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Journal of Geometry and Physics 22 (1997) 255–258

JOURNAL OF
GEOMETRY AND
PHYSICS

Space of conformal blocks in 4D WZW Theory

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Abstract

We give an algebro-geometric definition of the space of conformal blocks in the 4D WZW theory which has been introduced recently. We calculate the dimension of this vector space for certain algebraic surfaces.

Keywords: Conformal blocks; Stable bundles; Moduli spaces

1. Introduction

Two-dimensional (2D) conformal field theory has attracted the attention of many mathematicians in the last few years. Especially the Verlinde conjecture giving the dimensions of the spaces of conformal blocks has been at the center of interest for its remarkable connection with various fields of mathematics including algebraic geometry and representation theory. At present this conjecture has been proved rigorously [1].

Recently, a 4D analog of Wess–Zumino–Witten (WZW) theory has been proposed [3]. As in 2D CFT, one of the basic objects of study in this theory is the space of conformal blocks, which is described as the space of sections of a certain line bundle on the moduli of holomorphic vector bundles on an algebraic surface. In loc. cit. an analog of the Verlinde conjecture for the dimensions of these spaces has also been presented. Mathematically, however, this conjecture is not well defined since the relevant moduli space is rarely compact. The purpose of this note is to interpret the space of 4D conformal blocks by means of natural line bundles on the moduli of Gieseker semistable torsion-free sheaves, which have been defined by the present author [4]. As an example, we calculate the dimension of the space of global sections of these line bundles for rational ruled surfaces.

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2. The space of conformal blocks

Let X be a smooth projective surface over \mathbb{C} and H an ample line bundle on X . We denote by $\mathcal{M} = \mathcal{M}_H(r, c_1, c_2)$ the moduli space of rank- r vector bundles with Chern classes c_1, c_2 which are H -stable in the sense of Mumford–Takemoto. Although this is not in general a projective variety, it has a compactification \mathcal{M}^{ss} , the moduli of Gieseker semistable torsion-free sheaves [2]. Let $\overline{\mathcal{M}}$ denote the closure of \mathcal{M} in \mathcal{M}^{ss} for Zariski topology, which can be regarded as a minimal compactification of \mathcal{M} . We recall below the construction of determinant line bundles on $\overline{\mathcal{M}}$ as defined in [4].

We fix a sufficiently large integer N and let $\mathcal{O}_X(N) = H^{\otimes N}$. For a suitable integer $P(N)$ depending on N , \mathcal{M}^{ss} is constructed as the geometric invariant theory quotient of a Quot scheme $\mathcal{Q} = \text{Quot}^{\text{ss}}(\mathcal{O}_X(-N) \otimes \mathbb{C}^{P(N)})$ under $\text{SL}(P(N))$ action. We recall that \mathcal{Q} parametrizes the quotient sheaves E ,

$$\mathcal{O}_X(-N) \otimes \mathbb{C}^{P(N)} \rightarrow E,$$

together with an isomorphism $\mathbb{C}^{P(N)} \cong H^0(X, E(N))$.

We fix a universal family \mathcal{F} on $X \times \mathcal{Q}$. If C is a smooth curve on X with $c_1 \cdot C$ divisible by r , choose a line bundle L on C satisfying $\mathcal{X}(\mathcal{F}_x^C \otimes L) = 0$ for every sheaf \mathcal{F}_x corresponding to a point $x \in \mathcal{Q}$. We define a line bundle $\text{Det}_{\mathcal{F}}(C)$ on \mathcal{Q} as follows:

$$\text{Det}_{\mathcal{F}}(C) = \det((p_{\mathcal{Q}})_!(\mathcal{F}^C \otimes p_C^* L))^{\vee},$$

where $\mathcal{F}^C = \mathcal{F}|_{C \times \mathcal{Q}}$ and $p_C, p_{\mathcal{Q}}$ denote the projections from $C \times \mathcal{Q}$ to C and \mathcal{Q} , respectively. Since $\text{PGL}(P(N))$ acts trivially on the fiber of $\text{Det}_{\mathcal{F}}(C)$ at every point with closed orbit, we can descend the line bundle to \mathcal{M}^{ss} which will be also denoted by $\text{Det}_{\mathcal{F}}(C)$. It can be easily seen that this construction does not depend on the choice of L . Let $\mathcal{L}_{\mathcal{F}}(C) = \text{Det}_{\mathcal{F}}(C)|_{\overline{\mathcal{M}}}$ be the restricted line bundle. We define the space of conformal blocks as follows:

$$Z_{\mathcal{F}, C}(r, c_1, c_2) = H^0(\overline{\mathcal{M}}, \mathcal{L}_{\mathcal{F}}(C)).$$

It there exists a universal family \mathcal{E} on $X \times \overline{\mathcal{M}}$, we set

$$\mathcal{L}_{\mathcal{E}}(C) = \det((p_{\overline{\mathcal{M}}})_!(\mathcal{E}^C \otimes p_C^* L))^{\vee}$$

and similarly we define the space $Z_{\mathcal{E}, C}(r, c_1, c_2)$.

To see the relation of these spaces with 4D WZW theory, we assume that X is a regular algebraic surface (i.e. $H^1(X, \mathcal{O}_X) = 0$), $c_1 = 0$ and that there exists a universal family \mathcal{E} on $X \times \overline{\mathcal{M}}$. Then the Grothendieck–Riemann–Roch theorem yields

$$c_1(\mathcal{L}_{\mathcal{E}}(C)) = c_2(\mathcal{E})/[C],$$

where $[C] \in H_2(X, \mathbb{Z})$ denotes the homology class of C and $/$ is the slant product. This may be considered as a higher rank generalization of Donaldson's μ map.

We denote by $\overline{\mathcal{M}}_{\text{red}}$ the scheme theoretic reduction of $\overline{\mathcal{M}}$. Let F be the curvature form of a hermitian connection on \mathcal{E} compatible with the complex structure. If $[C]$ is the Poincaré

dual of a holomorphic two-form ω , then $c_1(\mathcal{L}_{\mathcal{E}}(C))$ is represented on the smooth locus of $\overline{\mathcal{M}}_{\text{red}}$ by the following two-form [5]:

$$\frac{1}{8\pi^2} \int_X \text{Tr}(F \wedge F) \wedge \omega,$$

which essentially reproduces the line bundle considered by physicists in [3]. This suggests that $Z_{\mathcal{F},C}(r, c_1, c_2)$ should be a mathematical formulation of the space of conformal blocks in 4D WZW theory. An interesting problem of computing its dimension remains open at present, although for bundles with $c_1 = 0$ a formula of Verlinde type has been presented [3]. Inspired by the conjecture, we shall compute in Section 3 the dimension of $Z_{\mathcal{F},C}(2, c_1, c_2)$ for certain bundles with $c_1 \neq 0$ on a rational ruled surface.

3. An example

Let $\pi : X \rightarrow \mathbb{P}^1$ be a ruled surface, namely the \mathbb{P}^1 -bundle associated to some rank-2 vector bundle on \mathbb{P}^1 . We denote by Σ, f the divisor class of a section and a fiber, respectively. Let $e = -\Sigma^2$ and assume that $c > \max(-\frac{1}{4}e, 0)$. Then by [6], we can find an ample line bundle H_c such that a rank-2 bundle E with $c_1(E) = \Sigma, c_2(E) = c$ is H_c -stable if and only if E is given as the following nontrivial extension of line bundles:

$$0 \rightarrow \mathcal{O}(\Sigma - cf) \rightarrow E \rightarrow \mathcal{O}(cf) \rightarrow 0.$$

Thus, the moduli space $\mathcal{M} = \mathcal{M}_{H_c}(2, \Sigma, c)$ is particularly simple since H -stability and Gieseker semistability coincide and torsion-free sheaves do not appear: we have $\mathcal{M} = \overline{\mathcal{M}} = \mathcal{M}^{\text{ss}}$. Since the extensions as above are parametrized by the vector space of dimension $4c + e - 2$

$$V_c \simeq H^1(X, \mathcal{O}(\Sigma - 2cf)),$$

we conclude that \mathcal{M} is isomorphic to the projective space $\mathbb{P}(V_c^\vee) \cong \mathbb{P}^n$ where $n = 4c + e - 3$. Furthermore, it is easy to see that there exists a universal rank-2 bundle \mathcal{E} on $X \times \mathcal{M}$ which is given as the following extension:

$$0 \rightarrow p_X^* \mathcal{O}(\Sigma - cf) \rightarrow \mathcal{E} \rightarrow p_X^* \mathcal{O}(cf) \otimes p_{\mathcal{M}}^* \mathcal{O}(1)^\vee \rightarrow 0,$$

where $p_X, p_{\mathcal{M}}$ denote the projections from $X \times \mathcal{M}$ to X, \mathcal{M} and $\mathcal{O}(1)$ is the tautological line bundle on $\mathcal{M} = \mathbb{P}^n$.

Let C be a smooth curve in X such that $C \cdot \Sigma$ is even. Using the universal extension above, a straightforward calculation shows that $\mathcal{L}_{\mathcal{E}}(C)$ is isomorphic to $\mathcal{O}(m)$ where m is the following intersection number:

$$m = \left(cf - \frac{1}{2} \Sigma \right) \cdot C.$$

Therefore, we obtain

$$\dim Z_{\mathcal{E},C}(2, \Sigma, c) = \dim H^0(\mathbb{P}^n, \mathcal{O}(m)) = \binom{n+m}{n}.$$

Acknowledgements

The author is thankful to Y. Takeda for numerous stimulating discussions.

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