Journal of Geometry and Physics 146 (2019) 103492

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Noncommutative Kähler structure on C^*-dynamical systems

Satyajit Guin

Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Uttar Pradesh 208016, India

ARTICLE INFO

Article history:
Received 12 January 2019
Received in revised form 3 July 2019
Accepted 15 August 2019
Available online 19 August 2019

MSC:
58B34
46L87

Keywords:
Noncommutative geometry
Complex structure
Kähler structure
Spectral triple
C^*-dynamical system
Noncommutative tori

ABSTRACT

Notions of noncommutative complex and Kähler structure have been introduced by Fröhlich et al. (1999), in the context of supersymmetric quantum theory. Here we show that whenever a C^*-dynamical system $(\mathcal{A}, G, \alpha, \tau)$ equipped with a faithful G-invariant trace τ, where G is an even dimensional abelian Lie group, determines a spectral triple, the smooth dense subalgebra $\mathcal{A}^\infty$ inherits a noncommutative Kähler structure. In particular, whenever $\mathbb{T}^{2n}$ acts ergodically on the algebra, it inherits a noncommutative Kähler structure. This produces a class of examples of noncommutative Kähler manifolds. As a corollary, we obtain that all the noncommutative even dimensional tori are noncommutative Kähler manifolds. We explicitly compute the space of complex differential forms and study holomorphic vector bundles on all noncommutative even dimensional tori.

1. Introduction

Classical differential geometry was extended to the noncommutative world of C^*-algebras in the early 80s by Connes in [7], and subsequently in [8]. Many highly singular (and classically intractable) objects such as the dual of a discrete group, Penrose tilings or quantum groups may be analyzed by applying cyclic cohomology, K-theory and other tools of noncommutative geometry. Apart from its own mathematical beauty, several fruitful applications of noncommutative geometry in physics (see for e.g. [10,15,37]) have been observed. Despite much progress in noncommutative geometry in past 30 years, noncommutative complex geometry is not developed that much yet. Connes–Cuntz first outlined a possible approach to the idea of a complex structure in noncommutative geometry based on the notion of positive Hochschild cocycle on an involutive algebra [14]. In ([8], Section VI.2) Connes shows that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. However, the corresponding problem of characterizing holomorphic structures on n-dimensional manifolds via positive Hochschild cocycles is still open.

Coming to concrete examples, a detail study of complex structure on noncommutative two-torus and holomorphic vector bundles on them is carried out in [32], taking motivation from [18,36]. Complex structure on the Podleś sphere is studied in [31] using a frame bundle approach, and simultaneously but independently in [24,25] using a classification of the covariant first order differential calculi of the irreducible quantum flag manifolds. Latter in [29], properties of the q-Dolbeault complex of [31] are formalized and it was shown to resemble in many aspects the analogous structure on the classical Riemann sphere. See [30] for the case of higher dimensional quantum projective spaces. A more comprehensive

E-mail address: sguin@iitk.ac.in.

https://doi.org/10.1016/j.geomphys.2019.103492
0393-0440/

version of noncommutative complex structure appeared latter in [2] and complex structure on quantum homogeneous spaces is studied in [3]. The main tool used in all these examples is the Woronowicz's differential calculus for quantum groups [39]. In this algebraic setting, recently the notion of Kähler structure has been introduced in [4] for quantum homogeneous spaces, taking the quantum flag manifolds as motivating family of examples. However, our approach (based on [20]) in this article is different from this, taking the noncommutative torus as motivating example. We discuss it now.

In noncommutative geometry, a (noncommutative) manifold is described by a triple called spectral triple. That the notion of spectral triple is the correct noncommutative generalization of classical manifolds is shown by Connes [13]. However, it turns out that the notion of spectral triple is not quite appropriate to describe the higher geometric structures, e.g. complex, Hermitian, Kähler or hyper-Kähler structures, even in the classical setting. Around '98, a decent approach to noncommutative complex, Hermitian, Kähler and hyper-Kähler geometry has been initiated by Fröhlich et al. [19,20] in the context of supersymmetric quantum theory. Unlike the case of above discussed examples, where the approach is algebraic, methods of Fröhlich et al. are geometric and analytic in the sense that spectral triple lies at the heart of it and integration theory is built-in using β-KMS state. Taking inspiration from Witten's supersymmetric approach to the Morse inequalities [38] and the work of Jaffe et al. on connections between cyclic cohomology and supersymmetry [28], Fröhlich et al. obtained the supersymmetric algebraic formulation of Riemannian, spin, symplectic, complex, Hermitian, Kähler and hyper-Kähler geometry in [19], which then readily generalizes to the noncommutative geometry framework of spectral triples in [20]. See also §3.B in [27] for discussion. It is important to mention here that there are well known links between supersymmetric σ-models and the geometry of manifolds [1]. The approach of Fröhlich et al. starts with a spectral triple and detects the precise analytic conditions required to obtain the complex, Hermitian, Kähler and hyper-Kähler structures on it. They have denoted these various higher geometric structures by $N = 1$, $N = 2$ and $N = (n, n)$ with $n = 1, 2, 4$, along the line of supersymmetry. We denote the relationship among these geometric structures vaguely by $N = (4, 4) \preccurlyeq N = (2, 2) \preccurlyeq N = (1, 1) \preccurlyeq N = 1$ to mean that the former is obtained from the latter by imposing certain additional conditions. Among these, our concern in this article are the $N = 1$, $N = (1, 1)$ and $N = (2, 2)$ geometries. We believe that our results will extend to the $N = (4, 4)$ case also but this needs further investigation. Note that the $N = 1$ data is specified by a Θ-summable even spectral triple in noncommutative geometry, and the $N = (2, 2)$ data extends the notion of Hermitian and Kähler manifolds to noncommutative geometry. For precise definitions see Section 2 Definitions 2.2–2.6. We will call these various higher geometric structures as the $N = \bullet$ or $N = (\bullet\,, \bullet)$ spectral data in this article. In the classical case of a spin manifold $\mathbb{M}$, from the $N = (1, 1)$ spectral data one may recover the graded algebra of differential forms on $\mathbb{M}$ and in particular, the exterior differential.

We now briefly describe our results. Let G be an even dimensional abelian Lie group acting strongly (by action $\alpha : G \curvearrowright \mathcal{A}$) on a unital C^*-algebra $\mathcal{A}$ and τ be a G-invariant faithful trace on $\mathcal{A}$, so that the quadruple $(\mathcal{A}, G, \alpha, \tau)$ forms a C^*-dynamical system equipped with a faithful G-invariant trace. This is in line with [7,16,35]. Let $\mathcal{A}^\infty$ be the smooth dense unital subalgebra of $\mathcal{A}$ under the action of G. We prove that whenever this dynamical system determines a $N = 1$ spectral data (i.e. an even spectral triple) on $\mathcal{A}^\infty$ (there is always a candidate which we explicitly mention), then it always extends to $N = (2, 2)$ Kähler spectral data i.e. $\mathcal{A}^\infty$ inherits a Kähler structure. Moreover, there are at least $\prod_{j=1,\, j\, odd}^{dim(G)}(dim(G) - j)$ different Kähler structures. In particular, whenever $G = \mathbb{T}^{2n}$ acts ergodically on the algebra, it inherits a Kähler structure. This produces a class of examples of noncommutative Kähler manifolds. As a corollary, we obtain that all the noncommutative even dimensional tori, like their classical counterpart the complex tori, are indeed noncommutative Kähler manifolds. Note that in the noncommutative situation, noncommutative two-torus was the only known example of noncommutative Kähler manifold [20] (apart from the ones recently produced in [4] by taking a different approach). As an application, we consider the particular case of noncommutative even dimensional tori and explicitly compute the associated space of complex differential forms. At the end we study holomorphic vector bundles on these and explain how the earlier set-up of Polishchuk–Schwarz [32] for the case of noncommutative two-torus follows as a special case of our general framework for C^*-dynamical systems. For a $4n$-dimensional abelian Lie group whether the Kähler structure obtained here extends further to a hyper-Kähler structure is left as an open question.

The organization of the paper is as follows. In Section 2 we recall from [20] few essential definitions and a procedure to extend a $N = 1$ spectral data to $N = (1, 1)$ spectral data over the same noncommutative base space using a suitable connection on a finitely generated projective module equipped with Hermitian structure. Using this extension procedure we prove the following theorem in Section 3.

Theorem 1.1. *Let G be an even dimensional abelian Lie group and $(\mathcal{A}, G, \alpha, \tau)$ be a C^*-dynamical system equipped with a faithful G-invariant trace τ. Whenever it determines a Θ-summable even spectral triple, the smooth dense subalgebra $\mathcal{A}^\infty$ inherits a Kähler structure.*

As corollaries we obtain the following results.

Corollary 1.2. *If $(\mathcal{A}, \mathbb{T}^{2k}, \alpha)$ is a C^*-dynamical system such that the action of $\mathbb{T}^{2k}$ is ergodic, then the smooth dense subalgebra $\mathcal{A}^\infty$ inherits a Kähler structure.*

Corollary 1.3. *For n even, the noncommutative n-torus $\mathcal{A}_\Theta$ satisfies the $N = (2, 2)$ Kähler spectral data, i.e. these are noncommutative Kähler manifolds.*

In Section 4, we explicitly compute the space of complex differential forms on all noncommutative even dimensional tori and finally, in Section 5 we study holomorphic vector bundles.

2. Preliminaries

All algebras considered in this article will be assumed unital.

Definition 2.1. A triple $(\mathcal{A}, \mathcal{H}, D)$ is called a spectral triple if

(1) $\mathcal{A}$ is a unital associative $*$-algebra represented faithfully on the separable Hilbert space $\mathcal{H}$ by bounded operators;
(2) D is an unbounded self-adjoint operator acting on $\mathcal{H}$ such that for each $a \in \mathcal{A}$

  (a) the commutator $[D, a]$ extends uniquely to a bounded operator on $\mathcal{H}$,
  (b) D has compact resolvent.

If there is a $\mathbb{Z}_2$-grading operator on $\mathcal{H}$ such that $[\gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\{\gamma, D\} = 0$ then the spectral triple is called *even*, and otherwise *odd*. Note that D has compact resolvent is equivalent to saying that $exp(-\varepsilon D^2)$ is a compact operator for all $\varepsilon > 0$. If $|D|^{-p}$ is in the Dixmier ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ then the spectral triple is called p-summable.

Definition 2.2. A quadruple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is called a set of $N = 1$ spectral data if

(1) $\mathcal{A}$ is a unital associative $*$-algebra represented faithfully on the separable Hilbert space $\mathcal{H}$ by bounded operators;
(2) D is an unbounded self-adjoint operator acting on $\mathcal{H}$ such that for each $a \in \mathcal{A}$

  (a) the commutator $[D, a]$ extends uniquely to a bounded operator on $\mathcal{H}$,
  (b) the operator $exp(-\varepsilon D^2)$ is trace class for all $\varepsilon > 0$;

(3) γ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$ such that $[\gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\{\gamma, D\} = 0$.

Remark 2.3. Observe that the $N = 1$ spectral data represents a Θ-summable even spectral triple in noncommutative geometry [9]. In particular, any p-summable even spectral triple is a $N = 1$ spectral data since finite summability implies Θ-summability. The associated space of differential forms, called the Dirac dga, extends the classical de-Rham dga on manifolds to noncommutative framework [10].

Definition 2.4. A quintuple $(\mathcal{A}, \mathcal{H}, d, \gamma, \star)$ is called a set of $N = (1, 1)$ spectral data if

(1) $\mathcal{A}$ is a unital associative $*$-algebra represented faithfully on the separable Hilbert space $\mathcal{H}$ by bounded operators;
(2) d is a densely defined closed operator on $\mathcal{H}$ such that

  (a) $d^2 = 0$,
  (b) the commutator $[d, a]$ extends uniquely to a bounded operator on $\mathcal{H}$ for each $a \in \mathcal{A}$,
  (c) the operator $exp(-\varepsilon\triangle)$, with $\triangle = dd^* + d^*d$, is trace class for all $\varepsilon > 0$;

(3) γ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$ such that $[\gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\{\gamma, d\} = 0$;
(4) $\star$ is a unitary operator acting on $\mathcal{H}$ such that

  (a) $\star d = \zeta d^* \star$ for some phase $\zeta \in \mathbb{S}^1 \subseteq \mathbb{C}$,
  (b) $[\star, a] = 0$ for all $a \in \mathcal{A}$.

Remark 2.5.

(1) In analogy with the classical case, the operator $\star$ is called the Hodge operator.
(2) As is always achievable in the classical case of manifolds, the Hodge operator can be taken to satisfy $\star^2 = 1$ and $[\star, \gamma] = 0$, and the phase $\zeta = -1$ (see discussion in Page 139 in [20]).
(3) The associated space of $N = (1, 1)$ differential forms is given in (Section 2.2.2 in [20]) and the notion of integration is described in (Section 2.2.3 in [20]).

Definition 2.6. An octuple $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T}, \gamma, \star)$ is called a set of $N = (2, 2)$ Kähler spectral data if

(1) the quintuple $(\mathcal{A}, \mathcal{H}, \partial + \overline{\partial}, \gamma, \star)$ forms a set of $N = (1, 1)$ spectral data;
(2) T, $\overline{T}$ are bounded self-adjoint operators on $\mathcal{H}$, and $\partial, \overline{\partial}$ are densely defined closed operators on $\mathcal{H}$ such that the following relations hold :

$$
\begin{array}{llll}
(a)\ \partial^2 = \overline{\partial}^2 = 0 & , (b)\ \{\partial, \overline{\partial}\} = 0 & , (c)\ [T, \overline{T}] = 0 & , \\
(d)\ [T, \partial] = \partial & , (e)\ [T, \overline{\partial}] = 0 & , (f)\ [\overline{T}, \partial] = 0 & , (g)\ [\overline{T}, \overline{\partial}] = \overline{\partial}\ ;
\end{array}
$$

(3) $[T, a] = [\overline{T}, a] = 0\ \forall a \in \mathcal{A}$, and $[\partial, a]$, $[\overline{\partial}, a]$, $\{\partial, [\overline{\partial}, a]\}$ extends uniquely to bounded operators on $\mathcal{H}$;
(4) the $\mathbb{Z}_2$-grading operator γ satisfies

  (a) $\{\gamma, \partial\} = \{\gamma, \overline{\partial}\} = 0$,
  (b) $[\gamma, T] = [\gamma, \overline{T}] = 0$;

(5) for some phase $\zeta \in \mathbb{S}^1$, the Hodge operator $\star \in \mathcal{U}(\mathcal{H})$ satisfies

  (a) $\star\partial = \zeta\overline{\partial}^*\star$,
  (b) $\star\overline{\partial} = \zeta\partial^*\star$;

(6) the following Kähler conditions are satisfied

  (a) $\{\partial, \overline{\partial}^*\} = \{\overline{\partial}, \partial^*\} = 0$,
  (b) $\{\partial, \partial^*\} = \{\overline{\partial}, \overline{\partial}^*\}$.

Remark 2.7.

(1) Condition 6(a) is consequence of 6(b) in classical complex geometry but has to be imposed as a separate condition in noncommutative framework [20]. This says that the Laplacian $\triangle = 2\triangle_{\overline{\partial}}$ like in the case of classical Kähler manifolds.
(2) In the classical case, T and $\overline{T}$ represent the holomorphic and the anti-holomorphic $\mathbb{Z}$-grading of complex differential forms (Page 538 in [19]). The presence of T and $\overline{T}$ in the $N = (2, 2)$ spectral data implies few crucial properties not enjoyed by the $N = (1, 1)$ spectral data (Propn. 2.32 and 2.35 in [20]).
(3) An octuple $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T}, \gamma, \star)$ satisfying conditions (1)–(5) above is called a *Hermitian spectral data* generalizing the classical notion of Hermitian manifolds. Condition (6) is precisely the Kähler condition on a noncommutative Hermitian manifold.
(4) The associated space of complex differential forms is described in Section 2.3.2 in [20] (in particular see Propn. 2.32), and for the notion of integration see Section 2.3.3.

Definition 2.4 of $N = (1, 1)$ spectral data has an alternative description. One can introduce the following two unbounded operators

$$\mathfrak{D} = d + d^* \quad , \quad \overline{\mathfrak{D}} = i(d - d^*)$$

(Caution: $\overline{\mathfrak{D}}$ is not the closure of $\mathfrak{D}$) which satisfy the relations

$$\mathfrak{D}^2 = \overline{\mathfrak{D}}^2 \quad , \quad \{\mathfrak{D}, \overline{\mathfrak{D}}\} = 0$$

making the notion of $N = (1, 1)$ spectral data an immediate generalization of a classical $N = (1, 1)$ Dirac bundle [19,20]. Conversely, starting with $\mathfrak{D}$, $\overline{\mathfrak{D}}$ satisfying the above relations, one can define

$$d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}}) \quad , \quad d^* = \frac{1}{2}(\mathfrak{D} + i\overline{\mathfrak{D}}) .$$

For all $\varepsilon > 0$, the condition $exp(-\varepsilon(dd^* + d^*d))$ is a trace class operator is equivalent with $exp(-\varepsilon\mathfrak{D}^2)$ is a trace class operator.

Lemma 2.8. *If the Hodge operator satisfy $\star^2 = 1$ and $[\star, \gamma] = 0$, and the phase $\zeta = -1$, then*

(1) $\{\gamma, d\} = 0$ if and only if $\{\gamma, \mathfrak{D}\} = \{\gamma, \overline{\mathfrak{D}}\} = 0$;
(2) $\star d = -d^*\star$ if and only if $\{\star, \mathfrak{D}\} = [\star, \overline{\mathfrak{D}}] = 0$.

Proof. Straightforward verification. □

Any $N = (1, 1)$ spectral data gives rise to a $N = 1$ spectral data over the same algebra by taking $D = d + d^*$. The converse, i.e. whether a $N = 1$ spectral data can be extended to $N = (1, 1)$ spectral data, is true for the classical case of manifolds [19]. However, in the noncommutative situation this is not obvious. Guided by the classical case of manifolds a procedure of extension is suggested by Fröhlich et al. in [20], which we discuss now.

Let $\mathcal{E}$ be a finitely generated projective left module over $\mathcal{A}$ and $\mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$. Clearly, $\mathcal{E}^*$ is also a left $\mathcal{A}$-module by the rule $(a\,.\,\phi)(\xi) := \phi(\xi)a^*$, $\forall \xi \in \mathcal{E}$. Throughout the article, we will always write f.g.p. for notational brevity to mean finitely generated projective. In the noncommutative situation a f.g.p. module represents a vector bundle over noncommutative space.

Definition 2.9. A Hermitian structure on $\mathcal{E}$ is an $\mathcal{A}$-valued positive-definite map $\langle .\,, .\rangle_{\mathcal{A}}$ such that :

(a) $\langle \xi, \xi' \rangle_{\mathcal{A}}^* = \langle \xi', \xi \rangle_{\mathcal{A}}\,, \quad \forall \xi, \xi' \in \mathcal{E}$.

(b) $\langle a\,.\,\xi\,,\,b\,.\,\xi'\rangle_{\mathcal{A}} = a(\langle\xi\,,\,\xi'\rangle_{\mathcal{A}})b^*\,,\quad \forall\,\xi,\xi'\in\mathcal{E},\ \forall\,a,b\in\mathcal{A}.$
(c) The map $g:\xi\longmapsto\Phi_\xi$ from $\mathcal{E}$ to $\mathcal{E}^*$, given by $\Phi_\xi(\eta)=\langle\eta,\xi\rangle_{\mathcal{A}}\,,\ \forall\,\eta\in\mathcal{E}$, gives a conjugate linear left $\mathcal{A}$-module isomorphism between $\mathcal{E}$ and $\mathcal{E}^*$, i.e. g can be regarded as a metric on $\mathcal{E}$. This property is referred as the self-duality of $\mathcal{E}$.

Any free $\mathcal{A}$-module $\mathcal{E}_0=\mathcal{A}^n$ has a Hermitian structure on it, given by $\langle\,\xi\,,\eta\,\rangle_{\mathcal{A}}=\sum_{j=1}^n\xi_j\eta_j^*$ for all $\xi=(\xi_1,\ldots,\xi_q)\in\mathcal{E}_0\,,\ \eta=(\eta_1,\ldots,\eta_q)\in\mathcal{E}_0$. We refer this as the *canonical Hermitian structure* on $\mathcal{E}_0$. Let $\Omega^1_D(\mathcal{A})$ be the $\mathcal{A}$-bimodule $\{\sum a_j[D,b_j]\,:\,a_j,b_j\in\mathcal{A}\}$ of noncommutative 1-forms and $d:\mathcal{A}\longrightarrow\Omega^1_D(\mathcal{A})$, given by $a\longmapsto[D,a]$, be the Dirac dga differential [10]. Note that $(da)^*=-da^*$ by definition.

Definition 2.10. Let $\mathcal{E}$ be a f.g.p. left $\mathcal{A}$-module equipped with a Hermitian structure $\langle.\,,.\rangle_{\mathcal{A}}$. A compatible connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla:\mathcal{E}\longrightarrow\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}$ satisfying

(a) $\nabla(a\xi)=a(\nabla\xi)+da\otimes\xi,\quad\forall\,\xi\in\mathcal{E},\ a\in\mathcal{A}\,;$
(b) $\langle\,\nabla\xi,\eta\,\rangle-\langle\,\xi,\nabla\eta\,\rangle=d\langle\,\xi,\eta\,\rangle_{\mathcal{A}}\quad\forall\,\xi,\eta\in\mathcal{E}\,.$

Any connection extends uniquely to a $\mathbb{C}$-linear map $\nabla:\Omega^\bullet_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}\longrightarrow\Omega^{\bullet+1}_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}$ satisfying $\nabla(\omega\otimes\xi)=(-1)^{deg(\omega)}\omega\nabla(\xi)+d\omega\otimes\xi$. The associated curvature of a connection is the $\mathcal{A}$-linear map $\Theta_\nabla:\mathcal{E}\longrightarrow\Omega^2_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}$ given by the composition $\nabla\circ\nabla$.

The meaning of the equality (b) in $\Omega^1_D(\mathcal{A})$ is, if $\nabla(\eta)=\sum\omega_j\otimes\eta_j\in\Omega^1_D(\mathcal{A})\otimes\mathcal{E}$, then $\langle\,\xi\,,\nabla\eta\,\rangle=\sum\langle\xi,\eta_j\rangle_{\mathcal{A}}\,\omega_j^*$ and $\langle\nabla\eta\,,\,\xi\,\rangle=\sum\omega_j\langle\eta_j,\xi\rangle_{\mathcal{A}}\,.$

A procedure to extend a $N=1$ spectral data to a $N=(1,1)$ spectral data :

Start with a $N=1$ spectral data $(\mathcal{A},\mathcal{H},D,\gamma)$ equipped with a real structure J [11,12]. That is, there exists an anti-unitary operator J on $\mathcal{H}$ such that

$$J^2=\varepsilon I\quad,\quad JD=\varepsilon' DJ\quad,\quad J\gamma=\varepsilon''\gamma J$$

for some signs $\varepsilon,\varepsilon',\varepsilon''=\pm1$ depending on KO-dimension $n\in\mathbb{Z}_8$ and satisfying

$$[JaJ^*,b]=[JaJ^*,[D,b]]=0\quad\forall\,a,b\in\mathcal{A}\,.$$

The real structure J now enables us to equip the Hilbert space $\mathcal{H}$ with an $\mathcal{A}$-bimodule structure

$$a\,.\,\xi\,.\,b:=aJb^*J^*(\xi)\,.$$

We can extend this to a right action of $\Omega^1_D(\mathcal{A}):=\{\sum_j a_j[D,b_j]:a_j,b_j\in\mathcal{A}\}$ on $\mathcal{H}$ by the rule

$$\xi\,.\,\omega:=J\omega^*J^*(\xi)\,.$$

Assume that $\mathcal{H}$ contains a dense f.g.p. left $\mathcal{A}$-module $\mathcal{E}$ equipped with a Hermitian structure $\langle.\,,.\rangle_{\mathcal{A}}$, and is stable under J and γ. In particular, $\mathcal{E}$ is itself an $\mathcal{A}$-bimodule. We make $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ into an inner-product space by the following rule :

$$\langle\xi\otimes\eta\,,\,\xi'\otimes\eta'\rangle:=\langle\eta\,,\,\langle J\xi,J\xi'\rangle_{\mathcal{A}}(\eta')\rangle_{\mathcal{H}}\,.\tag{2.1}$$

Note that this is indeed an inner-product because J is an anti-linear map and $\langle.\,,.\rangle_{\mathcal{A}}$ is linear in the first entry while $\langle.\,,.\rangle_{\mathcal{H}}$ is linear in the second entry. Let $\widetilde{\mathcal{H}}:=\overline{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}^{\langle\,,\,\rangle}$. Define the anti-linear flip operator

$$\begin{aligned}\Psi:\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}&\longrightarrow\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})\\ \omega\otimes\xi&\longmapsto J\xi\otimes\omega^*\,.\end{aligned}$$

It is easy to verify that Ψ is well-defined and satisfies $\Psi(as)=\Psi(s)a^*,\ \forall\,s\in\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}$. Consider a compatible connection

$$\nabla:\mathcal{E}\longrightarrow\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}$$

such that ∇ commutes with the grading γ on $\mathcal{E}\subseteq\mathcal{H}$, i.e. $\nabla\gamma\xi=(1\otimes\gamma)\nabla\xi$ for all $\xi\in\mathcal{E}$. For each such connection ∇ on $\mathcal{E}$, there is the following associated right-connection

$$\begin{aligned}\overline{\nabla}:\mathcal{E}&\longrightarrow\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})\\ \xi&\longmapsto-\Psi(\nabla J^*\xi)\end{aligned}$$

Thus, we get a $\mathbb{C}$-linear map (the so called "tensored connection")

$$\begin{aligned}\widetilde{\nabla}:\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}&\longrightarrow\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}\\ \xi_1\otimes\xi_2&\longmapsto\overline{\nabla}\xi_1\otimes\xi_2+\xi_1\otimes\nabla\xi_2\end{aligned}$$

Note that $\widetilde{\nabla}$ is not a connection in the usual sense because of the position of $\Omega^1_D(\mathcal{A})$. Define the following two $\mathbb{C}$-linear maps

$$c\,,\,\overline{c}\;:\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}\longrightarrow\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$$
$$c:\xi_1\otimes\omega\otimes\xi_2\longmapsto\xi_1\otimes\omega\,.\,\xi_2\,,$$
$$\overline{c}:\xi_1\otimes\omega\otimes\xi_2\longmapsto\xi_1\,.\,\omega\otimes\gamma\xi_2\,.$$

Now, introduce the following densely defined unbounded operators on $\widetilde{\mathcal{H}}$

$$\mathfrak{D}:=c\circ\widetilde{\nabla}\quad,\quad\overline{\mathfrak{D}}:=\overline{c}\circ\widetilde{\nabla}$$

(Caution: $\overline{\mathfrak{D}}$ is not the closure of $\mathfrak{D}$). In order to obtain a set of $N=(1,1)$ spectral data on $\mathcal{A}$, one has to find a specific connection ∇ on a suitable dense Hermitian f.g.p. left $\mathcal{A}$-module $\mathcal{E}$ such that

(a) The operators $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ become essentially self-adjoint on $\widetilde{\mathcal{H}}$,
(b) The relations $\mathfrak{D}^2=\overline{\mathfrak{D}}^2$ and $\{\mathfrak{D},\overline{\mathfrak{D}}\}=0$ are satisfied.

The $\mathbb{Z}_2$-grading on $\widetilde{\mathcal{H}}$ is simply the tensor product grading $\widetilde{\gamma}:=\gamma\otimes\gamma$, and the Hodge operator is taken to be $\star:=1\otimes\gamma$ (In [20], this is mistakenly taken as $\star=\gamma\otimes 1$). The sextuple $(\mathcal{A},\widetilde{\mathcal{H}},\mathfrak{D},\overline{\mathfrak{D}},\widetilde{\gamma},\star)$ is a candidate of $N=(1,1)$ spectral data extending the $N=1$ spectral data $(\mathcal{A},\mathcal{H},D,\gamma)$. This Hodge operator additionally satisfies $\star^2=1$ and $[\star,\gamma]=0$. Hence, Lemma 2.8 holds for this extension procedure for the phase $\zeta=-1$.

Apart from the classical case of manifolds, existence of such suitable connection ∇ is known for the cases of noncommutative 2-torus and fuzzy 3-sphere [20]. However, the general case remains open. This extension procedure is recently studied in [23] in order to define the tensor product of $N=(1,1)$ spectral data. In the next section, we shall see a class of examples arising from certain C^*-dynamical systems which satisfy this extension procedure.

3. Kähler structure on C^*-dynamical systems

Definition 3.1. A C^*-dynamical system is a tuple $(\mathcal{A},G,\alpha)$ where $\mathcal{A}$ is a unital C^*-algebra, G is a real Lie group and $\alpha:G\longrightarrow Aut(\mathcal{A})$ is a strongly continuous group homomorphism (i.e. for all $a\in\mathcal{A}$, the map $g\mapsto\alpha_g(a)$ is continuous).

We will work with C^*-dynamical systems $(\mathcal{A},G,\alpha)$ equipped with a faithful G-invariant trace τ, i.e. $\tau(\alpha_g(a))=\tau(a)$ for all $g\in G$. This is in line with [7,16,35]. Note that if the Lie group is compact and the action is ergodic then the unique G-invariant state is a faithful trace on $\mathcal{A}$ [26]. We say that $a\in\mathcal{A}$ is smooth if the map $g\mapsto\alpha_g(a)$ is in $C^\infty(G,\mathcal{A})$. The involutive algebra $\mathcal{A}^\infty=\{a\in\mathcal{A}:\,a\text{ is smooth}\}$ is a norm dense subalgebra of $\mathcal{A}$, called the smooth subalgebra. Note that this is unital as well. One crucial property enjoyed by this subalgebra is that it is closed under the holomorphic function calculus inherited from the ambient C^*-algebra $\mathcal{A}$ [22]. Henceforth, we will always work with the smooth subalgebra $\mathcal{A}^\infty$ and denote it simply by $\mathcal{A}$ for notational brevity.

To begin with we recall a result in [17] which provides an explicit set of generators of the irreducible representations of $\mathbb{C}l(n)$ for all n, together with an explicit involution J and (if n is even) a grading operator γ. This is summarized in below.

Proposition 3.2 ([17]). *Consider a positive integer n and an irreducible representation of $\mathbb{C}l(n)$ on a vector space $\mathbb{V}$. Up to unitary equivalence, it is determined by n many matrices γ_j such that*

$$\gamma_j^*=-\gamma_j\quad,\quad\gamma_j\gamma_k+\gamma_k\gamma_j=-2\delta_{jk}.$$

If n is even, there is a $\mathbb{Z}_2$ grading operator γ_V satisfying $\gamma_V\gamma_j=-\gamma_j\gamma_V$ for all $j=1,\ldots,n$. Moreover, there is an explicit anti-isometry J_V (charge conjugation) satisfying

$$(J_V)^2=\varepsilon\quad,\quad J_V\gamma_j=\varepsilon'\gamma_jJ_V\quad,\quad J_V\gamma_V=\varepsilon''\gamma_VJ_V$$

for some signs $\varepsilon,\varepsilon',\varepsilon''\in\{1,-1\}$ depending on n modulo 8 :

n	0	2	4	6	1	3	5	7
ε	+	−	−	+	+	−	−	+
ε'	+	+	+	+	−	+	−	+
ε''	+	−	+	−				

Candidate of a $N=1$ spectral data associated with C^*-dynamical systems:

Let $(\mathcal{A},G,\alpha,\tau)$ be a C^*-dynamical system equipped with a G-invariant faithful trace τ. Let $dim(G)=n$ and $N=2^{\lfloor n/2\rfloor}$. Let $\{X_1,\ldots,X_n\}$ be a basis of the Lie algebra $\mathfrak{g}$ of the Lie group G. Letting $\mathcal{H}=L^2(\mathcal{A},\tau)$ the G.N.S Hilbert space, we obtain a covariant representation of $(\mathcal{A},G,\alpha)$ on $\mathcal{H}$. Note that there is a bijective correspondence between covariant representations of $(\mathcal{A}, G, \alpha)$ and non-degenerate $\star$-representations of $\mathcal{A} \rtimes_\alpha G$. We obtain the following densely defined symmetric operator on $\widetilde{\mathcal{H}} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N$,

$$D := \sum_{j=1}^{n} \partial_j \otimes \gamma_j$$

acting on the domain $dom(D) = \mathcal{A} \otimes \mathbb{C}^N$ where, $\partial_j(a) := \frac{d}{dt}|_{t=0}\,\alpha_{exp(tX_j)}(a)$ and γ_j as in the above Proposition. Note that the map $\partial : \mathfrak{g} \longrightarrow \mathcal{D}er(\mathcal{A})$ given by $X_j \longmapsto \partial_j$, where $\mathcal{D}er(\mathcal{A})$ is the Lie algebra of derivations on $\mathcal{A}$, is a Lie algebra homomorphism i.e. $[\partial_j, \partial_\ell] = \partial_{[j,\ell]}$. Moreover, there is always a real structure $J = J_0 \otimes J_N$, where J_0 is the anti-linear operator $a \mapsto a^*$ and $J_N = J_V$ as in the above Proposition 3.2, satisfying

$$J^2 = \varepsilon I \quad , \quad JD = \varepsilon' DJ \quad , \quad J\gamma_N = \varepsilon'' \gamma_N J$$

(if grading $\gamma_N = \gamma_V$ exists) for some signs $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ depending on $n \in \mathbb{Z}_8$ and satisfying the above mentioned table. We also have

$$[JaJ^*, b] = [JaJ^*, [D, b]] = 0 \quad \forall\, a, b \in \mathcal{A}\,.$$

It is known that D, defined above, admits a self-adjoint extension [21]. But the summability and compactness of the resolvent of D is not guaranteed. So, if D is essentially self-adjoint with compact resolvent and gives the Θ-summability (note that any finitely summable spectral triple is Θ-summable [9]), then we obtain a Θ-summable even spectral triple $(\mathcal{A}, \widetilde{\mathcal{H}}, D, \gamma_N)$ if n is even; otherwise odd spectral triple $(\mathcal{A}, \widetilde{\mathcal{H}}, D)$ if n is odd. However, existence of such a self-adjoint extension of D is an intricate question and that is why we only get a candidate of a $N = 1$ spectral data.

Remark 3.3.

(a) It is known that if the Lie group G is compact then D, defined above, is essentially self-adjoint (Propn. 4.1 in [21]).
(b) If the Lie group G is compact and acts ergodically then we obtain a $dim(G)$-summable (and hence Θ-summable) spectral triple (Thm. 5.4 in [21]), independent of the choice of the Lie algebra basis. This is the case for the noncommutative n-torus $\mathcal{A}_\Theta$.
(c) Compactness and ergodicity is a sufficient condition only. Recall the case of quantum Heisenberg manifolds [34] where the Lie group acting is noncompact namely, the Heisenberg group. It is known [5,6] that one gets an honest 3-summable spectral triple in this case also, for a suitable choice of the Heisenberg Lie algebra basis.
(d) There is no characterization of C^*-dynamical systems known yet which gives genuine finite or Θ-summable spectral triples by the above discussed method.

Guided by these, we start with a C^*-dynamical system $(\mathcal{A}, G, \alpha, \tau)$ equipped with a G-invariant faithful trace τ, where G is an even dimensional abelian Lie group, such that the candidate discussed above determines an honest $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \sigma)$ with σ the $\mathbb{Z}_2$-grading. Since the Lie algebra $\mathfrak{g}$ is abelian, i.e. $[X_j, X_\ell] = 0$ for all $j, \ell \in \{1, \ldots, dim(G)\}$, we have $[\partial_j, \partial_\ell] = 0$. Our first objective is to show that this $N = 1$ spectral data always extends to $N = (1, 1)$ spectral data over $\mathcal{A}$ by the procedure of extension discussed in Section 2. Then we produce Kähler structure on $\mathcal{A}$. The key ingredient is the *Grassmannian connection* as we shall see. Let $dim(G) = 2k$ and $N = 2^{\lfloor dim(G)/2 \rfloor} = 2^k$. Consider the dense finitely generated free left $\mathcal{A}$-module $\mathcal{E} := \mathcal{A} \otimes \mathbb{C}^N \subseteq \mathcal{H} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N$ equipped with the canonical Hermitian structure. Clearly, $\mathcal{E}$ is stable under the real structure $J = J_0 \otimes J_N$ and the grading operator σ. Consider the following $\mathbb{C}$-linear map

$$\begin{aligned} \nabla : \mathcal{E} &\longrightarrow \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \\ \xi &\longmapsto (d\xi_1, \ldots\ldots, d\xi_N) \end{aligned}$$

for $\xi = (\xi_1, \ldots, \xi_N) \in \mathcal{E}$, where $d : \mathcal{A} \longrightarrow \Omega^1_D(\mathcal{A})$, given by $a \mapsto [D, a]$, is the Dirac dga differential. This is the Grassmannian connection on the free left $\mathcal{A}$-module $\mathcal{E}$ and is easily seen to be compatible with the canonical Hermitian structure given by $\langle \xi, \eta \rangle := \sum_{j=1}^{N} \xi_j \eta_j^*$. We fix the standard canonical free $\mathcal{A}$-module basis $\{e_1, \ldots, e_N\}$ of $\mathcal{E} = \mathcal{A} \otimes \mathbb{C}^N$. By abuse of notation, the same denotes the canonical linear basis of $\mathbb{C}^N$ if no confusion arise.

Lemma 3.4. *The Grassmannian connection $\nabla : \mathcal{E} \longrightarrow \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ and its associated right connection $\overline{\nabla} : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ satisfy $\nabla e_j = \overline{\nabla} e_j = 0\ \forall j \in \{1, \ldots, N\}$, and it commutes with the $\mathbb{Z}_2$-grading σ.*

Proof. Clearly, $\nabla e_j = 0\ \forall j \in \{1, \ldots, N\}$ by its definition. Note that, $\overline{\nabla}(e_j) = -\Psi(\nabla J^* e_j)$. Since $J = J_0 \otimes J_N$, $e_j = 1 \otimes (0, \ldots, 1, \ldots, 0) \in \mathcal{A} \otimes \mathbb{C}^N$ and $J_N^* = \varepsilon J_N$, we get

$$\begin{aligned} J^* e_j &= \varepsilon 1 \otimes J_N (0, \ldots, 1, \ldots, 0)^T \\ &= \varepsilon 1 \otimes ((J_N)_{1j}, \ldots, (J_N)_{Nj}) \\ &= \sum_{\ell=1}^{N} \varepsilon (J_N)_{\ell j} e_\ell\,. \end{aligned}$$

Since, $(J_N)_{\ell j}$ are scalars for all ℓ and ∇ is $\mathbb{C}$-linear map satisfying $\nabla e_\ell = 0$, our claim follows. Finally, commutation of ∇ with the $\mathbb{Z}_2$-grading operator is easy to observe. □

Note that any element $ae_i \otimes be_j$ of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ can be written as $e_i.(J^*a^*J) \otimes be_j$ (recall the right $\mathcal{A}$-module structure on $\mathcal{E}$), and since the tensor is over $\mathcal{A}$, this is same as $e_i \otimes ce_j$ for some $c \in \mathcal{A}$. Now, for any arbitrary element $e_i \otimes a_{ij}e_j$ of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, the tensored connection $\widetilde{\nabla}$ becomes

$$\begin{aligned}\widetilde{\nabla}(e_i \otimes a_{ij}e_j) &= \overline{\nabla}e_i \otimes a_{ij}e_j + e_i \otimes \nabla(a_{ij}e_j)\\ &= e_i \otimes da_{ij} \otimes e_j\,.\end{aligned}$$

So, we have

$$\begin{aligned}\mathfrak{D}(e_i \otimes a_{ij}e_j) &:= c \circ \widetilde{\nabla}(e_i \otimes a_{ij}e_j) = e_i \otimes (da_{ij})\,.\,e_j\\ \overline{\mathfrak{D}}(e_i \otimes a_{ij}e_j) &:= \bar{c} \circ \widetilde{\nabla}(e_i \otimes a_{ij}e_j) = e_i\,.\,(da_{ij}) \otimes \sigma e_j\end{aligned} \tag{3.2}$$

where, σ is the $\mathbb{Z}_2$-grading operator.

Proposition 3.5. *The Hilbert space $\overline{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ is isomorphic to $L^2(\mathcal{A}, \tau)^{N^2} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^{N^2}$.*

Proof. Since $\mathcal{E} = \mathcal{A} \otimes \mathbb{C}^N$, we have $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is isomorphic with $\mathcal{A} \otimes \mathbb{C}^{N^2}$. Because $\mathcal{E}$ has the canonical Hermitian structure on it, from the inner-product defined in Eq. (2.1), it follows that

$$\begin{aligned}\langle \xi \otimes \eta\,,\, \xi' \otimes \eta' \rangle &= \langle \eta\,,\, \langle J\xi, J\xi' \rangle_{\mathcal{A}}(\eta') \rangle_{\mathcal{H}}\\ &= \sum_{\ell,j} \langle \eta_\ell\,,\, \xi_j^* \xi_j' \eta_\ell' \rangle\\ &= \sum_{\ell,j} \tau\left(\eta_\ell^* \xi_j^* \xi_j' \eta_\ell'\right)\,.\end{aligned}$$

This is precisely the inner-product on $\mathcal{A} \otimes \mathbb{C}^{N^2}$ given by the inner-product $\langle a, b\rangle := \tau(a^*b)$ on $\mathcal{A}$ and the usual inner-product on $\mathbb{C}^{N^2}$. The completion is the Hilbert space $L^2(\mathcal{A}, \tau)^{N^2} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^{N^2}$, and this concludes the proof. □

Lemma 3.6. *$\mathfrak{D}$ and $\overline{\mathfrak{D}}$ are densely defined symmetric operators acting on the Hilbert space $\overline{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$.*

Proof. We have

$$\begin{aligned}&\langle \mathfrak{D}(e_i \otimes a_{ij}e_j), e_m \otimes a_{m\ell}e_\ell \rangle - \langle e_i \otimes a_{ij}e_j, \mathfrak{D}(e_m \otimes a_{m\ell}e_\ell) \rangle\\ &= \langle e_i \otimes (da_{ij}).e_j, e_m \otimes a_{m\ell}e_\ell \rangle - \langle e_i \otimes a_{ij}e_j, e_m \otimes (da_{m\ell}).e_\ell \rangle\\ &= \langle (da_{ij}).e_j, \langle e_i, e_m \rangle_{\mathcal{A}}(a_{m\ell}e_\ell) \rangle - \langle a_{ij}e_j, \langle e_i, e_m \rangle_{\mathcal{A}}(da_{m\ell}).e_\ell \rangle\\ &= \delta_{im}\left(\langle (da_{ij}).e_j, a_{m\ell}e_\ell \rangle - \langle a_{ij}e_j, (da_{m\ell}).e_\ell \rangle \right)\\ &= \langle (da_{ij}).e_j, a_{i\ell}e_\ell \rangle - \langle a_{ij}e_j, (da_{i\ell}).e_\ell \rangle\\ &= \left\langle \sum_{r=1}^{2k} \partial_r(a_{ij}) \otimes (\gamma_{r1j}, \ldots, \gamma_{rNj})\,,\ a_{i\ell}e_\ell \right\rangle - \left\langle a_{ij}e_j\,,\ \sum_{r=1}^{2k} \partial_r(a_{i\ell}) \otimes (\gamma_{r1\ell}, \ldots, \gamma_{rN\ell}) \right\rangle\\ &= \sum_{r=1}^{2k} \langle (\partial_r(a_{ij})\gamma_{r1j}, \ldots, \partial_r(a_{ij})\gamma_{rNj}), (0, \ldots, a_{i\ell}, \ldots, 0) \rangle\\ &\qquad - \langle (0, \ldots, a_{ij}, \ldots, 0), (\partial_r(a_{i\ell})\gamma_{r1\ell}, \ldots, \partial_r(a_{i\ell})\gamma_{rN\ell}) \rangle\\ &= \sum_{r=1}^{2k} \tau((\partial_r(a_{ij})\gamma_{r\ell j})^* a_{i\ell}) - \tau(a_{ij}^* \partial_r(a_{i\ell})\gamma_{rj\ell})\\ &= \sum_{r=1}^{2k} \tau\left(\partial_r(a_{ij}^*)\overline{\gamma_{r\ell j}}a_{i\ell}\right) + \tau\left(a_{ij}^* \partial_r(a_{i\ell}\overline{\gamma_{r\ell j}})\right)\\ &= \sum_{r=1}^{2k} \tau\Big(\partial_r(a_{ij}^* \overline{\gamma_{r\ell j}} a_{i\ell})\Big)\end{aligned}$$

Here, we are using the fact that for all $r \in \{1, \ldots, 2k\}$, $\gamma_r^* = -\gamma_r$. Hence, $\overline{(\gamma_r)_{\ell j}} = -(\gamma_r)_{j\ell}$. Now, for any $a \in \mathcal{A}$,

$$\tau(\partial_r(a)) = \tau\Big(\frac{d}{dt}\big|_{t=0}\, \alpha_{exp(tX_r)}(a)\Big) = 0$$

for all $r \in \{1, \ldots, 2k\}$, because τ is a G-invariant trace. This proves that $\mathfrak{D}$ is a symmetric operator. Similarly, one can show for $\overline{\mathfrak{D}}$. □

Proposition 3.7. *Both $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ are essentially self-adjoint operators acting on the Hilbert space $\overline{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} \cong L^2(\mathcal{A},\tau)^{N^2}$.*

Proof. Observe that for any $\xi = e_i \otimes a_{ij}e_j \in \mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$, we can write

$$\xi = \underbrace{(0,\dots,\underbrace{(0,\dots,a_{ij},\dots,0)}_{ith\ place\ ,\ N\ tuple},\dots,0)}_{N\ tuple} \in \mathcal{A}^{N^2}$$

and hence,

$$\begin{aligned}\mathfrak{D}(\xi) &= (0,\dots,\underbrace{\sum_{r=1}^{2k}\partial_r(a_{ij})\otimes\gamma_r(e_j)}_{\in L^2(\mathcal{A},\tau)^N},\dots,0) \in L^2(\mathcal{A},\tau)^{N^2}\\ &= e_i\otimes D(a_{ij}e_j)\end{aligned}$$

Now,

$$\begin{aligned}\overline{\mathfrak{D}}(\xi) &= e_i\,.\,da_{ij}\otimes\sigma e_j\\ &= -\varepsilon Jd(a_{ij}^*)Je_i\otimes\sigma e_j\\ &= -\varepsilon'\Big(\sum_{r=1}^{2k}\partial_r(a_{ij})\otimes\gamma_r(e_i)\Big)\otimes\sigma e_j\\ &= -\varepsilon' D(a_{ij}e_i)\otimes\sigma e_j\end{aligned}$$

and observe that

$$\begin{aligned}e_i\otimes a_{ij}e_j &= e_i\,.\,a_{ij}\otimes e_j\\ &= Ja_{ij}^*J^*e_i\otimes e_j\\ &= \varepsilon Ja_{ij}^*(1\otimes J_N e_i)\otimes e_j\\ &= \varepsilon(a_{ij}\otimes J_N^2 e_i)\otimes e_j\\ &= a_{ij}e_i\otimes e_j\,.\end{aligned}$$

Since, $\overline{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} \cong L^2(\mathcal{A},\tau)^{N^2}$ (Proposition 3.5), we see that the operator $\mathfrak{D}$ is of the form $1_N\otimes D$ and the operator $\overline{\mathfrak{D}}$ is of the form $-\varepsilon' D\otimes\sigma$, both acting on $\mathcal{A}^{N^2}\subseteq L^2(\mathcal{A},\tau)^{N^2}$. That is,

$$\mathfrak{D} = \sum_{j=1}^{2k}\partial_j\otimes 1_N\otimes\gamma_j \quad\text{and}\quad \overline{\mathfrak{D}} = -\varepsilon'\sum_{j=1}^{2k}\partial_j\otimes\gamma_j\otimes\sigma$$

acting on $L^2(\mathcal{A},\tau)^{N^2}\cong L^2(\mathcal{A},\tau)\otimes\mathbb{C}^N\otimes\mathbb{C}^N$. Since, we have assumed that the C^*-dynamical system $(\mathcal{A},G,\alpha,\tau)$ gives us an honest $N=1$ spectral data $\Big(\mathcal{A},L^2(\mathcal{A},\tau)\otimes\mathbb{C}^N, D=\sum_{j=1}^{2k}\partial_j\otimes\gamma_j\Big)$; D is essentially self-adjoint on $\mathcal{H}=L^2(\mathcal{A},\tau)\otimes\mathbb{C}^N$. Since the domain $\mathcal{A}^{N^2}$ is a core for the essentially self-adjoint operator $1_N\otimes D$ it follows that $\mathfrak{D}$ and similarly $\overline{\mathfrak{D}}$ are essentially self-adjoint operators. □

Remark 3.8. Since we are dealing with even dimensional Lie groups, $\varepsilon'=+1$ by the table mentioned in Proposition 3.2. However, we intend not to discard ε' in the expression of $\overline{\mathfrak{D}}$ for the time being for a specific reason. This will be explained towards the end of this section.

Lemma 3.9. *We have the relations $\mathfrak{D}^2=\overline{\mathfrak{D}}^2$ and $\{\mathfrak{D},\overline{\mathfrak{D}}\}=0$.*

Proof. Since σ is a $\mathbb{Z}_2$-grading operator on $(\mathcal{A},\mathcal{H},D)$, we have $\{D,\sigma\}=0$. This gives $\{\mathfrak{D},\overline{\mathfrak{D}}\}=0$. Now,

$$\begin{aligned}\mathfrak{D}^2 &= -\Big(\sum_{r=1}^{2k}\partial_r^2\Big)\otimes 1_N\otimes 1_N + \sum_{i<j}[\partial_i,\partial_j]\otimes 1_N\otimes\gamma_i\gamma_j\\ &= -\Big(\sum_{r=1}^{2k}\partial_r^2\Big)\otimes 1_N\otimes 1_N + \sum_{i<j}\partial_{[i,j]}\otimes 1_N\otimes\gamma_i\gamma_j\\ &= -\Big(\sum_{r=1}^{2k}\partial_r^2\Big)\otimes 1_N\otimes 1_N\end{aligned}$$

because $\mathfrak{g}$ is abelian. One gets exactly equal expression for $\overline{\mathfrak{D}}^2$. Hence, as operators on $L^2(\mathcal{A},\tau)^{N^2}$ we get $\mathfrak{D}^2=\overline{\mathfrak{D}}^2$. □

Remark 3.10. This is the place where we use the fact that $\mathfrak{g}$ is abelian to conclude $\mathfrak{D}^2 = \overline{\mathfrak{D}}^2$. Unless this equality holds we can not have $d^2 = 0$ for $d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$.

Lemma 3.11. *We have the following :*

(i) *For all $a \in \mathcal{A}$, $[d, a]$ extends to a bounded operator acting on the Hilbert space $\overline{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$, where $d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$;*
(ii) *$exp(-\varepsilon\mathfrak{D}^2)$ is a trace class operator for all $\varepsilon > 0$.*

Proof. Both these facts follow from our assumption that the C^*-dynamical system $(\mathcal{A}, G, \alpha, \tau)$ gives us an honest $N = 1$ spectral data $\left(\mathcal{A}, \mathcal{H} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N, D = \sum_{j=1}^{2k} \partial_j \otimes \gamma_j\right)$, and the explicit expressions of $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ in Proposition 3.7. Note that $Tr(exp(-\varepsilon\mathfrak{D}^2)) = NTr(exp(-\varepsilon D^2))$ for all $\varepsilon > 0$. □

Proposition 3.12. *Let G be an even dimensional abelian Lie group and $(\mathcal{A}, G, \alpha, \tau)$ be a C^*-dynamical system equipped with a faithful G-invariant trace τ. Whenever it determines a $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \sigma)$, it always extends to $N = (1, 1)$ spectral data over $\mathcal{A}$.*

Proof. Combining Proposition 3.7 and Lemmas 3.9, 3.11 we see that the only remaining part is to produce a $\mathbb{Z}_2$-grading and a Hodge operator. We have two self-adjoint unitaries $\gamma := 1 \otimes \sigma \otimes 1_N$ and $\gamma' := 1 \otimes 1_N \otimes \sigma$ acting on the Hilbert space $L^2(\mathcal{A}, \tau)^{N^2} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N \otimes \mathbb{C}^N$, satisfying

$$\{\mathfrak{D}, \gamma'\} = \{\overline{\mathfrak{D}}, \gamma\} = 0 \quad , \quad [\mathfrak{D}, \gamma] = [\overline{\mathfrak{D}}, \gamma'] = 0 \ .$$

The $\mathbb{Z}_2$-grading is obtained by taking $\widetilde{\gamma} := \gamma\gamma' = 1 \otimes \sigma \otimes \sigma$. Clearly, $\{\widetilde{\gamma}, \mathfrak{D}\} = \{\widetilde{\gamma}, \overline{\mathfrak{D}}\} = 0$. Finally, the Hodge operator is given by $\star := \gamma' = 1 \otimes 1_N \otimes \sigma$ acting on $\overline{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} = L^2(\mathcal{A}, \tau)^{N^2}$, and it satisfies $\{\star, \mathfrak{D}\} = [\star, \overline{\mathfrak{D}}] = 0$, $\star^2 = 1$, $[\star, \widetilde{\gamma}] = 0$. This concludes the proof in view of Lemma 2.8 by taking the phase $\zeta = -1$. □

An immediate corollary worth mentioning is the following.

Corollary 3.13. *Let G be an even dimensional abelian Lie group and $(\mathcal{A}, G, \alpha, \tau)$ be a C^*-dynamical system equipped with a faithful G-invariant trace τ. Whenever it determines a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where the Hilbert space $\mathcal{H} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N$ with $N = 2^{\lfloor dim(G)/2 \rfloor}$, the Dirac operator $D \otimes 1$ acting on $\mathcal{H} \otimes \mathbb{C}^N$ decomposes as $D = d + d^*$ with $d^2 = 0$.*

We now state and prove our main theorem.

Theorem 3.14. *Let G be an even dimensional abelian Lie group and $(\mathcal{A}, G, \alpha, \tau)$ be a C^*-dynamical system equipped with a faithful G-invariant trace τ. Whenever it determines a $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \sigma)$, it always extends to $N = (2, 2)$ Kähler spectral data over $\mathcal{A}$. That is, $\mathcal{A}$ inherits a (noncommutative) Kähler structure.*

The proof is a bit long and to make it transparent we first break it into the following three Lemmas.

Lemma 3.15. *The following bounded self-adjoint operator*

$$\mathcal{T} : L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N \otimes \mathbb{C}^N \longrightarrow L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N \otimes \mathbb{C}^N$$

$$\mathcal{T} := \sum_{j=1}^{2k} \frac{i\varepsilon'}{2} 1 \otimes \gamma_j \otimes \gamma_j\sigma$$

commutes with all elements of $\mathcal{A} \subseteq \mathcal{B}\left(L^2(\mathcal{A}, \tau)^{N^2}\right)$ and $[\mathcal{T}, d] = d$, where $d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$.

Proof. Recall that $L^2(\mathcal{A}, \tau)^{N^2} = \overline{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ (Proposition 3.5) and $\mathcal{A}$ is represented on $\overline{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ by its left action on $\mathcal{E}$. Clearly, $\mathcal{T}$ then commutes with $\mathcal{A} \subseteq \mathcal{B}\left(L^2(\mathcal{A}, \tau)^{N^2}\right)$. Recall the expressions of $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ from Proposition 3.7. We now have the following,

$$\begin{aligned}
&[-\varepsilon'\mathcal{T}\,,\ \mathfrak{D} - i\overline{\mathfrak{D}}\,] \\
&= \sum_{j=1}^{2k} \left[\frac{1}{2i} 1 \otimes \gamma_j \otimes \gamma_j\sigma\ ,\ \mathfrak{D} - i\overline{\mathfrak{D}}\right] \\
&= \sum_{j=1}^{2k} \frac{1}{2i}[1 \otimes \gamma_j \otimes \gamma_j\sigma\ ,\ 1 \otimes D] + \frac{\varepsilon'}{2}[1 \otimes \gamma_j \otimes \gamma_j\sigma\ ,\ D \otimes \sigma]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,r=1}^{2k} \frac{1}{2i}[1 \otimes \gamma_j \otimes \gamma_j \sigma\ ,\ \partial_r \otimes 1_N \otimes \gamma_r] + \frac{\varepsilon'}{2}[1 \otimes \gamma_j \otimes \gamma_j \sigma\ ,\ \partial_r \otimes \gamma_r \otimes \sigma] \\
&= \sum_{j,r=1}^{2k} \frac{1}{2i}\partial_r \otimes \gamma_j \otimes (\gamma_j \sigma \gamma_r - \gamma_r \gamma_j \sigma) + \frac{\varepsilon'}{2}\partial_r \otimes (\gamma_j \gamma_r + \gamma_r \gamma_j) \otimes \gamma_j \\
&= \sum_{j,r=1}^{2k} \frac{1}{2i}\partial_r \otimes \gamma_j \otimes (-\gamma_j \gamma_r - \gamma_r \gamma_j)\sigma + \frac{\varepsilon'}{2}\partial_r \otimes (\gamma_j \gamma_r + \gamma_r \gamma_j) \otimes \gamma_j \\
&= \sum_{j=1}^{2k} \frac{1}{i}\partial_j \otimes \gamma_j \otimes \sigma - \varepsilon' \partial_j \otimes 1_N \otimes \gamma_j \\
&= \frac{1}{i}(D \otimes \sigma) - \varepsilon'(1 \otimes D) \\
&= i\varepsilon' \overline{\mathfrak{D}} - \varepsilon' \mathfrak{D} \\
&= -\varepsilon'(\mathfrak{D} - i\overline{\mathfrak{D}}) .
\end{aligned}
$$

Hence, for $d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$ we see that $[\mathcal{T}, d] = d$. □

Lemma 3.16. *If there exists a skew-Hermitian matrix $\widetilde{\mathcal{I}} \in M_{N^2}(\mathbb{C})$ such that the bounded skew-adjoint operator $\mathcal{I} = 1 \otimes \widetilde{\mathcal{I}}$ acting on $L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^{N^2}$ satisfies the following,*

(1) $[\mathcal{I}, \mathcal{T}] = 0$
(2) $[\mathcal{I}, \widetilde{\gamma}] = 0$
(3) $[\mathcal{I}, \star] = 0$
(4) $[\mathcal{I}, [\mathcal{I}, d]] = -d$

then the $N = (1, 1)$ spectral data obtained in Proposition 3.12 extends to Hermitian spectral data over $\mathcal{A}$, i,e. $\mathcal{A}$ inherits a complex structure.

Proof. We want to write $d := \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$ as $\partial + \overline{\partial}$ where both $\partial, \overline{\partial}$ are differentials and $\mathcal{T} = T + \overline{T}$ such that all the conditions in Definition 2.6 except (6) are satisfied. Our idea of extending a $N = (1, 1)$ spectral data to a Hermitian spectral data is the following: if there exists such an $\mathcal{I}$, we try to decompose $\mathcal{T}$ as $T + \overline{T}$ while $\mathcal{I} = i(T - \overline{T})$. Then, define a new differential $d_2 = [\mathcal{I}, d]$. This will impose certain constraints on $\mathcal{I}$. We also have Propn. 2.27 in [20] as a reference. Now, let us verify that these ideas actually work.

Consider the densely defined operator $d_2 = [\mathcal{I}, d]$ such that $[\mathcal{I}, d_2] = -d$. This gives $\mathcal{I}^2 d - 2\mathcal{I}d\mathcal{I} + d\mathcal{I}^2 = -d$. Hence, $\mathcal{I}d\mathcal{I}d = \frac{1}{2}d\mathcal{I}^2 d = d\mathcal{I}d\mathcal{I}$. Then,

$$
\begin{aligned}
d_2^2 &= [\mathcal{I}, d][\mathcal{I}, d] \\
&= \mathcal{I}d\mathcal{I}d - d\mathcal{I}^2 d + d\mathcal{I}d\mathcal{I} \\
&= 0
\end{aligned}
$$

i,e. d_2 is a differential. Now, define

$$
\partial := \frac{1}{2}(d - id_2) \quad \text{and} \quad \overline{\partial} := \frac{1}{2}(d + id_2) .
$$

Then, $d = \partial + \overline{\partial}$ and part (1) in Definition 2.6 holds. Observe that $\{d, d_2\} = 0$. Both d and d_2 are anticommuting differentials which show that both ∂ and $\overline{\partial}$ are differentials. It is easy to check that $\{\partial, \overline{\partial}\} = 0$. Now, define

$$
T := \frac{1}{2}(\mathcal{T} - i\mathcal{I}) \quad \text{and} \quad \overline{T} := \frac{1}{2}(\mathcal{T} + i\mathcal{I}) .
$$

Then $\mathcal{T} = T + \overline{T}$ and $[T, \overline{T}] = \frac{i}{2}[\mathcal{T}, \mathcal{I}] = 0$. Now,

$$
\begin{aligned}
[T, \partial] &= \frac{1}{4}([\mathcal{T}, d] - i[\mathcal{T}, d_2] - i[\mathcal{I}, d] - [\mathcal{I}, d_2]) \\
&= \frac{1}{4}(d - id_2 - i[\mathcal{T} - i\mathcal{I}, d_2]) \\
&= \frac{1}{2}\partial - \frac{i}{2}[T, d_2] .
\end{aligned}
$$

Similarly, one can show that

$$[\overline{T},\partial]=\frac{1}{2}\overline{\partial}-\frac{i}{2}[\overline{T},d_2]\,,$$
$$[T,\overline{\partial}]=\frac{1}{2}\partial+\frac{i}{2}[T,d_2]\,,$$
$$[\overline{T},\overline{\partial}]=\frac{1}{2}\overline{\partial}+\frac{i}{2}[\overline{T},d_2]\,.$$

Now, by Lemma 3.15 we know that $[\mathcal{T},d]=d$. Hence,

$$\begin{aligned}[T,d_2] &= \frac{1}{2}(\mathcal{T}\mathcal{I}d-\mathcal{T}d\mathcal{I}-\mathcal{I}d\mathcal{T}+d\mathcal{I}\mathcal{T}-i[\mathcal{I},[\mathcal{I},d]])\\ &= \frac{1}{2}(\mathcal{I}[\mathcal{T},d]-[\mathcal{T},d]\mathcal{I}-i[\mathcal{I},[\mathcal{I},d]])\\ &= \frac{1}{2}(\mathcal{I}d-d\mathcal{I}-i[\mathcal{I},[\mathcal{I},d]])\\ &= \frac{1}{2}(d_2-i[\mathcal{I},[\mathcal{I},d]])\,.\end{aligned}$$

Similarly, one can show that

$$[\overline{T},d_2]=\frac{1}{2}(d_2+i[\mathcal{I},[\mathcal{I},d]])\,.$$

Hence, the following two relations

$$[T,d_2]=i\partial\quad\text{and}\quad[\,\overline{T},d_2]=-i\overline{\partial}$$

together is equivalent to

$$[\,\mathcal{I},[\,\mathcal{I},d]]=-d\,.$$

This shows that part (2) in Definition 2.6 holds. Both $\mathcal{I}$ and $\mathcal{T}$ commuting with $\mathcal{A}$ proves that $[T,a]=[\overline{T},a]=0$ for all $a\in\mathcal{A}$. Now,

$$\begin{aligned}[d_2,a] &= [[\,\mathcal{I},d],a]\\ &= [\,\mathcal{I},[d,a]]\end{aligned}$$

Since $[d,a]$ extends to a bounded operator, we get that both $[\partial,a]$ and $[\overline{\partial},a]$ extend to bounded operators for all $a\in\mathcal{A}$. This shows that part (3) in Definition 2.6 holds. Now,

$$\begin{aligned}\{\widetilde{\gamma},d_2\} &= \widetilde{\gamma}[\mathcal{I},d]+[\mathcal{I},d]\widetilde{\gamma}\\ &= \mathcal{I}\{\widetilde{\gamma},d\}-\{\widetilde{\gamma},d\}\mathcal{I}\\ &= 0\end{aligned}$$

since, $\{\widetilde{\gamma},d\}=0$. This shows that $\{\widetilde{\gamma},\partial\}=\{\widetilde{\gamma},\overline{\partial}\}=0$ i.e. part (4) in Definition 2.6 holds. Finally, observe that

$$\star\partial+\overline{\partial}^*\star=-i(\star d_2+d_2^*\star)$$
$$\star\overline{\partial}+\partial^*\star=i(\star d_2+d_2^*\star)$$

Now, using the fact that $\mathcal{I}$ is skew-adjoint we see that

$$\begin{aligned}\star d_2+d_2^*\star &= \star[\mathcal{I},d]+[\mathcal{I},d]^*\,\star\\ &= \star\mathcal{I}d-\star d\mathcal{I}+d^*\mathcal{I}^*\,\star-\mathcal{I}^*d^*\,\star\\ &= \mathcal{I}(\star d+d^*\star)-(\star d+d^*\star)\mathcal{I}\\ &= 0\end{aligned}$$

which shows that part (5) in Definition 2.6 holds for the phase $\zeta=-1$. Hence, existence of such suitable skew-adjoint operator $\mathcal{I}$ guarantees that the $N=(1,1)$ spectral data obtained in Proposition 3.12 extends to Hermitian spectral data over $\mathcal{A}$, i.e. $\mathcal{A}$ inherits a complex structure. □

Lemma 3.17. *The Hermitian spectral data obtained in previous Lemma 3.16 is a $N=(2,2)$ Kähler spectral data over $\mathcal{A}$, i.e. $\mathcal{A}$ inherits a Kähler structure, if and only if $\{d,d_2^*\}=\{d^*,d_2\}=0$ with $d_2=[\,\mathcal{I},d\,]$.*

Proof. Recall part (6) in Definition 2.6 which is precisely the Kähler condition. Observe that

$$\begin{aligned}\{\partial, \partial^*\} &= \partial\partial^* + \partial^*\partial \\ &= (d - id_2)(d^* + id_2^*) + (d^* + id_2^*)(d - id_2) \\ &= \{d, d^*\} + \{d_2, d_2^*\} + i\{d, d_2^*\} - i\{d^*, d_2\}\end{aligned}$$

Similarly,

$$\begin{aligned}\{\overline{\partial}, \overline{\partial}^*\} &= \{d, d^*\} + \{d_2, d_2^*\} - i\{d, d_2^*\} + i\{d^*, d_2\} \\ \{\partial, \overline{\partial}^*\} &= \{d, d^*\} - \{d_2, d_2^*\} - i\{d, d_2^*\} - i\{d^*, d_2\} \\ \{\overline{\partial}, \partial^*\} &= \{d, d^*\} - \{d_2, d_2^*\} + i\{d, d_2^*\} + i\{d^*, d_2\}\end{aligned}$$

This shows that the following conditions

(1) $\{d, d_2^*\} = \{d^*, d_2\} = 0$
(2) $\{d, d^*\} = \{d_2, d_2^*\}$

are necessary and sufficient for the complex structure obtained in Lemma 3.16 to extend to a Kähler structure on $\mathcal{A}$. However, condition (2) follows from condition (1) because

$$\begin{aligned}\{d, d^*\} &= dd^* + d^*d \\ &= -[\mathcal{I}, [\mathcal{I}, d]]d^* - d^*[\mathcal{I}, [\mathcal{I}, d]] \\ &= -\mathcal{I}d_2d^* + d_2\mathcal{I}d^* - d^*\mathcal{I}d_2 + d^*d_2\mathcal{I} \\ &= (d_2\mathcal{I}d^* - d_2d^*\mathcal{I}) + d_2d^*\mathcal{I} + (\mathcal{I}d^*d_2 - d^*\mathcal{I}d_2) - \mathcal{I}d^*d_2 + d^*d_2\mathcal{I} - \mathcal{I}d_2d^* \\ &= (d_2d_2^* + d_2^*d_2) + \{d_2, d^*\}\mathcal{I} - \mathcal{I}\{d^*, d_2\} \\ &= \{d_2, d_2^*\}\end{aligned}$$

if $\{d^*, d_2\} = 0$. Hence, the condition $\{d, d_2^*\} = \{d^*, d_2\} = 0$, with $d_2 = [\mathcal{I}, d]$, is necessary and sufficient for the complex structure obtained in Lemma 3.16 to become a Kähler structure on $\mathcal{A}$. □

Proof of Theorem 3.14. Let $dim(G) = 2k$ and $N = 2^k$. We first produce a skew-Hermitian matrix $\widetilde{\mathcal{I}} \in M_{N^2}(\mathbb{C})$ such that the skew-adjoint operator $\mathcal{I} = 1 \otimes \widetilde{\mathcal{I}}$ acting on $L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^{N^2}$ satisfy all the conditions of Lemma 3.16. Note that $M_{N^2}(\mathbb{C}) = M_N(\mathbb{C}) \otimes_{\mathbb{C}} M_N(\mathbb{C})$. Consider the Clifford algebra $\mathbb{C}l(2k)$ and suppose that $\{e_1, \ldots, e_{2k}\}$ be a generating set. Consider the following elements

$$A(\ell, j) := 1 \otimes e_\ell e_j + e_\ell e_j \otimes 1 \tag{3.3}$$

in $\mathbb{C}l(2k) \otimes \mathbb{C}l(2k)$ for each pair (ℓ, j) with $\ell < j$ and $\ell, j \in \{1, \ldots, 2k\}$. Each $A(\ell, j)$ commutes with the elements $e_1 \ldots e_{2k} \otimes e_1 \ldots e_{2k}$ and $1 \otimes e_1 \ldots e_{2k}$ of $\mathbb{C}l(2k) \otimes \mathbb{C}l(2k)$, as $e_1 \ldots e_{2k}$ lies in the center of $\mathbb{C}l(2k)$. Now, it is easy to verify that for each such pair (ℓ, j), the element $A(\ell, j)$ commutes with $\sum_{r \neq \ell, j} e_r \otimes e_r$ in $\mathbb{C}l(2k) \otimes \mathbb{C}l(2k)$. Observe that $A(\ell, j)$ also commutes with $e_\ell \otimes e_\ell + e_j \otimes e_j$. Hence, for each such pair (ℓ, j), $A(\ell, j)$ will commute with $\sum_{r=1}^{2k} e_r \otimes e_r$ in $\mathbb{C}l(2k) \otimes \mathbb{C}l(2k)$. Now, let $\pi : \mathbb{C}l(2k) \longrightarrow M_N(\mathbb{C})$, given by $\pi : e_r \mapsto \gamma_r$, be the irreducible representation in Proposition 3.2. The element $\prod_{j=1}^{2k} e_j \in \mathbb{C}l(2k)$ corresponds to the grading operator σ if k is even and $-i\sigma$ if k is odd under the representation π (see [17] for detail). Hence,

$$(\pi \otimes \pi)(A(\ell, j)) = 1 \otimes \gamma_\ell\gamma_j + \gamma_\ell\gamma_j \otimes 1$$

are skew-Hermitian matrices in $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ such that the skew-adjoint operators $1 \otimes \mathcal{I}_{(\ell,j)} := 1 \otimes (\pi \otimes \pi)(A(\ell, j))$ commute with $\mathcal{T}, \widetilde{\gamma}$ and $\star$ (recall the expression of $\mathcal{T}$ from Lemma 3.15 and that of $\widetilde{\gamma}$ and $\star$ from Proposition 3.12). Observe that

$$\begin{aligned}[1 \otimes \mathcal{I}_{(\ell,j)}, d] &= \sum_{r=1}^{2k} \frac{1}{2}\partial_r \otimes 1 \otimes [\gamma_\ell\gamma_j, \gamma_r] + \frac{i\varepsilon'}{2}\partial_r \otimes [\gamma_\ell\gamma_j, \gamma_r] \otimes \sigma \\ &= \partial_\ell \otimes 1 \otimes \gamma_j - \partial_j \otimes 1 \otimes \gamma_\ell + i\varepsilon'(\partial_\ell \otimes \gamma_j \otimes \sigma - \partial_j \otimes \gamma_\ell \otimes \sigma)\end{aligned}$$

and hence,

$$[1 \otimes \mathcal{I}_{(\ell,j)}, [1 \otimes \mathcal{I}_{(\ell,j)}, d]] = -2(\partial_\ell \otimes 1 \otimes \gamma_\ell + \partial_j \otimes 1 \otimes \gamma_j) - 2i\varepsilon'(\partial_\ell \otimes \gamma_\ell \otimes \sigma + \partial_j \otimes \gamma_j \otimes \sigma).$$

Hence, if we consider $\mathcal{I} = 1 \otimes \widetilde{\mathcal{I}}$ with

$$\widetilde{\mathcal{I}} = \frac{1}{2}\left(\mathcal{I}_{(1,2)} + \mathcal{I}_{(3,4)} + \cdots + \mathcal{I}_{(2k-1,2k)}\right)$$

then we have $[\mathcal{I}, [\mathcal{I}, d]] = -d$ along with $[\mathcal{I}, \mathcal{T}] = [\mathcal{I}, \widetilde{\gamma}] = [\mathcal{I}, \star] = 0$. Hence, by Lemma 3.16, the $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \sigma)$ extends to Hermitian spectral data over $\mathcal{A}$, i.e. $\mathcal{A}$ inherits a complex structure.

We now show that the condition in Lemma 3.17 is also satisfied. For $d_2 := [\mathcal{I}, d]$, note that

$$d_2 = \sum_{j=1,\, j\, odd}^{2k} \frac{1}{2}(\partial_j \otimes 1 \otimes \gamma_{j+1} - \partial_{j+1} \otimes 1 \otimes \gamma_j) + \frac{i\varepsilon'}{2}(\partial_j \otimes \gamma_{j+1} \otimes \sigma - \partial_{j+1} \otimes \gamma_j \otimes \sigma)$$

and recall that $d^* = \sum_{\ell=1}^{2k} \frac{1}{2}(\partial_\ell \otimes 1 \otimes \gamma_\ell - i\varepsilon'\partial_\ell \otimes \gamma_\ell \otimes \sigma)$. Then,

$$\begin{aligned}
&4\{d^*, d_2\} \\
&= \sum_{j=1,\, j\, odd}^{2k} \sum_{\ell=1}^{2k} (\partial_\ell\partial_j \otimes 1 \otimes \{\gamma_\ell, \gamma_{j+1}\} - \partial_\ell\partial_{j+1} \otimes 1 \otimes \{\gamma_\ell, \gamma_j\} + \partial_\ell\partial_j \otimes \{\gamma_\ell, \gamma_{j+1}\} \otimes 1 \\
&\qquad - \partial_\ell\partial_{j+1} \otimes \{\gamma_\ell, \gamma_j\} \otimes 1) \\
&= -4 \sum_{j=1,\, j\, odd}^{2k} \partial_{j+1}\partial_j \otimes 1 \otimes 1 + 4 \sum_{j=1,\, j\, odd}^{2k} \partial_j\partial_{j+1} \otimes 1 \otimes 1 \\
&= 0
\end{aligned}$$

since, $\mathfrak{g}$ is abelian. Similarly, one can verify that $\{d, d_2^*\} = 0$. Hence, the $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \sigma)$ extends to $N = (2, 2)$ Kähler spectral data over $\mathcal{A}$, i.e. $\mathcal{A}$ inherits Kähler structure. This completes the proof. □

Corollary 3.18. *If $(\mathcal{A}, \mathbb{T}^{2k}, \alpha)$ is a C^*-dynamical system such that the action of $\mathbb{T}^{2k}$ is ergodic, then $\mathcal{A}$ inherits a Kähler structure.*

Proof. Since $\mathbb{T}^{2k}$ is compact and the action is ergodic, the unique $\mathbb{T}^{2k}$-invariant state becomes a faithful trace [26], and we have a $2k$-summable (and hence Θ-summable) even spectral triple (Thm. 5.4 in [21]). Conclusion now follows from Theorem 3.14. □

Corollary 3.19. *For n even, the noncommutative n-torus $\mathcal{A}_\Theta$ satisfies the $N = (2, 2)$ Kähler spectral data, i.e. these are noncommutative Kähler manifolds.*

Proof. It is well known that the C^*-dynamical system $(\mathcal{A}_\Theta, \mathbb{T}^n, \alpha)$ on the noncommutative n-torus $\mathcal{A}_\Theta$, where $\alpha_{\mathbf{z}}(U_k) := z_k U_k,\ k = 1, \ldots, n$, equipped with a unique $\mathbb{T}^n$-invariant faithful trace given by

$$\tau\Big(\sum \alpha_{(m_1,\ldots,m_n)} U_1^{m_1} \ldots U_n^{m_n}\Big) := \alpha_{\mathbf{0}}$$

with $\alpha_{(m_1,\ldots,m_n)} \in \mathbb{S}(\mathbb{Z}^n)$, gives a n-summable (and hence Θ-summable) spectral triple

$$\Big(\mathcal{A}_\Theta\,,\ \ell^2(\mathbb{Z}^n) \otimes \mathbb{C}^{2^{\lfloor n/2 \rfloor}},\ D := \sum_{j=1}^{n} \partial_j \otimes \gamma_j\Big).$$

This spectral triple is even if n is even and we obtain a $N = 1$ spectral data on $\mathcal{A}_\Theta$. Conclusion now follows from Theorem 3.14. □

Remark 3.20. As mentioned earlier in the Introduction, characterizing holomorphic structures on n-dimensional manifolds, with $n > 2$, via positive Hochschild cocycles is still open. That is why methods in [10] do not extend to noncommutative higher dimensional tori.

Proposition 3.21. *There can be obtained at least $\prod_{j=1,\, j\, odd}^{dim(G)}(dim(G) - j)$ different Kähler structures in Theorem 3.14.*

Proof. Let $dim(G) = 2k$. In the previous Theorem 3.14, we produced the differential $d_2 = [\mathcal{I}, d]$ by taking a particular $\mathcal{I} = 1 \otimes \widetilde{\mathcal{I}}$ where $\widetilde{\mathcal{I}} = \frac{1}{2}\left(\mathcal{I}_{(1,2)} + \mathcal{I}_{(3,4)} + \ldots + \mathcal{I}_{(2k-1,2k)}\right)$. We now show that there are $\prod_{j=1,\, j\, odd}^{2k-1}(2k - j)$ different choice for $\widetilde{\mathcal{I}}$ built out of $\mathcal{I}_{(\ell,j)}$ with $\ell < j$ and $\ell, j \in \{1, \ldots, 2k\}$. First choose $\mathcal{I}_{(1,j)}$ with $j > 1$. Total number of choice is $2k - 1$.
Case 1: If $j = 2$, next choose $\mathcal{I}_{(3,r)}$ with $r > 3$.
Case 2: If $j > 2$, next choose $\mathcal{I}_{(2,r)}$ so that $r > 2$ and $r \in \{1, 2, \ldots, 2k\} \setminus \{1, 2, j\}$.
Hence for each $\mathcal{I}_{(1,j)}$, we get a total $2k - 3$ different choice to consider the next $\mathcal{I}_{(3,r)}$ or $\mathcal{I}_{(2,r)}$ accordingly as $j = 2$ or $j > 2$ respectively. Now,
Case 1: If $j = 2$ and $\mathcal{I}_{(3,r)}$ with $r > 3$ have been chosen, next consider $\mathcal{I}_{(s,t)}$ with $s = min\{\{1, 2, \ldots, 2k\} \setminus \{1, 2, 3, r\}\}$ and $t > s$ with $t \in \{1, 2, \ldots, 2k\} \setminus \{1, 2, 3, r\}$.
Case 2: If $j > 2$ and $\mathcal{I}_{(2,r)}$ with $r > 2$ have been chosen, next consider $\mathcal{I}_{(s,t)}$ with $s = min\{\{1, 2, \ldots, 2k\} \setminus \{1, 2, j, r\}\}$ and $t > s$ with $t \in \{1, 2, \ldots, 2k\} \setminus \{1, 2, j, r\}$.
Hence for each $\mathcal{I}_{(3,r)}$, we get a total $2k - 5$ different choice to choose the next $\mathcal{I}_{(s,t)}$, and similar choice for each $\mathcal{I}_{(2,r)}$.

Until this we get a total $(2k-1)(2k-3)(2k-5)$ different choice. Proceed like this to choose the next $\mathcal{I}_{(p,q)}$ with $\mathcal{I}_{(s,t)}$ being chosen. We will finally get total $\prod_{j=1,\, j\, odd}^{2k-1}(2k-j)$ different choice for $\widetilde{\mathcal{I}}$. It is a purely algebraic verification that all these choice of $\mathcal{I} = 1 \otimes \widetilde{\mathcal{I}}$ give us different d_2 satisfying $[\mathcal{I}, [\mathcal{I}, d]] = -d$ and $\{d^*, d_2\} = \{d, d_2^*\} = 0$. Thus, one can obtain $\prod_{j=1,\, j\, odd}^{2k-1}(2k-j)$ different Kähler structures in previous Theorem 3.14.

These various choice of indices (m, n) in $\mathcal{I}_{(m,n)}$ at each stage is best understood by a directed tree. For example, if $dim(G) = 2k = 2$ then there is a unique choice of $\widetilde{\mathcal{I}}$ namely, $\widetilde{\mathcal{I}} = \frac{1}{2}\mathcal{I}_{(1,2)}$. If $dim(G) = 2k = 4$ then we have the following tree for various choice of $\mathcal{I}_{(\ell,j)}$ at each stage,

$$\begin{array}{ccc}
(1,2) & (1,3) & (1,4) \\
| & | & | \\
(3,4) & (2,4) & (2,3)
\end{array}$$

Here, the top index represents different possible choice of $\mathcal{I}_{(1,j)}$ and we get total three different choice of $\widetilde{\mathcal{I}}$ namely, $\widetilde{\mathcal{I}} = \frac{1}{2}(\mathcal{I}_{(1,2)} + \mathcal{I}_{(3,4)})$, $\widetilde{\mathcal{I}} = \frac{1}{2}(\mathcal{I}_{(1,3)} + \mathcal{I}_{(2,4)})$ and $\widetilde{\mathcal{I}} = \frac{1}{2}(\mathcal{I}_{(1,4)} + \mathcal{I}_{(2,3)})$. If $dim(G) = 2k = 6$ then we have the following tree for various choice of $\mathcal{I}_{(i,j)}$ at each stage,

$$\begin{tikzcd}[column sep=small]
 & (1,2) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & & (1,3) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & & (1,4) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(3,4) \arrow[d, dash] & (3,5) \arrow[d, dash] & (3,6) \arrow[d, dash] & (2,4) \arrow[d, dash] & (2,5) \arrow[d, dash] & (2,6) \arrow[d, dash] & (2,3) \arrow[d, dash] & (2,5) \arrow[d, dash] & (2,6) \arrow[d, dash] \\
(5,6) & (4,6) & (4,5) & (5,6) & (4,6) & (4,5) & (5,6) & (3,6) & (3,5)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & (1,5) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & & (1,6) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(2,3) \arrow[d, dash] & (2,4) \arrow[d, dash] & (2,6) \arrow[d, dash] & (2,3) \arrow[d, dash] & (2,4) \arrow[d, dash] & (2,5) \arrow[d, dash] \\
(4,6) & (3,6) & (3,4) & (4,5) & (3,5) & (3,4)
\end{tikzcd}$$

The top index represents different possible choice of $\mathcal{I}_{(1,j)}$ and we get total fifteen different choice of $\widetilde{\mathcal{I}}$ given by half times the addition of each vertical row along their prescribed path. Observe that the $\widetilde{\mathcal{I}}$ given by half times the addition of the first vertical row, namely $\widetilde{\mathcal{I}} = \frac{1}{2}(\mathcal{I}_{(1,2)} + \mathcal{I}_{(3,4)} + \mathcal{I}_{(5,6)})$, is the one considered in Theorem 3.14. □

Remark 3.22. We do not know yet whether some or all of these different Kähler structures are unitary equivalent.

Explanation of Remark 3.8. We now explain why we did not discard ε' in every places from Proposition 3.7 up to Theorem 3.14. Reason is that as pointed out in [17], in the even case there are actually two possible real structures $J_\pm$ that differ by multiplication by the grading operator. None of them should be preferred as they are perfectly on the same footing. The table mentioned in Proposition 3.2 has the following extension :

n	0	2	4	6	0	2	4	6	1	3	5	7
ε	$+$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
ε'	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$
ε''	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$				

The first column represents the real structure J_+ and the second is for J_-. To accommodate both the possible real structures we did not discard ε'. Hence, accordingly as $\varepsilon' = +1$ or -1, both ∂ and $\overline{\partial}$ change and we actually obtain two different Kähler structures in Theorem 3.14, and therefore $2\prod_{j=1,\, j\, odd}^{dim(G)}(dim(G)-j)$ different Kähler structures in view of Proposition 3.21. However, it turns out that these two set of Kähler differentials corresponding to $J_\pm$ are unitary conjugate to each other. If we denote the Kähler differentials obtained in Theorem 3.14 by $\partial_\pm$ and $\overline{\partial_\pm}$ corresponding to the real structures $J_\pm$, then one can verify the following relationship

$$(1 \otimes \sigma \otimes 1)\,\partial_+ = \partial_-\,(1 \otimes \sigma \otimes 1) \quad and \quad (1 \otimes \sigma \otimes 1)\,\overline{\partial_+} = \overline{\partial_-}\,(1 \otimes \sigma \otimes 1)\,.$$

Here, the operator $1 \otimes \sigma \otimes 1$, which is a self-adjoint unitary acting on $L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N \otimes \mathbb{C}^N$, is precisely the product of the $\mathbb{Z}_2$-grading and the Hodge operator obtained in Proposition 3.12. Being unitary equivalent we do not distinguish the Kähler structures $\{\partial_+, \overline{\partial_+}\}$ and $\{\partial_-, \overline{\partial_-}\}$.

Corollary 3.23. *For the noncommutative two-torus $\mathcal{A}_\theta$, with irrational θ, represented faithfully on the Hilbert space $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2 \bigoplus \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2 \cong \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$ by diagonal operator, one has*

$$\gamma_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma_2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac operator is given by

$$D = \begin{pmatrix} 0 & i\partial_1 + \partial_2 & 0 & 0 \\ i\partial_1 - \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\partial_1 + \partial_2 \\ 0 & 0 & i\partial_1 - \partial_2 & 0 \end{pmatrix}$$

with the grading operator $\widetilde{\gamma}$ *and the Hodge operator* $\star$ *as*

$$\widetilde{\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \star = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad .$$

Two set of unitary equivalent Kähler differentials are given by

$$\partial = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2}(i\partial_1 - \partial_2) & 0 & 0 & 0 \\ \frac{i\varepsilon'}{2}(i\partial_1 - \partial_2) & 0 & 0 & 0 \\ 0 & -\frac{i\varepsilon'}{2}(i\partial_1 - \partial_2) & \frac{1}{2}(i\partial_1 - \partial_2) & 0 \end{pmatrix}$$

$$\overline{\partial} = \begin{pmatrix} 0 & \frac{1}{2}(i\partial_1 + \partial_2) & \frac{i\varepsilon'}{2}(i\partial_1 + \partial_2) & 0 \\ 0 & 0 & 0 & -\frac{i\varepsilon'}{2}(i\partial_1 + \partial_2) \\ 0 & 0 & 0 & \frac{1}{2}(i\partial_1 + \partial_2) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\varepsilon' = \pm 1$, *and the nilpotent differential is* $d := \partial + \overline{\partial}$ *with* $d + d^* = D$.

Proof. The matrices γ_1, γ_2 and σ are obtained from the explicit representation of $\mathbb{C}l(2)$ on $M_2(\mathbb{C})$ (see Proposition 3.2). From Proposition 3.12, since $d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$, we get

$$d = \frac{1}{2}(\partial_1 \otimes 1 \otimes \gamma_1 + \partial_2 \otimes 1 \otimes \gamma_2) + \frac{i\varepsilon'}{2}(\partial_1 \otimes \gamma_1 \otimes \sigma + \partial_2 \otimes \gamma_2 \otimes \sigma)$$

and from Theorem 3.14, we get

$$d_2 = \frac{1}{2}(\partial_1 \otimes 1 \otimes \gamma_2 - \partial_2 \otimes 1 \otimes \gamma_1) + \frac{i\varepsilon'}{2}(\partial_1 \otimes \gamma_2 \otimes \sigma - \partial_2 \otimes \gamma_1 \otimes \sigma)$$

The expression for the Dirac operator D is then clear since, $D = d + d^* = \mathfrak{D}$. Two set of unitary equivalent Kähler differentials are given by $\partial = \frac{1}{2}(d - id_2)$ and $\overline{\partial} = \frac{1}{2}(d + id_2)$ with $\varepsilon' = \pm 1$. □

Remark 3.24. The differential $\overline{\partial}$ in Corollary 3.23 coincides with the complex structure obtained in [10] from cyclic cohomology and using the equivalence of conformal and complex structures in two dimensions. This is further considered in [32]. We will come back to it again towards the end of Section 5.

4. Space of complex differential forms on noncommutative 2*n*-tori

In this section, we work with the complex (in fact Kähler) structure obtained in Theorem 3.14 to compute the space of complex differential forms on all noncommutative even dimensional tori. Recall the space of $N = (1, 1)$ differential forms (Section 2.2.2 in [20]) and complex differential forms (Section 2.3.2 in [20]).

Definition 4.1 (*[33]*). Let $\mathscr{A}$ be the universal C^*-algebra generated by $2n$ unitaries $U_1, \ldots, U_{2n}$ satisfying $U_jU_\ell = exp(2\pi i\Theta_{\ell j})U_\ell U_j$, where Θ is a real $2n \times 2n$ skew-symmetric matrix such that the lattice $\wedge_\Theta$ generated by its columns makes $\wedge_\Theta + \mathbb{Z}^{2n}$ dense in $\mathbb{R}^{2n}$. The compact connected Lie group $\mathbb{T}^{2n}$ acts on $\mathscr{A}$ by $\alpha_{\mathbf{z}}(U_\ell) = z_\ell U_\ell,\ \ell = 1, \ldots, 2n$. Let $\mathcal{A}_\Theta$ denote the smooth subalgebra of $\mathscr{A}$ under this action. Via Fourier transform one obtains

$$\mathcal{A}_\Theta := \left\{ \sum \alpha_{(j_1,\ldots,j_{2n})} U_1^{j_1} \ldots U_{2n}^{j_{2n}} \ :\ \alpha_{(j_1,\ldots,j_{2n})} \in \mathbb{S}(\mathbb{Z}^{2n}) \right\} .$$

Then, $\mathcal{A}_\Theta$ is a unital spectrally invariant subalgebra of $\mathscr{A}$, called the noncommutative $2n$-torus.

Proposition 4.2. *For the noncommutative $2n$-torus $\mathcal{A}_\Theta$, as an $\mathcal{A}_\Theta$-bimodule we have,*

(1) $\Omega^0_d(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$,
(2) $\Omega^\ell_d(\mathcal{A}_\Theta) := span\{a\prod_{j=1}^{\ell}[d, b_j] : a, b_j \in \mathcal{A}_\Theta\} \cong \mathcal{A}_\Theta^{\frac{2n!}{\ell!(2n-\ell)!}}$ $\forall\, 1 \le \ell \le 2n$,
(3) $\Omega^\ell_d(\mathcal{A}_\Theta) \cong \{0\}$ $\forall\, \ell > 2n$;

where, d is the operator constructed along the lines of the previous section.

Proof. Part (1) is obvious. Recall that $d = \frac{1}{2}(\mathfrak{D} - i\overline{\mathfrak{D}})$, where $\mathfrak{D} = 1 \otimes D$ and $\overline{\mathfrak{D}} = -\varepsilon' D \otimes \sigma$ acting on $\mathcal{H} = \overline{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} \cong L^2(\mathcal{A}, \tau)^{N^2}$ (see Propositions 3.5 and 3.7). Hence, for

$$d = \frac{1}{2}\sum_{j=1}^{2n}(\partial_j \otimes 1 \otimes \gamma_j + i\varepsilon'\partial_j \otimes \gamma_j \otimes \sigma)$$

we see that

$$\begin{aligned}[d, a] &= \frac{1}{2}\sum_{j=1}^{2n}\Big(\partial_j(a) \otimes 1 \otimes \gamma_j + i\varepsilon'\partial_j(a) \otimes \gamma_j \otimes \sigma\Big)\\ &= \sum_{j=1}^{2n}\partial_j(a) \otimes \Big(1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma\Big).\end{aligned}$$

We claim that the set $\{1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma : j = 1, \ldots, 2n\}$ is a linearly independent subset of $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. Consider

$$\sum_{j=1}^{2n}\alpha_j(1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma) = 0 \tag{4.4}$$

with $\alpha_j \in \mathbb{C}$ for all j. Multiplying this Eq. (4.4) by $1 \otimes \sigma$ from the right, and then again from the left, we get

$$\sum_{j=1}^{2n}\alpha_j(-1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma) = 0\,. \tag{4.5}$$

Now, (4.4)–(4.5) gives us

$$\sum_{j=1}^{2n} 1 \otimes \alpha_j\gamma_j = 0$$

in $M_N(\mathbb{C})\otimes M_N(\mathbb{C})$. Since, $\{\gamma_1, \ldots, \gamma_{2n}\}$ is a linearly independent subset of $M_N(\mathbb{C})$ we get $\alpha_j = 0$ for all j proving our claim. Hence, the following map

$$\begin{aligned}\Phi \ :\ \Omega^1_d(\mathcal{A}_\Theta) &\longrightarrow \mathcal{A}_\Theta^{2n}\\ a[d, b] &\longmapsto (a\partial_1(b), \ldots, a\partial_{2n}(b))\end{aligned}$$

is an injective $\mathcal{A}_\Theta$-bimodule map. Now, for any $(0, \ldots, a, \ldots, 0) \in \mathcal{A}_\Theta^{2n}$ with a in the jth place, the element $aU_j^*\delta U_j \in \Omega^1(\mathcal{A}_\Theta)$, δ being the universal differential, descends to $aU_j^*[d, U_j] \in \Omega^1_d(\mathcal{A}_\Theta)$ and $\Phi(aU_j^*[d, U_j]) = (0, \ldots, a, \ldots, 0)$, proving surjectivity of Φ. This concludes part (2) for $\ell = 1$. For arbitrary $1 \le \ell \le 2n$, first observe that

$$\Big(1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma\Big)^2 = 0 \quad , \quad \Big\{(1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma), (1 \otimes \frac{1}{2}\gamma_r + \frac{i\varepsilon'}{2}\gamma_r \otimes \sigma)\Big\} = 0$$

for any $1 \le j \ne r \le 2n$. Hence, for $a, b \in \mathcal{A}_\Theta$,

$$\begin{aligned}&[d, a][d, b]\\ &= \sum_{1\le j<r\le 2n}(\partial_j(a)\partial_r(b) - \partial_r(a)\partial_j(b)) \otimes \Big(1 \otimes \frac{1}{2}\gamma_j + \frac{i\varepsilon'}{2}\gamma_j \otimes \sigma\Big)\Big(1 \otimes \frac{1}{2}\gamma_r + \frac{i\varepsilon'}{2}\gamma_r \otimes \sigma\Big).\end{aligned}$$

Same argument as in the case of $\ell = 1$ will now show that $\Omega^2_d(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^{\frac{2n!}{2(2n-2)!}}$. By induction on $1 \le \ell \le 2n$ one concludes Part (2) and Part (3) simultaneously. □

Lemma 4.3. *For the noncommutative $2n$-torus $\mathcal{A}_\Theta$ with $n > 1$, as an $\mathcal{A}_\Theta$-bimodule, we have*

(1) $\Omega^{0,0}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$,

(2) $\Omega^{1,0}_{\partial,\bar\partial}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^n$,
(3) $\Omega^{0,1}_{\partial,\bar\partial}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^n$,
(4) $\Omega^{2,0}_{\partial,\bar\partial}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^{\frac{n(n-1)}{2}}$,
(5) $\Omega^{0,2}_{\partial,\bar\partial}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^{\frac{n(n-1)}{2}}$,
(6) *The product map* $\Omega^{0,1}_{\partial,\bar\partial}(\mathcal{A}_\Theta) \times \Omega^{0,1}_{\partial,\bar\partial}(\mathcal{A}_\Theta) \longrightarrow \Omega^{0,2}_{\partial,\bar\partial}(\mathcal{A}_\Theta)$ *is given by*

$$(a_1,\dots,a_n).(b_1,\dots,b_n) \longmapsto ((a_pb_q - a_qb_p)_{1\le p<q\le n}).$$

Proof. Part (1) is obvious. For part (2), from Theorem 3.14 we get that

$$\begin{aligned}
[\partial, a] &= \frac{1}{2}[d - id_2, a]\\
&= \frac{1}{4}\Big(\sum_{j=1}^{2n} \partial_j(a)\otimes 1\otimes\gamma_j - \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_\ell(a)\otimes 1\otimes\gamma_{\ell+1} + \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_{\ell+1}(a)\otimes 1\otimes\gamma_\ell\Big)\\
&\quad + \frac{i\varepsilon'}{4}\Big(\sum_{j=1}^{2n} \partial_j(a)\otimes\gamma_j\otimes\sigma - \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_\ell(a)\otimes\gamma_{\ell+1}\otimes\sigma + \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_{\ell+1}(a)\otimes\gamma_\ell\otimes\sigma\Big)\\
&= \frac{1}{4}\Big(\sum_{\ell=1,\,\ell\, odd}^{2n} (\partial_{\ell+1} - i\partial_\ell)(a)\otimes 1\otimes\gamma_{\ell+1} + (\partial_\ell + i\partial_{\ell+1})(a)\otimes 1\otimes\gamma_\ell\Big)\\
&\quad + \frac{i\varepsilon'}{4}\Big(\sum_{\ell=1,\,\ell\, odd}^{2n} (\partial_{\ell+1} - i\partial_\ell)(a)\otimes\gamma_{\ell+1}\otimes\sigma + (\partial_\ell + i\partial_{\ell+1})(a)\otimes\gamma_\ell\otimes\sigma\Big)\\
&= \sum_{\ell=1,\,\ell\, odd}^{2n} \frac{1}{4}(\partial_{\ell+1} - i\partial_\ell)(a)\otimes 1\otimes(\gamma_{\ell+1} + i\gamma_\ell) + \frac{i\varepsilon'}{4}(\partial_{\ell+1} - i\partial_\ell)(a)\otimes(\gamma_{\ell+1} + i\gamma_\ell)\otimes\sigma
\end{aligned}$$

for all $a \in \mathcal{A}_\Theta$. It can be verified (same way as in Proposition 4.2) that the set $\{1\otimes\frac{1}{4}(\gamma_{\ell+1} + i\gamma_\ell) + \frac{i\varepsilon'}{4}(\gamma_{\ell+1} + i\gamma_\ell)\otimes\sigma : \ell\in\{1,\dots,2n\},\ \ell \text{ is odd}\}$ is a linearly independent subset of $M_N(\mathbb{C})\otimes M_N(\mathbb{C})$. Hence, the following map

$$\begin{aligned}
\Phi : \Omega^{1,0}_{\partial,\bar\partial}(\mathcal{A}_\Theta) &\longrightarrow \mathcal{A}_\Theta^n\\
a[\partial, b] &\longmapsto \sum_{\ell=1,\,\ell\, odd}^{2n} \Big(0,\dots,\underbrace{a\partial_{\ell+1}(b) - ia\partial_\ell(b)}_{\frac{\ell+1}{2}\,th\ place},\dots,0\Big)
\end{aligned}$$

is an injective $\mathcal{A}_\Theta$-bimodule map. For arbitrary $\xi = (0,\dots,a,\dots,0) \in \mathcal{A}_\Theta^n$ with a in the $(\ell+1)/2$th place, $\Phi(aU^*_{\ell+1}[\partial, U_{\ell+1}]) = \xi$. This shows that Φ is surjective, concluding Part (2). Part (3) follows similarly since,

$$\begin{aligned}
[\bar\partial, a] &= \frac{1}{2}[d + id_2, a]\\
&= \frac{1}{4}\Big(\sum_{j=1}^{2n} \partial_j(a)\otimes 1\otimes\gamma_j + \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_\ell(a)\otimes 1\otimes\gamma_{\ell+1} - \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_{\ell+1}(a)\otimes 1\otimes\gamma_\ell\Big)\\
&\quad + \frac{i\varepsilon'}{4}\Big(\sum_{j=1}^{2n} \partial_j(a)\otimes\gamma_j\otimes\sigma + \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_\ell(a)\otimes\gamma_{\ell+1}\otimes\sigma - \sum_{\ell=1,\,\ell\, odd}^{2n} i\partial_{\ell+1}(a)\otimes\gamma_\ell\otimes\sigma\Big)\\
&= \sum_{\ell=1,\,\ell\, odd}^{2n} \frac{1}{4}(\partial_{\ell+1} + i\partial_\ell)(a)\otimes 1\otimes(\gamma_{\ell+1} - i\gamma_\ell) + \frac{i\varepsilon'}{4}(\partial_{\ell+1} + i\partial_\ell)(a)\otimes(\gamma_{\ell+1} - i\gamma_\ell)\otimes\sigma
\end{aligned}$$

for all $a \in \mathcal{A}_\Theta$.

For Part (5), denote $\delta_j := \partial_{2j} + i\partial_{2j-1}$ and $\eta_j := \gamma_{2j} - i\gamma_{2j-1}$ for $j = 1,\dots,n$. Observe that

$$\eta_j^2 = 0 \quad , \quad \{\eta_p, \eta_q\} = 0\ \forall\, p \neq q\,.$$

So by part (3) we see that

$$[\bar\partial, a] = \sum_{j=1}^{n} \frac{1}{4}\delta_j(a)\otimes 1\otimes\eta_j + \frac{i\varepsilon'}{4}\delta_j(a)\otimes\eta_j\otimes\sigma\,.$$

Hence, for arbitrary $a, b \in \mathcal{A}_\Theta$,

$$
\begin{aligned}
&[\,\overline{\partial}, a][\,\overline{\partial}, b]\\
&= \frac{1}{16}\sum_{\ell<r}(\delta_r(a)\delta_\ell(b) - \delta_\ell(a)\delta_r(b)) \otimes (1 \otimes \eta_\ell\eta_r - \eta_\ell\eta_r \otimes 1)\\
&\quad + \frac{i\varepsilon'}{16}\sum_{\ell\neq r}(\delta_r(a)\delta_\ell(b)) \otimes (\eta_\ell \otimes \eta_r - \eta_r \otimes \eta_\ell)(1 \otimes \sigma)\\
&= \sum_{\ell<r}(\delta_r(a)\delta_\ell(b) - \delta_\ell(a)\delta_r(b)) \otimes \left(\frac{1}{16}(1 \otimes \eta_\ell\eta_r - \eta_\ell\eta_r \otimes 1) + \frac{i\varepsilon'}{16}(\eta_\ell \otimes \eta_r\sigma - \eta_r \otimes \eta_\ell\sigma)\right)
\end{aligned}
$$

The set $\left\{\frac{1}{16}(1\otimes\eta_\ell\eta_r - \eta_\ell\eta_r\otimes 1) + \frac{i\varepsilon'}{16}(\eta_\ell\otimes\eta_r\sigma - \eta_r\otimes\eta_\ell\sigma) : 1 \leq \ell < r \leq n\right\}$ can be easily seen to be a linearly independent subset of $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. Hence, the following map

$$
\begin{aligned}
&\Phi : \Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \longrightarrow \mathcal{A}_\Theta^{\frac{n(n-1)}{2}}\\
&a[\,\overline{\partial}, b\,][\,\overline{\partial}, c\,] \longmapsto \Big((a\delta_r(b)\delta_\ell(c) - a\delta_\ell(b)\delta_r(c))_{1\leq\ell<r\leq n}\Big)
\end{aligned}
$$

is an injective $\mathcal{A}_\Theta$-bimodule map. To see surjectivity, observe that for any $a \in \mathcal{A}_\Theta$ in (ℓ, r)-position with $\ell < r$,

$$
\Phi : aU^*_{2\ell}U^*_{2r}[\,\overline{\partial}, U_{2r}][\,\overline{\partial}, U_{2\ell}] \longmapsto a\,.
$$

This completes Part (5), and Part (4) follows similarly.

For Part (6), starting with $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in \mathcal{A}^n_\Theta$ first obtain their respective inverse image in $\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)$ using Part (3), then take the product to get an element in $\Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)$ and finally use the isomorphism in Part (5) to find its image in $\mathcal{A}_\Theta^{\frac{n(n-1)}{2}}$. We left this for the reader to verify. $\square$

Corollary 4.4. *If $\{e_1, \ldots, e_n\}$ denotes the standard free module basis of $\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \cong \mathcal{A}^n_\Theta$, then $\{e_\ell e_r : 1 \leq \ell < r \leq n\}$ is a free module basis of $\Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^{\frac{n(n-1)}{2}}$. Moreover, $e_\ell e_r + e_r e_\ell = e_\ell^2 = 0$ for all $1 \leq \ell < r \leq n$.*

Proof. Follows from Part (3), (4) and (5) in the previous Lemma 4.3. $\square$

Theorem 4.5. *For the noncommutative $2n$-torus $\mathcal{A}_\Theta$, as an $\mathcal{A}_\Theta$-bimodule, one has*

(1) $\Omega^{\ell,0}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^{\frac{n!}{\ell!(n-\ell)!}}\ \forall\, 1 \leq \ell \leq n$,
(2) $\Omega^{0,\ell}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^{\frac{n!}{\ell!(n-\ell)!}}\ \forall\, 1 \leq \ell \leq n$,
(3) $\Omega^{\ell,0}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) = \Omega^{0,\ell}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) = \{0\}\ \forall\, \ell > n$,
(4) $\Omega^r_d(\mathcal{A}_\Theta) \cong \bigoplus_{p+q=r} \Omega^{p,q}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)$.

Proof. The case of $n = 1$ should be treated separately. In this case of $\mathcal{A}_\theta$,

$$
[\,\overline{\partial}, a] = \frac{1}{4}(\partial_2 + i\partial_1)(a) \otimes 1 \otimes (\gamma_2 - i\gamma_1) + \frac{i\varepsilon'}{4}(\partial_2 + i\partial_1)(a) \otimes (\gamma_2 - i\gamma_1) \otimes \sigma
$$

for all $a \in \mathcal{A}_\theta$. Since, $(\gamma_2 - i\gamma_1)^2 = \{\gamma_2 - i\gamma_1, \sigma\} = 0$, one gets that $[\,\overline{\partial}, a][\,\overline{\partial}, b] = 0$ for all $a, b \in \mathcal{A}_\theta$. Part (1, 2, 3) now follows by induction on ℓ in Lemma 4.3, similarly as in Proposition 4.2. To show Part (4) recall from Propn. 2.33 in [20] that it is enough to show $[T, \omega] \in \Omega^1_d(\mathcal{A}_\Theta)$ for all $\omega \in \Omega^1_d(\mathcal{A}_\Theta)$, where $T = \frac{1}{2}(\mathcal{T} - i\mathcal{I})$ is as in Lemma 3.16. Observe that if $\omega = a[d, b]$ then $[T, \omega] = a[\partial, b] \in \Omega^{1,0}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)$. By Proposition 4.2 and Lemma 4.3 we see that

$$
\Omega^1_d(\mathcal{A}_\Theta) = \Omega^{1,0}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \bigoplus \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)
$$

as $\mathcal{A}_\Theta$-bimodules. Hence, we conclude that $[T, \omega] \in \Omega^1_d(\mathcal{A}_\Theta)$ for all $\omega \in \Omega^1_d(\mathcal{A}_\Theta)$. This concludes Part (4). $\square$

5. Holomorphic vector bundles

5.1. Holomorphic vector bundle

Let $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T}, \gamma, \star)$ be a Hermitian (or in particular, $N = (2, 2)$ Kähler) spectral data over the unital algebra $\mathcal{A}$. Recall the space of complex differential forms from Section 2.3.2 and notion of integration from Section 2.3.3 in [20].

A crucial orthogonality property is mentioned in Propn. 2.35 in [20]. However, for this section it is enough to recall the following:

$$\Omega^{1,0}_{\partial,\overline{\partial}}(\mathcal{A}) := span\{a[\partial, b] : a, b \in \mathcal{A}\} \quad , \quad \Omega^{2,0}_{\partial,\overline{\partial}}(\mathcal{A}) := span\{a[\partial, b][\partial, c] : a, b, c \in \mathcal{A}\} \ ,$$
$$\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) := span\{a[\overline{\partial}, b] : a, b \in \mathcal{A}\} \quad , \quad \Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}) := span\{a[\overline{\partial}, b][\overline{\partial}, c] : a, b, c \in \mathcal{A}\} \ .$$

Definition 5.1 ([29]). The algebra of holomorphic elements in $\mathcal{A}$ is defined as

$$\mathcal{O}(\mathcal{A}) := Ker\left\{\overline{\partial} : \mathcal{A} \longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A})\right\} .$$

This is a $\mathbb{C}$-subalgebra of $\mathcal{A}$.

Definition 5.2 ([29]). A holomorphic structure on a f.g.p. left $\mathcal{A}$-module $\mathcal{E}$ is a flat $\overline{\partial}$-connection, i.e. connection $\nabla : \mathcal{E} \longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ such that the associated $\overline{\partial}$-curvature $\Theta \in \mathcal{H}om_{\mathcal{A}}\left(\mathcal{E}\, , \Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}\right)$ vanishes. The pair $(\mathcal{E}, \nabla)$ is called a holomorphic vector bundle over $\mathcal{A}$.

Definition 5.3 ([29]). If $(\mathcal{E}, \nabla)$ is a holomorphic vector bundle over $\mathcal{A}$ then

$$H^0(\mathcal{E}, \nabla) := ker\left\{\nabla : \mathcal{E} \longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}\right\}$$

is called the space of holomorphic sections on $\mathcal{E}$.

Remark 5.4.

(1) It follows from the definition of connection that $H^0(\mathcal{E}, \nabla)$ is a left $\mathcal{O}(\mathcal{A})$-module.
(2) Recall that in the classical case, a vector bundle on a complex manifold is holomorphic if and only if it admits a flat $\overline{\partial}$-connection.

Consider a f.g.p. left module $\mathcal{E}$ over $\mathcal{A}$. Then there exists a positive integer m and a left $\mathcal{A}$-module homomorphism $pr : \mathcal{A}^m \longrightarrow \mathcal{E}$. By definition, there exists a left $\mathcal{A}$-module $\mathcal{F}$ such that $\mathcal{E} \oplus \mathcal{F} \cong \mathcal{A}^m$ and denote $i : \mathcal{E} \longrightarrow \mathcal{A}^m$ to be the inclusion map determined by this isomorphism. We have $pr \circ i = id$ on $\mathcal{E}$.

Lemma 5.5. *Any $\overline{\partial}$-connection $\widetilde{\nabla}$ on the free module $\mathcal{A}^m$ induces a $\overline{\partial}$-connection ∇ on $\mathcal{E}$.*

Proof. Given such $\widetilde{\nabla}$, define

$$\nabla : \mathcal{E} \longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$
$$\nabla = (id \otimes pr) \circ \widetilde{\nabla} \circ i\, .$$

Clearly, ∇ is a $\mathbb{C}$-linear map. Now, for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$,

$$\begin{aligned}
\nabla(a\xi) &= (id \otimes pr) \circ \widetilde{\nabla}(ai(\xi)) \\
&= (id \otimes pr)\Big([\overline{\partial}, a] \otimes i(\xi) + a\widetilde{\nabla} \circ i(\xi)\Big) \\
&= [\overline{\partial}, a] \otimes \xi + a\nabla(\xi)
\end{aligned}$$

proving ∇ is a $\overline{\partial}$-connection on $\mathcal{E}$. □

Moreover, the converse is also true.

Proposition 5.6. *Any $\overline{\partial}$-connection ∇ on $\mathcal{E}$ is induced by a $\overline{\partial}$-connection $\widetilde{\nabla}$ on the free module $\mathcal{A}^m$.*

Proof. Start with a $\overline{\partial}$-connection $\widetilde{\nabla}$ on the free module $\mathcal{A}^m$. By previous Lemma 5.5, we get a $\overline{\partial}$-connection ∇ on $\mathcal{E}$ by the formula $\nabla = (id \otimes pr) \circ \widetilde{\nabla} \circ i$. Now, let ∇' be any other $\overline{\partial}$-connection on $\mathcal{E}$. Then $\nabla' - \nabla \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E})$. Since,

$$id \otimes pr : \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^m \longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

is surjective and $\mathcal{E}$ is a projective module, there exists a module map

$$\phi : \mathcal{E} \longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^m$$

such that $\nabla' - \nabla = (id \otimes pr) \circ \phi$. Then, $\widetilde{\phi} = \phi \circ pr \in \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^m, \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^m)$ and hence, $\widetilde{\nabla} + \widetilde{\phi}$ is a $\overline{\partial}$-connection on $\mathcal{A}^m$. The associated connection on $\mathcal{E}$ is

$$\begin{aligned} (id \otimes pr) \circ (\widetilde{\nabla} + \widetilde{\phi}) \circ i &= \nabla + (id \otimes pr) \circ \phi \\ &= \nabla' \end{aligned}$$

i,e. ∇' is induced by the $\overline{\partial}$-connection $\widetilde{\nabla} + \widetilde{\phi}$ on the free module $\mathcal{A}^m$. □

Proposition 5.7. *Any free module over $\mathcal{A}$ is a holomorphic vector bundle.*

Proof. Let $\mathcal{A}^m$ be a free module over $\mathcal{A}$ of rank m. Since $\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^m \cong \left(\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A})\right)^m$, define

$$\begin{aligned} \nabla_0 : \mathcal{A}^m &\longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^m \\ (a_1, \ldots, a_m) &\longmapsto \big([\,\overline{\partial}, a_1], \ldots, [\,\overline{\partial}, a_m]\big) \end{aligned}$$

It is easy to check that ∇_0 is a $\overline{\partial}$-connection. Let $\{e_1, \ldots, e_m\}$ denote the standard free $\mathcal{A}$-module basis of $\mathcal{A}^m$. Then, the associated curvature becomes

$$\begin{aligned} \Theta_{\nabla_0}(a_1, \ldots, a_m) &= \nabla_0\Big([\,\overline{\partial}, a_1], \ldots, [\,\overline{\partial}, a_m]\Big) \\ &= \sum_{j=1}^{m} \nabla_0([\,\overline{\partial}, a_j] \otimes e_j) \\ &= \sum_{j=1}^{m} -[\,\overline{\partial}, a_j]\nabla_0(e_j) + [\,\overline{\partial}, 1][\,\overline{\partial}, a_j] \otimes e_j \\ &= 0 \end{aligned}$$

since, $\nabla_0(e_j) = 0$. Hence, ∇_0 is flat $\overline{\partial}$-connection. This shows that $(\mathcal{A}^m, \nabla_0)$ is a holomorphic vector bundle over $\mathcal{A}$. □

Corollary 5.8. *The space of holomorphic sections of any free module $\mathcal{E}_0 = \mathcal{A}^m$ over $\mathcal{A}$ is a free $\mathcal{O}(\mathcal{A})$-module of rank m.*

Proof. Let $\mathcal{E}_0 = \mathcal{A}^m$ be a free module over $\mathcal{A}$ of rank m. Then $(\mathcal{E}_0, \nabla_0)$ is a holomorphic vector bundle over $\mathcal{A}$ by previous Proposition 5.7. Now, from Definition 5.3 we get that

$$\begin{aligned} H^0(\mathcal{E}_0, \nabla_0) &= Ker\left\{\nabla_0 : \mathcal{A}^m \longrightarrow \left(\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A})\right)^m\right\} \\ &= \{(a_1, \ldots, a_m) : [\,\overline{\partial}, a_j] = 0 \ \forall j = 1, \ldots, m\} \\ &= \{(a_1, \ldots, a_m) : \ a_j \in \mathcal{O}(\mathcal{A}) \forall j = 1, \ldots, m\} \\ &\cong \mathcal{O}(\mathcal{A})^m \end{aligned}$$

i,e. $H^0(\mathcal{E}_0, \nabla_0)$ is a free $\mathcal{O}(\mathcal{A})$-module of rank$=$ $rank(\mathcal{E}_0)$. □

5.2. Holomorphic vector bundles over noncommutative 2n-tori

Proposition 5.9. *The algebra $\mathcal{O}(\mathcal{A}_\Theta)$ of holomorphic elements in $\mathcal{A}_\Theta$ is $\mathbb{C}$.*

Proof. From Lemma 4.3,

$$\begin{aligned} \overline{\partial} : \mathcal{A}_\Theta &\longrightarrow \Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta) \cong \mathcal{A}^n_\Theta \\ a &\longmapsto ((\partial_2 + i\partial_1)(a), \ldots, (\partial_{2n} + i\partial_{2n-1})(a)) . \end{aligned}$$

Hence, by Definition 5.1,

$$\mathcal{O}(\mathcal{A}_\Theta) = \Big\{ a \in \mathcal{A}_\Theta : (\partial_{j+1} + i\partial_j)(a) = 0 \ \forall j \in \{1, \ldots, 2n\} \,;\, j \text{ is odd} \Big\} .$$

Arbitrary $a \in \mathcal{A}_\Theta$ is of the form $\sum_{(\ell_1,\ldots,\ell_{2n})\in\mathbb{Z}^{2n}} \alpha_{\ell_1,\ldots,\ell_{2n}} U_1^{\ell_1} \ldots U_{2n}^{\ell_{2n}}$ where $\alpha_{\ell_1,\ldots,\ell_{2n}} \in \mathbb{S}(\mathbb{Z}^{2n})$ and hence, for any odd $j \in \{1, \ldots, 2n\}$ we have

$$(\partial_{j+1} + i\partial_j)(a) = \sum (\ell_{j+1} + i\ell_j)\alpha_{\ell_1,\ldots,\ell_{2n}} U_1^{\ell_1} \ldots U_{2n}^{\ell_{2n}} .$$

This expression is equal to zero implies that $(\ell_{j+1} + i\ell_j)\alpha_{\ell_1,\ldots,\ell_{2n}} = 0$. Hence, $\ell_{j+1} = \ell_j = 0$. Thus, a is of the form $\sum \alpha_{\ell_1,\ldots,\ell_{2n}} U_1^{\ell_1} \ldots \widehat{U_j^{\ell_j}} \widehat{U_{j+1}^{\ell_{j+1}}} \ldots U_{2n}^{\ell_{2n}}$. This is true for all $j \in \{1, \ldots, 2n\}$, j is odd. Hence, we conclude that $a \in \mathbb{C}1$, which proves $\mathcal{O}(\mathcal{A}_\Theta) \cong \mathbb{C}$. □

Corollary 5.10. *Space of holomorphic sections of any free module $\mathcal{E}_0 = \mathcal{A}_\Theta^m$ over $\mathcal{A}_\Theta$ is $\mathbb{C}^m$.*

Proof. Follows from Corollary 5.8 and previous Proposition 5.9. □

Lemma 5.11. *$\Omega_d^r(\mathcal{A}_\Theta)$, $\Omega_{\partial,\overline{\partial}}^{\ell,0}(\mathcal{A}_\Theta)$, $\Omega_{\partial,\overline{\partial}}^{0,\ell}(\mathcal{A}_\Theta)$ all are holomorphic vector bundles over $\mathcal{A}_\Theta$.*

Proof. Follows from Propositions 5.7, 4.2 and Theorem 4.5. □

Theorem 5.12. *A necessary and sufficient condition for existence of holomorphic structure on a f.g.p. left module $\mathcal{E}$ over $\mathcal{A}_\Theta$ is that there exists n-tuple $(\nabla_1, \ldots, \nabla_n)$ of $\mathbb{C}$-linear maps $\nabla_j : \mathcal{E} \longrightarrow \mathcal{E}$ such that the following conditions are satisfied*

(1) $\nabla_j(a\xi) = a\nabla_j(\xi) + \delta_j(a)\xi \quad \forall\, a \in \mathcal{A}_\Theta$,
(2) $[\nabla_\ell, \nabla_r] = 0 \quad \forall\, 1 \le l < r \le n.$

where, $\delta_j = \partial_{2j} + i\partial_{2j-1}$.

Proof. Recall from Lemma 4.3 that $\Omega_{\partial,\overline{\partial}}^{0,1}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^n$. Hence, any $\overline{\partial}$-connection $\nabla : \mathcal{E} \longrightarrow \Omega_{\partial,\overline{\partial}}^{0,1}(\mathcal{A}_\Theta)\otimes\mathcal{E}$ on $\mathcal{E}$ is implemented by n-tuple of $\mathbb{C}$-linear maps $(\nabla_1, \ldots, \nabla_n)$ with each $\nabla_j : \mathcal{E} \longrightarrow \mathcal{E}$. Since, ∇ is a $\overline{\partial}$-connection, it is easy to verify that

$$\nabla_j(a\xi) = a\nabla_j(\xi) + \delta_j(a)\xi \quad \forall\, a \in \mathcal{A}_\Theta \text{ and } \xi \in \mathcal{E}$$

where, $\delta_j = \partial_{2j} + i\partial_{2j-1}$. If ∇ induces a holomorphic structure on $\mathcal{E}$ then $\Theta_\nabla = 0$. Now, if $\{e_1, \ldots, e_n\}$ denotes the standard free module basis of $\Omega_{\partial,\overline{\partial}}^{0,1}(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta^n$ then observe from Lemma 4.3 that the map $\overline{\partial}' : \Omega_{\partial,\overline{\partial}}^{0,1}(\mathcal{A}_\Theta) \longrightarrow \Omega_{\partial,\overline{\partial}}^{0,2}(\mathcal{A}_\Theta)$, given by $\overline{\partial}' : a[\overline{\partial}, b] \longmapsto [\overline{\partial}, a][\overline{\partial}, b]$, satisfies $\overline{\partial}'(e_j) = 0$. Hence, for any $\xi \in \mathcal{E}$, we have

$$\begin{aligned}
\Theta_\nabla(\xi) &= \sum_{j=1}^{n} \nabla(e_j \otimes \nabla_j(\xi)) \\
&= \sum_{j=1}^{n} -e_j\nabla(\nabla_j(\xi)) + \overline{\partial}'(e_j) \otimes \nabla_j(\xi) \\
&= \sum_{\ell,j=1}^{n} -e_je_\ell \otimes \nabla_\ell(\nabla_j(\xi)) \\
&= \sum_{\ell<j} e_\ell e_j \otimes [\nabla_\ell, \nabla_j](\xi)
\end{aligned}$$

because $e_pe_q + e_qe_p = 0$ for $p \neq q$ and $e_p^2 = 0$ (Corollary 4.4). Since, $\{e_\ell e_j : 1 \le \ell < j \le n\}$ is the standard free module basis of $\Omega_{\partial,\overline{\partial}}^{0,2}(\mathcal{A}_\Theta)$ we get $\Theta_\nabla = 0$ if and only if $[\nabla_\ell, \nabla_r] = 0$ for all $1 \le \ell < r \le n$. This fulfills our claim. □

Observe from Proposition 4.2 and Lemma 4.3 that $\Omega_d^1(\mathcal{A}_\Theta) \cong \Omega_{\partial,\overline{\partial}}^{1,0}(\mathcal{A}_\Theta) \bigoplus \Omega_{\partial,\overline{\partial}}^{0,1}(\mathcal{A}_\Theta)$. This is in fact an orthogonal direct sum by Propn. 2.35 in [20]. Hence, any $\mathbb{C}$-linear map $\nabla : \mathcal{E} \longrightarrow \Omega_d^1(\mathcal{A}_\Theta) \otimes_{\mathcal{A}_\Theta} \mathcal{E}$ satisfying $\nabla(a\xi) = a\nabla(\xi) + [d, a] \otimes \xi$, i.e. a d-connection, can be written as $\nabla^{1,0} + \nabla^{0,1}$. Let $\pi^{1,0}$ and $\pi^{0,1}$ be the orthogonal projections onto $\Omega_{\partial,\overline{\partial}}^{1,0}$ and $\Omega_{\partial,\overline{\partial}}^{0,1}$ respectively. Note that these are $\mathcal{A}_\Theta$-module maps.

Proposition 5.13. *Let $\mathcal{E}$ be a f.g.p. left module over $\mathcal{A}_\Theta$ and $\nabla : \mathcal{E} \longrightarrow \Omega_d^1(\mathcal{A}_\Theta) \otimes_{\mathcal{A}_\Theta} \mathcal{E}$ be a d-connection whose curvature has vanishing $(0, 2)$-component. Then, ∇ induces a holomorphic structure on $\mathcal{E}$. In particular, any flat d-connection induces a holomorphic structure on $\mathcal{E}$.*

Proof. Let ∇ be a d-connection and define

$$\begin{aligned}
\nabla' : \mathcal{E} &\longrightarrow \Omega_{\partial,\overline{\partial}}^{0,1}(\mathcal{A}_\Theta) \otimes_{\mathcal{A}_\Theta} \mathcal{E} \\
\xi &\longmapsto (\pi^{0,1} \otimes id)\nabla(\xi)\,.
\end{aligned}$$

Since $\pi^{0,1}$ is a left $\mathcal{A}_\Theta$-module homomorphism, it is easy to observe that ∇' is a $\overline{\partial}$-connection. Observe that the associated curvature satisfies the following relation

$$\Theta_{\nabla'} = (\pi^{0,2} \otimes id)\Theta_\nabla\,.$$

Hence, if the $(0, 2)$-component of the curvature Θ_∇ vanishes then ∇' is a flat $\overline{\partial}$-connection. For detail verification follow the proof of Propn. 4.7 in [2]. In particular, if Θ_∇ itself is zero i.e. ∇ is d-flat then ∇' induces a holomorphic structure on $\mathcal{E}$. □

If $(\mathcal{E},\nabla)$ is a holomorphic vector bundle over $\mathcal{A}$ then

$$0\longrightarrow \mathcal{E}\xrightarrow{\nabla}\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}\xrightarrow{\nabla}\Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}\xrightarrow{\nabla}\ldots\ldots$$

is a cochain complex. The cohomology groups of this complex are denoted by $H^\bullet(\mathcal{E},\nabla)$. Recall from Definition 5.3 that the zero-th cohomology is the space of holomorphic sections on $\mathcal{E}$. It follows from the definition of connection that each $H^\bullet(\mathcal{E},\nabla)$ is a left $\mathcal{O}(\mathcal{A})$-module. Hence, for the case of noncommutative torus $\mathcal{A}=\mathcal{A}_\Theta$, they are $\mathbb{C}$-vector spaces (by Proposition 5.9).

Proposition 5.14. *Every short exact sequence*

$$0\longrightarrow(\mathcal{E},\nabla_{\mathcal{E}})\xrightarrow{\phi}(\mathcal{F},\nabla_{\mathcal{F}})\xrightarrow{\psi}(\mathcal{G},\nabla_{\mathcal{G}})\longrightarrow 0$$

of holomorphic vector bundles over $\mathcal{A}_\Theta$ induces a long exact sequence

$$0\longrightarrow H^0(\mathcal{E},\nabla_{\mathcal{E}})\xrightarrow{\phi^*}H^0(\mathcal{F},\nabla_{\mathcal{F}})\xrightarrow{\psi^*}H^0(\mathcal{G},\nabla_{\mathcal{G}})\xrightarrow{\overline{\delta}}H^1(\mathcal{E},\nabla_{\mathcal{E}})\xrightarrow{\phi^*}\ldots\ldots$$

in cohomology of $\mathbb{C}$-vector spaces.

Proof. Since, $\Omega^{0,\bullet}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)$ are free modules over $\mathcal{A}_\Theta$ (Theorem 4.5) we get

$$0\longrightarrow\Omega^{0,\bullet}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)\otimes_{\mathcal{A}_\Theta}\mathcal{E}\xrightarrow{id\otimes\phi}\Omega^{0,\bullet}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)\otimes_{\mathcal{A}_\Theta}\mathcal{F}\xrightarrow{id\otimes\psi}\Omega^{0,\bullet}_{\partial,\overline{\partial}}(\mathcal{A}_\Theta)\otimes_{\mathcal{A}_\Theta}\mathcal{G}\longrightarrow 0$$

is an exact sequence of cochain complexes which induces a long exact sequence in cohomology (See Propn. 4.6 in [2]). □

5.3. The case of noncommutative two-torus revisited

In this final subsection we revisit the case of noncommutative two-torus $\mathcal{A}_\theta$ studied earlier in [32] and obtain their framework as a special case of our results for general C^*-dynamical systems.

Recall from Part (3) in Theorem 4.5 that the noncommutative space of complex two forms

$$\Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}_\theta):=span\{a[\,\overline{\partial},b][\,\overline{\partial},c]:a,b,c\in\mathcal{A}_\theta\}$$

vanishes identically for the case of noncommutative two-torus. Because of this reason for any $\overline{\partial}$-connection $\nabla:\mathcal{E}\longrightarrow\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\theta)\otimes_{\mathcal{A}_\theta}\mathcal{E}$, the associated $\overline{\partial}$-curvature $\Theta_\nabla:\mathcal{E}\longrightarrow\Omega^{0,2}_{\partial,\overline{\partial}}(\mathcal{A}_\theta)\otimes_{\mathcal{A}_\theta}\mathcal{E}$ is always zero i.e. ∇ is always $\overline{\partial}$-flat. Also, as observed in Lemma 4.3, we have

$$\begin{aligned}\Phi:\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\theta)&\longrightarrow\mathcal{A}_\theta\\ a[\,\overline{\partial},b]&\longmapsto a(\partial_2+i\partial_1)(b)\end{aligned}$$

is an $\mathcal{A}_\theta$-bimodule isomorphism. Hence, for any f.g.p. left $\mathcal{A}_\theta$-module $\mathcal{E}$ we get $\Omega^{0,1}_{\partial,\overline{\partial}}(\mathcal{A}_\theta)\otimes_{\mathcal{A}_\theta}\mathcal{E}$ is canonically isomorphic with $\mathcal{E}$, since $\mathcal{A}_\theta$ is unital. Therefore, a holomorphic structure on $\mathcal{E}$ is given by a $\mathbb{C}$-linear map $\nabla:\mathcal{E}\longrightarrow\mathcal{E}$ such that

$$\nabla(a\xi)=a\nabla(\xi)+(\partial_2+i\partial_1)(a)\xi$$

for all $\xi\in\mathcal{E}$ and $a\in\mathcal{A}_\theta$. For arbitrary $a\in\mathcal{A}_\theta$ of the form $\sum_{(r_1,r_2)\in\mathbb{Z}^2}\alpha_{r_1,r_2}U_1^{r_1}U_2^{r_2}$ we see that

$$(\partial_2+i\partial_1)(a)=(r_2+ir_1)a\,.$$

If we denote τ to be the purely imaginary number i then the derivation on $\mathcal{A}_\theta$ defined by

$$\partial_\tau\left(\sum_{(r_1,r_2)\in\mathbb{Z}^2}\alpha_{r_1,r_2}U_1^{r_1}U_2^{r_2}\right):=2\pi i\sum_{(r_1,r_2)\in\mathbb{Z}^2}(r_1\tau+r_2)\alpha_{r_1,r_2}U_1^{r_1}U_2^{r_2}$$

is equal to $\Phi\circ[\,\overline{\partial},.]$. This is the complex structure considered in [32] for $\mathcal{A}_\theta$, and we see that the definition of holomorphic vector bundle given in [32] for the case of noncommutative two-torus $\mathcal{A}_\theta$ is a special case of the general definition given in (Definition 5.2).

Moreover, the complex $\left(\Omega^{0,\bullet}_{\partial,\overline{\partial}}(\mathcal{A}_\theta)\otimes_{\mathcal{A}_\theta}\mathcal{E},\nabla\right)$ becomes just

$$0\longrightarrow\mathcal{E}\xrightarrow{\nabla}\mathcal{E}\longrightarrow 0$$

and hence, the cohomology becomes

$$H^0(\mathcal{E},\nabla)=Ker\{\nabla:\mathcal{E}\longrightarrow\mathcal{E}\}\quad\text{and}\quad H^1(\mathcal{E},\nabla)=Coker\{\nabla:\mathcal{E}\longrightarrow\mathcal{E}\}.$$

We see that this is the definition of the cohomology given in [32].

Open question

Under which condition(s) the $N = (2, 2)$ Kähler spectral data obtained in Theorem 3.14 extends further to a $N = (4, 4)$ hyper-Kähler spectral data (Defn. 2.37 in [20])? One necessary condition should be $dim(G) = 4n$ but we are not sure yet whether this is also the sufficient condition. Note that in the classical case, the $4n$-dimensional tori are actually hyper-Kähler manifolds. We expect the same for the noncommutative $4n$-dimensional tori also.

Acknowledgments

Author gratefully acknowledges the support of DST INSPIRE (India) Faculty award grant (DST/INSPIRE/04/2015/000901). He thanks Anirban Bose for useful discussion and the anonymous referee for her/his comments and suggestions.

References

[1] L. Álvarez-Gaumé, D.Z. Freedman, Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model, Comm. Math. Phys. 80 (3) (1981) 443–451.
[2] E. Beggs, S.P. Smith, Noncommutative complex differential geometry, J. Geom. Phys. 72 (2013) 7–33.
[3] R.O. Buachalla, Noncommutative complex structures on quantum homogeneous spaces, J. Geom. Phys. 99 (2016) 154–173.
[4] R.O. Buachalla, Noncommutative Kähler structures on quantum homogeneous spaces, Adv. Math. 322 (2017) 892–939.
[5] P.S. Chakraborty, S. Guin, Yang-mills on quantum heisenberg manifolds, Comm. Math. Phys. 330 (3) (2014) 1327–1337.
[6] P.S. Chakraborty, K.B. Sinha, Geometry on the quantum heisenberg manifold, J. Funct. Anal. 203 (2) (2003) 425–452.
[7] A. Connes, C^*-algèbres et géométrie différentielle, C. R. Acad. Sci. Paris Sér. A-B 290 (13) (1980) 599–604.
[8] A. Connes, Noncommutative differential geometry, Inst. Hautes Études Sci. Publ. Math. (62) (1985) 257–360.
[9] A. Connes, Compact metric spaces, fredholm modules, and hyperfiniteness, Ergodic Theory Dynam. Syst. 9 (2) (1989) 207–220.
[10] A. Connes, Noncommutative Geometry, Academic Press, Inc, San Diego, CA, 1994.
[11] A. Connes, Noncommutative geometry and reality, J. Math. Phys. 36 (11) (1995) 6194–6231.
[12] A. Connes, Gravity coupled with matter and the foundation of noncommutative geometry, Comm. Math. Phys. 182 (1) (1996) 155–176.
[13] A. Connes, On the spectral characterization of manifolds, J. Noncommut. Geom. 7 (1) (2013) 1–82.
[14] A. Connes, J. Cuntz, Quasi homomorphismes, cohomologie cyclique et positivité, Comm. Math. Phys. 114 (3) (1988) 515–526.
[15] A. Connes, M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives, in: AMS Colloquium Publications, vol. 55, Hindustan Book Agency, 2008.
[16] A. Connes, M.A. Rieffel, Yang–mills for noncommutative two-tori, Contemp. Math. 62 (1987) 237–266.
[17] L. Dabrowski, G. Dossena, Product of real spectral triples, Int. J. Geom. Methods Mod. Phys. 8 (8) (2011) 1833–1848.
[18] M. Dieng, A. Schwarz, Differential and complex geometry of two-dimensional noncommutative tori, Lett. Math. Phys. 61 (2002) 263–270.
[19] J. Fröhlich, O. Grandjean, A. Recknagel, Supersymmetric quantum theory and differential geometry, Comm. Math. Phys. 193 (3) (1998) 527–594.
[20] J. Fröhlich, O. Grandjean, A. Recknagel, Supersymmetric quantum theory and non-commutative geometry, Comm. Math. Phys. 203 (1) (1999) 119–184.
[21] O. Gabriel, M. Grensing, Ergodic actions and spectral triples, J. Oper. Theory 76 (2) (2016) 307–334.
[22] J. Gracia-Bondía, J. Várilly, H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser Boston, Inc., Boston, MA, 2001.
[23] S. Guin, The tensor product of supersymmetric n=(1, 1) spectral data, Int. J. Geom. Methods Mod. Phys. 15 (12) (2018) 1850207.
[24] I. Heckenberger, S. Kolb, The locally finite part of the dual coalgebra of quantised irreducible flag manifolds, Proc. Lond. Math. Soc. 89 (3) (2004) 457–484.
[25] I. Heckenberger, S. Kolb, De rham complex for quantized irreducible flag manifolds, J. Algebra 305 (2) (2006) 704–741.
[26] R. Hoegh-Krohn, M.B. Landstad, E. Stormer, Compact ergodic groups of automorphisms, Ann. of Math. (2) 114 (1) (1981) 75–86.
[27] D. Huybrechts, Complex Geometry, an Introduction, in: Universitext, Springer-Verlag, Berlin, 2005.
[28] A.M. Jaffe, A. Lesniewski, K. Osterwalder, On super-KMS functionals and entire cyclic cohomology, K-Theory 2 (6) (1989) 675–682.
[29] M. Khalkhali, G. Landi, W.D. van Suijlekom, Holomorphic structures on the quantum projective line, Int. Math. Res. Not. IMRN (4) (2011) 851–884.
[30] M. Khalkhali, A. Moatadelro, Noncommutative complex geometry of the quantum projective space, J. Geom. Phys. 61 (2011) 2436–2452.
[31] S. Majid, Noncommutative Riemannian and spin geometry of the standard q-sphere, Comm. Math. Phys. 256 (2) (2005) 255–285.
[32] A. Polishchuk, A. Schwarz, Categories of holomorphic vector bundles on noncommutative two-tori, Comm. Math. Phys. 236 (1) (2003) 135–159.
[33] M.A. Rieffel, Projective modules over higher-dimensional noncommutative tori, Canad. J. Math. 40 (2) (1988) 257–338.
[34] M.A. Rieffel, Deformation quantization of heisenberg manifolds, Comm. Math. Phys. 122 (4) (1989) 531–562.
[35] M.A. Rieffel, Critical points of yang–mills for noncommutative two-tori, J. Differential Geom. 31 (2) (1990) 535–546.
[36] A. Schwarz, Theta functions on noncommutative tori, Lett. Math. Phys. 58 (2001) 81–90.
[37] N. Seiberg, E. Witten, String theory and noncommutative geometry, J. High Energy Phys. (9) (1999) Paper 32, 93 pp.
[38] E. Witten, Supersymmetry and morse theory, J. Differential Geom. 17 (4) (1982) 661–692.
[39] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), Comm. Math. Phys. 122 (1) (1989) 125–170.