{J}} = k(x)\alpha + k(Jx)x_0 + AJx, & [H, x_0]_{\mathbb{J}} = c\alpha, \end{array}$$

where $k \in \mathfrak{g}^*$ is given by (15).

From these equations we see that $\left(\mathfrak{d}_{\tilde{A}}\right)_{\mathbb{J}} = \mathbb{C}y_0 \ltimes (\mathbb{C}x_0 \times \mathfrak{g}_J)$ as complex Lie algebras, where

$$\operatorname{ad}_{\mathbb{J}}(y_0) = \left(\begin{array}{c|ccc} c & z_1 & \cdots & z_n \\ \hline 0 & & & \\ \vdots & & M + \mathrm{i}N & \\ 0 & & & \end{array}\right)$$

with respect to an ordered $\mathbb{C}$-basis $\{x_0, v_1, \ldots, v_n\}$ of $\mathbb{C}x_0 \times \mathfrak{g}_J$. Here, $z_l = k(v_l) - \mathrm{i}k(Jv_l)$, for $l = 1, \ldots, n$, and $A = \begin{pmatrix} M & -N \\ N & M \end{pmatrix}$ in the ordered $\mathbb{R}$-basis $\{v_1, \ldots, v_n, Jv_1, \ldots, Jv_n\}$ of $\mathfrak{g}$.

4.1. Lie bialgebras of complex type in low dimensions

In the next paragraphs we apply the method developed in Theorem 21, starting with $\mathfrak{g} = \{0\}$ and $\mathfrak{g} = \mathbb{R}^2$, to show that all solvable metric Lie algebras of dimension four and six with metric of signature $(2r, 2s)$ admit bialgebra structures of complex type.

(i) We apply first Theorem 21 in the case $\mathfrak{g} = \{0\}$. Hence $\tilde{\mathfrak{g}} = \mathbb{R}^{1,1}$ is spanned by $\{x_0, y_0\}$ with the neutral metric $g(x_0, y_0) = 1$ and the skew-symmetric endomorphism $\tilde{A}$ is given by $\tilde{A}x_0 = cx_0$, $\tilde{A}y_0 = -cy_0$. In this case, the double extension $\mathfrak{d}_{\tilde{A}} = \mathbb{R}\alpha \oplus \mathbb{R}^{1,1} \oplus \mathbb{R}H$ has the following Lie bracket relations

$$[x_0, y_0] = c\alpha, \qquad [H, x_0] = cx_0, \qquad [H, y_0] = -cy_0,$$

the ad-invariant metric is given by $g(x_0, y_0) = 1 = g(\alpha, H)$, and the skew-symmetric complex structure $\mathbb{J}$ is determined by $\mathbb{J}x_0 = \alpha$, $\mathbb{J}y_0 = H$.

If $c = 0$, then $\mathfrak{d}_{\tilde{A}}$ is the pseudo-Euclidean abelian Lie algebra with metric of signature $(2, 2)$ and its canonical skew-symmetric complex structure.

If $c \neq 0$, then $(\mathfrak{d}_{\tilde{A}}, \mathbb{J}, g)$ is equivalent to $(L_2(1, 1), J, g)$ from Example 14, considering the following basis of $\mathfrak{d}_{\tilde{A}}$:

$$e_1 := -c^{-1}H, \qquad e_2 := cy_0, \qquad e_3 := c^{-1}x_0, \qquad e_4 := -c\alpha.$$

(ii) Now we consider the case $\mathfrak{g} = \mathbb{R}^2$, the abelian 2-dimensional Euclidean Lie algebra. We take a basis $\{u, v\}$ of $\mathfrak{g}$ such that $g(u, u) = g(v, v) = 1$ and $Ju = v$, and hence $\tilde{\mathfrak{g}} \cong \mathbb{R}^{1,3}$ is spanned by $\{x_0, y_0, u, v\}$ with the ad-invariant metric $g(x_0, y_0) = g(u, u) = g(v, v) = 1$. Any skew-symmetric endomorphism $\tilde{A}$ of $\tilde{\mathfrak{g}}$ satisfying the conditions of Theorem 21 takes the following form, in the ordered basis $\{x_0, y_0, u, v\}$:

$$\tilde{A} = \begin{pmatrix} c & 0 & -a & -b \\ 0 & -c & 0 & 0 \\ 0 & a & 0 & -r \\ 0 & b & r & 0 \end{pmatrix}$$

for some $a, b, c, r \in \mathbb{R}$. For all values of these parameters, the corresponding ad-invariant metric will have signature $(2, 4)$. There are two possible cases: the derivation $\tilde{A}$ is nilpotent or not.

Suppose first that $\tilde{A}$ is nilpotent, that is, $c = r = 0$. This is equivalent to the Lie algebra $\mathfrak{d}_{\tilde{A}} = \mathbb{R}\alpha \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}H$ being nilpotent (see [6]). If $a = b = 0$, we obtain that $\mathfrak{d}_{\tilde{A}} = \mathbb{R}^{2,4}$. On the other hand, if $a^2 + b^2 \neq 0$, we set

$$e_1 := (a^2 + b^2)^{-1/4}H, \qquad e_2 := -(a^2 + b^2)^{-1/4}y_0, \qquad e_3 := (a^2 + b^2)^{-1/2}(-bu + av),$$
$$e_4 := (a^2 + b^2)^{-1/2}(au + bv), \qquad e_5 := (a^2 + b^2)^{1/4}x_0, \qquad e_6 := (a^2 + b^2)^{1/4}\alpha.$$

Thus for any choice of such a and b, the Lie bracket on $\mathfrak{d}_{\tilde{A}}$ in terms of the basis $\{e_1, \dots, e_6\}$ is given by

$$[e_1, e_2] = -e_4, \qquad [e_1, e_4] = -e_5, \qquad [e_2, e_4] = -e_6, \tag{16}$$

and the Hermitian structure (J, g) on $\mathfrak{d}_{\tilde{A}}$ becomes

$$Je_1 = e_2, \qquad Je_3 = -e_4, \qquad Je_5 = e_6,$$
$$g(e_3, e_3) = g(e_4, e_4) = g(e_1, e_6) = -g(e_2, e_5) = 1.$$

It can be seen that $(\mathfrak{d}_{\tilde{A}}, \mathbb{J}, g)$ is equivalent to the metric Lie algebra with skew-symmetric complex structure $(L_3(1, 2) \times \mathbb{R}, J, g)$ constructed in Section 3.3.

Remark 22. The Lie algebra $L_3(1, 2) \times \mathbb{R}$ can be found in Salamon's list of 6-dimensional nilpotent Lie algebras admitting a complex structure (see [26]). In his notation, this Lie algebra corresponds to (0, 0, 0, 12, 14, 24).

Assume next that $\tilde{A}$ is not nilpotent, so that $c^2 + r^2 \neq 0$. Let us suppose that $a = b = 0$, and hence $\tilde{y} = 0$ (we are using the notation from Theorem 21). The Lie bracket on the double extension $\mathfrak{d}_{\tilde{A}} = \mathbb{R}\alpha \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}H$ is given by

$$[x_0, y_0] = c\alpha, \qquad [H, x_0] = cx_0, \qquad [H, y_0] = -cy_0,$$
$$[H, u] = rv, \qquad [H, v] = -ru, \qquad [u, v] = r\alpha,$$

the ad-invariant metric is obtained by adding the relation $g(\alpha, H) = 1$ to the ad-invariant metric on $\tilde{\mathfrak{g}}$, and the skew-symmetric complex structure $\mathbb{J}$ is defined as follows

$$\mathbb{J}u = v, \qquad \mathbb{J}x_0 = \alpha, \qquad \mathbb{J}y_0 = H.$$

We have the following possibilities, depending on the conditions satisfied by the parameters c and r.

◇ If $c = 0$, $r \neq 0$, set

$$e_1 := -r^{-1}H, \qquad e_2 := u, \qquad e_3 := v, \qquad e_4 := r\alpha, \qquad e_5 := r^{-1}y_0, \qquad e_6 := rx_0.$$

The Lie bracket on $\mathfrak{d}_{\tilde{A}}$ in terms of the basis $\{e_1, \dots, e_6\}$ is given by

$$[e_1, e_2] = -e_3, \qquad [e_1, e_3] = e_2, \qquad [e_2, e_3] = e_4,$$

and the Hermitian structure is

$$Je_1 = e_5, \qquad Je_2 = e_3, \qquad Je_4 = -e_6,$$
$$-g(e_1, e_4) = g(e_5, e_6) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Therefore $(\mathfrak{d}_{\tilde{A}}, \mathbb{J}, g)$ is equivalent to the metric Lie algebra with skew-symmetric complex structure $(\mathfrak{osc} \times \mathbb{R}^{1,1}, J, g)$ constructed in Section 3.2.

◇ If $c \neq 0$, $r = 0$, set

$$e_1 := -c^{-1}H, \qquad e_2 := c^{-1}y_0, \qquad e_3 := cx_0, \qquad e_4 := -c\alpha, \qquad e_5 := u, \qquad e_6 := v.$$

The Lie bracket on $\mathfrak{d}_{\tilde{A}}$ in terms of the basis $\{e_1, \dots, e_6\}$ is given by

$$[e_1, e_2] = e_2, \qquad [e_1, e_3] = -e_3, \qquad [e_2, e_3] = e_4,$$

and the Hermitian structure is

$$Je_1 = e_2, \qquad Je_3 = -e_4, \qquad Je_5 = e_6,$$
$$g(e_1, e_4) = g(e_2, e_3) = g(e_5, e_5) = g(e_6, e_6) = 1.$$

Hence, $(\mathfrak{d}_{\tilde{A}}, \mathbb{J}, g)$ is equivalent to the metric Lie algebra with skew-symmetric complex structure $(L_2(1, 1) \times \mathbb{R}^2, J, g)$ constructed in Section 3.3.

◇ If $c \neq 0$, $r \neq 0$, we will consider two cases, according to the sign of r/c. If $r/c < 0$, then we set

$$e_1 := -c^{-1}H, \qquad e_2 := c^{-1}y_0, \qquad e_3 := cx_0, \qquad e_4 := u, \qquad e_5 := v, \qquad e_6 := -c\alpha.$$

On the other hand, if $r/c > 0$, we set

$$e_1 := -c^{-1}H, \qquad e_2 := c^{-1}y_0, \qquad e_3 := cx_0, \qquad e_4 := v, \qquad e_5 := u, \qquad e_6 := -c\alpha.$$

In both cases, $\{e_1, \dots, e_6\}$ is a basis of $\mathfrak{d}_{\tilde{A}}$ such that its Lie bracket is given by

$$\begin{aligned} &[e_1, e_2] = e_2, \qquad [e_1, e_3] = -e_3, \qquad [e_1, e_4] = \lambda e_5, \qquad [e_1, e_5] = -\lambda e_4, \\ &[e_2, e_3] = e_6, \qquad [e_4, e_5] = \lambda e_6, \end{aligned} \tag{17}$$

for $\lambda = |r/c|$ and it can be shown that $\mathfrak{d}_{\tilde{A}}$ is isomorphic to $\mathfrak{g}_\lambda := L_{2,\lambda}(1, 3)$ from Section 3.3. The ad-invariant metric g is the same for both cases and it is given by

$$g(e_1, e_6) = g(e_2, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1,$$

whereas the skew-symmetric complex structures J_1 and J_2 corresponding to each case are given by

$$J_1e_1 = e_2, \qquad J_1e_3 = -e_6, \qquad J_1e_4 = e_5,$$

for $r/c < 0$ and

$$J_2e_1 = e_2, \qquad J_2e_3 = -e_6, \qquad J_2e_4 = -e_5,$$

for $r/c > 0$. Recall that for any $\mu \in \mathbb{C}$, the 3-dimensional complex Lie algebra $\mathfrak{r}_{3,\mu}(\mathbb{C})$ has a basis $\{x_0, x_1, x_2\}$ with Lie brackets given by: $[x_0, x_1] = x_1$, $[x_0, x_2] = \mu x_2$, and moreover, $\mathfrak{r}_{3,\mu}(\mathbb{C})$ is isomorphic to $\mathfrak{r}_{3,1/\mu}(\mathbb{C})$ for $\mu \neq 0$ (see [14]). It turns out that $(\mathfrak{g}_\lambda)_{J_1} \cong \mathfrak{r}_{3,-i\lambda}(\mathbb{C})$ and $(\mathfrak{g}_\lambda)_{J_2} \cong \mathfrak{r}_{3,i\lambda}(\mathbb{C})$ as complex Lie algebras.

Remark 23. (i) Since $(\mathfrak{g}_\lambda)_{J_1}$ and $(\mathfrak{g}_\lambda)_{J_2}$ are not isomorphic for $\lambda \neq 1$, J_1 and J_2 are not equivalent and the corresponding bialgebras $(\mathfrak{g}, \delta_{J_i})$, $i = 1, 2$, are not isomorphic. When $\lambda = 1$, it can be shown that J_1 is still not equivalent to J_2 and the corresponding bialgebras are not isomorphic, even though, in this case, $(\mathfrak{g}_1)_{J_1} \cong (\mathfrak{g}_1)_{J_2}$. Therefore, there exist two non-isomorphic Lie bialgebra structures on $\mathfrak{g}_1$ such that they induce isomorphic Lie algebra structures on $\mathfrak{g}_1^*$ (compare with Theorem 1.12 in [21]).
(ii) Concerning the behaviour of the Lie bialgebras $(\mathfrak{g}_\lambda, \delta_{J_i})$ for $i = 1, 2$ as λ approaches 0 or ∞, we point out that when $\lambda \to 0$, we obtain in both cases ($i = 1, 2$) the Lie bialgebra $L_2(1, 1) \times \mathbb{R}^2$ with the structure constructed above. On the other hand, if $\lambda \to \infty$, then the resulting Lie bialgebra is $\mathfrak{osc} \times \mathbb{R}^{1,1}$. Therefore, $(\mathfrak{g}_\lambda, \delta_{J_1})$ and $(\mathfrak{g}_\lambda, \delta_{J_2})$ are non-isomorphic Lie bialgebras which converge to the same bialgebra when $\lambda \to 0$ or $\lambda \to \infty$.

In [6], the indecomposable metric Lie algebras of dimension ≤ 6 have been classified. From this classification we can determine all (not necessarily indecomposable) solvable metric Lie algebras with metric of signature $(2r, 2s)$ in these dimensions. We list below the non-abelian ones:

$$L_2(1, 1), \qquad L_3(1, 2) \times \mathbb{R}, \qquad L_2(1, 1) \times \mathbb{R}^2, \qquad \mathfrak{osc} \times \mathbb{R}^{1,1}, \qquad L_{2,\lambda}(1, 3)\ (\lambda > 0). \tag{18}$$

Therefore, the results in this section can be summarized as follows:

Table 1

$\mathfrak{g}$	$\mathfrak{g}^*$
$L_2(1,1)$	$\mathfrak{aff}(\mathbb{C})$
$L_3(1,2)\times\mathbb{R}$	$\mathfrak{h}_3(\mathbb{C})$
$L_2(1,1)\times\mathbb{R}^2$	$\mathfrak{aff}(\mathbb{C})\times\mathbb{C}$
$\mathfrak{osc}\times\mathbb{R}^{1,1}$	$\mathfrak{aff}(\mathbb{C})\times\mathbb{C}$
$L_{2,\lambda}(1,3)$	$\begin{cases}\mathfrak{r}_{3,-i\lambda}(\mathbb{C}) \text{ for the complex structure } J_1\\ \mathfrak{r}_{3,i\lambda}(\mathbb{C}) \text{ for the complex structure } J_2\end{cases}$

Theorem 24. *Every solvable metric Lie algebra $(\mathfrak{g}, g)$, $\dim\mathfrak{g}\le 6$, such that g has signature $(2r, 2s)$ admits a skew-symmetric complex structure and hence, a Lie bialgebra structure of complex type.*

According to Theorem 10, for any Lie bialgebra of complex type $(\mathfrak{g}, \delta_J)$, its dual $\mathfrak{g}^*$ inherits a complex Lie algebra structure. Table 1 exhibits $\mathfrak{g}^*$ when $\mathfrak{g}$ is one of the algebras in (18) with the Hermitian structures considered above. The complex Lie algebras $\mathfrak{aff}(\mathbb{C})$ and $\mathfrak{r}_{3,\pm i\lambda}(\mathbb{C})$ in Table 1 have already been introduced, and $\mathfrak{h}_3(\mathbb{C})$ denotes the 3-dimensional complex Heisenberg Lie algebra.

Remark 25. We point out that Theorem 24 is no longer true in dimension 8. In fact, it follows from [1] that the cotangent algebra $(T^*\mathfrak{r}_4, \langle\cdot,\cdot\rangle)$ endowed with its natural neutral ad-invariant metric $\langle\cdot,\cdot\rangle$ of signature $(4, 4)$ does not admit any skew-symmetric complex structure, where the Lie algebra $\mathfrak{r}_4 = \operatorname{span}\{e_0, e_1, e_2, e_3\}$ is defined by

$$[e_0, e_1] = e_1, \qquad [e_0, e_2] = e_1 + e_2, \qquad [e_0, e_3] = e_2 + e_3.$$

5. Symplectic foliations on some Poisson Lie groups of complex type

In this section we exhibit some examples of low dimensional simply connected Poisson Lie groups of complex type. The underlying Lie groups are either solvable or nilpotent and thus they are diffeomorphic to Euclidean space. We also determine the corresponding symplectic foliations in some cases.

In what follows we will denote $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{i,j} := \frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$. Also, for any function $h\in C^\infty(\mathbb{R}^n)$ we set $h_i := \frac{\partial h}{\partial x_i}$ and the Hamiltonian vector field X_h associated to h is defined by $X_h(g) = \{h, g\}$. If Π is the Poisson structure corresponding to $\{\cdot,\cdot\}$, the characteristic distribution associated to Π is given by

$$\mathcal{C}_x = \{X_h|_x : h\in C^\infty(\mathbb{R}^n)\}\subset T_x\mathbb{R}^n.$$

Example 26. We consider the Lie algebra $\mathfrak{g} := L_2(1,1)$ from Example 14. Let G denote the simply connected Lie group with Lie algebra $\mathfrak{g}$; it is diffeomorphic to $\mathbb{R}^4$ and it can be realized as the following matrix group:

$$G = \left\{\begin{pmatrix}1 & x_2 & x_4\\ 0 & \mathrm{e}^{-x_1} & x_3\\ 0 & 0 & 1\end{pmatrix} : x_i\in\mathbb{R}\right\}.$$

Identifying the matrix above with the point $(x_1, x_2, x_3, x_4)\in\mathbb{R}^4$, then $\mathbb{R}^4$ acquires a group structure given by

$$(x_1, x_2, x_3, x_4) * (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2\mathrm{e}^{-y_1} + y_2, y_3\mathrm{e}^{-x_1} + x_3, x_4 + y_4 + x_2y_3).$$

The r-matrix $J : \mathfrak{g}\to\mathfrak{g}$ can be considered as an element $R\in\bigwedge^2\mathfrak{g}$ via the ad-invariant metric $g : \mathfrak{g}\to\mathfrak{g}^*$, by setting $R := J\circ g^{-1} : \mathfrak{g}^*\to\mathfrak{g}$. In this case, one can easily see that $R = e_1\wedge e_3 - e_2\wedge e_4$, where $e_1, \ldots, e_4$ are as in (8). According to Remark 2, the corresponding multiplicative Poisson tensor Π_R determined by R making G a Poisson Lie group is given by $\Pi_R = \overleftarrow{R} - \overrightarrow{R}$. Performing standard computations, we arrive at the following expression for Π_R:

$$\Pi_R = (\mathrm{e}^{-x_1} - 1)\partial_{1,3} + x_2\partial_{1,4} - x_2\mathrm{e}^{-x_1}\partial_{2,3} + (\mathrm{e}^{-x_1} - x_2^2 - 1)\partial_{2,4}.$$

Equivalently, the Poisson bracket of two functions $f, g\in C^\infty(\mathbb{R}^4)$ is given by

$$\{f, g\} = (\mathrm{e}^{-x_1} - 1)(f_1g_3 - f_3g_1) + x_2(f_1g_4 - f_4g_1) - x_2\mathrm{e}^{-x_1}(f_2g_3 - f_3g_2) + (\mathrm{e}^{-x_1} - x_2^2 - 1)(f_2g_4 - f_4g_2).$$

From this, we obtain that the Hamiltonian vector field X_f associated to $f\in C^\infty(\mathbb{R}^4)$ can be expressed as

$$\begin{aligned} X_f = &-\left((\mathrm{e}^{-x_1} - 1)f_3 + x_2f_4\right)\partial_1 + \left(x_2\mathrm{e}^{-x_1}f_3 - (\mathrm{e}^{-x_1} - x_2^2 - 1)f_4\right)\partial_2\\ &+ \left((\mathrm{e}^{-x_1} - 1)f_1 - x_2\mathrm{e}^{-x_1}f_2\right)\partial_3 + \left(x_2f_1 + (\mathrm{e}^{-x_1} - x_2^2 - 1)f_2\right)\partial_4. \end{aligned}$$

Let us consider now any point $p = (x_1, x_2, x_3, x_4)\in\mathbb{R}^4$. It is easy to see that $\Pi_R(p) = 0$ if and only if $x_1 = x_2 = 0$, whereas if $x_1^2 + x_2^2 \neq 0$, then $\Pi_R(p)$ has rank 4, where we consider $\Pi_R(p)$ as a linear transformation $\Pi_R(p) : T_p^*\mathbb{R}^4 \to$

$T_p\mathbb{R}^4$. Consequently, the symplectic foliation on $\mathbb{R}^4$ determined by Π_R can be described in the following way: each point $\{(0, 0, x_3, x_4)\}$ is a 0-dimensional leaf, there are no 2-dimensional leaves, and the open set

$$\mathcal{U} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 \neq 0\}$$

is the remaining 4-dimensional symplectic leaf. The corresponding symplectic form on $\mathcal{U}$, $\omega = (\Pi_R|_{\mathcal{U}})^{-1}$, is given by

$$\omega = \frac{1}{\Delta}\left\{-(e^{-x_1} - x_2^2 - 1)\, dx_1 \wedge dx_3 - x_2 e^{-x_1}\, dx_1 \wedge dx_4 + x_2\, dx_2 \wedge dx_3 - (e^{-x_1} - 1)\, dx_2 \wedge dx_4\right\},$$

where $\Delta = (e^{-x_1} - 1)^2 + x_2^2$.

Let

$$\Pi_R^{(1)} = -x_1\partial_{1,3} + x_2\partial_{1,4} - x_2\partial_{2,3} - x_1\partial_{2,4}$$

be the linear part of Π_R in a neighbourhood of $(0, 0, x_3, x_4)$. It follows that the Lie algebra determined by $\Pi_R^{(1)}$ is isomorphic to $\mathfrak{aff}(\mathbb{C})$. Therefore, according to [12], Π_R is analytically linearizable.

Example 27. Let $\mathfrak{n} := L_3(1, 2) \times \mathbb{R}$ be the nilpotent Lie algebra considered in Section 4.1, and denote by $\mathcal{N}$ the corresponding simply connected Lie group, which is diffeomorphic to $\mathbb{R}^6$ and can be realized as the following matrix group:

$$\mathcal{N} = \left\{\begin{pmatrix} 1 & 0 & -\frac{2}{3}x_2^2 & 0 & \frac{1}{6}x_1x_2 + \frac{1}{3}x_4 & -\frac{1}{6}x_2^2 & x_6 \\ 0 & 1 & -\frac{2}{3}x_1 & 0 & \frac{1}{6}x_1^2 & -\frac{1}{6}x_1x_2 + \frac{1}{3}x_4 & x_5 \\ 0 & 0 & 1 & 0 & -\frac{1}{2}x_1 & \frac{1}{2}x_2 & x_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_i \in \mathbb{R}\right\}.$$

Identifying the matrix above with the point $(x_1, \ldots, x_6) \in \mathbb{R}^6$, then $\mathbb{R}^6$ acquires a group structure given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 - \frac{1}{2}(x_1y_2 - x_2y_1) \\ x_5 + y_5 + \frac{1}{6}x_1^2y_2 - \frac{2}{3}x_1y_4 + \left(-\frac{1}{6}x_1x_2 + \frac{1}{3}x_4\right)y_1 \\ x_6 + y_6 - \frac{1}{6}x_2^2y_1 - \frac{2}{3}x_2y_4 + \left(\frac{1}{6}x_1x_2 + \frac{1}{3}x_4\right)y_2 \end{pmatrix}.$$

The r-matrix $J : \mathfrak{n} \to \mathfrak{n}$ determines a bivector $R \in \bigwedge^2 \mathfrak{n}$ by setting $R := J \circ g^{-1} : \mathfrak{n}^* \to \mathfrak{n}$. In this case, one can easily see that $R = e_4 \wedge e_3 + e_5 \wedge e_1 + e_6 \wedge e_2$, where $e_1, \ldots, e_6$ are as in (16), and the multiplicative Poisson tensor $\Pi_R = \overleftarrow{R} - \overrightarrow{R}$ determined by R making $\mathcal{N}$ a Poisson Lie group is given by:

$$\Pi_R = x_1\partial_{3,5} + x_2\partial_{3,6} - x_2\partial_{4,5} + x_1\partial_{4,6} - \frac{1}{6}(x_1^2 + x_2^2)\partial_{5,6}.$$

Note that $\Pi_R(x_1, \ldots, x_6) = 0$ if and only if $x_1 = x_2 = 0$. The Poisson bracket of two functions $f, g \in C^\infty(\mathbb{R}^6)$ induced by Π_R is given by

$$\{f, g\} = x_1(f_3g_5 - f_5g_3) + x_2(f_3g_6 - f_6g_6) - x_2(f_4g_5 - f_5g_4) + x_1(f_4g_6 - f_6g_4) - \frac{1}{6}(x_1^2 + x_2^2)(f_5g_6 - f_6g_5).$$

Therefore, the Hamiltonian vector field X_f associated to $f \in C^\infty(\mathbb{R}^6)$ can be expressed as

$$X_f = (-x_1f_5 - x_2f_6)\,\partial_3 + (x_2f_5 - x_1f_6)\,\partial_4 + \left(x_1f_3 - x_2f_4 + \frac{1}{6}(x_1^2 + x_2^2)f_6\right)\partial_5 + \left(x_2f_3 + x_1f_4 - \frac{1}{6}(x_1^2 + x_2^2)f_5\right)\partial_6.$$

The symplectic foliation on $\mathbb{R}^6$ corresponding to Π_R is the singular foliation associated to the characteristic distribution $\mathcal{C}_x$, and in what follows we are to determine its symplectic leaves. Clearly, $\mathcal{C}_x = \{0\}$ if and only if $x_1 = x_2 = 0$. Let us denote by

p_j, $j = 1, \ldots, 6$, the projections $p_j(x_1, \ldots, x_6) = x_j$. Then we have $X_{p_1} = X_{p_2} = 0$ and also

$$\begin{aligned} X_{p_3} &= x_1\partial_5 + x_2\partial_6, \\ X_{p_4} &= -x_2\partial_5 + x_1\partial_6, \\ X_{p_5} &= -x_1\partial_3 + x_2\partial_4 - \frac{1}{6}\left(x_1^2 + x_2^2\right)\partial_6, \\ X_{p_6} &= -x_2\partial_3 - x_1\partial_4 + \frac{1}{6}\left(x_1^2 + x_2^2\right)\partial_5. \end{aligned}$$

Observe now that if $x_1^2+x_2^2 \neq 0$, then we have that $\{X_{p_3}\big|_x, X_{p_4}\big|_x, X_{p_5}\big|_x, X_{p_6}\big|_x\}$ is a basis of $\mathcal{C}_x$. Moreover, from the expressions obtained for $X_{p_j}(x)$ above, we can easily see that $\{\partial_3|_x, \partial_4|_x, \partial_5|_x, \partial_6|_x\}$ is another basis for $\mathcal{C}_x$. From all these considerations we obtain the following description of the symplectic leaves:

- the 0-dimensional leaves are the points $\{(0, 0, c_3, c_4, c_5, c_6)\}$;
- there are no 2-dimensional leaves;
- the 4-dimensional leaves are the affine subspaces

$$\mathcal{F}_{c_1,c_2} = \{(c_1, c_2, x_3, x_4, x_5, x_6) : x_3, \ldots, x_6 \in \mathbb{R}\},$$

where c_1, c_2 are real constants such that $c_1^2 + c_2^2 \neq 0$. The symplectic form ω_{c_1,c_2} on $\mathcal{F}_{c_1,c_2}$ induced by Π_C is given by

$$\omega_{c_1,c_2} = -\frac{1}{6}\,\mathrm{d}x_3 \wedge \mathrm{d}x_4 - (c_1^2 + c_2^2)^{-1}(c_1\,\mathrm{d}x_3 \wedge \mathrm{d}x_5 + c_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_6 - c_2\,\mathrm{d}x_4 \wedge \mathrm{d}x_5 + c_1\,\mathrm{d}x_4 \wedge \mathrm{d}x_6),$$

where $(x_3, \ldots, x_6)$ are global coordinates on $\mathcal{F}_{c_1,c_2}$.

It follows from [9, Proposition 3.3] that Π_R is analytically linearizable. Let

$$\Pi_R^{(1)} = x_1\partial_{3,5} + x_2\partial_{3,6} - x_2\partial_{4,5} + x_1\partial_{4,6}$$

be the linear part of Π_R in a neighbourhood of $(0, 0, x_3, x_4, x_5, x_6)$. Observe that the Lie algebra determined by $\Pi_R^{(1)}$ is isomorphic to the complex Heisenberg Lie algebra $\mathfrak{h}_3(\mathbb{C})$, considered as a real Lie algebra.

Example 28. We consider the Lie group G_λ with Lie algebra $\mathfrak{g}_\lambda := L_{2,\lambda}(1, 3)$ from Section 4.1, for $\lambda > 0$. This group has the following matrix realization:

$$G_\lambda = \left\{ \begin{pmatrix} 1 & x_2 & 0 & x_4\sqrt{\frac{\lambda}{2}} & x_5\sqrt{\frac{\lambda}{2}} & x_6 \\ 0 & \mathrm{e}^{-x_1} & 0 & 0 & 0 & x_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\lambda x_1) & -\sin(\lambda x_1) & \sqrt{\frac{\lambda}{2}}\,(x_4\sin(\lambda x_1) + x_5\cos(\lambda x_1)) \\ 0 & 0 & 0 & \sin(\lambda x_1) & \cos(\lambda x_1) & \sqrt{\frac{\lambda}{2}}\,(-x_4\cos(\lambda x_1) + x_5\sin(\lambda x_1)) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

for $x_1, \ldots, x_6 \in \mathbb{R}$. This induces the following multiplication on $\mathbb{R}^6$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ y_2 + x_2\mathrm{e}^{-y_1} \\ x_3 + y_3\mathrm{e}^{-x_1} \\ y_4 + x_4\cos(\lambda y_1) + x_5\sin(\lambda y_1) \\ y_5 - x_4\sin(\lambda y_1) + x_5\cos(\lambda y_1) \\ x_6 + y_6 + x_2y_3 + \frac{\lambda}{2}\sin(\lambda y_1)(x_4y_4 + x_5y_5) + \frac{\lambda}{2}\cos(\lambda y_1)(x_4y_5 - x_5y_4) \end{pmatrix}.$$

Recall from Section 4.1 the two inequivalent Hermitian structures (J_1, g) and (J_2, g) on $\mathfrak{g}_\lambda$. The r-matrices $J_i : \mathfrak{g}_\lambda \to \mathfrak{g}_\lambda$ determine a bivector $R_i \in \bigwedge^2 \mathfrak{g}_\lambda$ by setting $R_i := J_i \circ g^{-1} : \mathfrak{g}_\lambda^* \to \mathfrak{g}_\lambda$, for $i = 1, 2$. One can easily see that

$$\begin{aligned} R_1 &= e_1 \wedge e_3 - e_2 \wedge e_6 + e_4 \wedge e_5, \\ R_2 &= e_1 \wedge e_3 - e_2 \wedge e_6 - e_4 \wedge e_5, \end{aligned}$$

where $e_1, \ldots, e_6$ are as in (17), and the multiplicative Poisson tensors $\Pi_i^\lambda = \overleftarrow{R_i} - \overrightarrow{R_i}$ determined by R_i are given by:

$$\begin{aligned}
\Pi_1^\lambda &= (\mathrm{e}^{-x_1} - 1)\partial_{1,3} + x_2\partial_{1,6} - x_2\mathrm{e}^{-x_1}\partial_{2,3} + (\mathrm{e}^{-x_1} - x_2^2 - 1)\partial_{2,6} \\
&\quad - \lambda x_5\mathrm{e}^{-x_1}\partial_{3,4} + \lambda x_4\mathrm{e}^{-x_1}\partial_{3,5} + \lambda(x_4 + x_2x_5)\partial_{4,6} + \lambda(x_5 - x_2x_4)\partial_{5,6}, \\
\Pi_2^\lambda &= (\mathrm{e}^{-x_1} - 1)\partial_{1,3} + x_2\partial_{1,6} - x_2\mathrm{e}^{-x_1}\partial_{2,3} + (\mathrm{e}^{-x_1} - x_2^2 - 1)\partial_{2,6} \\
&\quad - \lambda x_5\mathrm{e}^{-x_1}\partial_{3,4} + \lambda x_4\mathrm{e}^{-x_1}\partial_{3,5} - \lambda(x_4 - x_2x_5)\partial_{4,6} - \lambda(x_5 + x_2x_4)\partial_{5,6}.
\end{aligned}$$

Note that $\Pi_i^\lambda(x_1, \ldots, x_6) = 0$ if and only if $x_1 = x_2 = x_4 = x_5 = 0$. The characteristic distribution $\mathcal{C}_i^\lambda$ associated to Π_i^λ satisfies $(\mathcal{C}_i^\lambda)_x = \{0\}$ if and only if $x_1 = x_2 = x_4 = x_5 = 0$. For the remaining points, $\dim(\mathcal{C}_i^\lambda)_x = 4$ and

$$(\mathcal{C}_i^\lambda)_x = \operatorname{span}\{V|_x,\ W_i|_x,\ \partial_3|_x,\ \partial_6|_x\},$$

where V and W_i are the following vector fields on $\mathbb{R}^6$:

$$\begin{aligned}
V &= -\left(\mathrm{e}^{-x_1} - 1\right)\partial_1 + x_2\mathrm{e}^{-x_1}\partial_2 - \lambda x_5\mathrm{e}^{-x_1}\partial_4 + \lambda x_4\mathrm{e}^{-x_1}\partial_5, \\
W_1 &= -x_2\partial_1 - \left(\mathrm{e}^{-x_1} - x_2^2 - 1\right)\partial_2 - \lambda(x_4 + x_2x_5)\partial_4 - \lambda(x_5 - x_2x_4)\partial_5, \\
W_2 &= -x_2\partial_1 - \left(\mathrm{e}^{-x_1} - x_2^2 - 1\right)\partial_2 + \lambda(x_4 - x_2x_5)\partial_4 + \lambda(x_5 + x_2x_4)\partial_5.
\end{aligned}$$

Note that $[V, W_i] = 0$ for $i = 1, 2$. We conclude that in both cases the 0-dimensional leaves are the points $\{(0, 0, c_3, 0, 0, c_6)\}$ and the open set $\mathcal{U} = \{x \in \mathbb{R}^6 : x_1^2 + x_2^2 + x_4^2 + x_5^2 \neq 0\}$ is foliated by 4-dimensional symplectic manifolds, the integral manifolds of the above distributions. Consider the disjoint union $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$, where $\mathcal{U}_i$, $i = 1, 2, 3$, are the sets

$$\begin{aligned}
\mathcal{U}_1 &= \{x \in \mathbb{R}^6 : x_1^2 + x_2^2 \neq 0, x_4 = x_5 = 0\}, \\
\mathcal{U}_2 &= \{x \in \mathbb{R}^6 : x_4^2 + x_5^2 \neq 0, x_1 = x_2 = 0\}, \\
\mathcal{U}_3 &= \{x \in \mathbb{R}^6 : x_1^2 + x_2^2 \neq 0, x_4^2 + x_5^2 \neq 0\}.
\end{aligned}$$

The set $\mathcal{U}_i$, $i = 1$ or 2, has four connected components diffeomorphic to $\mathbb{R}^4$, and each of them is a leaf of both distributions, since $(\mathcal{C}_1^\lambda)_x = (\mathcal{C}_2^\lambda)_x$ for $x \in \mathcal{U}_1 \cup \mathcal{U}_2$. On the other hand, these distributions satisfy $(\mathcal{C}_1^\lambda)_x \neq (\mathcal{C}_2^\lambda)_x$ for each $x \in \mathcal{U}_3$, and therefore, their integral manifolds through these points do not coincide. Nevertheless, the integral manifolds of $\mathcal{C}_1^\lambda$ and $\mathcal{C}_2^\lambda$ through $c = (c_1, \ldots, c_6) \in \mathcal{U}_3$ are diffeomorphic to $S_c \times \mathbb{R}^2$, where S_c is a surface contained in $\{x \in \mathcal{U}_3 : x_3 = c_3, x_6 = c_6\}$.

Let

$$\begin{aligned}
\left(\Pi_1^\lambda\right)^{(1)} &= -x_1\partial_{1,3} + x_2\partial_{1,6} - x_2\partial_{2,3} - x_1\partial_{2,6} - \lambda x_5\partial_{3,4} + \lambda x_4\partial_{3,5} + \lambda x_4\partial_{4,6} + \lambda x_5\partial_{5,6} \\
\left(\Pi_2^\lambda\right)^{(1)} &= -x_1\partial_{1,3} + x_2\partial_{1,6} - x_2\partial_{2,3} - x_1\partial_{2,6} - \lambda x_5\partial_{3,4} + \lambda x_4\partial_{3,5} - \lambda x_4\partial_{4,6} - \lambda x_5\partial_{5,6},
\end{aligned}$$

be the linear parts of Π_i^λ in a neighbourhood of $(0, 0, x_3, 0, 0, x_6)$. It follows that the Lie algebra determined by $(\Pi_1^\lambda)^{(1)}$ (resp. $(\Pi_2^\lambda)^{(1)}$) is isomorphic to $\mathfrak{r}_{3,-i\lambda}(\mathbb{C})$ (resp. $\mathfrak{r}_{3,i\lambda}(\mathbb{C})$), considered as real Lie algebras.

Remark 29. (i) Replacing x_2 by $-x_2$, the vector field W_1 is transformed into W_2 while the other generators of the distributions $\mathcal{C}_1^\lambda$ and $\mathcal{C}_2^\lambda$ remain unchanged. Thus, the corresponding leaves of both distributions are diffeomorphic. However, these Lie–Poisson structures are not equivalent since the associated Lie bialgebras are non-isomorphic. Furthermore, Π_1^λ and Π_2^λ, with $\lambda \neq 1$, are not equivalent as Poisson structures since their linear parts give rise to non-isomorphic Lie algebras.

(ii) The Lie algebras $\mathfrak{r}_{3,-i\lambda}(\mathbb{C})$ and $\mathfrak{r}_{3,i\lambda}(\mathbb{C})$, considered as six-dimensional real Lie algebras, are real analytically degenerate. In fact, the Poisson tensor

$$\Pi = \left(\Pi_1^\lambda\right)^{(1)} + (x_1\partial_1 + x_2\partial_2) \wedge (x_4\partial_4 + x_5\partial_5)$$

is non-linearizable since it has rank 6 almost everywhere, while its linear part $\left(\Pi_1^\lambda\right)^{(1)}$ has rank at most 4. Therefore, $\mathfrak{r}_{3,-i\lambda}(\mathbb{C})$ is degenerate. A similar argument works for the Lie algebra $\mathfrak{r}_{3,i\lambda}(\mathbb{C})$.

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