Journal of Geometry and Physics 88 (2015) 30–51

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Almost α-paracosymplectic manifolds

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ARTICLE INFO

Article history:
Received 10 April 2014
Received in revised form 23 September 2014
Accepted 30 September 2014
Available online 27 October 2014

MSC:
primary 53B30
53C15
53C25
secondary 53D10

Keywords:
Almost paracontact metric manifold
Almost paracosymplectic manifold
Almost para-Kenmotsu manifold
Para-Kaehler manifold

ABSTRACT

This paper is a complete study of almost α-paracosymplectic manifolds. Basic properties of such manifolds are obtained and general curvature identities are proved. The manifolds with para-Kaehler leaves are characterized. It is proved that, for dimensions greater than 3, almost α-paracosymplectic manifolds are locally conformal to almost paracosymplectic manifolds and locally D-homothetic to almost para-Kenmotsu manifolds. Furthermore, it is proved that characteristic (Reeb) vector field ξ is harmonic on almost α-para-Kenmotsu manifold if and only if it is an eigenvector of the Ricci operator. It is showed that almost α-para-Kenmotsu (κ, μ, ν)-space has para-Kaehler leaves. 3-dimensional almost α-para-Kenmotsu manifolds are classified. As an application, it is obtained that 3-dimensional almost α-para-Kenmotsu manifold is (κ, μ, ν)-space on an every open and dense subset of the manifold if and only if Reeb vector field is harmonic. Furthermore, examples are constructed.

1. Introduction

The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [1] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy, [2]. However such manifolds were studied earlier, [3–6]. These authors called such structures almost para-coHermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [7,8,2].

Considering the recent stage of the developments in the theory, there is an impression that the geometers are focused on problems in almost paracontact metric geometry which are created ad hoc. Recently, a long awaited survey article, [9], concerning almost cosymplectic manifolds as Blair's monograph [10] about contact metric manifolds appeared.

Almost (para)contact metric structure is given by a pair (η, Φ), where η is a 1-form, Φ is a 2-form and $\eta \wedge \Phi^n$ is a volume element. It is well known that then there exists a unique vector field ξ, called the characteristic (Reeb) vector field, such that $i_\xi \eta = 1 i_\xi \Phi = 0$. The Riemannian or pseudo-Riemannian geometry appears if we try to introduce *a compatible* structure which is a metric or pseudo-metric g and an affinor ϕ ((1, 1)-tensor field), such that

$$\Phi(X, Y) = g(\phi X, Y), \qquad \phi^2 = \epsilon(Id - \eta \otimes \xi). \tag{1.1}$$

We have almost paracontact metric structure for $\epsilon = +1$ and almost contact metric for $\epsilon = -1$. Then, the triple (ϕ, ξ, η) is called almost paracontact structure or almost contact structure, resp.

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http://dx.doi.org/10.1016/j.geomphys.2014.09.011
0393-0440/

Combining the assumption concerning the forms η and Φ, we obtain many different types of almost (para)contact manifolds, e.g. (para)contact if η is contact form and $d\eta = \Phi$, almost (para)cosymplectic if $d\eta = 0 d\Phi = 0$, almost (para)Kenmotsu if $d\eta = 0 d\Phi = 2\eta \wedge \Phi$.

Recently geometers discovered many similarities between almost contact metric and almost paracontact metric manifolds. Simply, we could transliterate some properties from the Riemannian geometry to pseudo-Riemannian geometry. However, the situation is more delicate: there are examples of almost paracontact metric manifolds without Riemannian counterparts e.g. [11–13].

In the geometry of almost (para)contact metric manifold it is often convenient to define tensor fields

$$\mathcal{A} = -\nabla\xi, \qquad h = \frac{1}{2}\mathcal{L}_\xi\phi. \tag{1.2}$$

The meaning of these tensors depends on particular class of manifolds. In general almost (para)contact metric manifold is called (κ, μ, ν)-space if

$$R(X, Y)\xi = \eta(X)lY - \eta(Y)lX, \qquad lX = R(X, \xi)\xi = \kappa\phi^2 + \mu h + \nu\phi h, \tag{1.3}$$

for some functions $\kappa\mu\nu$. Classifications are obtained for contact metric, almost cosymplectic, almost α-Kenmotsu and almost α-cosymplectic manifolds, e.g. in [14–20]. Note that manifolds with constant sectional curvature c are $(c, 0, 0)$-spaces.

The paper is organized in the following way.

Section 2 is preliminary section, where we recall the definition of almost paracontact metric manifold.

In Section 3 we introduce the class of almost paracontact metric manifolds which is defined by

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi, \tag{1.4}$$

where α is a function. These manifolds are called almost α-paracosymplectic. They contain properly almost paracosymplectic, $\alpha = 0$, and almost α-para-Kenmotsu, $\alpha = const. \neq 0$ manifolds. In this section, we study basic properties of such manifolds.

In the short auxiliary Section 4 we recall the notion of para-Kaehler manifold.

There is natural foliation of almost α-paracosymplectic manifold where each leaf carries a structure of almost para-Kaehler manifold. In Section 5 we prove the following characterization: almost α-paracosymplectic manifold has para-Kaehler leaves if and only if

$$(\nabla_X\phi)Y = \alpha g(\phi X, Y)\xi + g(hX, Y)\xi - \alpha\eta(Y)\phi X - \eta(Y)hX. \tag{1.5}$$

In Section 6 we prove the following fundamental identity

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y = \eta(Y)\mathcal{A}\phi X + 2\alpha(g(X, \phi Y)\xi + \eta(Y)\phi X).$$

We derive a formula for the curvature $R(X, Y)\xi$, which is a cornerstone for the further study of (κ, μ, ν)-spaces.

In Section 7 we study almost α-para-Kenmotsu manifolds. We obtain curvature identities for such manifolds. We get a formula for the Ricci operator Q, cf. Theorem 5.

$$[Q, \phi] = [l, \phi] - 4\alpha(1-n)h - \eta \otimes (\phi Q\xi) + (\eta \circ Q\phi) \otimes \xi, \quad l = R(\cdot, \xi)\xi, \tag{1.6}$$

under assumption that leaves are para-Kaehler. We prove that if h vanishes everywhere then almost α-para-Kenmotsu manifold is a locally warped product $\mathbb{R} \times_f M$, of real line and almost para-Kaehler manifold.

In Section 8 we study almost α-para-Kenmotsu manifolds with harmonic Reeb vector field. The main result here is that characteristic vector field ξ is harmonic if and only if it is an eigenvector of the Ricci operator.

In Sections 9, 10 we study conformal and D-homothetic deformations of almost paracontact metric manifolds. We prove that almost α-paracosymplectic manifold M1) is locally conformal to almost paracosymplectic manifold if $\dim M \geqslant 5$2) is locally $D_{1,\alpha}$-homothetic to almost para-Kenmotsu manifold near the points where $\alpha \neq 0$. We obtain that almost α-para-Kenmotsu (κ, μ, ν)-space has para-Kaehler leaves which being combined with (1.6) gives following identity

$$[Q, \phi] = 2\mu h\phi - 2(\nu + 2(1-n)\alpha)h. \tag{1.7}$$

In the last section, we classify locally 3-dimensional almost α-para-Kenmotsu manifolds. We obtain possible forms of the Ricci operator and subsequently, on the base of the result of Section 8, we show that almost α-para-Kenmotsu 3-manifold is (κ, μ, ν)-space on an every open and dense subset of manifold if and only if ξ is harmonic.

2. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold and ϕ is a $(1, 1)$ tensor field, ξ is a vector field and η is a one-form on M. Then (ϕ, ξ, η) is called an almost paracontact structure on M if

(i) $\phi^2 = Id - \eta \otimes \xi, \ \eta(\xi) = 1$,
(ii) the tensor field ϕ induces an almost paracomplex structure on the distribution $D = \ker\eta$, that is the eigendistributions $D^{\pm}$, corresponding to the eigenvalues ± 1, have equal dimensions, $\dim D^+ = \dim D^- = n$.

The manifold M is said to be an almost paracontact manifold if it is endowed with an almost paracontact structure [2].

Let M be an almost paracontact manifold. M will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.1}$$

For such manifold, we have

$$\eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \ \eta \circ \phi = 0. \tag{2.2}$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X, Y) = g(\phi X, Y), \tag{2.3}$$

usually called fundamental form. For an almost α-paracosymplectic manifold, there exists an orthogonal basis $\{X_1, \dots, X_n, Y_1, \dots Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}, g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \dots, n\}$. Such basis is called a ϕ-basis.

On an almost paracontact manifold, one defines the $(1, 2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be normal [2]. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$

$$J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X + \lambda\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable.

3. Almost α-paracosymplectic manifolds

An almost paracontact metric manifold M^{2n+1}, with a structure (ϕ, ξ, η, g) is said to be an almost α-paracosymplectic manifold, if

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi, \tag{3.1}$$

where α may be a constant or function on M. Although α is arbitrary, we will prove that if dimension $2n+1 \geqslant 5$, then $d\alpha = f\eta$, for a (smooth) function f.

For a particular choices of the function α we have the following subclasses,

- almost α-para-Kenmotsu manifolds, $\alpha = const. \neq 0$,
- almost paracosymplectic manifolds, $\alpha = 0$.

If additionally normality condition is fulfilled, then manifolds are called α-para-Kenmotsu or paracosymplectic, resp.

Definition 1. For an almost α-paracosymplectic manifold, define the $(1, 1)$-tensor field $\mathcal{A}$ by

$$\mathcal{A}X = -\nabla_X \xi. \tag{3.2}$$

Lemma 1. *Let ω be a 2-form on a manifold $\bar{M}$ $\dim(\bar{M}) = n \geqslant 4$ and ω has maximal rank at every point, equivalently $\omega^{\wedge[\frac{n}{2}]}$ is non-zero at every point. If for a 1-form β on $\bar{M}$ $\beta \wedge \omega = 0$ at a point $p \in \bar{M}$, then $\beta = 0$ at p. Particularly β vanishes everywhere on $\bar{M}$, if $\beta \wedge \omega$ is everywhere zero.*

Proof. Let $\beta \wedge \omega = 0$ at p and $\beta_p \neq 0$. Then there is a vector v at p, such that $\beta_p(v) = 1$ and $i_v(\beta \wedge \omega)_p = \omega_p - \beta_p \wedge \gamma_p \gamma_p = i_v\omega_p$. Hence $\omega_p = \beta_p \wedge \gamma_p$ and $\omega_p^{\wedge 2} = 0$. In consequence as $[\frac{n}{2}] \geqslant 2\omega_p^{\wedge[\frac{n}{2}]} = 0$ which contradicts our assumption that ω is of maximal rank. □

Proposition 1. *For an almost α-paracosymplectic manifold M^{2n+1}, we have*

$$\begin{aligned}
&\text{(i) } \mathcal{L}_\xi \eta = 0, \qquad \text{(ii) } g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y), \qquad \text{(iii) } \mathcal{A}\xi = 0, \\
&\text{(iv) } \mathcal{L}_\xi \Phi = 2\alpha\Phi, \qquad \text{(v) } (\mathcal{L}_\xi g)(X, Y) = -2g(\mathcal{A}X, Y), \\
&\text{(vi) } \eta(\mathcal{A}X) = 0, \qquad \text{(vii) } d\alpha = f\eta \quad \text{if } n \geqslant 2
\end{aligned} \tag{3.3}$$

where $\mathcal{L}$ indicates the operator of the Lie differentiation, XY are arbitrary vector fields on M^{2n+1} and $f = i_\xi d\alpha$.

Proof. To prove (i) and (iv) we use the coboundary formula

$$\mathcal{L}_\xi \eta = d \circ i_\xi + i_\xi \circ d,$$

then we obtain

$$\mathcal{L}_\xi = d(i_\xi \eta) = 0, \tag{3.4}$$

$$\mathcal{L}_\xi \Phi = i_\xi d\Phi = i_\xi(2\alpha\eta \wedge \Phi) = 2\alpha(i_\xi \eta \wedge \Phi - \eta \wedge i_\xi \Phi) = 2\alpha\Phi. \tag{3.5}$$

To prove (ii) we note that η is closed, then

$$0 = 2d\eta(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = -g(\mathcal{A}X, Y) + g(X, \mathcal{A}Y). \tag{3.6}$$

Now

$$\begin{aligned}(\mathcal{L}_\xi g)(X, Y) &= \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]) \\ &= -g(\mathcal{A}X, Y) - g(X, \mathcal{A}Y) = -2g(\mathcal{A}X, Y),\end{aligned} \tag{3.7}$$

and (3.7) implies (v). For ξ is unit vector field, we have

$$0 = Xg(\xi, \xi) = 2g(\nabla_X \xi, \xi) = -2\eta(AX) = -2g(A\xi, X), \tag{3.8}$$

for arbitrary vector field X, which yields (iii) and (vi). Finally to proof (vii), we use Lemma 1.

We put $\beta = 2\alpha\eta$. So $d\Phi = \beta \wedge \Phi$, applying exterior differential to this equation and taking interior product with i_ξ in the result, we obtain $0 = \gamma \wedge \Phi (i_\xi \Phi = 0)$ everywhere, $\gamma = i_\xi d\beta$. If $\dim(M^{2n+1}) \geqslant 5$ ($n \geqslant 2$) by the above lemma, γ vanishes identically on M^{2n+1}. Notice $\gamma = i_\xi d\beta = 2i_\xi(d\alpha \wedge \eta)$ as $d\eta = 0$ and $0 = (i_\xi d\alpha)\eta - d\alpha$, $(i_\xi \eta = 1)$. Hence $d\alpha = f\eta f = i_\xi d\alpha$. □

Proposition 2. *For an almost α-paracosymplectic manifold, we have*

$$\mathcal{A}\phi + \phi\mathcal{A} = -2\alpha\phi, \tag{3.9}$$

$$\nabla_\xi \phi = 0. \tag{3.10}$$

Proof. From $(\mathcal{L}_\xi \Phi)(X, Y) = \xi(\Phi(X, Y)) - \Phi([\xi, X], Y) - \Phi(X, [\xi, Y])$ and definition of Φ, we get

$$(\mathcal{L}_\xi \Phi)(X, Y) = g((\nabla_\xi \phi)X - \phi\mathcal{A}X - \mathcal{A}\phi X, Y). \tag{3.11}$$

In virtue of (iv), Proposition 1, and (3.11), we obtain

$$2\alpha\phi X = (\nabla_\xi \phi)X - \phi\mathcal{A}X - \mathcal{A}\phi X. \tag{3.12}$$

As $\nabla_\xi \eta$ and $\nabla_\xi \xi$ vanish identically, $\nabla_\xi \phi^2 = \nabla_\xi (Id - \eta \otimes \xi) = 0$. Hence we obtain

$$0 = (\nabla_\xi \phi^2)X = \phi(\nabla_\xi \phi)X + (\nabla_\xi \phi)\phi X. \tag{3.13}$$

If X is a field of $+1$-eigenvectors of ϕ, then by (3.12) and (3.13), we have

$$2\alpha X = (\nabla_\xi \phi)X - \phi\mathcal{A}X - \mathcal{A}X,$$

$$2\alpha X = 2\alpha\phi X = -(\nabla_\xi \phi)X - \mathcal{A}X - \phi\mathcal{A}X.$$

From the last identities, we get $(\nabla_\xi \phi)X = 0$. Using the same arguments, one can prove $(\nabla_\xi \phi)X = 0$, if $\phi X = -X$. By $\nabla_\xi \xi = 0$, we have $(\nabla_\xi \phi)\xi = 0$. Therefore $\nabla_\xi \phi = 0$ identically, as near each point there is a frame of vector fields, consisting only from ξ and eigenvector fields of ϕ. □

Let define $h = \frac{1}{2}\mathcal{L}_\xi \phi$. In the following proposition we establish some properties of the tensor field h.

Proposition 3. *For an almost α-paracosymplectic manifold, we have the following relations*

$$g(hX, Y) = g(X, hY), \tag{3.14}$$

$$h \circ \phi + \phi \circ h = 0, \tag{3.15}$$

$$h\xi = 0, \tag{3.16}$$

$$\nabla \xi = \alpha\phi^2 + \phi \circ h = -\mathcal{A}. \tag{3.17}$$

Proof. Similarly as in Proposition 2, we have

$$(\mathcal{L}_\xi \phi^2)X = \phi(\mathcal{L}_\xi \phi)X + (\mathcal{L}_\xi \phi)\phi X = 2\phi hX + 2h\phi X. \tag{3.18}$$

and

$$\mathcal{L}_\xi \phi^2 = -(\mathcal{L}_\xi \eta) \otimes \xi = 0. \tag{3.19}$$

From (3.18) and (3.19), we get (3.15). By using the formula

$$(\mathcal{L}_\xi \phi)X = [\xi, \phi X] - \phi [\xi, X] = \nabla_\xi \phi X - \nabla_{\phi X} \xi - \phi(\nabla_\xi X - \nabla_X \xi),$$

we obtain

$$h = \frac{1}{2}(\mathcal{A}\phi - \phi\mathcal{A}). \tag{3.20}$$

Properties of ϕ, $\mathcal{A}$ and (3.20) follow, that h is symmetric tensor field, $g(hX, Y) = g(X, hY)$. Moreover, $h\xi = 0$ and $\eta \circ h = 0$. Using (3.5), (3.7) and the following identity

$$(\mathcal{L}_\xi \Phi)(X, Y) = (\mathcal{L}_\xi g)(\phi X, Y) + g((\mathcal{L}_\xi \phi)X, Y),$$

we obtain

$$\alpha\phi = -\mathcal{A}\phi + h. \tag{3.21}$$

From (3.15) and (3.21), we obtain

$$\alpha\phi^2 + \phi \circ h = -\mathcal{A} = \nabla\xi. \quad \square$$

Corollary 1. *All the above propositions imply the following formulas for the traces*

$$\begin{aligned} &tr(\mathcal{A}\phi) = tr(\phi\mathcal{A}) = 0, \qquad tr(h\phi) = tr(\phi h) = 0, \\ &tr(\mathcal{A}) = -2\alpha n, \qquad tr(h) = 0. \end{aligned} \tag{3.22}$$

4. Para-Kaehler manifolds

This is an auxiliary section. The general reference for the notions which appear here is [21]. We recall basic concepts of a para-Hermitian geometry. An almost para-Hermitian manifold is a manifold M endowed with an almost para-complex structure J and a semi-Riemannian metric g such that

$$J^2 = I, \qquad g(JX, JY) = -g(X, Y), \tag{4.1}$$

for X, Y tangent to M, where I is the identity map. The dimension of M is even. The almost para-Hermitian manifold M is para-Kaehler if the almost para-complex structure J is a covariant constant $\nabla J = 0$, with respect to the Levi-Civita connection. An almost para-complex structure is integrable if and only if the Nijenhuis torsion of J vanishes identically

$$N_J(X, Y) = J^2 [X, Y] + [JX, JY] - J [JX, Y] - J [X, JY] = 0.$$

An almost para-complex structure of a para-Kaehler manifold is always integrable. In the terms of the local coordinates maps, integrability is equivalent to the existence of a set of maps, covering the manifold, the para-complex structure has constant coefficients in the local map coordinates. If $p \in M$ is a point, then near p, there are coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, the local components $J_i^k = const.$ are constants.

5. Almost α-paracosymplectic manifolds with para-Kaehler leaves

The idea is to restrict further our consideration to the particular class of manifolds. However this class of manifolds is wide enough to provide interesting results and examples. In fact each 3-dimensional manifold belongs to this class. Let $M^{2n+1} = (M, \phi, \xi, \eta, g)$ be an almost α-paracosymplectic manifold. By the definition, the form η is closed, therefore distribution $\mathcal{D} : \eta = 0$ is completely integrable. Each leaf of the foliation, determined by $\mathcal{D}$, carries an almost para-Kaehler structure $(J, <, >)$

$$J\bar{X} = \phi\bar{X}, \qquad \langle \bar{X}, \bar{Y} \rangle = g(\bar{X}, \bar{Y}),$$

$\bar{X}$, $\bar{Y}$ are vector fields tangent to the leaf. If this structure is para-Kaehler, leaf is called a para-Kaehler leaf.

Lemma 2. *An almost α-paracosymplectic manifold M^{2n+1} has para-Kaehler leaves if and only if*

$$(\nabla_X \phi)Y = g(\mathcal{A}X, \phi Y)\xi + \eta(Y)\phi\mathcal{A}X, \quad \mathcal{A} = -\nabla\xi.$$

Proof. Let $\mathcal{F}_a$ be a leaf passing through a point $a \in M$. The characteristic vector, restricted to $\mathcal{F}_a$, is normal of $\mathcal{F}_a$ and the restriction $\mathcal{A}\,|_{\mathcal{F}} = -\nabla\xi\,|_{\mathcal{F}}$ is the Weingarten operator (the shape tensor) of $\mathcal{F}_a$. Let II denote the second fundamental form of $\mathcal{F}_a$. The Gauss equation yields

$$(\nabla_{\bar X}\phi)\bar Y = \nabla_{\bar X}\phi\bar Y - \phi\nabla_{\bar X}\bar Y = \bar\nabla_{\bar X}J\bar Y + II(\bar X,\phi\bar Y)\xi - \phi(\bar\nabla_{\bar X}\bar Y + II(\bar X,\bar Y)\xi) \tag{5.1}$$

$$= (\bar\nabla_{\bar X}J)\bar Y + II(\bar X,\phi\bar Y)\xi = II(\bar X,\phi\bar Y)\xi, \tag{5.2}$$

where, we have used $\bar\nabla J = 0$. The above identity implies $(\nabla_X\phi)Y = g(\mathcal{A}X,\phi Y)\xi$, for vector fields X, Y on M, such that $\eta(X) = \eta(Y) = 0$. For arbitrary vector fields X, Y, we have a decomposition $X = (X - \eta(X)\xi) + \eta(X)\xi$. To finish the proof, we recall that $\nabla_\xi\phi = 0$ and $(\nabla_X\phi)\xi = \phi\mathcal{A}X$. □

Proposition 4. *Let $M^{2n+1} = (M,\phi,\xi,\eta,g)$ be an almost α-paracosymplectic manifold. Then, for $\alpha = 0$ (resp. $\alpha \neq 0$), foliation $\mathcal{F}$ is totally geodesic (resp. totally umbilical) if and only if $h = 0$.*

Proof. Using the Gauss equation and (3.17), we obtain

$$II(\bar X,\bar Y) = -g(\bar Y,\alpha\phi^2\bar X + \phi h\bar X) = -\alpha g(\bar X,\bar Y) - g(\bar X,\phi h\bar Y), \tag{5.3}$$

for all $\bar X,\bar Y \in \Gamma(D)$. This completes the proof. □

Proposition 5. *An almost α-paracosymplectic manifold M has para-Kaehler leaves if and only if*

$$(\nabla_X\phi)Y = \alpha g(\phi X,Y)\xi + g(hX,Y)\xi - \alpha\eta(Y)\phi X - \eta(Y)hX, \tag{5.4}$$

for $\alpha = 0$ it is a formula known for almost paracosymplectic manifolds, [7].

Proof. If we use Lemma 2 and the identity (3.17), we have

$$\begin{aligned}(\nabla_X\phi)Y &= -g(\alpha\phi^2X + \phi hX,\phi Y)\xi - \eta(Y)\phi(\alpha\phi^2X + \phi hX)\\ &= -\alpha g(\phi^2X,\phi Y)\xi - g(\phi hX,\phi Y)\xi - \alpha\eta(Y)\phi X - \eta(Y)hX.\end{aligned}$$

By means of (2.1), we get the requested equation. □

As a direct consequence, we have the following result.

Theorem 1. *Let M^{2n+1} be an almost α-para-Kenmotsu manifold with para-Kaehler leaves. Then M^{2n+1} is para-Kenmotsu ($\alpha = 1$) if and only $\mathcal{A} = -\phi^2$.*

Remark 1. For a similar notion in contact metric geometry see e.g. [22,17,18]. There are many other papers, where this notion appears explicitly or implicitly. Compare the references in [9]. We also note that in almost contact metric geometry there is more general idea of manifold carrying CR-structure. All almost contact metric manifolds with Kaehler leaves are Levi-flat CR-manifolds.

6. Basic structure and curvature identities

Lemma 3. *For an almost α-paracosymplectic manifold (M,ϕ,ξ,η,g) with its fundamental 2-form Φ, the following equations hold*

$$(\nabla_X\Phi)(Z,\phi Y) + (\nabla_X\Phi)(Y,\phi Z) = -\eta(Y)g(\mathcal{A}X,Z) - \eta(Z)g(\mathcal{A}X,Y), \tag{6.1}$$

$$(\nabla_X\Phi)(\phi Y,\phi Z) - (\nabla_X\Phi)(Y,Z) = \eta(Y)g(\mathcal{A}X,\phi Z) - \eta(Z)g(\mathcal{A}X,\phi Y). \tag{6.2}$$

Proof. Differentiating the identity $\phi^2 = I - \eta\otimes\xi$ covariantly, we obtain

$$(\nabla_X\phi)\phi Y + \phi(\nabla_X\phi)Y = g(Y,\mathcal{A}X)\xi + \eta(Y)\mathcal{A}X. \tag{6.3}$$

If we take the inner product with Z, we obtain (6.1). Replacing Z by ϕZ in (6.1), using the anti-symmetry of Φ, we get (6.2). □

Proposition 6. *For any almost α-paracosymplectic manifold, we have*

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y - \eta(Y)\mathcal{A}\phi X - 2\alpha(g(X,\phi Y)\xi + \eta(Y)\phi X) = 0. \tag{6.4}$$

Proof. Let us define $(0,3)$-tensor field $\mathcal{B}$ as follows

$$\mathcal{B}(X,Y,Z) = g((\nabla_{\phi X}\phi)\phi Y,Z) - g((\nabla_X\phi)Y,Z) - \eta(Y)g(\mathcal{A}\phi X,Z) - 2\alpha(g(X,\phi Y)\eta(Z) + \eta(Y)g(\phi X,Z)).$$

Anti-symmetrizing $\mathcal{B}$ with respect to X, Y we have

$$\begin{aligned}\mathcal{B}(X,Y,Z)-\mathcal{B}(Y,X,Z) = {} & (\nabla_{\phi X}\Phi)(\phi Y,Z)-(\nabla_{\phi Y}\Phi)(\phi X,Z)-(\nabla_X\Phi)(Y,Z)+(\nabla_Y\Phi)(X,Z)\\ & -\eta(Y)g(\mathcal{A}\phi X,Z)+\eta(X)g(\mathcal{A}\phi Y,Z)-2\alpha((g(X,\phi Y)-g(Y,\phi X))\eta(Z)\\ & +\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)).\end{aligned} \tag{6.5}$$

Recalling the well known formula

$$\begin{aligned}3d\Phi(X,Y,Z) &= (\nabla_X\Phi)(Y,Z)+(\nabla_Z\Phi)(X,Y)+(\nabla_Y\Phi)(Z,X)\\ &= 2\alpha(\eta(X)\Phi(Y,Z)+\eta(Z)\Phi(X,Y)+\eta(Y)\Phi(Z,X)).\end{aligned}$$

and applying this in (6.5), we obtain

$$\mathcal{B}(X,Y,Z)-\mathcal{B}(Y,X,Z)=-(\nabla_Z\Phi)(\phi X,\phi Y)+(\nabla_Z\Phi)(X,Y)-\eta(Y)g(\mathcal{A}\phi X,Z)+\eta(X)g(\mathcal{A}\phi Y,Z).$$

By (6.2), the right hand side of this equality vanishes identically, so $\mathcal{B}$ is symmetric with respect to X, Y.

Symmetrizing $\mathcal{B}$ with respect to Y, Z, we find

$$\mathcal{B}(X,Y,Z)+\mathcal{B}(X,Z,Y)=(\nabla_{\phi X}\Phi)(\phi Y,Z)+(\nabla_{\phi X}\Phi)(\phi Z,Y)-\eta(Y)g(\mathcal{A}\phi X,Z)-\eta(Z)g(\mathcal{A}\phi X,Y).$$

By means of (6.1), we obtain $\mathcal{B}(X,Y,Z)+\mathcal{B}(X,Z,Y)=0$, i.e. $\mathcal{B}$ is antisymmetric with respect to Y, Z. The tensor $\mathcal{B}$ having such symmetries must vanish identically, which implies (6.4). □

Lemma 4. *For an almost α-paracosymplectic manifold, we also have*

$$(\nabla_{\phi X}\phi)Y-(\nabla_X\phi)\phi Y+\eta(Y)\mathcal{A}X-2\alpha(g(X,Y)\xi-\eta(Y)X)=0, \tag{6.6}$$

$$(\nabla_{\phi X}\phi)Y+\phi(\nabla_X\phi)Y-g(\mathcal{A}X,Y)\xi-2\alpha(g(X,Y)\xi-\eta(Y)X)=0. \tag{6.7}$$

Proof. Putting ϕY instead of Y in (6.4), we obtain

$$(\nabla_{\phi X}\phi)Y-\eta(Y)(\nabla_{\phi X}\phi)\xi-(\nabla_X\phi)\phi Y-2\alpha(g(X,Y)-\eta(Y)\eta(X)\xi)=0. \tag{6.8}$$

Using (3.17) and $(\nabla_{\phi X}\phi)\xi=-\mathcal{A}X-2\alpha\phi^2X$, in (6.8), we get (6.6). Eq. (6.7) comes from (6.3) and (6.6). □

With the help of (6.7), one can easily get the following proposition.

Proposition 7. *For an almost α-paracosymplectic manifold, we have*

$$\phi(\nabla_{\phi X}\phi)Y+(\nabla_X\phi)Y=-2\alpha\eta(Y)\phi X+g(\alpha\phi X+hX,Y)\xi. \tag{6.9}$$

Theorem 2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost α-paracosymplectic manifold. Then, for any $X, Y\in\chi(M^{2n+1})$,*

$$\begin{aligned}R(X,Y)\xi &= d\alpha(X)(Y-\eta(Y)\xi)-d\alpha(Y)(X-\eta(X)\xi)\\ &\quad +\alpha\eta(X)(\alpha Y+\phi hY)-\alpha\eta(Y)(\alpha X+\phi hX)+(\nabla_X\phi h)Y-(\nabla_Y\phi h)X.\end{aligned} \tag{6.10}$$

Proof. We notice that $\nabla_{X,Y}\xi=-(\nabla_X\mathcal{A})Y$. Now, if we substitute $\mathcal{A}$, according to (3.17) and apply the covariant derivative to the all summands in the result, we obtain

$$\nabla_{X,Y}\xi=d\alpha(X)(Y-\eta(Y)\xi)+\alpha\eta(X)(\alpha Y+\phi hY)+(\nabla_X\phi h)Y. \tag{6.11}$$

Eq. (6.10) follows from (6.11) and Ricci identity. □

The formula for the curvature $R(X,Y)\xi$, simplifies if $2n+1\geqslant 5$, according to Proposition 1,(vii).

Corollary 2. *For an almost α-paracosymplectic manifold $M^{2n+1}n\geqslant 2$*

$$R(X,Y)\xi=(f+\alpha^2)(\eta(X)Y-\eta(Y)X)+\alpha(\eta(X)\phi hY-\eta(Y)\phi hX)+(\nabla_X\phi h)Y-(\nabla_Y\phi h)X, \tag{6.12}$$

where $f=i_\xi d\alpha$.

7. Almost α-para-Kenmotsu manifolds

In this section, we study curvature properties of an almost α-para-Kenmotsu manifold.

Theorem 3. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y\in\chi(M^{2n+1})$,*

$$R(X,Y)\xi=\alpha\eta(X)(\alpha Y+\phi hY)-\alpha\eta(Y)(\alpha X+\phi hX)+(\nabla_X\phi h)Y-(\nabla_Y\phi h)X. \tag{7.1}$$

Proof. It is direct consequence of Theorem 2 for constant α. $\square$

Theorem 4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X \in \chi(M^{2n+1})$ we have*

$$R(\xi, X)\xi = \alpha^2\phi^2X + 2\alpha\phi hX - h^2X + \phi(\nabla_\xi h)X, \tag{7.2}$$

$$(\nabla_\xi h)X = -\alpha^2\phi X - 2\alpha hX + \phi h^2X - \phi R(X, \xi)\xi, \tag{7.3}$$

$$\frac{1}{2}(R(\xi, X)\xi + \phi R(\xi, \phi X)\xi) = \alpha^2\phi^2X - h^2X, \tag{7.4}$$

$$S(X, \xi) = -2n\alpha^2\eta(X) + g(div(\phi h), X), \tag{7.5}$$

$$S(\xi, \xi) = -2n\alpha^2 + trh^2. \tag{7.6}$$

Proof. If we replace X by ξ and Y by X in (7.1) and use (3.17) we obtain (7.2). For the proof of (7.3), we apply the tensor field ϕ both sides of (7.2) and recall $\nabla_\xi\phi = 0$. Hence we have

$$-\phi R(X, \xi)\xi = \alpha^2\phi X + 2\alpha hX - \phi h^2X + (\nabla_\xi h)X - g((\nabla_\xi h)X, \xi)\xi.$$

Replacing X by ϕX in (7.2) we get

$$R(\xi, \phi X)\xi = \alpha^2\phi^3X + 2\alpha\phi h\phi X - h^2\phi X + \phi(\nabla_\xi h)\phi X.$$

If we apply ϕ to the last equation we have

$$\phi R(\xi, \phi X)\xi = \alpha^2\phi^2X + 2\alpha h\phi X - h^2X + (\nabla_\xi h)\phi X. \tag{7.7}$$

One can easily show that $\phi(\nabla_\xi h)X = -(\nabla_\xi h)\phi X$. Combining (7.2) with (7.7) we get (7.4).

Taking into account ϕ-basis and (7.1), Ricci curvature $S(X, \xi)$ can be given by

$$\begin{aligned} S(X, \xi) &= \sum_{i=1}^{n}[g(R(e_i, X)\xi, e_i) - g(R(\phi e_i, X)\xi, \phi e_i)] \\ &= -2n\alpha^2\eta(X) - \sum_{i=1}^{n}(g((\nabla_X\phi h)e_i, e_i) - g((\nabla_X\phi h)\phi e_i, \phi e_i)) \\ &\quad + \sum_{i=1}^{n}(g((\nabla_{e_i}\phi h)X, e_i) - g((\nabla_{\phi e}\phi h)X, \phi e_i)). \end{aligned} \tag{7.8}$$

After some calculations we have

$$\sum_{i=1}^{n}(g((\nabla_X\phi h)e_i, e_i) - g((\nabla_X\phi h)\phi e_i, \phi e_i)) = 0,$$

$$\sum_{i=1}^{n}(g((\nabla_{e_i}\phi h)X, e_i) - g((\nabla_{\phi e_i}\phi h)X, \phi e_i)) = g(div(\phi h), X).$$

Using the last two equations in (7.8), we obtain

$$S(X, \xi) = -2n\alpha^2\eta(X) + g(div(\phi h), X).$$

By direct calculation, we find

$$S(\xi, \xi) = -2n\alpha^2 + trh^2. \quad \square$$

Proposition 8. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y, Z \in \chi(M^{2n+1})$ we have*

$$\begin{aligned} &g(R(\xi, X)Y, Z) + g(R(\xi, X)\phi Y, \phi Z) - g(R(\xi, \phi X)\phi Y, Z) - g(R(\xi, \phi X)Y, \phi Z) \\ &\quad = 2(\nabla_{hX}\Phi)(Y, Z) + 2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) - 2\alpha\eta(Z)g(\phi hX, Y) + 2\alpha\eta(Y)g(\phi hX, Z). \end{aligned} \tag{7.9}$$

Proof. The symmetries of the curvature tensor give $g(R(\xi, X)Y, Z) = g(X, R(Y, Z)\xi)$ and then, using (7.1), the left hand side can be written as

$$2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) + \mathcal{F}(X, Y, Z) - \mathcal{F}(X, Z, Y), \tag{7.10}$$

where

$$\mathcal{F}(X,Y,Z)=g(X,(\nabla_Y\phi h)Z+\phi(\nabla_Y\phi h)\phi Z)+g(X,(\nabla_{\phi Y}\phi h)\phi Z)-g(\phi X,(\nabla_{\phi Y}\phi h)Z).$$

By direct computations

$$\phi(\nabla_Y\phi h)\phi Z+(\nabla_Y\phi h)Z=(\nabla_Y\phi)hZ-h(\nabla_Y\phi)Z, \tag{7.11}$$

and

$$\begin{aligned} g(X,(\nabla_{\phi Y}\phi h)\phi Z)-g(\phi X,(\nabla_{\phi Y}\phi h)Z) &= -g(\phi X,\phi((\nabla_{\phi Y}\phi h)\phi Z)) \\ &\quad +\eta(X)\eta((\nabla_{\phi Y}\phi h)\phi Z)-g(\phi X,(\nabla_{\phi Y}\phi h)Z). \end{aligned} \tag{7.12}$$

With the help of (7.11), (7.12), (6.9) and $\eta((\nabla_{\phi Y}\phi h)\phi Z)=g(hZ,\alpha\phi Y-hY)$, we compute a formula for $\mathcal{F}(X,Y,Z)$ and after using it we obtain

$$\begin{aligned} \mathcal{F}(X,Y,Z)-\mathcal{F}(X,Z,Y) &= 2(\nabla_{hX}\Phi)(Y,Z)-6d\Phi(Y,Z,hX)+2\alpha\eta(Z)g(\phi hX,Y)-2\alpha\eta(Y)g(\phi hX,Z) \\ &= 2(\nabla_{hX}\Phi)(Y,Z)-2\alpha\eta(Z)g(\phi hX,Y)+2\alpha\eta(Y)g(\phi hX,Z). \end{aligned} \tag{7.13}$$

where the last equality holds by $d\Phi=2\alpha\eta\wedge\Phi$. From (7.10) and (7.13), we get the required formula. □

Theorem 5. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold with para-Kaehler leaves. Then the following identity holds*

$$Q\phi-\phi Q=l\phi-\phi l-4\alpha(1-n)h-\eta\otimes\phi Q\xi+(\eta\circ Q\phi)\otimes\xi, \tag{7.14}$$

where l denotes the Jacobi operator, defined by $l=R(\cdot,\xi)\xi$.

Proof. We recall the formula (7.1)

$$R(X,Y)\xi=\alpha\eta(X)(\alpha Y+\phi hY)-\alpha\eta(Y)(\alpha X+\phi hX)+(\nabla_X\phi h)Y-(\nabla_Y\phi h)X.$$

On the other hand the following formula is valid

$$(\nabla_X\phi h)Y=(\nabla_X\phi)hY+\phi((\nabla_X h)Y). \tag{7.15}$$

Using (5.4) and (7.15), we obtain

$$R(X,Y)\xi=\alpha\eta(X)(\alpha Y+\phi hY)-\alpha\eta(Y)(\alpha X+\phi hX)+\phi((\nabla_X h)Y-(\nabla_Y h)X)+(\nabla_X\phi)hY-(\nabla_Y\phi)hX. \tag{7.16}$$

By assumption M^{2n+1} has para-Kaehler leaves thus by (5.4) $(\nabla_X\phi)hY=\alpha g(\phi X,hY)\xi+g(hX,hY)\xi$ in consequence, as $h\phi$ is symmetric, $(\nabla_X\phi)hY-(\nabla_Y\phi)hX$ vanishes identically. Since h is a symmetric operator, we easily get

$$g((\nabla_X h)Y-(\nabla_Y h)X,\xi)=g((\nabla_X h)\xi,Y)-g((\nabla_Y h)\xi,X). \tag{7.17}$$

Using the formulas (3.17), $h\xi=0$ and $\phi h+h\phi=0$ in (7.17), we find

$$g((\nabla_X h)Y-(\nabla_Y h)X,\xi)=2g(\phi h^2X,Y). \tag{7.18}$$

By applying ϕ to (7.16) and using $\phi^2=I-\eta\otimes\xi$ and (7.18) we obtain

$$(\nabla_X h)Y-(\nabla_Y h)X=\phi R(X,Y)\xi+2g(\phi h^2X,Y)\xi-\alpha^2(\eta(X)\phi Y-\eta(Y)\phi X)-\alpha(\eta(X)hY-\eta(Y)hX). \tag{7.19}$$

Now we suppose that P is a fixed point of M and X,Y,Z are vector fields such that $(\nabla X)_P=(\nabla Y)_P=(\nabla Z)_P=0$. The Ricci identity for ϕ

$$R(X,Y)\phi Z-\phi R(X,Y)Z=(\nabla_X\nabla_Y\phi)Z-(\nabla_Y\nabla_X\phi)Z-(\nabla_{[X,Y]}\phi)Z,$$

at the point P, reduces to the form

$$R(X,Y)\phi Z-\phi R(X,Y)Z=\nabla_X(\nabla_Y\phi)Z-\nabla_Y(\nabla_X\phi)Z.$$

Due to our assumption that M^{2n+1} has para-Kaehler leaves from (5.4), we obtain at P

$$\begin{aligned} R(X,Y)\phi Z-\phi R(X,Y)Z &= \nabla_X(\nabla_Y\phi)Z-\nabla_Y(\nabla_X\phi)Z \\ &= \alpha\,(g((\nabla_X\phi)Y-(\nabla_Y\phi)X,Z)\xi-\eta(Z)((\nabla_X\phi)Y-(\nabla_Y\phi)X)) \\ &\quad +g((\nabla_X h)Y-(\nabla_Y h)X,Z)\xi-\eta(Z)((\nabla_X h)Y-(\nabla_Y h)X) \\ &\quad +g(\alpha\phi Y+hY,Z)(\alpha\phi^2X+\phi hX)-g(\alpha\phi X+hX,Z)(\alpha\phi^2Y+\phi hY) \\ &\quad -g(Z,\alpha\phi^2X+\phi hX)(\alpha\phi Y+hY)+g(Z,\alpha\phi^2Y+\phi hY)(\alpha\phi X+hX). \end{aligned} \tag{7.20}$$

Using (5.4) and (7.19) in (7.20), we find

$$\begin{aligned} R(X,Y)\phi Z-\phi R(X,Y)Z =\ & g(\phi R(X,Y)\xi ,Z)\xi -\eta (Z)\phi R(X,Y)\xi \\ & +g(\alpha \phi Y+hY,Z)(\alpha \phi ^{2}X+\phi hX)-g(\alpha \phi X+hX,Z)(\alpha \phi ^{2}Y+\phi hY) \\ & -g(Z,\alpha \phi ^{2}X+\phi hX)(\alpha \phi Y+hY)+g(Z,\alpha \phi ^{2}Y+\phi hY)(\alpha \phi X+hX). \end{aligned} \tag{7.21}$$

Using (2.1) and the curvature tensor properties, we get

$$g(\phi R(\phi X,\phi Y)Z,\phi W)=-g(R(Z,W)\phi X,\phi Y)+\eta (R(\phi X,\phi Y)Z)\eta (W). \tag{7.22}$$

Then by (7.21) and (7.22), we obtain

$$\begin{aligned} g(\phi R(\phi X,\phi Y)Z,\phi W) =\ & -g(\phi R(Z,W)X,\phi Y)+\eta (R(\phi X,\phi Y)Z)\eta (W)+\eta (X)g(\phi R(Z,W)\xi ,\phi Y) \\ & -g(\alpha \phi W+hW,X)(g(\alpha \phi ^{2}Z,\phi Y)+g(\phi hZ,\phi Y)) \\ & +g(\alpha \phi Z+hZ,X)(g(\alpha \phi ^{2}W,\phi Y)+g(\phi hW,\phi Y)) \\ & +g(X,\alpha \phi ^{2}Z+\phi hZ)(g(\alpha \phi W,\phi Y)+g(hW,\phi Y)) \\ & -g(X,\alpha \phi ^{2}W+\phi hW)(g(\alpha \phi Z,\phi Y)+g(hZ,\phi Y)). \end{aligned} \tag{7.23}$$

Replacing in (7.21) X, Y by ϕX, ϕY respectively, and taking the inner product with ϕW, we get

$$\begin{aligned} & g(R(\phi X,\phi Y)\phi Z,\phi W)-g(\phi R(\phi X,\phi Y)Z,\phi W) \\ & \quad =-\eta (Z)g(\phi R(\phi X,\phi Y)\xi ,\phi W)+g(\alpha \phi ^{2}Y+h\phi Y,Z)g(\alpha \phi ^{3}X+\phi h\phi X,\phi W) \\ & \qquad -g(\alpha \phi ^{2}X+h\phi X,Z)g(\alpha \phi ^{3}Y+\phi h\phi Y,\phi W)-g(Z,\alpha \phi ^{3}X+\phi h\phi X)g(\alpha \phi ^{2}Y+h\phi Y,\phi W) \\ & \qquad +g(Z,\alpha \phi ^{3}Y+\phi h\phi Y)g(\alpha \phi ^{2}X+h\phi X,\phi W). \end{aligned} \tag{7.24}$$

Comparing (7.23) to (7.24), we get by direct computation

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) =\ & g(R(Z,W)X,Y)-\eta (R(Z,W)X)\eta (Y) \\ & -\eta (X)g(R(Z,W)\xi ,Y)+\eta (R(\phi X,\phi Y)Z)\eta (W) \\ & -\eta (Z)g(\phi R(\phi X,\phi Y)\xi ,\phi W)-2\alpha g(X,Z)g(Y,\phi hW)+2\alpha \eta (X)\eta (Z)g(\phi hW,Y) \\ & +2\alpha g(Y,Z)g(X,\phi hW)-2\alpha \eta (Y)\eta (Z)g(\phi hW,X) \\ & +2\alpha g(X,W)g(Y,\phi hZ)-2\alpha \eta (X)\eta (W)g(\phi hZ,Y) \\ & +2\alpha \eta (Y)\eta (W)g(X,\phi hZ)-2\alpha g(Y,W)g(\phi hZ,X). \end{aligned} \tag{7.25}$$

Let $\{e_i,\phi e_i,\xi\}$, $i\in\{1,\ldots,n\}$, be a local ϕ-basis. Setting $Y=Z=e_i$ in (7.25), we have

$$\begin{aligned} \sum_{i=1}^{n} g(R(\phi X,\phi e_i)\phi e_i,\phi W) =\ & \sum_{i=1}^{n}(g(R(e_i,W)X,e_i)-\eta (X)g(R(e_i,W)\xi ,e_i)+\eta (R(\phi X,\phi e_i)e_i)\eta (W) \\ & -2\alpha g(X,e_i)g(e_i,\phi hW)+2\alpha g(e_i,e_i)g(\phi hW,X) \\ & +2\alpha g(X,W)g(e_i,\phi he_i)-2\alpha \eta (X)\eta (W)g(\phi he_i,e_i)-2\alpha g(e_i,W)g(\phi he_i,X)). \end{aligned} \tag{7.26}$$

On the other hand, putting $Y=Z=\phi e_i$ in (7.25), we get

$$\begin{aligned} \sum_{i=1}^{n} g(R(\phi X,e_i)e_i,\phi W) =\ & \sum_{i=1}^{n}(g(R(\phi e_i,W)X,\phi e_i)-\eta (X)g(R(\phi e_i,W)\xi ,\phi e_i)+\eta (R(\phi X,e_i)\phi e_i)\eta (W) \\ & -2\alpha g(X,\phi e_i)g(\phi e_i,\phi hW)+2\alpha g(\phi e_i,\phi e_i)g(\phi hW,X) \\ & +2\alpha g(X,W)g(\phi e_i,\phi h\phi e_i)-2\alpha \eta (X)\eta (W)g(\phi h\phi e_i,\phi e_i) \\ & -2\alpha g(\phi e_i,W)g(\phi h\phi e_i,X)). \end{aligned} \tag{7.27}$$

Using the definition of the Ricci operator, (7.26) and (7.27), one can easily get

$$\begin{aligned} -\phi Q\phi X+\phi l\phi X+QX-lX =\ & \eta (X)Q\xi +4\alpha (1-n)\phi hX \\ & +\sum_{i=1}^{n}(g(R(\phi X,e_i)\phi e_i,\xi)-g(R(\phi X,\phi e_i)e_i,\xi))\xi . \end{aligned} \tag{7.28}$$

Finally, applying ϕ to (7.28) and using $\phi^2=I-\eta\otimes\xi$, we obtain the requested equation. □

Theorem 6. *Let M^{2n+1} be an almost α-para-Kenmotsu manifold of constant sectional curvature c. Then $c=-\alpha^2$ and $h^2=0$.*

Proof. By assumptions

$$R(\xi, X)\xi = c(\eta(X)\xi - X) = \phi R(\xi, \phi X)\xi, \tag{7.29}$$

for any $X \in \Gamma(M)$. Using the last relation in (7.4), we obtain

$$h^2X = (\alpha^2 + c)\phi^2 X \tag{7.30}$$

By (3.10), (7.30), we find $\nabla_\xi h^2 = 0$. In virtue of (7.3), (7.29) and (7.30), we have

$$0 = \nabla_\xi h^2 = h(\nabla_\xi h) + (\nabla_\xi h)h = -4\alpha h^2 = -4\alpha(\alpha^2 + c)\phi^2.$$

Hence $c = -\alpha^2$ and $h^2 = 0$. □

The proof of the following theorem is exactly same with almost Kenmotsu manifolds [23], therefore we omit the proof.

Theorem 7. *Let M^{2n+1} be an almost α-para-Kenmotsu manifold with $h = 0$. Then M^{2n+1} is a locally warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kaehler manifold, M_1 is an open interval with coordinate t, and $f^2 = we^{2\alpha t}$ for some positive constant.*

Remark 2. Almost Kenmotsu manifolds in almost contact metric geometry appeared in [23–25]. These manifolds were extensively studied e.g. [23,17,19,20]. Arbitrary almost Kenmotsu manifold can be locally deformed conformally to almost cosymplectic manifold. Almost Kenmotsu manifolds were generalized to almost α-Kenmotsu, $\alpha = const.$, and subsequently to almost α-cosymplectic manifolds.

8. Harmonic vector fields

Let (M, g) be a smooth, oriented, connected pseudo-Riemannian manifold and (TM, g^S) its tangent bundle endowed with the Sasaki metric g^S (also referred to as Kaluza–Klein metric in Mathematical Physics). By definition, the *energy* of a smooth vector field V on M is the energy of the corresponding $V : (M, g) \to (TM, g^s)$. When M is compact, the *energy* of V is determined by

$$E(V) = \frac{1}{2}\int_M (tr_g V^* g^s)dv = \frac{n}{2} vol(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 \, dv.$$

The non-compact case, one can take into account over relatively compact domains. It can be shown that $V : (M, g) \to (TM, g^s)$ is harmonic map if and only if

$$tr\,[R(\nabla.V, V).] = 0, \;\; \nabla^*\nabla V = 0, \tag{8.1}$$

where

$$\nabla^*\nabla V = -\sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V) \tag{8.2}$$

is the rough Laplacian with respect to a pseudo-orthonormal local frame $\{e_1, \dots, e_n\}$ on (M, g) with $g(e_i, e_i) = \varepsilon_i = \pm 1$ for all indices $i = 1, \dots, n$.

If (M, g) is a compact Riemannian manifold, only parallel vector fields define harmonic maps.

Next, for any real constant $\rho \neq 0$, let $\chi^\rho(M) = \left\{W \in \chi(M) : \|W\|^2 = \rho\right\}$. We consider vector fields $V \in \chi^\rho(M)$ which are critical points for the energy functional $E\,|_{\chi^\rho(M)}$, restricted to vector fields of the same length. The Euler–Lagrange equations of this variational condition yield that V is a harmonic vector field if and only if

$$\nabla^*\nabla V \text{ is collinear to } V. \tag{8.3}$$

This characterization is well known in the Riemannian case [10,1,12]. Using same arguments in pseudo-Riemannian case, G. Calvaruso [26] proved that same result is still valid for vector fields of constant length, if it is not lightlike.

Let T_1M denote the unit tangent sphere bundle over M, and again by g^S the metric induced on T_1M by the Sasaki metric of TM. Then, it is shown that in [27], the map on $V : (M, g) \to (T_1M, g^s)$ is harmonic if V is a harmonic vector field and the additional condition

$$tr[R(\nabla.V, V)\cdot] = 0 \tag{8.4}$$

is satisfied. G. Calvaruso [26] also investigated harmonicity properties for left-invariant vector fields on three-dimensional Lorentzian Lie groups, obtaining several classification results and new examples of critical points of energy functionals.

In the non-compact case, conditions (8.1) and (8.3) are respectively taken as definitions of harmonic vector fields and of vector fields defining harmonic maps.

Recently, D. Perrone proved that the characteristic vector field of an almost cosymplectic three-manifold is minimal if and only if it is an eigenvector of the Ricci operator.

Theorem 8. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost* α*-para-Kenmotsu manifold. Then*

$$\bar{\Delta}\xi = -\nabla^*\nabla\xi = (2n\alpha^2 - tr(h^2))\xi - Q\xi_{|\ker\eta}.$$

Proof. Now, let $(e_i, \phi e_i, \xi)$, $i = 1, \ldots, n$, be a local orthogonal ϕ-basis. Then we obtain

$$\begin{aligned}
\bar{\Delta}\xi &= -\sum_{i=1}^{n}(\nabla_{e_i}\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}e_i}\xi - \nabla_{\phi e_i}\nabla_{\phi e_i}\xi + \nabla_{\nabla_{\phi e_i}\phi e_i}\xi) \\
&= -\sum_{i=1}^{n}((\nabla_{e_i}\nabla\xi)e_i - (\nabla_{\phi e_i}\nabla\xi)\phi e_i) \\
&\overset{(3.17)}{=} \sum_{i=1}^{n}((\nabla_{e_i}A)e_i - (\nabla_{\phi e_i}A)\phi e_i) \\
&= -div\phi h + 2n\alpha^2\xi.
\end{aligned}$$

By (7.5) and (7.6), we get

$$\bar{\Delta}\xi = (2n\alpha^2 - tr(h^2))\xi - Q\xi_{|\ker\eta}.$$

This ends the proof. □

As an immediate consequence of Theorem 8 we obtain following theorem.

Theorem 9. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost* α*-para-Kenmotsu manifold. Characteristic vector field* ξ *is harmonic if and only if it is an eigenvector of the Ricci operator.*

9. Conformal and *D*-homothetic deformations

Let M^{2n+1} be an almost α-paracosymplectic manifold and (ϕ, ξ, η, g) be an almost α-paracosymplectic structure. Let $\mathcal{R}_\eta(M^{2n+1})$ be the set of the locally defined smooth functions f on M^{2n+1} such that $df \wedge \eta = 0$, whenever df is defined.

Let M^{2n+1} be an almost paracontact metric manifold. Let f be a function on $M^{2n+1} f > 0$ everywhere. Consider a deformation of the structure

$$\phi \mapsto \phi' = \phi, \qquad \xi \mapsto \xi' = \frac{1}{f}\xi, \qquad \eta \mapsto \eta' = f\eta, \qquad g \mapsto g' = fg, \tag{9.1}$$

we call (ϕ', ξ', η', g') the conformal deformation of (ϕ, ξ, η, g). Respectively we say that almost paracontact metric manifold $(M^{2n+1}, \phi', \xi', \eta')$ is conformal to $(M^{2n+1}, \phi, \xi, \eta, g)$. Almost paracontact metric manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$ and $(M^{2n+1}, \phi', \xi', \eta', g')$ are called locally conformal if there is an open covering $(U_i)_{i\in I}M^{2n+1} = \bigcup U_i$, such that almost paracontact metric manifolds $(U_i, \phi|_{U_i}, \xi|_{U_i}, \eta|_{U_i}, g|_{U_i})$ and $(U_i, \phi'|_{U_i}, \xi'|_{U_i}, \eta'|_{U_i}, g'|_{U_i})$ are conformal.

Theorem 10. *Arbitrary almost* α*-paracosymplectic manifold* $(M^{2n+1}, \phi, \xi, \eta, g)n \geqslant 2$ *is locally conformal to an almost paracosymplectic manifold. In the other words, near each point* $p \in M^{2n+1}$*, there is defined function u, such that structure* (ϕ', ξ', η', g')

$$\phi' = \phi, \qquad \xi' = e^{2u}\xi, \qquad \eta' = e^{-2u}\eta, \qquad g' = e^{-2u}g, \tag{9.2}$$

is almost paracosymplectic. The function u is unique up to additive constant and $\alpha\eta = du$*.*

Proof. From Proposition 1(vii), form $\beta = \alpha\eta$ is closed, $d\beta = 0$. Hence there is an open covering (U_ι) of M^{2n+1}, and functions $u_\iota : U_\iota \to \mathbb{R}$, such that $du_\iota = \beta|_{U_\iota}$. We define an almost paracontact metric structure on U_ι as in (9.2), with $f = u_\iota$. Then

$$\begin{aligned}
&d\Phi' = 2e^{-2u}(-du + \alpha\eta) \wedge \Phi = 0, \\
&d\eta'^{-2u}du \wedge \eta = -2e^{-2u}\alpha\eta \wedge \eta = 0. \quad \square
\end{aligned}$$

Consider a $D_{\gamma,u}$-homothetic deformation of (ϕ, ξ, η, g) into an almost paracontact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, defined as

$$\tilde{\phi} = \phi, \qquad \tilde{\xi} = \frac{1}{u}\xi, \qquad \tilde{\eta} = u\eta, \qquad \tilde{g} = \gamma g + (u^2 - \gamma)\eta \otimes \eta, \tag{9.3}$$

where γ is a positive constant and $u \in \mathcal{R}_\eta(M^{2n+1})$, $u \neq 0$ at any point of M^{2n+1}. We have $\tilde{\Phi} = \gamma\Phi$. Because of $du \wedge \eta = 0$,

$$d\tilde{\eta} = du \wedge \eta + u\, d\eta = 0 \quad \text{and} \quad d\tilde{\Phi} = 2\left(\frac{\alpha}{u}\right)\tilde{\eta} \wedge \tilde{\Phi}. \tag{9.4}$$

Thus, $D_{\gamma,u}$-homothety of almost α-paracosymplectic structure (ϕ, ξ, η, g) gives almost $(\frac{\alpha}{u})$-paracosymplectic structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on the same manifold.

Following the definition of locally conformal almost paracontact metric manifolds we define the notion of locally $D_{\gamma,\beta}$-homothetic almost α-paracosymplectic manifolds.

Theorem 11. *An almost α-paracosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g) n \geqslant 2$ is locally $D_{1,\alpha}$-homothetic to an almost para-Kenmotsu manifold on the set $U : \alpha \neq 0$.*

Proof. We remark that according to Proposition 1(vii), $\alpha \in \mathcal{R}_\eta(M)$. We set $u = \alpha$ in (9.3), then (9.4) follows that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is almost para-Kenmotsu on a connected component of U. □

Proposition 9. *Let $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be an almost α-paracosymplectic structure obtained from (ϕ, ξ, η, g) by a $D_{\gamma,\beta}$-homothetic deformation. Then we have the following relation between the Levi-Civita connections $\tilde{\nabla}$ and ∇.*

$$\tilde{\nabla}_X Y = \nabla_X Y - \left(\frac{\beta^2 - \gamma}{\beta^2}\right) g(\mathcal{A}X, Y)\xi + \frac{d\beta(\xi)}{\beta}\eta(Y)\eta(X)\xi. \tag{9.5}$$

Proof. By the Koszul formula we have

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = X\tilde{g}(Y, Z) + Y\tilde{g}(X, Z) - Z\tilde{g}(X, Y) + \tilde{g}([X, Y], Z) + \tilde{g}([Z, X], Y) + \tilde{g}([Z, Y], X),$$

for any vector fields X, Y, Z. Using $\tilde{g} = \gamma g + (\beta^2 - \gamma)\eta \otimes \eta$ in the last equation, we obtain

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2\gamma g(\nabla_X Y, Z) + 2\beta d\beta(\xi)\eta(X)\eta(Y)\eta(Z) + 2(\beta^2 - \gamma)\left[\eta(\nabla_X Y)\eta(Z) + g(Y, \nabla_X \xi)\eta(Z)\right]. \tag{9.6}$$

Moreover, $\tilde{g}(\tilde{\nabla}_X Y, Z)$ is equal to

$$\gamma g(\tilde{\nabla}_X Y, Z) + (\beta^2 - \gamma)\eta(\tilde{\nabla}_X Y)\eta(Z) \tag{9.7}$$

and

$$\eta(\tilde{\nabla}_X Y) = \frac{1}{\beta^2}\tilde{g}(\tilde{\nabla}_X Y, \xi). \tag{9.8}$$

Substituting $\tilde{g}(\tilde{\nabla}_X Y, Z)$ and $\eta(\tilde{\nabla}_X Y)$ in (9.6) by (9.7), (9.8), we obtain

$$\begin{aligned}\gamma g(\tilde{\nabla}_X Y, Z) + \frac{(\beta^2 - \gamma)}{\beta^2}\tilde{g}(\tilde{\nabla}_X Y, \xi)\eta(Z) &= \gamma g(\nabla_X Y, Z) + \beta d\beta(\xi)\eta(X)\eta(Y)\eta(Z) \\ &\quad + (\beta^2 - \gamma)\left[\eta(\nabla_X Y)\eta(Z) + g(Y, \nabla_X \xi)\eta(Z)\right].\end{aligned} \tag{9.9}$$

For $Z = \xi$, (9.6) gives

$$\tilde{g}(\tilde{\nabla}_X Y, \xi) = \gamma g(\nabla_X Y, \xi) + \beta d\beta(\xi)\eta(X)\eta(Y) + (\beta^2 - \gamma)\left[\eta(\nabla_X Y) + g(Y, \nabla_X \xi)\right]. \tag{9.10}$$

Using (9.9), (9.10), by direct computations we obtain (9.5). □

Proposition 10. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ and $(\tilde{M}^{2n+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are locally $D_{\gamma,\beta}$-homothetic almost α-paracosymplectic manifolds. Then following identities hold:*

$$\tilde{\mathcal{A}}X = \frac{1}{\beta}\mathcal{A}X, \tag{9.11}$$

$$\tilde{h}X = \frac{1}{\beta}hX, \tag{9.12}$$

$$\tilde{R}(X, Y)\tilde{\xi} = \frac{1}{\beta}R(X, Y)\xi + \frac{1}{\beta^2}d\beta(\xi)\left[\eta(X)\mathcal{A}Y - \eta(Y)\mathcal{A}X\right], \tag{9.13}$$

for any vector fields X, Y, Z.

Proof. From (3.17), (9.3) and (9.5), we obtain (9.11). Formula (9.12) follows from (9.3), if we use the properties of h.

From (9.3) and (9.5), we have

$$\tilde{\nabla}_X \tilde{\nabla}_Y \tilde{\xi} = \nabla_X \tilde{\nabla}_Y \tilde{\xi} - \frac{(\beta^2 - \gamma)}{\beta^2} g(\mathcal{A}X, \tilde{\nabla}_Y \tilde{\xi})\xi + \frac{1}{\beta}d\beta(\xi)\eta(X)\eta(\tilde{\nabla}_Y \tilde{\xi})\xi, \tag{9.14}$$

$$\tilde{\nabla}_Y \tilde{\xi} = \frac{1}{\beta}\nabla_Y \xi. \tag{9.15}$$

By (9.14), (9.15) and properties of $\mathcal{A}$, we get

$$\tilde{\nabla}_X \tilde{\nabla}_Y \tilde{\xi} = \frac{X(\beta)}{\beta^2}\mathcal{A}Y + \frac{1}{\beta}\nabla_X\nabla_Y\xi + \frac{(\beta^2-\gamma)}{\beta^3}g(\mathcal{A}X, \mathcal{A}Y)\xi. \tag{9.16}$$

We compute (9.13) from the curvature formula

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{9.17}$$

using (9.15) and (9.16). □

10. Almost α-para-Kenmotsu (κ, μ, ν)-spaces

In this section we study almost α-para-Kenmotsu manifolds under assumption that the curvature satisfies (κ, μ, ν)-nullity condition

$$R(X, Y)\xi = \eta(Y)BX - \eta(X)BY, \tag{10.1}$$

where B is Jacobi operator of ξ, $BX = R(X, \xi)\xi$, and

$$BX = \kappa\phi^2X + \mu hX + \nu\phi hX, \tag{10.2}$$

for $\kappa, \mu, \nu \in \mathcal{R}_\eta(M^{2n+1})$. Particularly $B\xi = 0$.

If an almost α-paracosymplectic manifold satisfies (10.1), then the manifold is said to be almost α-paracosymplectic (κ, μ, ν)-space.

Using (9.13) and after some calculations one can prove the following proposition.

Proposition 11. *Let M be an almost α-paracosymplectic manifold. If M satisfies nullity conditions, then manifold $\tilde{M}$ obtained by $D_{\gamma,\beta}$-homothety of M also satisfies nullity conditions. In other words if M is a (κ, μ, ν)-space, then $\tilde{M}$ is $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-space and*

$$\tilde{\kappa} = \frac{\kappa}{\beta^2} + \frac{\alpha}{\beta^3}d\beta(\xi), \qquad \tilde{\mu} = \frac{\mu}{\beta}, \qquad \tilde{\nu} = \frac{\nu}{\beta} + \frac{d\beta(\xi)}{\beta^2}.$$

For an almost α-paracosymplectic (κ, μ, ν)-space we may consider scalar invariant with respect to the $D_{\gamma,\beta}$-homotheties, that is function $I(\alpha, \kappa, \mu, \nu)$ with the property that $I(\alpha, \kappa, \mu, \nu) = I(\alpha', \kappa', \mu', \nu')$ for arbitrary $D_{\gamma,\beta}$-homothety. In the case $\mu \neq 0$ by direct computations we find that

$$I_0(\alpha, \kappa, \mu, \nu) = \frac{\kappa - \alpha\nu}{\mu^2}, \tag{10.3}$$

is an invariant. An almost α-paracosymplectic space will be called of constant I_0-type if $\mu \neq 0$ and $I_0 = const$. We may ask the question about number of functionally independent invariants. The answer at this moment is unknown. Also we do not know the geometric interpretation of I_0.

Proposition 12. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-para-Kenmotsu (κ, μ, ν)-space. Then the following identities hold:*

$$l = \kappa\phi^2 + \mu h + \nu\phi h, \tag{10.4}$$
$$l\phi - \phi l = 2\mu h\phi - 2\nu h, \tag{10.5}$$
$$h^2 = (\kappa + \alpha^2)\phi^2, \tag{10.6}$$
$$\nabla_\xi h = -(2\alpha + \nu)h + \mu h\phi, \tag{10.7}$$
$$\nabla_\xi h^2 = -2(2\alpha + \nu)(\kappa + \alpha^2)\phi^2, \tag{10.8}$$
$$\xi(\kappa) = -2(2\alpha + \nu)(\kappa + \alpha^2), \tag{10.9}$$
$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(X, hY)\xi - \eta(Y)hX) + \nu(g(X, \phi hY)\xi - \eta(Y)\phi hX), \tag{10.10}$$
$$Q\xi = 2n\kappa\xi, \tag{10.11}$$
$$(\nabla_X\phi)Y = g(Y, hX + \alpha\phi X)\xi - \eta(Y)(hX + \alpha\phi X), \tag{10.12}$$
$$\begin{aligned}(\nabla_X\phi h)Y - (\nabla_Y\phi h)X = {} & (\kappa + \alpha^2)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ & + (\nu + \alpha)(\eta(Y)\phi hX - \eta(X)\phi hY),\end{aligned} \tag{10.13}$$
$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X = {} & (\kappa + \alpha^2)(\eta(Y)\phi X - \eta(X)\phi Y + 2g(Y, \phi X)\xi) + \mu(\eta(Y)\phi hX - \eta(X)\phi hY) \\ & + (\nu + \alpha)(\eta(Y)hX - \eta(X)hY),\end{aligned} \tag{10.14}$$

for all vector fields X, Y on M^{2n+1}.

Proof. From (10.1) we get

$$lX = R(X,\xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX + \nu\phi hX, \tag{10.15}$$

which gives (10.4). By replacing X by ϕX in (10.15) and next applying ϕ to the result, we obtain

$$l\phi X = \kappa\phi X + \mu h\phi X - \nu hX,$$
$$\phi lX = \kappa\phi X - \mu h\phi X + \nu hX,$$

and (10.5) comes from the last two equations. From (10.15), we easily get

$$\phi l\phi X = \kappa\phi^2 X - \mu hX - \nu\phi hX. \tag{10.16}$$

Then by (10.15) and (10.16), we obtain

$$-lX - \phi l\phi X = 2(\alpha^2\phi^2 X - h^2 X).$$

Comparing this equation with (7.4), we have (10.6). (10.7) can be easily get from (7.3) and (10.6). From (10.6), (10.7) and $\nabla_\xi h^2 = (\nabla_\xi h)h + h(\nabla_\xi h)$, we obtain (10.8). One can easily get (10.9) by differentiating (10.6) along ξ.

In virtue of (10.1), we have

$$\begin{aligned} g(R(Y,Z)\xi, X) = {} & \kappa(\eta(Z)g(X,Y) - \eta(Y)g(X,Z)) \\ & + \mu(\eta(Z)g(X,hY) - \eta(Y)g(X,hZ)) + \nu(\eta(Z)g(X,\phi hY) - \eta(Y)g(X,\phi hZ)). \end{aligned} \tag{10.17}$$

The last equation completes the proofs of (10.10) and (10.11).

Taking into account (10.17), the left hand side of (7.9) can be rewritten as

$$2\kappa(\eta(Z)g(X,Y) - \eta(Y)g(X,Z)).$$

So, (7.9) simplifies to

$$(\nabla_{hX}\Phi)(Y,Z) = (\kappa + \alpha^2)(\eta(Z)g(X,Y) - \eta(Y)g(X,Z)) + \alpha(\eta(Z)g(\phi hX, Y) - \eta(Y)g(\phi hX, Z)).$$

Replacing X by hX in the last equation and next using (10.6) in the result, we get

$$g((\nabla_X\phi)Y, Z) = (\eta(Z)g(hX,Y) - \eta(Y)g(hX,Z)) + \alpha(\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z)). \tag{10.18}$$

Then (10.12) follows from (10.18). On the other hand (10.12) can be written as

$$(\nabla_X\phi)Y = -g(\phi\mathcal{A}X, Y)\xi + \eta(Y)\phi\mathcal{A}X.$$

From (7.1), (10.1) we obtain (10.13). One can easily show that

$$(\nabla_X\phi h)Y - (\nabla_Y\phi h)X = (\nabla_X\phi)hY - (\nabla_Y\phi)hX + \phi((\nabla_X h)Y - (\nabla_Y h)X). \tag{10.19}$$

By (10.12)

$$(\nabla_X\phi)hY = g(hY, hX + \alpha\phi X)\xi. \tag{10.20}$$

by applying ϕ to the obtained result, we have (10.14). □

Theorem 12. *If $(M^{2n+1}, \phi, \xi, \eta, g)$ is an almost α-para-Kenmotsu (κ, μ, ν)-space and the set of points where $\alpha \neq 0$ is dense in M^{2n+1}, then M^{2n+1} has para-Kaehler leaves.*

Proof. We only need to prove this if $n \geqslant 2$, for arbitrary 3-dimensional α-para-Kenmotsu manifold has para-Kaehler leaves. According to Theorem 11, α-paracosymplectic manifold $(U_0, \phi|_{U_0}, \xi|_{U_0}, \eta|_{U_0}, g|_{U_0})$ is $D_{1,\alpha}$-homothetic to almost para-Kenmotsu manifold $(U_0, \phi', \xi', \eta', g')$, (cf. (9.3)), on a connected component $U_0 \subset U$. In virtue of Proposition 5, Eq. (10.12) tells us that U_0, viewed as almost para-Kenmotsu manifold, has para-Kaehler leaves. Moreover, we notice that arbitrary $D_{\gamma,\beta}$-homothety preserves this property, thus we may conclude, that the original structure (ϕ, ξ, η, g), restricted to U_0, also satisfies the para-Kaehler structure. Therefore $(\nabla_X\phi)Y$ satisfies (10.12) on U. As U is dense, (10.12) has to be satisfied everywhere on M^{2n+1}. Now, from Proposition 5, follows that M^{2n+1} has para-Kaehler leaves. □

In our terminology, manifolds of constant sectional curvature c, are almost paracosymplectic $(c, 0, 0)$-spaces.

Corollary 3. *An almost α-para-Kenmotsu manifold of constant sectional curvature has para-Kaehler leaves.*

Corollary 4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-para-Kenmotsu (κ, μ, ν)-space. Then*

$$Q\phi - \phi Q = 2\mu h\phi - 2(\nu + 2\alpha(1-n))h. \tag{10.21}$$

Proof. Using (10.1) and $\phi h = -h\phi$, we obtain $l\phi - \phi l = 2\mu h\phi - 2\nu h$. On the other hand, from (10.11) and Theorem 9 one can easily prove that both $\eta \otimes \phi Q$ and $(\eta \circ Q\phi) \otimes \xi$ vanish. So (10.21) follows from (7.14). $\square$

Remark 3. Manifolds which are conformal or locally conformal to cosymplectic manifolds were studied by many authors, e.g. [28–31,25,32].

Remark 4. $D_{\gamma,\beta}$-homotheties as they appear in almost contact metric geometry are particular class of deformations considered by S. Tanno [33]. The general deformation of a metric (Riemannian) has a form $g' = \alpha g + \omega \otimes \theta + \theta \otimes \omega + \beta\omega \otimes \omega$ where ω, β are one-forms and α, β are functions, $\alpha > 0\alpha + \beta > 0$. The work of Tanno seems to be nowadays completely forgotten within the framework of almost contact metric geometry.

11. Classification of the 3-dimensional almost α-para-Kenmotsu (κ, μ, ν)-spaces

In this section, different possibilities for the tensor field h are investigated. Thus we can comprehend the differences between the almost α-para-Kenmotsu and almost α-Kenmotsu cases by looking at the possible Jordan forms of the tensor field h.

It is well known that a self-adjoint linear operator Ψ of a Euclidean space is always diagonalizable, but this is not the case for a self-adjoint linear operator Ψ for a Lorentzian inner product. It is known [34, pp. 50–55] that self-adjoint linear operator of a vector space with a Lorentzian inner product can be put into four possible canonical forms. In particular, the matrix representation g of the induced metric on M_1^3 is of Lorentz type, so the self-adjoint linear Ψ of M_1^3 can be put into one of the following four forms with respect to frames $\{e_1, e_2, e_3\}$ at $T_pM_1^3$ where $T_pM_1^3$ is a tangent space to M at p [35,36].

$$(\mathfrak{h}_1\text{-type})\ \Psi = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}, \qquad g = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$(\mathfrak{h}_2\text{-type})\ \Psi = \begin{pmatrix}\lambda & 0 & 0\\ 1 & \lambda & 0\\ 0 & 0 & \lambda_3\end{pmatrix}, \qquad g = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$(\mathfrak{h}_3\text{-type})\ \Psi = \begin{pmatrix}\gamma & -\lambda & 0\\ \lambda & \gamma & 0\\ 0 & 0 & \lambda_3\end{pmatrix}, \qquad g = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \lambda \neq 0,$$

$$(\mathfrak{h}_4\text{-type})\ \Psi = \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 1\\ 1 & 0 & \lambda\end{pmatrix}, \qquad g = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The matrices g for types $(\mathfrak{h}_1)$ and $(\mathfrak{h}_3)$ are with respect to an orthonormal basis of $T_pM_1^3$, whereas for types $(\mathfrak{h}_2)$ and $(\mathfrak{h}_4)$ are with respect to a pseudo-orthonormal basis. This is a basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$ satisfying $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$.

Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-paracosymplectic manifold. Then operator h has following types. $(\mathfrak{h}_1\text{-type})$

$$U_1 = \{p \in M \mid h(p) \neq 0\} \subset M$$

$$U_2 = \{p \in M \mid h(p) = 0, \text{ in a neighborhood of } p\} \subset M$$

That h is a smooth function on M implies $U_1 \cup U_2$ is an open and dense subset of M, so any property satisfied in $U_1 \cup U_2$ is also satisfied in M. For any point $p \in U_1 \cup U_2$ there exists a local orthonormal ϕ-basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p, where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. On U_1 we put $he = \lambda e$, where λ is a non-vanishing smooth function. Since $trh = 0$, we have $h\phi e = -\lambda\phi e$. The eigenvalue function λ is continuous on M and smooth on $U_1 \cup U_2$. So, h has following form

$$\begin{pmatrix}\lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & 0\end{pmatrix} \tag{11.1}$$

respect to local orthonormal ϕ-basis $\{e, \phi e, \xi\}$.

$(\mathfrak{h}_2\text{-type})$ Using same methods in [12], one can construct a local pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ in a neighborhood of p where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$. Let $\mathcal{U}$ be the open subset of M where $h \neq 0$. For every $p \in \mathcal{U}$ there exists an open neighborhood of p such that $he_1 = e_2$, $he_2 = 0$, $he_3 = 0$ and $\phi e_1 = \pm e_1 \phi e_2 = \mp e_2$, $\phi e_3 = 0$ and also $\xi = e_3$. Thus the tensor h has the form

$$\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{11.2}$$

relative a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$.

($\mathfrak{h}_3$-type) We can find a local orthonormal ϕ-basis $\{e, \phi e, \xi\}$ in a neighborhood of p where $-g(e,e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. Now, let $\mathcal{U}_1$ be the open subset of M where $h \neq 0$ and let $\mathcal{U}_2$ be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open subset of M. For every $p \in \mathcal{U}_1$ there exists an open neighborhood of p such that $he = \lambda \phi e$, $h\phi e = -\lambda e$ and $h\xi = 0$ where λ is a non-vanishing smooth function. Since $trh = 0$, the matrix form of h is given by

$$\begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.3}$$

with respect to local orthonormal basis $\{e, \phi e, \xi\}$.

($\mathfrak{h}_4$-type) A local pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ is constructed in a neighborhood of p where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$. Since the tensor h is ($\mathfrak{h}_4$-type) (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$) then $he_1 = \lambda e_1 + e_3$, $he_2 = \lambda e_2$ and $he_3 = e_2 + \lambda e_3$. Since $0 = trh = g(he_1, e_2) + g(he_2, e_1) + g(he_3, e_3) = 3\lambda$, then $\lambda = 0$. We write $\xi = g(\xi, e_2)e_1 + g(\xi, e_1)e_2 + g(\xi, e_3)e_3$ respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3\}$. Since $h\xi = 0$, we have $0 = g(\xi, e_2)e_3 + g(\xi, e_3)e_2$. Hence we get $\xi = g(\xi, e_1)e_2$ which leads to a contradiction with $g(\xi, \xi) = 1$. Thus, this case does not occur.

Since the proof of the following lemma is similar to [12], we omit the proof.

Lemma 5. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold. Then a canonical form of h stays constant in an open neighborhood of any point for h.*

In a 3-dimensional pseudo-Riemannian manifold case, the curvature tensor can be written by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y), \tag{11.4}$$

for any $X, Y, Z \in \Gamma(TM)$.

Using same procedure with [37], we have

Lemma 6. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_1$ type. Then for the covariant derivative on $\mathcal{U}_1$ the following equations are valid*

$$\begin{aligned}
&\text{(i) } \nabla_e e = \frac{1}{2\lambda}[\sigma(e) - (\phi e)(\lambda)]\phi e + \alpha\xi, \qquad (11.5)\\
&\text{(ii) } \nabla_e \phi e = \frac{1}{2\lambda}[\sigma(e) - (\phi e)(\lambda)]e - \lambda\xi\\
&\text{(iii) } \nabla_e \xi = \alpha e + \lambda\phi e,\\
&\text{(iv) } \nabla_{\phi e} e = -\frac{1}{2\lambda}[\sigma(\phi e) + e(\lambda)]\phi e - \lambda\xi,\\
&\text{(v) } \nabla_{\phi e}\phi e = -\frac{1}{2\lambda}[\sigma(\phi e) + e(\lambda)]\phi e - \alpha\xi,\\
&\text{(vi) } \nabla_{\phi e}\xi = \alpha\phi e - \lambda e,\\
&\text{(vii) } \nabla_\xi e = a_1\phi e, \qquad \text{(viii) } \nabla_\xi \phi e = a_1 e,\\
&\text{(ix) } [e, \xi] = \alpha e + (\lambda - a_1)\phi e,\\
&\text{(x) } [\phi e, \xi] = -(\lambda + a_1)e + \alpha\phi e,\\
&\text{(xi) } [e, \phi e] = \frac{1}{2\lambda}[\sigma(e) - (\phi e)(\lambda)]e + \frac{1}{2\lambda}[\sigma(\phi e) + e(\lambda)]\phi e,\\
&\text{(xii) } \nabla_\xi h = \xi(\lambda)s - 2a_1 h\phi, \qquad \text{(xiii) } h^2 - \alpha^2\phi^2 = \frac{1}{2}S(\xi,\xi)\phi^2
\end{aligned}$$

where

$$a_1 = g(\nabla_\xi e, \phi e), \qquad \sigma = S(\xi, \cdot)_{\ker\eta}.$$

Lemma 7. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_1$ type. Then the Ricci operator Q is given by*

$$\begin{aligned}
Q = {}& \left(\frac{r}{2} + \alpha^2 - \lambda^2\right)I + \left(-\frac{r}{2} + 3(\lambda^2 - \alpha^2)\right)\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h)\\
&+ \sigma(\phi^2) \otimes \xi - \sigma(e)\eta \otimes e + \sigma(\phi e)\eta \otimes \phi e.
\end{aligned} \tag{11.6}$$

Lemma 8. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_2$ type. Then for the covariant derivative on $\mathcal{U}$ the following equations are valid*

$$\begin{array}{lll} \text{(i) } \nabla_{e_1} e_1 = -b_1 e_1 + \xi, & \text{(ii) } \nabla_{e_1} e_2 = b_1 e_2 - \alpha\xi, & \text{(iii) } \nabla_{e_1}\xi = \alpha e_1 - e_2, \\ \text{(iv) } \nabla_{e_2} e_1 = -b_2 e_1 - \alpha\xi, & \text{(v) } \nabla_{e_2} e_2 = b_2 e_2, & \text{(vi) } \nabla_{e_2}\xi = \alpha e_2, \\ \text{(vii) } \nabla_{\xi} e_1 = a_2 e_1, & \text{(viii) } \nabla_{\xi} e_2 = -a_2 e_2, & \\ \text{(ix) } [e_1, \xi] = (\alpha - a_2) e_1 - e_2, & \text{(x) } [e_2, \xi] = (\alpha + a_2) e_2, & \\ \text{(xi) } [e_1, e_2] = b_2 e_1 + b_1 e_2, & & \\ \text{(xii) } \nabla_{\xi} h = -2a_2 h\phi, & \text{(xiii) } h^2 = 0. & \end{array} \tag{11.7}$$

where $a_2 = g(\nabla_{\xi} e_1, e_2) b_1 = g(\nabla_{e_1} e_2, e_1)$ and $b_2 = g(\nabla_{e_2} e_2, e_1) = -\frac{1}{2}\sigma(e_1)$.

Proof. By $\nabla\xi = -\alpha^2\phi + \phi h$, we obtain (iii), (vi).

Using pseudo-orthonormal basis $\{e_1, e_2, e_3 = \xi\}$ with $\phi e_1 = e_1$, $\phi e_2 = -e_2$, $\phi e_3 = 0$ we have

$$\begin{aligned} \nabla_{e_1} e_2 &= g(\nabla_{e_1} e_2, e_2) e_1 + g(\nabla_{e_1} e_2, e_1) e_2 + g(\nabla_{e_1} e_2, \xi)\xi \\ &= g(\nabla_{e_1} e_2, e_1) e_2 - g(e_2, \nabla_{e_1}\xi)\xi \\ &\overset{\text{(iii)}}{=} g(\nabla_{e_1} e_2, e_1) e_2 - \alpha\xi \\ &= b_1 e_2 - \alpha\xi. \end{aligned}$$

The proofs of other covariant derivative equalities are similar to (ii).

Putting $X = e_1$, $Y = e_2$ and $Z = \xi$ in Eq. (11.4), we have

$$R(e_1, e_2)\xi = -\sigma(e_1) e_2 + \sigma(e_2) e_1. \tag{11.8}$$

On the other hand, by using (7.1), we get

$$\begin{aligned} R(e_1, e_2)\xi &= (\nabla_{e_1}\phi h) e_2 - (\nabla_{e_2}\phi h) e_1 \\ &= 2b_2 e_2. \end{aligned} \tag{11.9}$$

Comparing (11.9) with (11.8), we obtain

$$\sigma(e_1) = -2b_2, \qquad \sigma(e_2) = 0 = S(\xi, e_2). \tag{11.10}$$

Hence, the function b_2 is obtained from the last equation. □

Lemma 9. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_2$ type. Then the Ricci operator Q is given by*

$$Q = \left(\frac{r}{2} + \alpha^2\right) I - \left(\frac{r}{2} + 3\alpha^2\right) \eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_{\xi} h) + \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_2. \tag{11.11}$$

Proof. From (11.4), we obtain

$$R(X, \xi)\xi = S(\xi, \xi) X - S(X, \xi)\xi + QX - \eta(X) Q\xi - \frac{r}{2}(X - \eta(X)\xi),$$

for any vector field X. By (7.2) and (7.6) the last equation reduces to

$$QX = \frac{1}{2} S(\xi, \xi)\phi^2 X - 2\alpha\phi h X - \phi(\nabla_{\xi} h) X - S(\xi, \xi) X + S(X, \xi)\xi + \eta(X) Q\xi + \frac{r}{2}(X - \eta(X)\xi). \tag{11.12}$$

By setting $S(X, \xi) = S(\phi^2 X, \xi) + \eta(X) S(\xi, \xi)$ in (11.12), we have

$$\begin{aligned} QX = {} & \frac{S(\xi, \xi)}{2}\phi^2 X - 2\alpha\phi h X - \phi(\nabla_{\xi} h) X - S(\xi, \xi) X \\ & + S(\phi^2 X, \xi)\xi + \eta(X) S(\xi, \xi)\xi + \eta(X) Q\xi + \frac{r}{2}\phi^2 X. \end{aligned} \tag{11.13}$$

On the other hand, the Ricci tensor S can be written with respect to the orthonormal basis $\{e_1, e_2, \xi\}$ as follows

$$Q\xi = \sigma(e_1) e_2 + S(\xi, \xi)\xi. \tag{11.14}$$

Using (11.14) in (11.13), we get

$$QX = \frac{1}{2}\left(r + 2\alpha^2\right) X - \frac{1}{2}(6\alpha^2 + r)\eta(X)\xi - 2\alpha\phi h - \phi(\nabla_{\xi} h) X + \sigma(\phi^2 X)\xi + \eta(X)\sigma(e_1) e_2 \tag{11.15}$$

for arbitrary vector field X. This ends the proof. □

Lemma 10. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_3$ type. Then for the covariant derivative on $\mathcal{U}_1$ the following equations are valid*

$$\begin{aligned}
&\text{(i) } \nabla_e e = b_3\phi e + (\alpha+\lambda)\xi, \qquad \text{(ii) } \nabla_e \phi e = b_3 e, \qquad \text{(iii) } \nabla_e \xi = (\alpha+\lambda)e,\\
&\text{(iv) } \nabla_{\phi e} e = b_4\phi e, \qquad \text{(v) } \nabla_{\phi e}\phi e = b_4 e + (\lambda-\alpha)\xi, \qquad \text{(vi) } \nabla_{\phi e}\xi = -(\lambda-\alpha)\phi e,\\
&\text{(vii) } \nabla_\xi e = a_3\phi e, \qquad \text{(viii) } \nabla_\xi \phi e = a_3 e,\\
&\text{(ix) } [e,\xi] = (\alpha+\lambda)e - a_3\phi e, \qquad \text{(x) } [\phi e,\xi] = -a_3 e - (\lambda-\alpha)\phi e,\\
&\text{(xi) } [e,\phi e] = b_3 e - b_4\phi e,\\
&\text{(xii) } \nabla_\xi h = \xi(\lambda)s - 2a_3 h\phi, \qquad \text{(xiii) } h^2 - \alpha^2\phi^2 = \frac{1}{2}S(\xi,\xi)\phi^2,
\end{aligned} \tag{11.16}$$

where $a_3 = g(\nabla_\xi e, \phi e)$, $b_3 = -\frac{1}{2\lambda}\left[\sigma(\phi e) + (\phi e)(\lambda)\right]$ *and* $b_4 = \frac{1}{2\lambda}\left[\sigma(e) - e(\lambda)\right]$.

Proof. By $\nabla\xi = \alpha\phi^2 + \phi h$, we have (iii), (vi).

Using ϕ-basis, we have

$$\begin{aligned}
\nabla_\xi\phi e &= -g(\nabla_\xi\phi e, e)e + g(\nabla_\xi\phi e, \phi e)\phi e + g(\nabla_\xi\phi e, \xi)\xi\\
&= g(\phi e, \nabla_\xi e)e = a_3 e,
\end{aligned}$$

So we prove (viii). The proofs of other covariant derivative equalities are similar to (viii).

Setting $X = e, Y = \phi e, \ Z = \xi$ in Eq. (11.4), we have

$$R(e,\phi e)\xi = -g(Qe,\xi)\phi e + g(Q\phi e,\xi)e.$$

Since $\sigma(X) = g(Q\xi, X)$, we have

$$R(e,\phi e)\xi = -\sigma(e)\phi e + \sigma(\phi e)e. \tag{11.17}$$

On the other hand, by using (7.1), we have

$$\begin{aligned}
R(e,\phi e)\xi &= (\nabla_e\phi h)\phi e - (\nabla_{\phi e}\phi h)e\\
&= (-2b_3\lambda - (\phi e)(\lambda))e + (-2b_4\lambda - e(\lambda))\phi e.
\end{aligned} \tag{11.18}$$

Comparing (11.18) with (11.17), we get

$$\sigma(e) = e(\lambda) + 2b_4\lambda, \quad \sigma(\phi e) = -(\phi e)(\lambda) - 2b_3\lambda.$$

Hence, the functions b_3 and b_4 are obtained from the last equation. □

Lemma 11. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_3$ type. Then the Ricci operator Q is given by*

$$Q = aI + b\eta\otimes\xi - 2\alpha\phi h - \phi(\nabla_\xi h) + \sigma(\phi^2)\otimes\xi - \sigma(e)\eta\otimes e + \sigma(\phi e)\eta\otimes\phi e, \tag{11.19}$$

where a and b are smooth functions defined by $a = \alpha^2 + \lambda^2 + \frac{r}{2}$ and $b = -3(\lambda^2+\alpha^2) - \frac{r}{2}$, respectively.

Proof. Using (11.4), we get

$$R(X,\xi)\xi = S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi - \frac{r}{2}(X - \eta(X)\xi),$$

for any vector field X. By (7.2), the last equation reduces to

$$QX = -\alpha^2\phi^2X + h^2X - 2\alpha\phi hX - \phi(\nabla_\xi h)X - S(\xi,\xi)X + S(X,\xi)\xi + \eta(X)Q\xi + \frac{r}{2}(X - \eta(X)\xi). \tag{11.20}$$

By writing $S(X,\xi) = S(\phi^2X,\xi) + \eta(X)S(\xi,\xi)$ in (11.20), we obtain

$$\begin{aligned}
QX = {} & \frac{S(\xi,\xi)}{2}\phi^2X - 2\alpha\phi hX - \phi(\nabla_\xi h)X - S(\xi,\xi)X + S(\phi^2X,\xi)\xi\\
& + \eta(X)S(\xi,\xi)\xi + \eta(X)Q\xi + \frac{r}{2}\phi^2X.
\end{aligned} \tag{11.21}$$

On the other hand S can be written with respect to the orthonormal basis $\{e, \phi e, \xi\}$ as

$$Q\xi = -\sigma(e)e + \sigma(\phi e)\phi e + S(\xi, \xi)\xi. \tag{11.22}$$

Using (11.22) in (11.21), we have

$$\begin{aligned} QX = &\left(\alpha^2 + \lambda^2 + \frac{r}{2}\right)X + \left(-3(\lambda^2 + \alpha) - \frac{r}{2}\right)\eta(X)\xi - 2\alpha\phi hX \\ &- \phi(\nabla_\xi h)X + \sigma(\phi^2 X)\xi - \eta(X)\sigma(e)e + \eta(X)\sigma(\phi e)\phi e, \end{aligned} \tag{11.23}$$

for arbitrary vector field X. This completes the proof. □

Theorem 13. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold. If the characteristic vector field ξ is a harmonic map then almost α-paracosymplectic (κ, μ, ν)-manifold always exist on every open and dense subset of M. Conversely, if M is an almost α-paracosymplectic (κ, μ, ν)-manifold with constant α then the characteristic vector field ξ is a harmonic map.*

Proof. We will prove theorem for three cases respect to chosen (pseudo) orthonormal basis.

Case 1: We assume that h is $\mathfrak{h}_1$ type.

Since ξ is a harmonic vector field, ξ is an eigenvector of Q. Hence we deduce that $\sigma = 0$. Putting $s = \frac{1}{\lambda}h$ in (11.5)(xii), we find

$$Q = \left(\frac{r}{2} + \alpha^2 - \lambda^2\right)I + \left(-\frac{r}{2} + 3(\lambda^2 - \alpha^2)\right)\eta \otimes \xi - 2a_1 h - \left(2\alpha + \frac{\xi(\lambda)}{\lambda}\right)\phi h. \tag{11.24}$$

Setting $Z = \xi$ in (11.4) and using (11.24), we obtain

$$R(X, Y)\xi = (-\alpha^2 + \lambda^2)(\eta(Y)X - \eta(X)Y) - 2a_1(\eta(Y)hX - \eta(X)hY) - \left(2\alpha + \frac{\xi(\lambda)}{\lambda}\right)(\eta(Y)\phi hX - \eta(X)\phi hY),$$

where the functions κ, μ and ν defined by $\kappa = \frac{S(\xi,\xi)}{2} = (\lambda^2 - \alpha^2)$, $\mu = -2a_1$, $\nu = -(2\alpha + \frac{\xi(\lambda)}{\lambda})$, respectively. Moreover, using (11.24), we have $Q\phi - \phi Q = 2\mu h\phi - 2\nu h$.

Case 2: Secondly, let h be $\mathfrak{h}_2$ type.

Putting $\sigma = 0$ in (11.11) and using (11.7) (xii), we get

$$Q = \left(\frac{r}{2} + \alpha^2\right)I - \left(\frac{r}{2} + 3\alpha^2\right)\eta \otimes \xi - 2a_2 h - 2\alpha\phi hX. \tag{11.25}$$

When $\xi = Z$ in (11.4) we obtain

$$R(X, Y)\xi = -S(X, \xi) + S(Y, \xi) - \eta(X)QY + \eta(Y)QX + \frac{r}{2}(\eta(X)Y - \eta(Y)X), \tag{11.26}$$

for any vector fields X, Y. By applying (11.25) in (11.26), we have

$$R(X, Y)\xi = -\alpha^2(\eta(Y)X - \eta(X)Y) - 2a_2(\eta(Y)hX - \eta(X)hY) - 2\alpha(\eta(Y)\phi hX - \eta(X)\phi hY)$$

where the functions κ, μ and ν defined by $\kappa = \frac{S(\xi,\xi)}{2} = -\alpha^2$, $\mu = -2a_2$, $\nu = -2\alpha$, respectively. Furthermore, by (11.25), we have $Q\phi - \phi Q = 2\mu h\phi - 2\nu h$.

Case 3: Finally, we suppose that h is $\mathfrak{h}_3$ type.

Since ξ is a harmonic map, we have $\sigma = 0$. Putting $s = \frac{1}{\lambda}h$ in (11.19) we get

$$Q = aI + b\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h), \tag{11.27}$$

Setting $\xi = Z$ in (11.4) we again obtain

$$R(X, Y)\xi = -S(X, \xi) + S(Y, \xi) - \eta(X)QY + \eta(Y)QX + \frac{r}{2}(\eta(X)Y - \eta(Y)X), \tag{11.28}$$

for any vector fields X, Y. Using (11.27) in (11.28), we get

$$R(X, Y)\xi = -(\alpha^2 + \lambda^2)(\eta(Y)X - \eta(X)Y) - 2a_3(\eta(Y)hX - \eta(X)hY) - \left(2\alpha + \frac{\xi(\lambda)}{\lambda}\right)(\eta(Y)\phi hX - \eta(X)\phi hY),$$

where the functions κ, μ and ν are defined by $\kappa = -(\alpha^2 + \lambda^2)$, $\mu = -2a_3$, $\nu = -(2\alpha + \frac{\xi(\lambda)}{\lambda})$, respectively. With the help of (11.27), we get $Q\phi - \phi Q = 2\mu h\phi - 2\nu h$.

Conversely, let M be an almost α-para-Kenmotsu (κ, μ, ν)-manifold. Using Theorem 9 and (10.11), we conclude that ξ is harmonic.

This completes the proof. □

12. Examples

Example 1. We consider the 3-dimensional manifold

$$M = \left\{(x, y, z) \in R^3 \mid x \neq 0,\ y \neq 0\right\}$$

and the vector fields

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \phi e_1 = \frac{\partial}{\partial y}, \qquad e_3 = \xi = x\frac{\partial}{\partial x} + (y+2x)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dz$ and the fundamental 2-form $\Phi = -dx\wedge dy + (y+2x)dx\wedge dz - xdy\wedge dz$ defines an almost para-Kenmotsu manifold.

Let g, ϕ be the pseudo-Riemannian metric and the $(1, 1)$-tensor field given by

$$g = \begin{pmatrix} 1 & 0 & \frac{-x}{2} \\ 0 & -1 & \frac{y+2x}{2} \\ \frac{-x}{2} & \frac{y+2x}{2} & 1-3x^2-4xy-y^2 \end{pmatrix},$$

$$\phi = \begin{pmatrix} 0 & 1 & -(y+2x) \\ 1 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We easily get

$$[e_1, e_2] = 0,$$
$$[e_1, e_3] = e_1 + 2e_2,$$
$$[e_2, e_3] = e_2.$$

Moreover, the above example is an almost para-Kenmotsu $(\kappa, \mu, \nu) = (1, 1, -2)$-space.

Example 2. We consider the 3-dimensional manifold

$$M = \left\{(x, y, z) \in R^3 \mid x \neq 0,\ y \neq 0, z \neq 0\right\}$$

and the vector fields

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = \xi = x\,(1-z)\frac{\partial}{\partial x} + (-x + y(1+z))\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$g(e_1, e_1) = g(e_1, e_3) = g(e_2, e_2) = g(e_2, e_3) = 0, \qquad g(e_1, e_2) = g(e_3, e_3) = 1.$$

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The 1-form $\eta = dz$ and the fundamental 2-form $\Phi = dx\wedge dy + (x - y(1+z))dx\wedge dz + (x\,(1-z))dy\wedge dz$ defines an almost α-para-Kenmotsu manifold.

Let g, ϕ be the pseudo-Riemannian metric and the $(1, 1)$-tensor field given by

$$g = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2}(x - y(1+z)) \\ \frac{1}{2} & 0 & \frac{1}{2}(-x\,(1-z)) \\ \frac{1}{2}(x - y(1+z)) & \frac{1}{2}(-x\,(1-z)) & 1-2x(1-z)(x-y)(1+z) \end{pmatrix},$$

$$\phi = \begin{pmatrix} 1 & 0 & -x\,(1-z) \\ 0 & -1 & -x + y(1+z) \\ 0 & 0 & 0 \end{pmatrix}.$$

We easily get

$$\phi e_1 = e_1, \qquad \phi e_2 = -e_2, \qquad \phi\xi = 0,$$
$$[e_1, e_2] = 0,$$
$$[e_1, e_3] = (1 - z)e_1 - e_2,$$
$$[e_2, e_3] = (1 + z)e_2.$$

Moreover, the above example is an almost α-para-Kenmotsu $(\kappa, \mu, \nu) = (-1, -2z, -2)$-space.

Remark 5. To our knowledge, the above example is the first numerical example of almost α-para-Kenmotsu satisfying $\kappa = -1$ and $h \neq 0$ in R^3.

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