



The geodesic flow on a Riemannian supermanifold

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ARTICLE INFO

Article history:

Received 10 July 2011

Received in revised form 11 December 2011

Accepted 3 February 2012

Available online 11 February 2012

MSC:

primary 58A50

secondary 53C22

53D25

58C50

Keywords:

Supermanifolds

Riemannian metrics

Geodesics

Geodesic flow

ABSTRACT

We give a natural definition of geodesics on a Riemannian supermanifold (\mathcal{M}, g) and extend the usual geodesic flow on $T^*\mathcal{M}$ associated to the underlying Riemannian manifold (M, \tilde{g}) to a geodesic “superflow” on $T^*\mathcal{M}$. Integral curves of this flow turn out to be in natural bijection with geodesics on \mathcal{M} . We also construct the corresponding exponential map and generalize the well-known faithful linearization of isometries to Riemannian supermanifolds.

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1. Introduction

The geometry of supermanifolds is, after a first, pioneering, “Storm and Stress”-period in the 70s and 80s, still in its youth but with by now many mathematicians studying in detail the fundamentals as well as the applications of this theory. Our work on Riemannian supermanifolds is motivated by at least two developments originating in physics and relying on Riemannian geometry extended to the category of supermanifolds. First, we are interested in those aspects of the “Stolz–Teichner program” on the geometrization of elliptic cohomology that relate to cobordisms of Riemannian supermanifolds and to supermanifolds in general (see [1] for a detailed account). Of course, extending certain results of this article to super loop spaces (compare [2] for the ungraded case) would give a direct connection to the physics literature underlying elliptic cohomology, where the role of loop spaces is central (see, e.g., [3]). Second, work of Zirnbauer and collaborators (see, e.g., [4]) shows the surprising importance of the notion of “supersymmetric spaces” for physics, notably mesoscopic physics. In this context it is very natural to ask for a general theory of Riemannian supermanifolds with these spaces replacing the symmetric spaces of ordinary Riemannian geometry. The subject of supersymmetric spaces was, in fact, the main motivation of the work of Goertsches [5] on Riemannian supergeometry.

The first cornerstone of Riemannian geometry is – after the metric itself – the notion of a geodesic curve. Upon extending these notions, metrics and geodesics, to supermanifolds it looks highly reasonable to aim at fulfilling the following three conditions:

- (i) geodesic supercurves should reduce to the usual geodesics in the case of classical (ungraded) Riemannian manifolds

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- (ii) an isometry fixing a point and having a trivial linearization in this point should be the identity morphism
- (iii) geodesics should be closely related to integral curves of a “geodesic flow” on the (co-)tangent bundle of the given Riemannian manifold.

The definition of a Riemannian metric on a supermanifold is fairly straight forward and seems to be generally accepted in the literature. (See [Definition 2.7](#).) The notion of geodesic supercurves proposed by Goertsches fulfills (i) and (ii), but unfortunately we could not establish (iii) in this approach. Since the geodesic flow is a tool of great importance in the geometry and analysis on a Riemannian manifold, we found it crucial not to give up condition (iii). Furthermore, the discussion of initial conditions in [\[5\]](#) (see the beginning of Subsection 4.4 there) does not seem to be fully conclusive. Let us underline that the equations chosen for defining geodesics in this article translate in local coordinates to a system of second-order ODEs, whereas the geodesics of Goertsches give rise to second-order equations and first-order equations (for the even resp. odd coordinates). This explains, at least on an intuitive level, why there might be no geodesic flow related to Goertsches’ notion of geodesics.

Let us now proceed to a short description of the contents of this article.

First of all, we underline that we work throughout in the sheaf-theoretic framework, i.e., in the approach to supermanifolds going back to Kostant and the Russian school around Berezin, Bernstein, Leites and Manin.

We define, in the second section, geodesics as “supercurves” that are autoparallel with respect to the natural even time direction:

$$0 = \frac{\nabla}{dt}(T\Phi^*(\partial_t)) = (\Phi^*\nabla)_{\partial_t}(\partial_t \circ \Phi^*),$$

where $\Phi^*\nabla$ is the pull-back of the Levi-Civita connection, ∂_t is $\frac{\partial}{\partial t}$ and $\partial_t \circ \Phi^*$ is a canonically defined vector field along Φ (see the main text for more details). The preceding geodesic equations were already derived by Monterde and Muñoz Masqué by a variational principle (compare Thm. 7.1 and Cor. 7.1 in [\[6\]](#)). Apparently, the solutions of this equation were never studied in detail, excepting the observation that condition (i) above holds, already made in [\[6\]](#).

[Theorem 2.21](#) then gives existence and unicity of geodesics in a Riemannian supermanifold \mathcal{M} , with general initial condition given by \mathcal{J} in $\text{Mor}(\mathcal{S}, T\mathcal{M})$, with \mathcal{S} being an arbitrary auxiliary supermanifold.

We also recall there the relation between locally free sheaves and supervector bundles as objects in the category of supermanifolds, mainly in order to have a precise grip on (co-)tangent bundles.

In [Section 3](#), we first remind the reader of the class of superfunctions on the total space of a supervector bundle that are polynomial on the fibers, allowing us to define then a natural energy function H on the cotangent bundle $T^*\mathcal{M}$ of a Riemannian supermanifold (\mathcal{M}, g) . We give intrinsically (and in coordinates) the canonical symplectic form on $T^*\mathcal{M}$ and explicit the Hamiltonian vector field X_H associated to H . We proceed then to show that “integral supercurves” of this vector field (i.e., curves given by the geodesic flow) are in bijection with geodesics on (\mathcal{M}, g) , see [Theorems 3.13](#) and [3.14](#). We also give here a brief account of the analogous results in the case of odd Riemannian metrics.

The last section, [4](#), gives applications of the geodesic flow morphism constructed in the preceding section. First, we show the existence of an exponential map generalizing the ungraded case and such that $\exp_q : T_q\mathcal{M} \rightarrow \mathcal{M}$, for q in the body of \mathcal{M} , is a local diffeomorphism ([Theorem 4.4](#)). Finally, we prove the result that an isometry of a connected Riemannian supermanifold fixing a point of the body and having the identity as its tangent map in this point, is already the identity morphism of the supermanifold. This implies immediately the unicity of the “geodesic symmetries” in the sense of Riemannian symmetric spaces, fixing a point in the body and having there minus the identity as their tangent map.

2. Riemannian metrics on supermanifolds

In this section we briefly recall several fundamental notions, giving ideas of proof of certain folkloristic facts ([Propositions 2.12](#), [2.14](#) and [2.20](#)) only when we did not find a reference in the literature. We define supergeodesics ([Definition 2.17](#)) differently from [\[5\]](#), but our definition also reduces to the standard definition in the ungraded case. We end this section with the first important new result of the article, namely existence and unicity of a supergeodesic for a given initial condition ([Theorem 2.21](#)).

We use throughout this article the sheaf-theoretic approach to supermanifolds going back to Kostant [\[7\]](#) and Leites [\[8\]](#), thus basing our work on the following:

Definition 2.1. A supermanifold \mathcal{M} of dimension $m|n$ is a locally ringed space $(M, \mathcal{O}_{\mathcal{M}})$, where $\mathcal{O}_{\mathcal{M}}$ is a sheaf of commutative \mathbb{Z}_2 -graded algebras on M , such that

- (i) M is a Hausdorff second countable topological space,
- (ii) every point $m \in M$ has a neighborhood U such that the ringed space $(U, \mathcal{O}_{\mathcal{M}|U})$ is isomorphic to a superdomain $\mathcal{V}^{m|n} = (V, \mathcal{O}_{\mathbb{R}^{m|n}|V})$ with V open in \mathbb{R}^m .

Reminder 2.2. If $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}})$ are ringed spaces, a morphism $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ is a pair $(\tilde{\Phi}, \Phi^*)$, where $\tilde{\Phi} : X \rightarrow Y$ is a continuous mapping and $\Phi^* : \mathcal{O}_{\mathcal{Y}} \rightarrow \tilde{\Phi}_*\mathcal{O}_{\mathcal{X}}$ is a collection of homomorphisms of rings $\Phi^*(U) : \mathcal{O}_{\mathcal{Y}}(U) \rightarrow \mathcal{O}_{\mathcal{X}}(\tilde{\Phi}^{-1}(U))$, for each open subset U of Y , compatible with the restriction mappings. If the ringed spaces are locally ringed

spaces, i.e., all stalks have a unique maximal ideal, the morphism $\Phi_x^* : \mathcal{O}_{y, \tilde{\Phi}(x)} \rightarrow \mathcal{O}_{x, x}$ is supposed to be local for all $x \in X$, i.e., $\Phi_x^*(\mathfrak{m}_{\tilde{\Phi}(x)}) \subset \mathfrak{m}_x$.

We remark that the datum of such a collection Φ^* of homomorphisms is equivalent to $\Phi^* : \tilde{\Phi}^{-1}\mathcal{O}_y \rightarrow \mathcal{O}_x$ a collection of homomorphisms of rings $\Phi^*(V) : (\tilde{\Phi}^{-1}\mathcal{O}_y)(V) \rightarrow \mathcal{O}_x(V)$, for each open subset V of X , compatible with the restriction mappings. Recall that the inverse image $\tilde{\Phi}^{-1}\mathcal{O}_y$ of \mathcal{O}_y is defined as

$$(\tilde{\Phi}^{-1}\mathcal{O}_y)(V) := \lim_{\substack{W \supset \tilde{\Phi}(V) \\ W \text{ open in } Y}} \mathcal{O}_y(W).$$

This equivalence is due to the fact that $\tilde{\Phi}_*$ and $\tilde{\Phi}^{-1}$ are adjoint functors, i.e.

$$\text{Hom}(\mathcal{O}_y, \tilde{\Phi}_*\mathcal{O}_x) \xrightarrow{\cong} \text{Hom}(\tilde{\Phi}^{-1}\mathcal{O}_y, \mathcal{O}_x).$$

Proposition 2.3. Given supermanifolds $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, $\mathcal{N} = (N, \mathcal{O}_{\mathcal{N}})$ the following holds:

(i) there is a morphism of sheaves $\beta : \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{C}_M^\infty$, called the reduction mapping, such that the following sequence:

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}}^{\text{nil}} \longrightarrow \mathcal{O}_{\mathcal{M}} \xrightarrow{\beta} \mathcal{C}_M^\infty \longrightarrow 0$$

is exact, where $\mathcal{O}_{\mathcal{M}}^{\text{nil}}$ is the sheaf of nilpotent elements of $\mathcal{O}_{\mathcal{M}}$.

(ii) if $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ is a morphism of supermanifolds, then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}} & \xrightarrow{\Phi^*} & \tilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \\ \beta \downarrow & & \downarrow \beta \\ \mathcal{C}_M^\infty & \xrightarrow{(\tilde{\Phi})^*} & \tilde{\Phi}_*\mathcal{C}_N^\infty \end{array}$$

For every supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, one defines $\mathcal{T}_{\mathcal{M}}(U) := \text{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(U))$ for each open set U in M . Here $\text{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(U))$ denotes the $\mathcal{O}_{\mathcal{M}}(U)$ -module of derivations of $\mathcal{O}_{\mathcal{M}}(U)$, where a homogeneous derivation φ is by definition an element of $\text{End}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(U))_0 \cup \text{End}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(U))_1$ such that

$$\varphi(ab) = \varphi(a)b + (-1)^{|\varphi||a|}a\varphi(b) \quad \text{for all } a, b \in \mathcal{O}_{\mathcal{M}}(U).$$

$\mathcal{T}_{\mathcal{M}}$ is called the *tangent sheaf* and has the following structure (compare, e.g. [9], Satz 4.36 for the proof):

Theorem 2.4. $\mathcal{T}_{\mathcal{M}}$ is a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$ -modules of rank (m, n) .

The sections of $\mathcal{T}_{\mathcal{M}}$ are called *vector fields*. The \mathbb{R} -vector space $\mathcal{T}_{\mathcal{M}}(U)$ has a natural Lie superalgebra structure given by the following bracket $[\cdot, \cdot]$

$$[X, Y]f := X(Yf) - (-1)^{|X||Y|}Y(Xf),$$

satisfying the following graded Jacobi identity for each triple X, Y, Z of vector fields:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]].$$

On a supermanifold one also has the *numerical tangent space*, defined as follows:

For every point p in M , the numerical tangent space $T_p^{\text{num}}\mathcal{M}$ of \mathcal{M} at p is the space spanned by the homogeneous derivations $\varphi : \mathcal{O}_{\mathcal{M}, p} \rightarrow \mathbb{R}$ satisfying

$$\varphi(fg) = \varphi(f)\tilde{g}(p) + (-1)^{|\varphi||f|}\tilde{f}(p)\varphi(g),$$

where $\mathcal{O}_{\mathcal{M}, p} := \lim_{U \ni p} \mathcal{O}_{\mathcal{M}}(U)$ is the stalk of $\mathcal{O}_{\mathcal{M}}$ at p and $\tilde{f}(p) = \beta(f)(p)$.

For any $p \in U$, there is a natural reduction map $\mathcal{T}_{\mathcal{M}}(U) \rightarrow T_p^{\text{num}}\mathcal{M}$ sending a vector field $X \in \mathcal{T}_{\mathcal{M}}(U)$ to its value $X^{\text{num}}(p)$ at p . Remark that, contrary to the classic case, a vector field is not determined by the collection of its numerical values at all points.

There is a natural real vector bundle of rank $m + n$ over M having the numerical tangent spaces as its fibers, canonically splitting as the direct sum of $(T^{\text{num}}\mathcal{M})_0 \rightarrow M$ and $(T^{\text{num}}\mathcal{M})_1 \rightarrow M$. Then X^{num} is a smooth section of $T^{\text{num}}\mathcal{M} \rightarrow M$. If $\mathcal{M} = (M, \Gamma_{AE}^\infty)$ with E a smooth real vector bundle over M , the “odd part” of the numerical tangent bundle, $(T^{\text{num}}\mathcal{M})_1$, is in fact isomorphic to E .

We use again the letter β to denote the reduction map to the numerical bundle composed with the projection to the even part of the tangent space:

$$\beta : \mathcal{T}_{\mathcal{M}} \rightarrow (M, \Gamma_{(T^{\text{num}}\mathcal{M})_0}^\infty),$$

where $(M, \Gamma_{(T^{\text{num}}\mathcal{M})_0}^\infty)$ is naturally isomorphic to the sheaf (M, Γ_{TM}^∞) of vector fields on M .

For $f \in \mathcal{O}_{\mathcal{M}}(U)$ and $X \in \mathcal{T}_{\mathcal{M}}(U)$, we will often write \tilde{f} and \tilde{X} instead of $\beta(f)$ and $\beta(X)$, respectively.

Definition 2.5. The cotangent sheaf of a supermanifold \mathcal{M} is defined as the dual $\Omega_{\mathcal{M}}^1$ of $\mathcal{T}_{\mathcal{M}}$.

Proposition 2.6. The $\mathcal{O}_{\mathcal{M}}$ -module sheaf $\Omega_{\mathcal{M}}^1$ is locally free of rank (m, n) .

Definition 2.7. A (even) graded Riemannian metric on a supermanifold \mathcal{M} is a graded symmetric even non-degenerate $\mathcal{O}_{\mathcal{M}}$ -linear morphism of sheaves

$$g : \mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}},$$

where non-degeneracy means that the mapping $X \mapsto g(X, \cdot)$ is an isomorphism from $\mathcal{T}_{\mathcal{M}}$ to $\Omega_{\mathcal{M}}^1$.

A supermanifold equipped with a graded Riemannian metric is called a Riemannian supermanifold.

Remarks. Let $q \in M$, the morphism g defines a scalar superproduct g_q on the real supervector space $T_q^{\text{num}}\mathcal{M}$ as follows: Let u, v be two numerical tangent vectors in $T_q^{\text{num}}\mathcal{M}$. Choose vector fields $X, Y \in \mathcal{T}_{\mathcal{M}}(U)$ where U is an open neighborhood of q in M , fulfilling $X^{\text{num}}(q) = u$ and $Y^{\text{num}}(q) = v$. We then set $g_q(u, v) := \widetilde{g(X, Y)}(q)$. In general, this family of scalar superproducts does not determine the graded Riemannian metric. By restriction to the even numerical vector fields this scalar superproduct induces a pseudo-Riemannian metric \tilde{g} on M , whereas the restriction to the odd numerical vector fields gives a symplectic structure on the real vector bundle $(T^{\text{num}}\mathcal{M})_1 \rightarrow M$. It follows notably that the odd dimension of a Riemannian supermanifold is always even.

Our goal is to define the notion of supergeodesics, as “autoparallel curves” with respect to a connection. Let thus $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold and \mathcal{E} a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$ -modules of finite rank on M .

Definition 2.8. A connection on \mathcal{E} is an even morphism $\nabla : \mathcal{E} \rightarrow \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}$ of sheaves of \mathbb{R} -supervector spaces that satisfies the Leibniz rule

$$\nabla(f \cdot v) = df \otimes v + f \cdot \nabla(v)$$

for all sections $f \in \mathcal{O}_{\mathcal{M}}(U)$ and $v \in \mathcal{E}(U)$ where $df(X) := (-1)^{|X||f|}X(f)$ for all $X \in \mathcal{T}_{\mathcal{M}}(U)$, and on all open subsets U of M .

Given a vector field X on \mathcal{M} , we have a map $\nabla_X : \mathcal{E} \rightarrow \mathcal{E}$ defined via the canonical pairing $\langle \cdot, \cdot \rangle$ between the tangent and the cotangent sheaf, $\nabla_X(v) = \langle X, \nabla(v) \rangle$. Then ∇_X satisfies:

$$(\nabla_X)(f \cdot v) = X(f) \cdot v + (-1)^{|X||f|}f \cdot (\nabla_X)(v) \quad \text{and} \quad |(\nabla_X)(v)| = |X| + |v|.$$

In the case $\mathcal{E} = \mathcal{T}_{\mathcal{M}}$, we speak of a connection on \mathcal{M} .

Definition 2.9. We define the torsion of a connection ∇ on \mathcal{M} by

$$T_{\nabla}(X, Y) := \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X, Y].$$

Definition 2.10. If \mathcal{M} is equipped with an even graded Riemannian metric g , a connection ∇ is called metric if the connection respects the metric in the following sense

$$\nabla g = 0, \quad \text{i.e.} \quad X(g(Y \otimes Z)) = g(\nabla_X Y \otimes Z) + (-1)^{|X||Y|}g(Y \otimes \nabla_X Z) \quad \forall X, Y, Z \in \mathcal{T}_{\mathcal{M}}(U).$$

We also have the natural graded analogue of Christoffel symbols: If (q_i) is a system of coordinates on $U \subset M$, the expansion

$$\nabla_{\partial_{q_i}} \partial_{q_j} = \sum_k \Gamma_{ij}^k \partial_{q_k}$$

gives elements $\Gamma_{ij}^k \in \mathcal{O}_{\mathcal{M}}(U)$ of parity $|\Gamma_{ij}^k| = |q_i| + |q_j| + |q_k|$.

Theorem 2.11. On a supermanifold \mathcal{M} with a graded Riemannian metric g , there exists a unique torsion free and metric connection ∇ , which will be called the Levi-Civita connection of the metric.

The proof follows immediately upon interpreting the usual six-term-formula in a graded sense and sheaf-theoretically.

A connection on \mathcal{M} induces a connection on the vector bundle $TM \rightarrow M$ by reduction. Indeed, there exists a unique connection $\tilde{\nabla}$ on M such that

$$\tilde{\nabla}_{\beta(X)}(\beta(Y)) = \beta(\nabla_X Y) \quad \forall X, Y \in \mathcal{T}_{\mathcal{M}}(U) \text{ and for all } U \text{ open of } M.$$

In our case, the Levi-Civita connection associated to a graded Riemannian metric induces the usual Levi-Civita connection of the induced metric \tilde{g} on M . Moreover, in coordinates (q_i) , the reduction $\beta(\Gamma_{ij}^k)$ of a Christoffel symbol Γ_{ij}^k is the Christoffel symbol of the Levi-Civita connection on M with respect to the corresponding coordinates on M .

In the classical case, we have an explicit formula for Christoffel symbols. Adding appropriate signs this formula holds true in the super context:

Proposition 2.12. *In coordinates (q_i) on U , the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection on a Riemannian supermanifold are given by*

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l [\partial_{q_i} g_{jl} + (-1)^{|q_i||q_j|} \partial_{q_j} g_{il} - (-1)^{|q_l|(|q_i|+|q_j|)} \partial_{q_l} g_{ij}] g^{lk}, \quad (2.1)$$

where $g_{ij} := g(\partial_{q_i}, \partial_{q_j}) \in \mathcal{O}_{\mathcal{M}}(U)$ and $g^{lk} \in \mathcal{O}_{\mathcal{M}}(U)$ are defined by $\sum_k g^{ik} g_{kj} = \delta_{i,j}$.

Proof. Direct computation using the six-term-formula that defines the Levi-Civita connection.

A connection on \mathcal{M} allows to define a covariant derivative of vector fields along a curve. We develop these notions here in some detail for supermanifolds.

Definition 2.13. Let $\Phi = (\tilde{\Phi}, \Phi^*) : \mathcal{N} \rightarrow \mathcal{M}$ be a morphism of supermanifolds and U an open set of M . A vector field along $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ on U is a morphism of supervector spaces

$$X : \mathcal{O}_{\mathcal{M}}(U) \rightarrow (\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}})(U)$$

such that its homogeneous components X_0, X_1 satisfy the following derivation property

$$X_\alpha(fg) = X_\alpha(f) \Phi^*(g) + (-1)^{\alpha \cdot |f|} \Phi^*(f) X_\alpha(g) \quad \text{for } \alpha = 0, 1 \text{ and } f, g \in \mathcal{O}_{\mathcal{M}}(U).$$

The set $\text{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(U), (\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}})(U))$ of all vector fields along Φ will also be denoted by $\mathcal{T}_M^\Phi(U)$. Remark that \mathcal{T}_M^Φ is naturally a sheaf of $\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}$ -modules over M .

Recall that a morphism of supermanifolds can be viewed as a morphism $\Phi^* : \mathcal{O}_{\mathcal{M}} \rightarrow \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}$ of sheaves over M or as a morphism $\Phi^* : \tilde{\Phi}^{-1} \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{N}}$ of sheaves over N . Accordingly, vector fields along Φ can also be viewed as a sheaf of $\mathcal{O}_{\mathcal{N}}$ -modules over $N : \mathcal{T}_N^\Phi := \text{Der}_{\mathbb{R}}(\tilde{\Phi}^{-1} \mathcal{O}_{\mathcal{M}}, \mathcal{O}_{\mathcal{N}})$.

There are two standard ways of constructing vector fields along Φ . If X is a vector field on \mathcal{M} , then

$$\hat{X} := \Phi^* \circ X$$

is a vector field along Φ , and if Y is a vector field on \mathcal{N} , we get the following vector field along Φ :

$$(T\Phi^*)(Y) := Y \circ \Phi^*.$$

We thus obtain two morphisms of sheaves over N , namely $\tilde{\Phi}^{-1} \mathcal{T}_M^\Phi \xrightarrow{\Phi^*} \mathcal{T}_N^\Phi$ and $\mathcal{T}_N^\Phi \xrightarrow{T\Phi^*} \mathcal{T}_N^\Phi$, and their analogues over M . Since knowledge of this sheaves will be useful in the sequel, we give a more explicit description.

Proposition 2.14.

(i) \mathcal{T}_M^Φ is a locally free sheaf of $\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}$ -modules over M of the same rank as \mathcal{T}_M . More precisely, we have an isomorphism

$$\mathcal{T}_M^\Phi \xrightarrow{\cong} \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_M.$$

(ii) \mathcal{T}_N^Φ is a locally free sheaf of $\mathcal{O}_{\mathcal{N}}$ -modules over N of the same rank as \mathcal{T}_M . More precisely, we have an isomorphism

$$\mathcal{T}_N^\Phi \xrightarrow{\cong} \mathcal{O}_{\mathcal{N}} \otimes_{\tilde{\Phi}^{-1} \mathcal{O}_{\mathcal{M}}} \tilde{\Phi}^{-1} \mathcal{T}_M.$$

Proof. We will prove only (i). Part (ii) then follows from the fact that $\tilde{\Phi}_*$ and $\tilde{\Phi}^{-1}$ are adjoint functors.

We will construct two isomorphisms Q and S as below:

$$\text{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}, \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}) \xrightarrow{Q} \text{Hom}_{\mathcal{O}_{\mathcal{M}}}(\Omega_{\mathcal{M}}^1, \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}) \xrightarrow{S} \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_M.$$

Step 1:

We define Q on U , a chart domain of \mathcal{M} with local coordinates (q_i) . Let D be a vector field along Φ on U , $Q(U)(D)$ is then the morphism of $\mathcal{O}_{\mathcal{M}}(U)$ -module such that

$$Q(U)(D)(dq_i) = (-1)^{|i|} D(q_i) \quad \forall i,$$

where $dq_i(\partial_{q_j}) := \delta_{i,j}$.

Thus if $\omega = \sum_i \omega_i \cdot dq_i$, we have $Q(U)(D)(\omega) = \sum_i (-1)^{|D||\omega_i|+|i|} \Phi^*(\omega_i) \cdot D(q_i)$. It follows that $Q(U)$ is a morphism of $(\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}})(U)$ -modules and Q is a $\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}$ -module morphism.

Step 2:

Similarly, we define S on a chart domain U . Let $\varphi \in \text{Hom}_{\mathcal{O}_{\mathcal{M}}(U)}(\Omega_{\mathcal{M}}^1(U), \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}(U))$, then

$$S(U)(\varphi) = \sum_i (-1)^{|i|} \varphi(dq_i) \otimes_{\mathcal{O}_{\mathcal{N}}(U)} \partial_{q_i}, \quad \text{where } \partial_{q_i} := \frac{\partial}{\partial q_i}.$$

A direct computation shows that both maps Q and S are isomorphisms of $\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}$ -modules. \square

Remark. The proof of Proposition 2.14 shows that the isomorphism $L := S \circ Q$ looks as follows in local coordinates on U , for a $D \in \mathcal{T}_M^\Phi(U)$:

$$\begin{aligned} L(D) &= \sum_i (-1)^{|i|} Q(D)(dq_i) \otimes \partial_{q_i} \\ &= \sum_i D(q_i) \otimes \partial_{q_i}. \end{aligned}$$

Furthermore, if (q_i) are coordinates on \mathcal{M} , then $(\hat{\partial}_{q_i})$ is a basis of $\mathcal{T}_M^\Phi(U)$ and we have

$$T\Phi^*(Y) = Y \circ \Phi^* = \sum_i Y(\Phi^* q_i) \hat{\partial}_{q_i}$$

for all vector fields Y on N .

Given a connection on \mathcal{M} and a morphism $\Phi : \mathcal{N} \rightarrow \mathcal{M}$, we introduce the associated pull-back connection in order to get a covariant derivative along a curve. One has, in fact, the statement below:

Proposition/Definition 2.15. *There exists a unique connection $\hat{\nabla}$ on \mathcal{T}_M^Φ such that the diagram*

$$\begin{array}{ccc} \mathcal{T}_{\mathcal{M}} & \xrightarrow{\nabla} & \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \\ \Phi^* \downarrow & & \downarrow \Phi^* \\ \mathcal{T}_M^\Phi & \xrightarrow{\hat{\nabla}} & \tilde{\Phi}_* \Omega_{\mathcal{N}}^1 \otimes_{\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}} \mathcal{T}_M^\Phi \end{array}$$

commutes. This connection is usually called the pullback of ∇ via Φ and is denoted by $\Phi^ \nabla$.*

Remark. The vertical arrow is defined as follows. First there is a natural map from $\Omega_{\mathcal{M}}^1$ to $\tilde{\Phi}_* \Omega_{\mathcal{N}}^1$ induced by Φ^* which is also denoted by Φ^* . Given U an open set in M and $\omega \in \Omega_{\mathcal{M}}^1(U)$, we get an element ν of $\tilde{\Phi}_* \Omega_{\mathcal{N}}^1(U)$:

$$\nu(X) := \hat{\omega}(T\Phi^*(X)) \quad \text{for } X \in (\tilde{\Phi}_* \mathcal{T}_{\mathcal{N}})(U),$$

where $\hat{\omega}$ is defined by

$$\begin{aligned} \hat{\omega} : \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}(U) \otimes_{\mathcal{O}_{\mathcal{M}}(U)} \mathcal{T}_{\mathcal{M}}(U) &\longrightarrow \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}(U) \\ f \otimes Y &\longmapsto (-1)^{|f||\omega|} f \cdot \Phi^*(\omega(Y)) \end{aligned}$$

and the isomorphism $\mathcal{T}_M^\Phi \cong \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}}$. Using $\Phi^* : \mathcal{O}_{\mathcal{M}} \rightarrow \tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}$ the map $\Phi^* : \Omega_{\mathcal{M}}^1 \rightarrow \tilde{\Phi}_* \Omega_{\mathcal{N}}^1$ turns out to be a morphism of $\mathcal{O}_{\mathcal{M}}$ -modules, that can be tensored with $\Phi^* : \mathcal{T}_{\mathcal{M}} \rightarrow \mathcal{T}_M^\Phi$ to obtain the arrow

$$\Phi^* : \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \longrightarrow \tilde{\Phi}_* \Omega_{\mathcal{N}}^1 \otimes_{\tilde{\Phi}_* \mathcal{O}_{\mathcal{N}}} \mathcal{T}_M^\Phi.$$

Proof of Proposition 2.15. The proof follows immediately upon interpreting the usual proof in the classical case of ungraded manifolds sheaf-theoretically. \square

Remark. The diagram in Proposition/Definition 2.15 can be equivalently replaced by the following diagram:

$$\begin{array}{ccc} \tilde{\Phi}^{-1} \mathcal{T}_{\mathcal{M}} & \xrightarrow{\nabla} & \tilde{\Phi}^{-1} \Omega_{\mathcal{M}}^1 \otimes_{\tilde{\Phi}^{-1} \mathcal{O}_{\mathcal{M}}} \tilde{\Phi}^{-1} \mathcal{T}_{\mathcal{M}} \\ \Phi^* \downarrow & & \downarrow \Phi^* \\ \mathcal{T}_N^\Phi & \xrightarrow{\hat{\nabla}} & \Omega_{\mathcal{N}}^1 \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{T}_N^\Phi \end{array}$$

We can now define the covariant derivative along a supercurve $\Phi : \mathbb{R}^{1|1} \rightarrow \mathcal{M}$. Let (\mathcal{M}, g) be a Riemannian supermanifold and ∇ its Levi-Civita connection. Consider $\Phi^* \nabla$, the pullback connection of ∇ under Φ .

Definition 2.16. The even covariant derivative along $\Phi : \mathbb{R}^{1|1} \rightarrow \mathcal{M}$ is defined as

$$\frac{\nabla}{dt} := (\Phi^* \nabla)_{\partial_t} : \mathcal{T}_M^\Phi \longrightarrow \mathcal{T}_M^\Phi,$$

induced by the duality pairing of ∂_t with elements of $\Omega_{\mathbb{R}^{1|1}}^1$ where (t, θ) are the canonical coordinates on $\mathbb{R}^{1|1} \cong \mathbb{R} \times \mathbb{R}^{0|1}$.

Note that if (q_i) are local coordinates on $U \subset M$, we have for all vector fields $X \in \mathcal{T}_M^\Phi(U)$

$$\frac{\nabla}{dt}(X) = \sum_k \left[\partial_t X(q_k) + \sum_{i,j} X(q_i) \partial_t \Phi^*(q_j) \Phi^*(\Gamma_{ji}^k) \right] \otimes \partial_k.$$

Analogously, we can define the covariant derivative along a curve with respect to the odd coordinate θ on $\mathbb{R}^{1|1}$:

$$\frac{\nabla}{d\theta} := (\Phi^* \nabla)_{\partial_\theta} : \mathcal{T}_M^\Phi \longrightarrow \mathcal{T}_M^\Phi,$$

induced by the duality pairing of ∂_θ with elements of $\Omega_{\mathbb{R}^{1|1}}^1$.

In local coordinates, we have a similar formula with an additional sign:

$$\frac{\nabla}{d\theta}(X) = \sum_k \left[\partial_\theta X(q_k) + \sum_{i,j} (-1)^{|X|+|q_i|} X(q_i) \partial_\theta \Phi^*(q_j) \Phi^*(\Gamma_{ji}^k) \right] \otimes \partial_k.$$

We can now give a natural definition of supergeodesics.

Definition 2.17. A supergeodesic on (\mathcal{M}, ∇) , a supermanifold with a connection ∇ , is a supercurve $\Phi : \mathbb{R}^{1|1} \cong \mathbb{R} \times \mathbb{R}^{0|1} \rightarrow \mathcal{M}$ such that

$$\frac{\nabla}{dt}(T\Phi^*(\partial_t)) = 0.$$

The preceding condition is locally equivalent to

$$\partial_t^2 \Phi^*(q_k) + \sum_{i,j} \partial_t \Phi^*(q_i) \partial_t \Phi^*(q_j) \Phi^*(\Gamma_{ji}^k) = 0 \quad \forall k, \quad (2.2)$$

where (q_k) are local coordinates on \mathcal{M} . Note that the Eq. (2.2) restricted to those k corresponding to even coordinates are the usual geodesic equations of the underlying classical manifold equipped with the reduced connection $\tilde{\nabla}$ on M .

Remarks. (1) Of course, in general a supergeodesic is only defined on an open subsupermanifold of $\mathbb{R}^{1|1}$ with body an open interval in \mathbb{R} . Slightly abusing, we will often write $\mathbb{R}^{1|1}$ meaning such an open subsupermanifold of it.

(2) One can easily generalize Definitions 2.16 and 2.17 to the case $\mathbb{R} \times \mathcal{S}$, \mathcal{S} an arbitrary supermanifolds replacing $\mathbb{R}^{0|1}$.

A different definition was given by Goertsches [5]: a supergeodesic in the sense of Goertsches is a supercurve $\Phi : \mathbb{R}^{1|1} \rightarrow \mathcal{M}$ such that

$$\left(\frac{\nabla}{dt} T\Phi^*(\partial_t) \right)^{\text{num}} = \left(\frac{\nabla}{d\theta} T\Phi^*(\partial_\theta) \right)^{\text{num}} = 0. \quad (2.3)$$

Remark that such a curve automatically satisfies the following equations:

$$\left(\frac{\nabla}{d\theta} T\Phi^*(\partial_t) \right)^{\text{num}} = \left(\frac{\nabla}{d\theta} T\Phi^*(\partial_\theta) \right)^{\text{num}} = 0.$$

Locally, (2.3) is equivalent to

$$f_k'' + \sum_{\substack{i \text{ even,} \\ j \text{ even}}} f_i' \cdot f_j' \cdot \Phi^*(\Gamma_{ji}^k) = 0 \quad \forall k \text{ even,}$$

and

$$f_\delta' + \sum_{\substack{i \text{ even,} \\ \beta \text{ odd}}} f_\beta \cdot f_i' \cdot \Phi^*(\Gamma_{i\beta}^\delta) = 0 \quad \forall \delta \text{ odd,}$$

where $f_i, f_\delta \in \mathcal{C}_\mathbb{R}^\infty(\tilde{\Phi}^{-1}(U))$ are functions defined as follows:

$$\Phi^*(q_i) = f_i \quad \forall i \text{ even} \quad \text{and} \quad \Phi^*(q_\beta) = f_\beta \cdot \theta \quad \forall \beta \text{ odd.}$$

We call here and in the sequel an index $(i, \delta, k, \beta, \dots)$ “even” resp. “odd” if the corresponding coordinate has the property.

In particular, if $\mathcal{M} = \mathbb{R}^{1|2}$ is equipped with a flat metric (i.e., $\Gamma_{ij}^k = 0$), (2.3) is equivalent to

$$f_k'' = 0 \quad \forall k \text{ even, and } f_\delta' = 0 \quad \forall \delta \text{ odd.}$$

On the other hand, the geodesic equation (2.2) is equivalent to

$$f_k'' = 0 \quad \forall k \text{ even, and } f_\delta'' = 0 \quad \forall \delta \text{ odd.}$$

The Eqs. (2.2) and (2.3) are thus obviously non-equivalent, moreover initial conditions for these two notions of geodesics are not the same.

Let us explain in some detail what an initial condition is in the super context. For this we need to recall supervector bundles and the tangent map (in the category of supervector bundles) coming from a morphism of supermanifolds.

If $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ is a supermanifold and V an open subset of X , we denote by $\mathcal{X}|_V$ the open subsupermanifold given by $(V, \mathcal{O}_{\mathcal{X}|_V})$.

Definition 2.18. Let \mathcal{V} and \mathcal{M} be supermanifolds and let $\pi : \mathcal{V} \rightarrow \mathcal{M}$ be a surjective submersion.

\mathcal{V} is called a *supervector bundle* with base \mathcal{M} and typical fiber $\mathbb{R}^{p|q}$ if

- (i) for all $x \in M$, there exists a local trivialization (U, φ) , i.e. there is an open neighborhood U of x and an isomorphism $\varphi : \mathcal{V}_{|\tilde{\pi}^{-1}(U)} \rightarrow \mathbb{R}^{p|q} \times \mathcal{M}|_U$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_{|\tilde{\pi}^{-1}(U)} & \xrightarrow{\varphi} & \mathbb{R}^{p|q} \times \mathcal{M}|_U \\ \pi \downarrow & \swarrow \text{proj}_2 & \\ \mathcal{M}|_U & & \end{array}$$

- (ii) and there exists a covering of M by local trivializations $\{(U_\alpha, \varphi_\alpha)\}$ as in (i) such that for all α, β with $U_{\alpha\beta} := U_\alpha \cap U_\beta$ nonempty set, the transition function $\varphi_{\alpha\beta} := \varphi_\beta \circ (\varphi_\alpha)^{-1} : \mathbb{R}^{p|q} \times \mathcal{M}|_{U_{\alpha\beta}} \rightarrow \mathbb{R}^{p|q} \times \mathcal{M}|_{U_{\alpha\beta}}$ is determined by an element $\phi_{\alpha\beta} \in GL_{(p,q)}(\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta}))$, that is

$$(\varphi_{\alpha\beta})^*(f) = f \quad \forall f \in \mathcal{O}_{\mathcal{M}}(U_{\alpha\beta}),$$

$$(\varphi_{\alpha\beta})^*(v_i) = \sum_j v_j \cdot (\phi_{\alpha\beta})_{ji} \quad \forall i,$$

where (v_i) are the canonical coordinates on $\mathbb{R}^{p|q}$.

Theorem 2.19 ([10]). Let \mathcal{M} be a supermanifold. The category of locally free sheaves of $\mathcal{O}_{\mathcal{M}}$ -modules of finite rank is equivalent to the category of supervector bundles over \mathcal{M} .

Remark. Let now $T\mathcal{M} = (TM, \mathcal{O}_{T\mathcal{M}})$ and $T^*\mathcal{M} = (T^*M, \mathcal{O}_{T^*\mathcal{M}})$ be the supervector bundles associated by the preceding construction to the locally free sheaves $\mathcal{T}_{\mathcal{M}}$ and $\mathcal{Q}_{\mathcal{M}}^1$ respectively. If \mathcal{M} is of dimension $n|m$, then the total spaces of these bundles both have dimension $2n|2m$, which fits with the naïve idea of position and velocity resp. position and momentum variables. Of course, this point of view is also crucial for the existence of a canonical even symplectic form on $T^*\mathcal{M}$ (see below in Section 3; compare also [7]). Denoting the projection from $T\mathcal{M}$ to \mathcal{M} by π , one observes that $\Gamma(\mathcal{M}, T\mathcal{M}) := \{\Phi \in \text{Mor}(\mathcal{M}, T\mathcal{M}) \mid \pi \circ \Phi = \text{id}_{\mathcal{M}}\}$ is the space of even vector fields or derivations on \mathcal{M} . Instead of going to a “bigger tangent bundle” of dimension $2m + n|2n + m$ (compare, e.g., [11]), we prefer to stick to the above tangent bundle and note that $\Gamma(\mathcal{M}, T\mathcal{M}) \oplus \Gamma(\mathcal{M}, \Pi T\mathcal{M})$, with $\Pi T\mathcal{M}$ the vector bundle associated to the module sheaf $\Pi \mathcal{T}_{\mathcal{M}}$ obtained by applying parity change to $\mathcal{T}_{\mathcal{M}}$, is the space of all vector fields on \mathcal{M} (compare, e.g., the Sections 7 and 8 of [10]). In more modern language, we can formulate this via relative morphisms (see [12] for a recent and detailed account of this point of view). Let $p : \mathcal{V} \rightarrow \mathcal{M}$ be a fiber bundle in the category **Sman** of supermanifolds, then we have a functor to the category **Set** of sets:

$$\mathbf{Sman} \rightarrow \mathbf{Set}, \quad \mathcal{S} \mapsto \text{Mor}_{\mathcal{M}}(\mathcal{M} \times \mathcal{S}, \mathcal{V}) := \{\Phi \in \text{Mor}(\mathcal{M} \times \mathcal{S}, \mathcal{V}) \mid p \circ \Phi = \text{proj}_1\},$$

and vector fields on \mathcal{M} are given by taking $\mathcal{V} = T\mathcal{M}$ and $\mathcal{S} = \mathbb{R}^{0|1}$. This formalizes, in fact, the classical point of view of physicists saying that given an odd vector field X , multiplication by an “odd constant” (realized by the canonical odd coordinate τ on $\mathbb{R}^{0|1}$) yields an even “vector field” $\tau \cdot X$.

Proposition 2.20. Each morphism of supermanifolds $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ induces a morphism of supermanifolds $T\Phi : T\mathcal{N} \rightarrow T\mathcal{M}$ called the tangent map of Φ .

Sketch of the proof. Let $(q_i^{\mathcal{N}})$ and $(q_j^{\mathcal{M}})$ be coordinates on \mathcal{N} and \mathcal{M} . They induce natural coordinates $(q_i^{\mathcal{N}}, v_i^{\mathcal{N}})$ and $(q_j^{\mathcal{M}}, v_j^{\mathcal{M}})$ which trivialize $T\mathcal{N}$ and $T\mathcal{M}$, respectively. In these coordinates, the tangent map of Φ is given as follows

$$(T\Phi)^*(q_j^{\mathcal{M}}) = \Phi^*(q_j^{\mathcal{M}}) \quad \forall j,$$

$$(T\Phi)^*(v_j^{\mathcal{M}}) = \sum_i v_i^{\mathcal{N}} \cdot \partial_{q_i^{\mathcal{N}}} \Phi^*(q_j^{\mathcal{M}}) \quad \forall j. \quad \square$$

Let \mathcal{S} be a supermanifold and $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ a morphism of supermanifolds. Using the tangent map $T\Phi : T\mathbb{R} \times T\mathcal{S} \rightarrow T\mathcal{M}$, the section $s_1 : t \mapsto (1, t)$ of $T\mathbb{R}$ and the canonical zero section $s_0 : \mathcal{S} \rightarrow T\mathcal{S}$ of $T\mathcal{S}$, we define $\partial_t \Phi : \mathbb{R} \times \mathcal{S} \rightarrow T\mathcal{M}$, the partial derivative of Φ in the direction of \mathbb{R} , as follows:

$$\partial_t \Phi := T\Phi \circ (s_1 \times s_0) : \mathbb{R} \times \mathcal{S} \rightarrow T\mathcal{M}.$$

In local coordinates, this amounts to

$$\begin{aligned} (\partial_t \Phi)^*(q_i^{\mathcal{M}}) &= \Phi^*(q_i^{\mathcal{M}}) \\ (\partial_t \Phi)^*(v_i^{\mathcal{M}}) &= \partial_t \Phi^*(q_i^{\mathcal{M}}). \end{aligned}$$

We define for \mathcal{X}, \mathcal{Y} supermanifolds and $x_0 \in X$, $\text{inj}_{x=x_0} : \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y}$ by $\widetilde{\text{inj}_{x=x_0}}(y) = (x_0, y)$ and $(\text{inj}_{x=x_0})^*(f \otimes g) = \widetilde{f}(x_0) \cdot g$ for $f \in \mathcal{O}_{\mathcal{X}}$ and $g \in \mathcal{O}_{\mathcal{Y}}$. The pullback operation $(\text{inj}_{x=x_0})^*$ is often denoted by $\text{ev}_{x=x_0}$.

Theorem 2.21. Let $\mathcal{J} \in \text{Mor}(\mathbb{R}^{0|1}, T\mathcal{M})$. Then there exists a unique geodesic $\Phi : \mathbb{R}^{1|1} \cong \mathbb{R} \times \mathbb{R}^{0|1} \rightarrow \mathcal{M}$ such that $\partial_t \Phi \circ \text{inj}_{t=0} = \mathcal{J}$.

Proof. The “initial condition” \mathcal{J} renders the solution of the system of ODEs (2.2) unique, since $\partial_t \Phi \circ \text{inj}_{t=0} = \mathcal{J}$ reads in the local coordinates discussed before the theorem as follows:

$$\text{inj}_{t=0}^* \circ \Phi^*(q_i) = \Phi^*(q_i)|_{t=0} = \mathcal{J}^*(q_i) \quad \text{and} \quad \partial_t \Phi^*(q_i)|_{t=0} = \mathcal{J}^*(v_i). \quad \square$$

Remarks. (1) Of course, the morphism Φ is, in general, only defined on an open, connected subsupermanifold of $\mathbb{R}^{1|1}$, whose body contains the point 0.

(2) The preceding theorem can easily be generalized to cover the situation discussed in the remark after Definition 2.17: given a supermanifold \mathcal{S} and a morphism $\mathcal{J} \in \text{Mor}(\mathcal{S}, T\mathcal{M})$, there exists a unique geodesic $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ such that $\partial_t \Phi \circ \text{inj}_{t=0} = \mathcal{J}$.

3. The geodesic flow

In this section we first clarify the construction of superfunctions on supervector bundles coming from a locally free sheaf, as well as of the canonical two-form on the cotangent bundle of a supermanifolds (both already sketched in [7]). We get a natural “energy function” H on the (co-)tangent bundle of a supermanifold and an associated Hamiltonian vector field X_H . The flow and integral curves of the latter even vector field exist by results of Monterde and Montesinos resp. Dumitrescu. We then prove that integral curves of X_H on $T^*\mathcal{M}$ are in natural bijection with geodesics on \mathcal{M} , as asked for in the introduction.

Proposition 3.1. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold and \mathcal{F} be a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$ -modules and let $\mathcal{V} = (V, \mathcal{O}_{\mathcal{V}}) \xrightarrow{\pi} \mathcal{M}$ be the supervector bundle associated to \mathcal{F} . Then there exists a natural sheaf morphism $\widetilde{\pi}^{-1} \mathcal{F}^* \rightarrow \mathcal{O}_{\mathcal{V}}$ over V or equivalently a sheaf morphism $\mathcal{F}^* \rightarrow \widetilde{\pi}_* \mathcal{O}_{\mathcal{V}}$ over M .

Proof. The proof follows directly from the construction of the supervector bundle \mathcal{V} associated to \mathcal{F} . \square

Remark. Section of $\widetilde{\pi}^{-1} \mathcal{F}^*$ over an open subset in V are of course local superfunctions of $\mathcal{O}_{\mathcal{V}}(V)$ that are “linear in the fiber directions”.

Corollary 3.2. Let $\pi : \mathcal{V} \rightarrow \mathcal{M}$ be the canonical projection. Then we have a natural morphism of sheaves

$$\begin{array}{ccc} \widetilde{\pi}^{-1}(\text{Sym}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}^*)) & \xrightarrow{\chi} & \mathcal{O}_{\mathcal{V}} \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

Proof. By the preceding proposition we have a morphism of $\widetilde{\pi}^{-1} \mathcal{O}_{\mathcal{M}}$ -module sheaves $\widetilde{\pi}^{-1} \mathcal{F}^* \rightarrow \mathcal{O}_{\mathcal{V}}$. Since $\mathcal{O}_{\mathcal{V}}$ is a sheaf of unital supercommutative superalgebras, we have a canonical extension to $\widetilde{\pi}^{-1}(\text{Sym}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}^*)) \cong \text{Sym}_{\widetilde{\pi}^{-1} \mathcal{O}_{\mathcal{M}}}(\widetilde{\pi}^{-1} \mathcal{F}^*) \rightarrow \mathcal{O}_{\mathcal{V}}$, denoted by χ . \square

In the case that $\mathcal{F} = \Omega_{\mathcal{M}}^1$, we have, after identifying $(\Omega_{\mathcal{M}}^1)^*$ with $\mathcal{T}_{\mathcal{M}}$, notably the following morphism of $\mathcal{O}_{\mathcal{M}}(M)$ -modules:

$$(\text{Sym}_{\mathcal{O}_{\mathcal{M}}}^2(\mathcal{T}_{\mathcal{M}}))(M) \xrightarrow{\chi(T^*M)} \mathcal{O}_{T^*\mathcal{M}}(T^*M).$$

If \mathcal{M} is equipped with a graded Riemannian metric $g : \mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}}$, we have the isomorphisms $g^{\natural} : \mathcal{T}_{\mathcal{M}} \xrightarrow{\cong} \Omega_{\mathcal{M}}^1$ and $g^{\flat} : \Omega_{\mathcal{M}}^1 \xrightarrow{\cong} \mathcal{T}_{\mathcal{M}}$ induced by the metric, where

$$g^{\natural}(X)(Y) := g(X, Y) \quad \forall X, Y \in \mathcal{T}_{\mathcal{M}}(U).$$

Using the above identification and the metric we have

$$\Omega_{\mathcal{M}}^1(M) \otimes \Omega_{\mathcal{M}}^1(M) \xrightarrow[\cong]{g^b \otimes g^b} \mathcal{T}_{\mathcal{M}}(M) \otimes \mathcal{T}_{\mathcal{M}}(M) \xrightarrow{g} \mathcal{O}_{\mathcal{M}}(M),$$

which determines an element h of $(\Omega_{\mathcal{M}}^1(M) \otimes \Omega_{\mathcal{M}}^1(M))^*$. The standard theory of linear superalgebra gives the following canonical isomorphisms

$$(\Omega_{\mathcal{M}}^1(M) \otimes \Omega_{\mathcal{M}}^1(M))^* \cong (\Omega_{\mathcal{M}}^1(M))^* \otimes (\Omega_{\mathcal{M}}^1(M))^* \cong \mathcal{T}_{\mathcal{M}}(M) \otimes \mathcal{T}_{\mathcal{M}}(M).$$

Let us now define $H := \frac{1}{2} \chi(T^*M)(h) \in \mathcal{O}_{T^*\mathcal{M}}(T^*M)$, where h is viewed as a symmetric element of $\mathcal{T}_{\mathcal{M}}(M) \otimes \mathcal{T}_{\mathcal{M}}(M)$. In local coordinates, we obtain

$$H = \frac{1}{2} \sum_{i,j} p_i \cdot g^{ij} \cdot p_j,$$

where (q_i, p_i) are coordinates on $T^*\mathcal{M}$ induced by coordinates (q_i) on \mathcal{M} .

Recall that in the super context we have a natural notion of graded differential k -form $\Omega_{\mathcal{M}}^k$ defined, e.g., in [7]. Moreover, we have a canonical exterior differentiation d of bidegree $(1, 0)$ of the $(\mathbb{Z} \times \mathbb{Z}_2)$ -graded commutative algebra $\Omega_{\mathcal{M}}$ of graded differential forms.

Definition 3.3. A symplectic supermanifold is a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ together with a closed non-degenerate two-form $\omega \in \Omega_{\mathcal{M}}^2(M)$. We call the symplectic form and the corresponding symplectic manifold *even* (resp. *odd*) if $\omega \in \Omega_{\mathcal{M}}^2(M)_0$ (resp. $\omega \in \Omega_{\mathcal{M}}^2(M)_1$).

Proposition 3.4. Let \mathcal{M} be a supermanifold. Then the cotangent bundle $T^*\mathcal{M}$ of \mathcal{M} is equipped with a canonical (even) symplectic form $\omega \in \Omega_{T^*\mathcal{M}}^2(T^*M)_0$. As in the classical case of ungraded manifold, $\omega = -d\alpha$ with a canonical potential $\alpha \in \Omega_{T^*\mathcal{M}}^1(T^*M)_0$.

Proof. Using the notations of Proposition 2.14 and the remark after this proposition, we can define the canonical even one-form α on $T^*\mathcal{M}$ as follows:

$$\begin{array}{ccc} \text{Der}_{\mathbb{R}}(\mathcal{O}_{T^*\mathcal{M}}(T^*M)) & \xrightarrow{T(\pi^{T^*\mathcal{M}})^*} \text{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(M), \mathcal{O}_{T^*\mathcal{M}}(T^*M)) & \xrightarrow[\cong]{L} \mathcal{O}_{T^*\mathcal{M}}(T^*M) \otimes_{\mathcal{O}_{\mathcal{M}}(M)} \mathcal{T}_{\mathcal{M}}(M) \\ & \searrow \alpha & \downarrow 1 \otimes \chi \\ & & \mathcal{O}_{T^*\mathcal{M}}(T^*M). \end{array}$$

Here $T^*\mathcal{M} \xrightarrow{\pi^{T^*\mathcal{M}}} \mathcal{M}$ is the canonical projection, L is the map defined in the Proposition 2.14 and χ is the morphism given by the Proposition 3.1. The projection $\pi^{T^*\mathcal{M}}$ will simply be denoted by π in the rest of this proof. We obtain an element $\alpha \in \Omega_{T^*\mathcal{M}}^1(T^*M)_0$ described as follows in the local coordinates (q_i, p_i) :

$$\begin{aligned} \alpha(X) &= (1 \otimes \chi) \circ L \circ T\pi^*(X) \\ &= (1 \otimes \chi) \circ L(X \circ \pi^*) \\ &= (1 \otimes \chi) \left(\sum_i X \circ \pi^*(q_i) \otimes \partial_{q_i} \right) \\ &= \sum_i X(q_i) \cdot \chi(\partial_{q_i}), \quad \text{by identifying } \pi^*(q_i) \text{ with } q_i, \\ &= \sum_i X(q_i) \cdot p_i \\ &= \sum_i (-1)^{|p_i|(|q_i|+|X|)} p_i \cdot X(q_i) \\ &= \sum_i (-1)^{|p_i|} p_i \cdot d(q_i)(X) \quad \forall X \in \mathcal{T}_{T^*\mathcal{M}}(T^*M). \end{aligned}$$

We thus locally have $\alpha = \sum_i (-1)^{|q_i|} p_i \cdot d(q_i)$.

Now we set $\omega := -d\alpha \in \Omega_{T^*\mathcal{M}}^2(T^*M)_0$. To see that ω is non-degenerate, consider its local expression:

$$\begin{aligned} \omega &= -d\alpha = - \left[\sum_i (-1)^{|q_i|} d(p_i) \wedge d(q_i) + \sum_i (-1)^{|q_i|+0 \cdot 1+|p_i| \cdot 0} p_i \cdot d^2(q_i) \right] \\ &= - \sum_i (-1)^{|q_i|+1+|q_i||p_i|} d(q_i) \wedge d(p_i) \\ &= \sum_i d(q_i) \wedge d(p_i). \quad \square \end{aligned}$$

The non-degeneracy of ω allows us to give the following definition:

Definition 3.5. Let (\mathcal{M}, ω) be a symplectic supermanifold. For any $f \in \mathcal{O}_{\mathcal{M}}(M)$ let $X_f \in \mathcal{T}_{\mathcal{M}}(M)$ be the *Hamiltonian vector field* defined by the relation:

$$i(X_f)\omega = df,$$

where $(i(X)\omega)(Y) := (-1)^{|X||\omega|}\omega(X, Y)$.

Note that X_f has necessarily the same parity as f if ω is even. Direct calculation now shows:

Proposition 3.6. Let \mathcal{M} be a supermanifold and $(T^*\mathcal{M}, \omega)$ be its cotangent bundle equipped with the canonical symplectic form. The Hamiltonian vector field $X_f \in \mathcal{T}_{T^*\mathcal{M}}(T^*\mathcal{M})$ associated to an even function $f \in \mathcal{O}_{T^*\mathcal{M}}(T^*\mathcal{M})_0$ is an even vector field and it is locally given by the following equations:

$$\begin{aligned} X_f(q_i) &= (-1)^{|q_i|} \partial_{p_i} f \\ X_f(p_i) &= -\partial_{q_i} f. \end{aligned}$$

Corollary 3.7. In the situation of the preceding proposition, we have for H , the energy function, the following equations:

$$\begin{aligned} X_H(q_i) &= \sum_j p_j g^{ji} \\ X_H(p_i) &= -\frac{1}{2} \sum_{k,j} (-1)^{|q_i||q_k|} p_k \partial_{q_i} (g^{kj}) p_j. \end{aligned}$$

Definition 3.8. Let \mathcal{M} be a supermanifold and X be an even vector field on \mathcal{M} . A *flow* of X is a morphism $F : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$ of supermanifolds such that:

$$\widehat{\partial}_t \circ F^* = F^* \circ X \quad \text{and} \quad F \circ \text{inj}_{t=0} = \text{id}_{\mathcal{M}}, \quad (3.1)$$

where t is the canonical coordinate on \mathbb{R} , $\widehat{\partial}_t$ the lift of the derivation ∂_t on \mathbb{R} to a derivation on $\mathbb{R} \times \mathcal{M}$ acting just on the first factor, and $\text{inj}_{t=0} : \mathcal{M} \cong \{0\} \times \mathcal{M} \xrightarrow{0 \times \text{id}_{\mathcal{M}}} \mathbb{R} \times \mathcal{M}$ is the injection map at $t = 0$.

Remark. Of course, the domain of the flow map F is in general only an open subsupermanifold \mathcal{V} of $\mathbb{R} \times \mathcal{M}$, containing $\{0\} \times \mathcal{M}$. Abusively, we will often write $\mathbb{R} \times \mathcal{M}$ instead of \mathcal{V} .

The following result is due to Monterde and Montesinos [13]:

Theorem 3.9. Let X be an even vector field on a supermanifold \mathcal{M} , \widetilde{X} the reduced vector field on M , and $V_{\widetilde{X}} \subset \mathbb{R} \times M$ the flow domain of \widetilde{X} . Then there exists a unique map $F : \mathcal{V}_X \rightarrow \mathcal{M}$, where \mathcal{V}_X is the subsupermanifold of $\mathbb{R} \times \mathcal{M}$ having as body $V_{\widetilde{X}}$, satisfying (3.1).

Given the flow F of an even vector field X , we want to use an appropriate notion of initial condition to get “integral supercurves”. If we choose a point q_0 in M , we get a curve as follows:

$$\begin{array}{ccc} \mathbb{R} \cong \mathbb{R} \times \{q_0\} & \xrightarrow{\text{inj}_{q=q_0}} & \mathbb{R} \times \mathcal{M} \\ & \searrow \psi & \downarrow F \\ & & \mathcal{M} \end{array}$$

This type of initial condition leads to a poor notion of “integral supercurves”, since ψ factorizes over M , the body of the supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$. Thus we need a notion of initial condition that is more elaborated than simply choosing a point in the body of \mathcal{M} .

Following Dumitrescu [14] we give

Definition 3.10. Let \mathcal{M}, \mathcal{S} be supermanifolds, X an even vector field on \mathcal{M} and $\mathcal{I} \in \text{Mor}(\mathcal{S}, \mathcal{M})$. A morphism $\Psi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ such that

$$\widehat{\partial}_t \circ \Psi^* = \Psi^* \circ X \quad \text{and} \quad \Psi \circ \text{inj}_{t=0} = \mathcal{I}, \quad (3.2)$$

where t is the canonical even coordinate on \mathbb{R} and $\text{inj}_{t=0} : \mathcal{S} \rightarrow \mathbb{R} \times \mathcal{S}$ the evaluation map at $t = 0$, is called an “integral curve for X , parametrized by \mathcal{S} and satisfying the initial condition \mathcal{I} ”.

The case we are mostly concerned with will be $\mathcal{S} = \mathbb{R}^{0|1}$ but we need this more general “family notion” for integral curves notably in the proof of Theorem 4.6. We note that one can give the following geometrical description of the set of initial conditions (for $\mathcal{S} = \mathbb{R}^{0|1}$):

Proposition 3.11. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold and E be a real vector bundle over M such that $\mathcal{M} \cong (M, \Gamma_{\wedge E^*}^{\infty})$. Then there is a 1 : 1 correspondence between E and $\text{Mor}(\mathbb{R}^{0|1}, \mathcal{M})$.

Idea of the proof. The local coordinates given by local trivializations of the vector bundle E yield immediately a bijection between E and $\text{Mor}(\mathbb{R}^{0|1}, \mathcal{M})$. \square

Similarly to the standard ungraded case, integral curves are given by factorization over the flow map:

Proposition 3.12 ([14]). Let X be an even vector field on \mathcal{M} and F be the flow of X . Let $\mathcal{I} \in \text{Mor}(\mathcal{S}, \mathcal{M})$ be an initial condition. There exists a unique integral curve $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ of X with initial condition \mathcal{I} . Precisely, it is given by $\Phi = F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$.

We can now give the first fundamental results of this article, relating the integral curves of the geodesic flow on $T^*\mathcal{M}$ to the geodesics of a Riemannian supermanifold \mathcal{M} .

Theorem 3.13. Let \mathcal{M} be a supermanifold with an even Riemannian metric g , and let $(T^*\mathcal{M}, \omega, H)$ be the Hamiltonian dynamical system associated to (\mathcal{M}, g) , and let \mathcal{S} be a supermanifold. Furthermore, let X_H be the Hamiltonian vector field associated to H , \mathcal{I} an initial condition in $\text{Mor}(\mathcal{S}, T^*\mathcal{M})$ and Ψ the associated integral curve of X_H ($\Psi = F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$). Then the supercurve $\Phi := \pi^{T^*\mathcal{M}} \circ \Psi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ satisfies the geodesic equation:

$$\frac{\nabla}{dt}(T\Phi^*(\partial_t)) = 0,$$

subject to the initial condition $\partial_t \Phi \circ \text{inj}_{t=0} = g^b \circ \mathcal{I} \in \text{Mor}(\mathcal{S}, T\mathcal{M})$, where $g^b : T^*\mathcal{M} \xrightarrow{\cong} T\mathcal{M}$ is the isomorphism induced by the metric g .

Proof. Let Ψ be an integral curve of X_H . Then Ψ is given by $\Psi = F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$ using Proposition 3.12 where $F : \mathbb{R} \times T^*\mathcal{M} \rightarrow T^*\mathcal{M}$ is the flow of X_H and $\mathcal{I} \in \text{Mor}(\mathcal{S}, T^*\mathcal{M})$ is the initial condition.

Recall that F is the unique morphism such that

$$\widehat{\partial}_t \circ F^* = F^* \circ X_H \quad \text{and} \quad F \circ \text{inj}_{t=0} = \text{id}_{T^*\mathcal{M}}.$$

In local coordinates F is uniquely determined by the following system of equations (compare Corollary 3.7):

$$\widehat{\partial}_t (F^*(q_i)) = \sum_j F^*(p_j g^{ji})$$

$$\widehat{\partial}_t (F^*(p_i)) = -\frac{1}{2} \sum_{k,j} (-1)^{|i||k|} F^*(p_k \partial_{q_i}(g^{kj}) p_j)$$

$$(\text{inj}_{t=0})^* \circ F^*(q_i) = q_i$$

$$(\text{inj}_{t=0})^* \circ F^*(p_i) = p_i.$$

We note here and in the sequel the parity $|p_i| = |q_i|$ often by $|i|$.

Applying $(\text{id}_{\mathbb{R}} \times \mathcal{I})^*$ to the two first equations of the above system and \mathcal{I}^* to the remaining equations, we get

$$\partial_t (\Psi^*(q_i)) = \sum_j \Psi^*(p_j g^{ji}) \tag{3.3}$$

$$\partial_t (\Psi^*(p_i)) = -\frac{1}{2} \sum_{k,j} (-1)^{|i||k|} \Psi^*(p_k \partial_{q_i}(g^{kj}) p_j) \tag{3.4}$$

$$(\text{inj}_{t=0})^* \circ \Psi^*(q_i) = \mathcal{I}^*(q_i) \tag{3.5}$$

$$(\text{inj}_{t=0})^* \circ \Psi^*(p_i) = \mathcal{I}^*(p_i). \tag{3.6}$$

Eq. (3.3) gives now

$$\sum_i \partial_t \Psi^*(q_i) \cdot \Psi^*(g_{ij}) = \Psi^*(p_j). \tag{3.7}$$

Derivation along the “even time direction” t yields:

$$\begin{aligned} \partial_t \Psi^*(p_j) &= \sum_i [\partial_t^2 \Psi^*(q_i) \cdot \Psi^*(g_{ij}) + \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(g_{ij})] \\ &= \sum_i \partial_t^2 \Psi^*(q_i) \cdot \Psi^*(g_{ij}) + \sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_k} g_{ij}). \end{aligned}$$

Let us first consider the last summand of the last RHS:

$$\begin{aligned}
& \sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_k} g_{ij}) \\
&= \frac{1}{2} \left[\sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_k} g_{ij}) + \sum_{i,k} \partial_t \Psi^*(q_k) \cdot \partial_t \Psi^*(q_i) \cdot \Psi^*(\partial_{q_i} g_{kj}) \right] \\
&= \frac{1}{2} \sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \left(\Psi^*(\partial_{q_k} g_{ij}) + (-1)^{|i||k|} \Psi^*(\partial_{q_i} g_{kj}) \right).
\end{aligned}$$

After relabeling indices we arrive at

$$\partial_t \Psi^*(p_l) = \sum_j \partial_t^2 \Psi^*(q_j) \cdot \Psi^*(g_{jl}) + \frac{1}{2} \sum_{i,j} \partial_t \Psi^*(q_j) \cdot \partial_t \Psi^*(q_i) \cdot \left(\Psi^*(\partial_{q_i} g_{jl}) + (-1)^{|j||i|} \Psi^*(\partial_{q_j} g_{il}) \right). \quad (3.8)$$

Using Eq. (3.7), Eq. (3.4) yields:

$$\begin{aligned}
\partial_t (\Psi^*(p_i)) &= -\frac{1}{2} \sum_{k,j} (-1)^{|i||k|} \Psi^*(p_k) \cdot \Psi^*(\partial_{q_i} g^{kj}) \cdot \Psi^*(p_j) \\
&= -\frac{1}{2} \sum_{\substack{k,j \\ l,s}} (-1)^{|i||k|} \partial_t \Psi^*(q_l) \cdot \Psi^*(g_{lk}) \cdot \Psi^*(\partial_{q_i} g^{kj}) \cdot \partial_t \Psi^*(q_s) \cdot \Psi^*(g_{sj}).
\end{aligned}$$

Summing the following equality over k

$$\partial_{q_i} (g_{lk} \cdot g^{kj}) = \partial_{q_i} g_{lk} \cdot g^{kj} + (-1)^{|i|(|l|+|k|)} g_{lk} \cdot \partial_{q_i} g^{kj}$$

gives

$$\sum_k (-1)^{|i||l|} \partial_{q_i} g_{lk} \cdot g^{kj} + \sum_k (-1)^{|i||k|} g_{lk} \cdot \partial_{q_i} g^{kj} = 0.$$

Then

$$\begin{aligned}
\partial_t (\Psi^*(p_i)) &= \frac{1}{2} \sum_{\substack{k,j \\ l,s}} (-1)^{|i||l|} \partial_t \Psi^*(q_l) \cdot \Psi^*(\partial_{q_i} g_{lk}) \cdot \Psi^*(g^{kj}) \cdot \partial_t \Psi^*(q_s) \cdot \Psi^*(g_{sj}) \\
&= \frac{1}{2} \sum_{\substack{k,j \\ l,s}} (-1)^{|i||l|+|s|} \partial_t \Psi^*(q_l) \cdot \Psi^*(\partial_{q_i} g_{lk}) \cdot \Psi^*(g^{kj}) \cdot \Psi^*(g_{js}) \cdot \partial_t \Psi^*(q_s) \\
&= \frac{1}{2} \sum_{k,l} (-1)^{|i||l|+|k|} \partial_t \Psi^*(q_l) \cdot \Psi^*(\partial_{q_i} g_{lk}) \cdot \partial_t \Psi^*(q_k) \\
&= \frac{1}{2} \sum_{k,l} (-1)^{|k|(|l|+|l|)+|i||l|} \partial_t \Psi^*(q_l) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_i} g_{lk}).
\end{aligned}$$

After again relabeling indices, we have

$$\begin{aligned}
\partial_t \Psi^*(p_l) &= \frac{1}{2} \sum_{j,i} (-1)^{|j|(|l|+|i|)+|l||i|} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_j) \cdot \Psi^*(\partial_{q_l} g_{ij}) \\
&= \frac{1}{2} \sum_{j,i} (-1)^{|l|(|i|+|j|)} \partial_t \Psi^*(q_j) \cdot \partial_t \Psi^*(q_i) \cdot \Psi^*(\partial_{q_l} g_{ij}).
\end{aligned} \quad (3.9)$$

Multiplying (3.8) with $\Psi^*(g^{lk})$ and summing over l gives:

$$\sum_l \partial_t \Psi^*(p_l) \cdot \Psi^*(g^{lk}) = \partial_t^2 \Psi^*(q_k) + \frac{1}{2} \sum_{i,j,l} \partial_t \Psi^*(q_j) \cdot \partial_t \Psi^*(q_i) \cdot \Psi^*(\partial_{q_i} g_{jl} + (-1)^{|j||i|} \partial_{q_j} g_{il}) \cdot \Psi^*(g^{lk}). \quad (3.10)$$

Applying the same operations to (3.9) yields

$$\sum_l \partial_t \Psi^*(p_l) \cdot \Psi^*(g^{lk}) = \frac{1}{2} \sum_{j,i,l} (-1)^{|l|(|i|+|j|)} \partial_t \Psi^*(q_j) \cdot \partial_t \Psi^*(q_i) \cdot \Psi^*(\partial_{q_l} g_{ij}) \cdot \Psi^*(g^{lk}). \quad (3.11)$$

Now, using the equality $\Phi^*(q_i) = \Psi^* \circ (\pi^{T^* \mathcal{M}})^*(q_i) = \Psi^*(q_i)$ and equating the RHS of the equalities (3.10) and (3.11), we get exactly (2.2) since the Christoffel symbols are given by (2.1).

Let us check that $\partial_t \Phi \circ \text{inj}_{t=0} = g^\flat \circ \mathcal{I}$.

Recall the canonical sections s_0, s_1 defined before Theorem 2.21. Since $\Phi = \pi^{T^*\mathcal{M}} \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$, we can, in fact, show that $\partial_t \Phi = g^\flat \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$.

Firstly, we have

$$\begin{aligned} \partial_t \Phi &= T\Phi \circ (s_1 \times s_0) \\ &= T\pi^{T^*\mathcal{M}} \circ TF \circ (\text{id}_{T\mathbb{R}} \times T\mathcal{I}) \circ (s_1 \times s_0) \\ &= T\pi^{T^*\mathcal{M}} \circ TF \circ (s_1 \times (T\mathcal{I} \circ s_0)) : \mathbb{R} \times \mathcal{S} \rightarrow T\mathcal{M}. \end{aligned}$$

If (q_i, p_i, v_i, r_i) are the usual local coordinates on $T(T^*\mathcal{M})$ and (q_i, v_i) local coordinates on $T\mathcal{M}$, identify with the image of the tangent map associated to the zero section of $T^*\mathcal{M} \rightarrow \mathcal{M}$. Let further denote (v_t, t) the point $v_t \frac{d}{dt}|_t$ of $T\mathbb{R}$. We then have

$$\begin{aligned} (\partial_t \Phi)^*(q_i) &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \circ (TF)^* \circ (T\pi^{T^*\mathcal{M}})^*(q_i) \\ &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \circ (TF)^*(q_i) \\ &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) (F^*(q_i)) \\ &= (\text{id}_{\mathbb{R}}^* \times \mathcal{I}^*) (F^*(q_i)) \\ &= (\text{id}_{\mathbb{R}}^* \times \mathcal{I}^*) \circ F^* \circ (g^\flat)^*(q_i) \\ &= (g^\flat \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I}))^*(q_i), \end{aligned}$$

and

$$\begin{aligned} (\partial_t \Phi)^*(v_i) &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \circ (TF)^* \circ (T\pi^{T^*\mathcal{M}})^*(v_i) \\ &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \circ (TF)^*(v_i) \\ &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \left(\sum_j (v_j \cdot \partial_{q_j} F^*(q_i) + r_j \cdot \partial_{p_j} F^*(q_i)) + v_t \cdot \partial_t F^*(q_i) \right) \\ &= (\text{id}_{\mathbb{R}}^* \times \mathcal{I}^*) (\partial_t F^*(q_i)) \\ &= (\text{id}_{\mathbb{R}}^* \times \mathcal{I}^*) ((F^* \circ X_H)(q_i)), \quad \text{since } F \text{ is the flow of } X_H, \\ &= (\text{id}_{\mathbb{R}}^* \times \mathcal{I}^*) \left(F^* \left(\sum_j p_j g^{ji} \right) \right) \\ &= (\text{id}_{\mathbb{R}}^* \times \mathcal{I}^*) (F^* \circ (g^\flat)^*(v_i)) \\ &= (g^\flat \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I}))^*(v_i). \end{aligned}$$

Since we now have $\partial_t \Phi = g^\flat \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$, we can easily conclude the proof of the theorem:

$$\partial_t \Phi \circ \text{inj}_{t=0} = g^\flat \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I}) \circ \text{inj}_{t=0} = g^\flat \circ \mathcal{I}. \quad \square$$

Reciprocally, we have the following:

Theorem 3.14. Let $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ be a supergeodesic with initial condition $\partial_t \Phi \circ \text{inj}_{t=0} = \mathcal{J}$ in $\text{Mor}(\mathcal{S}, T\mathcal{M})$. Then $g^\sharp \circ \partial_t \Phi : \mathbb{R} \times \mathcal{S} \rightarrow T^*\mathcal{M}$ is an integral curve of X_H with initial condition $g^\sharp \circ \mathcal{J} \in \text{Mor}(\mathcal{S}, T^*\mathcal{M})$, where $g^\sharp : T\mathcal{M} \xrightarrow{\cong} T^*\mathcal{M}$ is the isomorphism induced by the metric g .

Proof. Let $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ be a supercurve such that $\frac{\nabla}{dt}(T\Phi^*(\partial_t)) = 0$ and $\partial_t \Phi \circ \text{inj}_{t=0} = \mathcal{J}$ in $\text{Mor}(\mathcal{S}, T\mathcal{M})$. We define $\mathcal{I} := g^\sharp \circ \mathcal{J}$ and $\Psi := F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$. By the previous theorem, we know that $\pi^{T^*\mathcal{M}} \circ \Psi$ is a geodesic of \mathcal{M} with initial condition $g^\flat \circ \mathcal{I} = \mathcal{J}$. By unicity, we have $\Phi = \pi^{T^*\mathcal{M}} \circ \Psi$. Now, using a calculation made in the proof of the preceding theorem, we have $\partial_t \Phi = g^\flat \circ F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$. Then $g^\sharp \circ \partial_t \Phi = F \circ (\text{id}_{\mathbb{R}} \times \mathcal{I})$, i.e., $g^\sharp \circ \partial_t \Phi$ is the integral curve of X_H with initial condition $\mathcal{I} = g^\sharp \circ \mathcal{J}$. \square

We briefly comment on the case of odd Riemannian metrics.

Definition 3.15. An odd graded Riemannian metric on a supermanifold \mathcal{M} is a graded symmetric odd non-degenerate $\mathcal{O}_{\mathcal{M}}$ -linear morphism of sheaves

$$g : \mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}},$$

where non-degeneracy means that the mapping $X \mapsto g(X, \cdot)$ is an (odd) isomorphism from $\mathcal{T}_{\mathcal{M}}$ to $\Omega_{\mathcal{M}}^1$.

Such a metric will simply be called an “odd Riemannian metric” in the sequel. It follows from the non-degeneracy that the even and the odd dimension of a supermanifold equipped with an odd Riemannian metric must be the same. Since g is odd the condition of metricity for a connection on \mathcal{M} reads as follows

$$(-1)^{|X|}X(g(Y \otimes Z)) = g(\nabla_X Y \otimes Z) + (-1)^{|X||Y|}g(Y \otimes \nabla_X Z) \quad \forall X, Y, Z \in \mathcal{T}_{\mathcal{M}}(U).$$

Using the following six-term-formula

$$\begin{aligned} 2g(\nabla_X Y \otimes Z) &= (-1)^{|X|}Xg(Y \otimes Z) - (-1)^{|Z|(|X|+|Y|)+|Z|}Zg(X \otimes Y) + (-1)^{|X|(|Y|+|Z|)+|Y|}Yg(Z \otimes X) \\ &\quad + g([X, Y] \otimes Z) - (-1)^{|X|(|Y|+|Z|)}g([Y, Z] \otimes X) + (-1)^{|Z|(|X|+|Y|)}g([Z, X] \otimes Y) \end{aligned}$$

one obtains

Proposition 3.16. *Let \mathcal{M} be a supermanifold with an odd Riemannian metric g . Then there exists a unique linear connection ∇ on \mathcal{M} that is metric and torsion free. This connection is automatically even as a map $\mathcal{T}_{\mathcal{M}} \rightarrow \Omega^1_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}$.*

Remark. We call the connection ∇ of the preceding proposition the “Levi-Civita connection of (\mathcal{M}, g) ”.

As in the even case we have the following definition (compare Definition 2.17):

Definition 3.17. An (odd) Riemannian supergeodesic on a supermanifold \mathcal{M} with an odd Riemannian metric g is a supercurve $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ such that

$$\frac{\nabla}{dt}(T\Phi^*(\partial_t)) = 0, \quad (3.12)$$

where \mathcal{S} is an arbitrary supermanifold and t is the natural coordinate of \mathbb{R} .

In order to have a local expression for the differential equation defining a supergeodesic we set as usual $\nabla_{\partial_i} \partial_j = \sum \Gamma_{ij}^k \partial_k$ (compare the text after Definition 2.10). Using the above six-term-formula for odd metrics a tedious but straight forward calculation yields:

$$(-1)^{|i|+|j|+|k|} \Gamma_{ij}^k = \frac{1}{2} \sum_l [(-1)^{|i|} \partial_i g_{jl} + (-1)^{|i||j|+|j|} \partial_j g_{il} - (-1)^{|l|(1+|i|+|j|)} \partial_l g_{ij}] g^{lk},$$

where $g_{ij} := g(\partial_i, \partial_j) \in \mathcal{O}_{\mathcal{M}}(U)$ and $g^{lk} \in \mathcal{O}_{\mathcal{M}}(U)$ are defined by $\sum_k g^{ik} g_{kj} = \delta_{ij}$.

As in the case of even metrics one has an existence and uniqueness result for geodesics.

Proposition 3.18. *Let \mathcal{M} be a supermanifold with an odd Riemannian metric g , and let $\mathcal{S} = (S, \mathcal{O}_{\mathcal{S}})$ be an arbitrary supermanifold and $\mathcal{J} \in \text{Mor}(\mathcal{S}, T\mathcal{M})$. Then there exists a unique supergeodesic $\Phi : \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{M}$ such that the initial condition $\mathcal{J} = \partial_t \Phi \circ \text{inj}_{t=0}$ is fulfilled.*

Remark. Of course, the supergeodesic is in general only defined on a (maximal) open subsupermanifold of $\mathbb{R} \times \mathcal{S}$ containing $\{0\} \times S$ in its body. Abusively, we will often not explicitly mention this maximal domain of definition.

In order to construct the corresponding “geodesic flow” one first has to observe that an odd metric canonically induces an isomorphism of super vector bundles $g^{\natural} : T\mathcal{M} \rightarrow \Pi T^*\mathcal{M}$, where $\Pi T^*\mathcal{M}$ is the super vector bundle associated to the $\mathcal{O}_{\mathcal{M}}$ -module sheaf $\Pi\Omega^1_{\mathcal{M}}$. We observe furthermore that the supermanifold $\Pi T^*\mathcal{M}$ has a canonical odd one-form $\alpha^{\Pi T^*\mathcal{M}}$ whose exterior derivative $\omega^{\Pi T^*\mathcal{M}} := -d(\alpha^{\Pi T^*\mathcal{M}})$ is an odd symplectic form on $\Pi T^*\mathcal{M}$. The construction of $\alpha^{\Pi T^*\mathcal{M}}$ closely parallels the construction of the one-form $\alpha = \alpha^{T^*\mathcal{M}}$ (compare the proof of Proposition 3.4). If (q_1, \dots, q_{m+n}) are local coordinates on \mathcal{M} and (p_1, \dots, p_{m+n}) are the associated fiber coordinates on $T^*\mathcal{M}$ with $|q_i| = |p_i| = 0$ resp. 1 for $1 \leq i \leq m$ resp. $m < i \leq m+n$, then $\Pi T^*\mathcal{M}$ has local coordinates (q_i, p_i^{Π}) , where p_i^{Π} is odd resp. even for $1 \leq i \leq m$ resp. $m < i \leq m+n$. In these coordinates we have

$$\omega^{\Pi T^*\mathcal{M}} = \sum_{i=1}^{n+m} (-1)^{|i|} d(q_i) \wedge d(p_i^{\Pi}).$$

We define the energy function H as in the even case (compare the text after Corollary 3.2) but it should be immediately stressed that H is an odd global superfunction on $\Pi T^*\mathcal{M}$. In the above local coordinates one gets

$$H = -\frac{1}{2} \sum_{i,j} p_i^{\Pi} g^{ij} p_j^{\Pi}.$$

Observe that the reduction \tilde{H} is the zero function on the body of $\Pi T^*\mathcal{M}$. Since $\omega^{\Pi T^*\mathcal{M}}$ and H both are odd, the Hamiltonian vector field X_H on $\Pi T^*\mathcal{M}$ fulfilling $d(H) = \omega^{\Pi T^*\mathcal{M}}(X_H, \cdot)$ is even. Using Theorem 3.9 and Proposition 3.12 we immediately get following

Proposition 3.19. *Let (\mathcal{M}, g) be an odd Riemannian supermanifold and X_H the Hamiltonian vector field associated to the energy function H on the (odd) symplectic supermanifold $(\Pi T^*\mathcal{M}, \omega^{\Pi T^*\mathcal{M}})$. Then*

- (i) *there exists a unique flow for X_H , $F : \mathbb{R} \times \Pi T^*\mathcal{M} \rightarrow \Pi T^*\mathcal{M}$,*
- (ii) *given a supermanifold \mathcal{S} and an element $\mathcal{J} \in \text{Mor}(\mathcal{S}, \Pi T^*\mathcal{M})$, there is a unique integral curve $\Psi : \mathbb{R} \times \mathcal{S} \rightarrow \Pi T^*\mathcal{M}$ of X_H such that the initial condition $\Psi \circ \text{inj}_{t=0} = \mathcal{J}$ is satisfied.*

As in the case of even metrics we finally obtain a bijection between integral curves of the geodesic flow and geodesics. Since the proof is very similar to the proof of [Theorem 3.13](#) we only announce the result without giving the proof.

Theorem 3.20. *Let (\mathcal{M}, g) be an odd Riemannian supermanifold and X_H the Hamiltonian vector field associated to the energy function H on $\Pi T^*\mathcal{M}$. Let furthermore \mathcal{S} be an arbitrary supermanifold. Then*

- (i) *given $\mathfrak{I} \in \text{Mor}(\mathcal{S}, \Pi T^*\mathcal{M})$, the integral curve Ψ of X_H with initial condition \mathfrak{I} projects to a geodesic $\Phi := \pi^{\Pi T^*\mathcal{M}} \circ \Psi$ with initial condition $g^\flat \circ \mathfrak{I}$, and*
- (ii) *given $\mathfrak{J} \in \text{Mor}(\mathcal{S}, T\mathcal{M})$, the supergeodesic Φ with initial condition \mathfrak{J} yields an integral curve $\Psi := g^\sharp \circ \partial_t \Phi$ of X_H with initial condition $g^\sharp \circ \mathfrak{J}$.*

4. The exponential map of a Riemannian supermanifold

In this section we study, for q in the body of a supermanifold \mathcal{M} , the fiber $T_q\mathcal{M}$ of the tangent bundle over q , the restriction of the tangent map of a morphism to this fiber, and its relation to the “numerical tangent map” ([Lemma 4.2](#)). The main result of this section is that a Riemannian supermanifold possesses a (Riemannian) exponential map and that $\exp_q : T_q\mathcal{M} \rightarrow \mathcal{M}$ is a diffeomorphism near 0 ([Theorem 4.4](#)). As an example of an application we show that linearization in a fixed point of an isometry of a Riemannian supermanifold is faithful ([Corollary 4.7](#)).

Let (\mathcal{M}, g) be an (even) Riemannian supermanifold, and X_H be the even Hamiltonian vector field on $T^*\mathcal{M}$ associated to the energy function H . Let furthermore $F : \mathbb{R} \times T^*\mathcal{M} \rightarrow T^*\mathcal{M}$ be the flow of X_H and $g^\sharp : T\mathcal{M} \rightarrow T^*\mathcal{M}$ be the isomorphism of supermanifolds induced by the metric g .

Consider the following composition of applications:

$$\mathbb{R} \times T\mathcal{M} \xrightarrow{\text{id}_{\mathbb{R}} \times g^\sharp} \mathbb{R} \times T^*\mathcal{M} \xrightarrow{F} T^*\mathcal{M} \xrightarrow{\pi^{T^*\mathcal{M}}} \mathcal{M}.$$

We want to take the fiber of $T\mathcal{M}$ above $q \in M$:

$$\begin{array}{ccc} T\mathcal{M} & & \\ \downarrow \pi^{T\mathcal{M}} & & \\ \{q\} & \xrightarrow{q} & \mathcal{M} \end{array}$$

Let us recall the morphism of rings of global sections $(\pi^{T\mathcal{M}})^*(TM) : \mathcal{O}_{\mathcal{M}}(M) \rightarrow \mathcal{O}_{T\mathcal{M}}(TM)$, associated to the sheaf morphism $(\pi^{T\mathcal{M}})^* : (\pi^{T\mathcal{M}})^{-1}(\mathcal{O}_{\mathcal{M}}) \rightarrow \mathcal{O}_{T\mathcal{M}}$ over TM .

Let \mathcal{I}_q be the ideal of $\mathcal{O}_{\mathcal{M}}(M)$ defined by

$$\mathcal{I}_q := \{f \in \mathcal{O}_{\mathcal{M}}(M) \mid q^*(f) = 0\},$$

where $q^*(f) = \tilde{f}(q)$. We denote the ideal in $\mathcal{O}_{T\mathcal{M}}(TM)$ generated by $(\pi^{T\mathcal{M}})^*(\mathcal{I}_q)$ by

$$\mathcal{I}_q^{\pi^{T\mathcal{M}}} := \langle (\pi^{T\mathcal{M}})^*(\mathcal{I}_q) \rangle_{\mathcal{O}_{T\mathcal{M}}(TM)}.$$

Then (compare [8] for this construction in general), there exists a subsupermanifold $\mathcal{S} \xrightarrow{i_q} T\mathcal{M}$ of $T\mathcal{M}$ such that

$$\mathcal{I}_{\mathcal{S}} := \{f \in \mathcal{O}_{T\mathcal{M}}(TM) \mid i_q^*(f) = 0\} = \mathcal{I}_q^{\pi^{T\mathcal{M}}}.$$

The supermanifold \mathcal{S} will be denoted by $T_q\mathcal{M}$.

Note that if $\mathcal{M} = \mathbb{R}^{m|n}$ and q is an element of \mathbb{R}^m , we have $T\mathcal{M} = \mathbb{R}^{m|n} \times \mathbb{R}^{m|n}$ and $T_q\mathcal{M} = \mathbb{R}^{m|n} \times \{q\} \cong \mathbb{R}^{m|n}$. (Compare the Remark after [Theorem 2.19](#).) In general, given a finite dimensional \mathbb{Z}_2 -graded \mathbb{R} -vector space $V = V_0 \oplus V_1$, we have of course an associated “linear supermanifold”. Then $T_q\mathcal{M}$ is the supermanifold associated to $T_q^{\text{num}}\mathcal{M}$.

We now define the map $\exp_q : T_q\mathcal{M} \rightarrow \mathcal{M}$ via the following diagram:

$$\begin{array}{ccccccc} \mathbb{R} \times T\mathcal{M} & \xrightarrow{\text{id}_{\mathbb{R}} \times g^\sharp} & \mathbb{R} \times T^*\mathcal{M} & \xrightarrow{F} & T^*\mathcal{M} & \xrightarrow{\pi^{T^*\mathcal{M}}} & \mathcal{M} \\ \uparrow \text{id}_{\mathbb{R}} \times i_q & & & & & \nearrow \exp_q & \\ \mathbb{R} \times T_q\mathcal{M} & & & & & & \\ \uparrow 1 \times \text{id}_{T_q\mathcal{M}} & & & & & & \\ \{1\} \times T_q\mathcal{M} \cong T_q\mathcal{M} & & & & & & \end{array}$$

Recall that the flow F is only defined on an open subsupermanifold of $\mathbb{R} \times T^*\mathcal{M}$ containing $\{0\} \times T^*\mathcal{M}$. Then \exp_q is in general only defined on an open subsupermanifold of $T_q\mathcal{M}$ containing in its body $(0, q)$. We can now show that \exp_q is a locally diffeomorphism near $0 \in T_q\mathcal{M}$.

Lemma 4.1. *Let $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of supermanifolds. Let $q \in M$, then we have a unique morphism of supermanifolds $T_q\Phi : T_q\mathcal{M} \rightarrow T_{\tilde{\Phi}(q)}\mathcal{N}$ such that $i_{\tilde{\Phi}(q)} \circ T_q\Phi = T\Phi \circ i_q$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccc} T_q\mathcal{M} & \xrightarrow{i_q} & T\mathcal{M} & \xrightarrow{T\Phi} & T\mathcal{N} & \xleftarrow{i_p} & T_p\mathcal{N} \\ \downarrow & & \downarrow \pi^{T\mathcal{M}} & & \downarrow \pi^{T\mathcal{N}} & & \downarrow \\ \{q\} & \longrightarrow & \mathcal{M} & \xrightarrow{\Phi} & \mathcal{N} & \longleftarrow & \{p\}, \end{array}$$

where $p := \tilde{\Phi}(q) \in N$. We will show that the morphism $T\Phi \circ i_q$ factorizes over the morphism i_p .

Recall that a morphism of real smooth supermanifolds is completely determined by the superalgebra morphism given by the pulling-back global sections.

Let us define the morphism $T_q\Phi : T_q\mathcal{M} \rightarrow T_p\mathcal{N}$ by setting its body map equal to $\tilde{T}_q\Phi := T_q\tilde{\Phi}$ and on the level of global sections by

$$\begin{aligned} (T_q\Phi)^* : \mathcal{O}_{T_p\mathcal{N}}(T_p\mathcal{N}) &\longrightarrow \mathcal{O}_{T_q\mathcal{M}}(T_q\mathcal{M}) \\ f &\longmapsto i_q^* \circ (T\Phi)^*(\hat{f}), \end{aligned}$$

where \hat{f} is an element of $\mathcal{O}_{T\mathcal{N}}(TN)$ such that $i_p^*(\hat{f}) = f$.

For all $\hat{f} \in \mathcal{O}_{T\mathcal{N}}(TN)$ such that $i_p^*(\hat{f}) = 0$, we have $\hat{f} \in \mathcal{J}_{T_p\mathcal{N}} = \mathcal{J}_p^{\pi^{T\mathcal{N}}}$ by the very construction of $T_p\mathcal{N}$ and then by definition of $\mathcal{J}_p^{\pi^{T\mathcal{N}}}$:

$$\hat{f} \in \langle (\pi^{T\mathcal{N}})^*(\mathcal{J}_p) \rangle_{\mathcal{O}_{T\mathcal{N}}(TN)}.$$

Thus it is enough to show that $i_q \circ (T\Phi)^*(\hat{f}) = 0$ whenever $\hat{f} = (\pi^{T\mathcal{N}})^*(g)$ with $g \in \mathcal{J}_p$ in order to assure that $(T_q\Phi)^*$ is well-defined.

For $g \in \mathcal{J}_p \triangleleft \mathcal{O}_{\mathcal{N}}(N)$ we have

$$(T\Phi)^*((\pi^{T\mathcal{N}})^*(g)) = (\pi^{T\mathcal{M}})^*(\Phi^*(g)) \in \langle (\pi^{T\mathcal{M}})^*(\mathcal{J}_q) \rangle_{\mathcal{O}_{T\mathcal{M}}(TM)},$$

since the identity $\tilde{\Phi}^*(h)(q) = \tilde{h}(p)$ for $h \in \mathcal{O}_{\mathcal{N}}(N)$ shows that $g \in \mathcal{J}_p$ implies that $\Phi^*(g) \in \mathcal{J}_q$.

Moreover, since $\mathcal{J}_{T_q\mathcal{M}} = \langle (\pi^{T\mathcal{M}})^*(\mathcal{J}_q) \rangle_{\mathcal{O}_{T\mathcal{M}}(TM)}$ we have $i_q^*((\pi^{T\mathcal{M}})^*(\Phi^*(g))) = 0$, i.e. $i_q^* \circ (T\Phi)^*(\hat{f}) = 0$. \square

Let us recall that for a morphism of supermanifolds $\Phi : \mathcal{M} \rightarrow \mathcal{N}$, and a point $q \in M$, we have a numerical analogue of the tangent map, i.e. a linear map $T_q^{\text{num}}\Phi : T_q^{\text{num}}\mathcal{M} \rightarrow T_{\tilde{\Phi}(q)}^{\text{num}}\mathcal{N}$ defined by

$$T_q^{\text{num}}\Phi(X)([f]_{\tilde{\Phi}(q)}) := X([\Phi^*(f)]_q), \quad \forall X \in T_q^{\text{num}}\mathcal{M}, \quad \forall [f]_{\tilde{\Phi}(q)} \in \mathcal{O}_{\mathcal{N}, \tilde{\Phi}(q)}.$$

It turns out that this map encodes exactly the same information as the morphism $T_q\Phi$.

Lemma 4.2. *Let $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of supermanifolds and $q \in M$. Then the following holds:*

- (i) $T_q\Phi : T_q\mathcal{M} \rightarrow T_{\tilde{\Phi}(q)}\mathcal{N}$ is the morphism of (linear) supermanifolds associated to the even \mathbb{R} -linear map $T_q^{\text{num}}\Phi : T_q^{\text{num}}\mathcal{M} \rightarrow T_{\tilde{\Phi}(q)}^{\text{num}}\mathcal{N}$, and
- (ii) the numerical tangent map $T_q^{\text{num}}\Phi$ is uniquely determined by the morphism of supermanifolds $T_q\Phi$.

Proof. A calculation in the local coordinates on $T\mathcal{M}$ and $T\mathcal{N}$, that are induced from local coordinates on \mathcal{M} and \mathcal{N} , directly shows that both maps $T_q\Phi$ and $T_q^{\text{num}}\Phi$ are uniquely determined by the evaluation of the super Jacobi matrix of Φ in the point q of the body of \mathcal{M} . \square

It is well-known that a morphism $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ is a local diffeomorphism near q if and only if $T_q^{\text{num}}\Phi$ is an isomorphism of \mathbb{Z}_2 -graded vector space (see, e.g., [15, Theorem 4.4.1]).

Using the preceding lemma, we thus immediately get:

Proposition 4.3. *The morphism $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ is locally a diffeomorphism near $q \in M$ if and only if $T_q\Phi : T_q\mathcal{M} \rightarrow T_{\tilde{\Phi}(q)}\mathcal{N}$ is an isomorphism of supermanifolds.*

Theorem 4.4. *The map $\exp_q : T_q\mathcal{M} \rightarrow \mathcal{M}$ is a local diffeomorphism near 0.*

Proof. We will show that the map $T_0(\exp_q) : T_0(T_q\mathcal{M}) \rightarrow T_q\mathcal{M}$ is an isomorphism. More precisely, after identifying $T_0(T_q\mathcal{M})$ with $T_q\mathcal{M}$ as in the classical case, we will see that $T_0(\exp_q) = \text{id}_{T_q\mathcal{M}}$.

Let us focus on the map:

$$\begin{aligned} T_0^{\text{num}}(\exp_q) : T_0^{\text{num}}(T_q\mathcal{M}) = \text{Der}((\mathcal{O}_{T_q\mathcal{M},0}), \mathbb{R}) &\longrightarrow \text{Der}((\mathcal{O}_{\mathcal{M},q}), \mathbb{R}) = T_q^{\text{num}}\mathcal{M} \\ X &\longmapsto X \circ \exp_q^*. \end{aligned}$$

If (q_i) is a system of local coordinates on \mathcal{M} around q , we denote (q_i, v_i) the induced local coordinates on $T\mathcal{M}$. Let us denote the natural coordinates on $T_q\mathcal{M}$ by (v_i) as well. Then $T_0^{\text{num}}(T_q\mathcal{M})$ is the \mathbb{Z}_2 -graded \mathbb{R} -vector space spanned by (∂_{v_i}) where $|\partial_{v_i}| = |v_i|$.

The classical theory of exponential maps on ungraded Riemannian manifolds gives the following equality

$$T_0^{\text{num}}(\exp_q)(\partial_{v_i}) = \partial_{q_i} \quad \forall i \text{ such that } |\partial_{v_i}| = 0.$$

Let us now deal with the odd case. If $X = \partial_{v_i}$ with $|v_i| = 1$, the “integral curve” $\gamma : \mathbb{R}^{0|1} \rightarrow T_q\mathcal{M}$ of X is defined by $\gamma^*(v_j) = \delta_{i,j} \cdot \theta$, where θ is the canonical coordinate of $\mathbb{R}^{0|1}$.

Since

$$X(f) = \frac{d}{d\theta} \gamma^*(f) \quad \forall f \in \mathcal{O}_{T_q\mathcal{M}},$$

we have

$$T_0^{\text{num}}(\exp_q)(X) = \frac{d}{d\theta} \gamma^* \circ \exp_q^*,$$

where $\gamma^* \circ \exp_q^* = \gamma^* \circ (\text{inj}_{t=1}^* \times (i_q^* \circ g^{\sharp*})) \circ F^* \circ (\pi^{T^*\mathcal{M}})^* = (\text{inj}_{t=1}^* \times (\gamma^* \circ i_q^* \circ g^{\sharp*})) \circ F^* \circ (\pi^{T^*\mathcal{M}})^*$.

Let us consider the map $\Gamma : \mathbb{R}^{1|1} \rightarrow \mathcal{M}$ defined by:

$$\Gamma^* := (\text{id}_{\mathbb{R}} \times (\gamma^* \circ i_q^* \circ g^{\sharp*})) \circ F^* \circ (\pi^{T^*\mathcal{M}})^*.$$

Then, using [Theorem 3.13](#), Γ is the unique geodesic of \mathcal{M} with initial condition $\eta = i_q \circ \gamma \in \text{Mor}(\mathbb{R}^{0|1}, T\mathcal{M})$, i.e. Γ is the supercurve such that

$$\begin{cases} \partial_t^2 \Gamma^*(q_k) + \sum_{|u|=|l|=0} \partial_t \Gamma^*(q_u) \partial_t \Gamma^*(q_l) \Gamma^*(\Gamma_{lu}^k) = 0 & \text{if } |q_k| = 0, \\ \partial_t^2 \Gamma^*(q_k) + \sum_{\substack{|u|=0 \\ |l|=0}} \partial_t \Gamma^*(q_u) \partial_t \Gamma^*(q_l) \Gamma^*(\Gamma_{lu}^k) + 2 \sum_{\substack{|u|=0 \\ |l|=1}} \partial_t \Gamma^*(q_u) \partial_t \Gamma^*(q_l) \Gamma^*(\Gamma_{lu}^k) = 0 & \text{if } |q_k| = 1, \end{cases}$$

satisfying the initial condition

$$\begin{cases} \Gamma^*(q_k)|_{t=0} = \tilde{q}_k(q) & \text{if } |q_k| = 0, \\ \Gamma^*(q_k)|_{t=0} = 0 & \text{if } |q_k| = 1, \end{cases} \quad \begin{cases} \partial_t \Gamma^*(q_k)|_{t=0} = \eta^*(v_k) = 0 & \text{if } |q_k| = 0, \\ \partial_t \Gamma^*(q_k)|_{t=0} = \eta^*(v_k) = \delta_{k,i} \cdot \theta & \text{if } |q_k| = 1. \end{cases}$$

Then the unique solution $\Gamma : \mathbb{R}^{1|1} \rightarrow \mathcal{M}$ of this system is

$$\begin{cases} \Gamma^*(q_k) = \tilde{q}_k(q) & \text{if } |q_k| = 0 \\ \Gamma^*(q_k) = t \cdot \theta \cdot \delta_{k,i} & \text{if } |q_k| = 1. \end{cases}$$

Now, we have

$$T_0^{\text{num}}(\exp_q)(X) = \frac{d}{d\theta} \text{inj}_{t=1}^* \circ \Gamma^* \in \text{Der}(\mathcal{O}_{\mathcal{M},q}, \mathbb{R}),$$

and then

$$T_0^{\text{num}}(\exp_q)(X)(q_k) = \frac{d}{d\theta} \Gamma^*(q_k)|_{t=1} = \begin{cases} 0 & \text{if } |q_k| = 0 \\ \delta_{k,i} & \text{if } |q_k| = 1, \end{cases}$$

that is

$$T_0^{\text{num}}(\exp_q)(X) = \partial_{q_i}.$$

This finishes the proof that $T_0(\exp_q)$ is the identity map. Therefore \exp_q is a local diffeomorphism near 0. \square

Remark. Already calculations with explicit Riemannian metrics on $\mathcal{M} = \mathbb{R}^{1|2}$ show that the most direct extension of the ungraded Gauss lemma cannot hold true, i.e., given a Riemannian supermanifold \mathcal{M} , then, in general, for $q \in M$, $v = v_0 + v_1 \in T_q^{\text{num}} \mathcal{M}$, and $w \in T_{v_0}^{\text{num}}(T_q \mathcal{M}) \cong T_q^{\text{num}} \mathcal{M}$,

$$g_{\widetilde{\exp}_q(v_0)}((T_{v_0}^{\text{num}} \exp_q)(v), (T_{v_0}^{\text{num}} \exp_q)(w)) \neq g_q(v, w).$$

We can now apply the properties of the above constructed super exponential map to generalize a fundamental linearization result on Riemannian isometries to the category of supermanifolds.

Let $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{N}, g^{\mathcal{N}})$ be Riemannian supermanifolds and $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ a morphism.

Recall that $g^{\mathcal{N}}$ can be viewed as an element of $\Omega_{\mathcal{N}}^1 \otimes_{\mathcal{O}_{\mathcal{N}}} \Omega_{\mathcal{N}}^1$, moreover $\Phi^* : \widetilde{\Phi}^{-1} \mathcal{O}_{\mathcal{N}} \rightarrow \mathcal{O}_{\mathcal{M}}$ induces a morphism of sheaves

$$\begin{array}{ccc} \widetilde{\Phi}^{-1} \Omega_{\mathcal{N}}^1 & \xrightarrow{\Phi^*} & \Omega_{\mathcal{M}}^1 \\ & \searrow & \swarrow \\ & M & \end{array}$$

Thus the expression $\Phi^*(\widetilde{\Phi}^{-1} g^{\mathcal{N}})$ makes sense as an element of $\Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \Omega_{\mathcal{M}}^1$.

Definition 4.5. Let $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{N}, g^{\mathcal{N}})$ be Riemannian supermanifolds. An *isometry* is an isomorphism of supermanifolds $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ such that the following equality is satisfied

$$\Phi^*(\widetilde{\Phi}^{-1} g^{\mathcal{N}}) = g^{\mathcal{M}}.$$

If $(q_i^{\mathcal{M}})$ and $(q_j^{\mathcal{N}})$ are local coordinates on \mathcal{M} and \mathcal{N} respectively, the above condition means that

$$g_{ij}^{\mathcal{M}} = \sum_{k,l} (-1)^{|q_k^{\mathcal{N}}|(|q_j^{\mathcal{M}}| + |q_l^{\mathcal{N}}|)} \partial_i^{\mathcal{M}}(\Phi^*(q_k^{\mathcal{N}})) \cdot \partial_j^{\mathcal{M}}(\Phi^*(q_l^{\mathcal{N}})) \cdot \Phi^*(g_{kl}^{\mathcal{N}}),$$

where $\partial_i^{\mathcal{M}}$ and $\partial_j^{\mathcal{N}}$ denote $\partial_{q_i^{\mathcal{M}}}$ and $\partial_{q_j^{\mathcal{N}}}$ respectively.

Theorem 4.6. Let $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ be an isometry and $q \in M$, then we have

$$\Phi \circ \exp_q^{\mathcal{M}} = \exp_{\widetilde{\Phi}(q)}^{\mathcal{N}} \circ T_q \Phi.$$

Proof. Let us define the following supercurves

$$\Lambda := \pi^{T^* \mathcal{M}} \circ F^{\mathcal{M}} \circ (\text{id}_{\mathbb{R}} \times (g^{\mathcal{M}^\sharp} \circ i_q)) : \mathbb{R} \times T_q \mathcal{M} \rightarrow \mathcal{M}$$

and

$$\Gamma := \Phi \circ \Lambda : \mathbb{R} \times T_q \mathcal{M} \rightarrow \mathcal{N}, \quad (4.1)$$

where $F^{\mathcal{M}}$ is the geodesic flow of the Hamiltonian dynamical system $(T^* \mathcal{M}, \omega, H)$.

We want to show that Γ is a geodesic of \mathcal{N} with initial condition $\eta = i_{\widetilde{\Phi}(q)} \circ T_q \Phi \in \text{Mor}(T_q \mathcal{M}, T_{\mathcal{N}})$. By the unicity of geodesics, it will then follow

$$\Gamma = \pi^{T^* \mathcal{N}} \circ F^{\mathcal{N}} \circ (\text{id}_{\mathbb{R}} \times (g^{\mathcal{N}^\sharp} \circ i_{\widetilde{\Phi}(q)} \circ T_q \Phi)). \quad (4.2)$$

The statement of the theorem will then be obtained by evaluating both expressions (4.1) and (4.2) for Γ at $t = 1$.

Since Φ is an isometry and the Levi-Civita connection is unique

$$(\Phi^* \nabla^{\mathcal{N}})(T \Phi^*) = T \Phi^*(\nabla^{\mathcal{M}})$$

and thus

$$\frac{\nabla^{\mathcal{N}}}{dt} (T \Gamma^*(\partial_t)) = T \Phi^* \left(\frac{\nabla^{\mathcal{M}}}{dt} (T \Lambda^*(\partial_t)) \right) = 0$$

because Λ is a geodesic of \mathcal{M} . It follows that Γ is a geodesic.

Furthermore

$$\begin{aligned} \partial_t \Gamma \circ \text{inj}_{t=0} &= \partial_t (\Phi \circ \Lambda) \circ \text{inj}_{t=0} \\ &= T \Phi \circ \partial_t \Lambda \circ \text{inj}_{t=0} \\ &= T \Phi \circ i_q \\ &= i_{\widetilde{\Phi}(q)} \circ T_q \Phi = \eta. \end{aligned}$$

We thus have shown that (4.2) holds and the claim of the theorem follows by evaluating both sides in $t = 1$. \square

Now, we can give the announced linearization result.

Corollary 4.7. Let \mathcal{M} be a connected Riemannian supermanifold and let $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ be an isometry such that there exists a point $q_0 \in M$ with

$$\tilde{\Phi}(q_0) = q_0 \quad \text{and} \quad T_{q_0}\Phi = \text{id}_{T_{q_0}\mathcal{M}}.$$

Then Φ is the identity morphism of \mathcal{M} .

Proof. Let U be the subset of M defined by

$$U := \{q \in M \mid \tilde{\Phi}(q) = q \text{ and } T_q\Phi = \text{id}_{T_q\mathcal{M}}\}.$$

Remark that if $\tilde{\Phi}(q) = q$ the equality $T_q\Phi = \text{id}_{T_q\mathcal{M}}$ is equivalent to the numerical version $T_q^{\text{num}}\Phi = \text{id}_{T_q^{\text{num}}\mathcal{M}}$. Thus U is a closed subset of M .

Obviously, the set U is nonempty since $q_0 \in U$. Let q be an element of U . By the preceding theorem we have

$$\Phi \circ \exp_q = \exp_{\tilde{\Phi}(q)} \circ T_q\Phi = \exp_q.$$

By Theorem 4.4, the exponential map is a local diffeomorphism and we thus have an open V in M containing q such that $\Phi|_V = \text{id}_{\mathcal{M}|_V}$, implying that U is open. By the connectedness of M , it follows that U equals the whole body M of the supermanifold \mathcal{M} . Now we know that $\tilde{\Phi}(q) = q$ and $T_q\Phi = \text{id}_{T_q\mathcal{M}}$ for all $q \in M$ and consequently we have $\Phi \circ \exp_q = \exp_q$ for all $q \in M$. The morphism Φ is then the identity near each $q \in M$. This prove that Φ is the identity morphism of \mathcal{M} . \square

Acknowledgments

The research for this article was partially supported by the SFB/TR 12, “Symmetries and Universality in Mesoscopic systems”, of the DFG. The first author would like to thank Roland Knevel for suggesting to use the idea of parametrized families of supervarieties and morphisms also in the context of geodesics. Both authors thank the referee for his/her remarks.

The first author was partially supported by the Université Franco-Allemande (DFH-UFA) via the IRTG 1133 “Geometry and Analysis of Symmetries”.

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