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The geometry of the two-component Camassa–Holm and Degasperis–Procesi equations

J. Escher [a], M. Kohlmann [a,*], J. Lenells [b]

[a] Institute for Applied Mathematics, University of Hannover, D-30167 Hannover, Germany

[b] Department of Mathematics, Baylor University, One Bear Place 97328, Waco, USA

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ABSTRACT

We use geometric methods to study two natural two-component generalizations of the periodic Camassa–Holm and Degasperis–Procesi equations. We show that these generalizations can be regarded as geodesic equations on the semidirect product of the diffeomorphism group of the circle $\mathrm{Diff}(S^1)$ with some space of sufficiently smooth functions on the circle. Our goals are to understand the geometric properties of these two-component systems and to prove local well-posedness in various function spaces. Furthermore, we perform some explicit curvature calculations for the two-component Camassa–Holm equation, giving explicit examples of large subspaces of positive curvature.

1. Introduction

In a seminal paper [1], Arnold pointed out that the Euler equations for the motion of a rotating rigid body and the Euler equations of hydrodynamics can both be viewed geometrically as geodesic equations on a Lie group endowed with an invariant metric. More recently, several other equations of physical interest have been found to arise in a similar way; examples include the Korteweg–de Vries, Burgers, Camassa–Holm (CH), and other Euler–Poincaré equations. This geometric viewpoint is not only aesthetically appealing, but is also useful in the study of well-posedness and stability issues. It is therefore of interest to find and study further examples of this type.

The CH equation is a re-expression of the geodesic flow on the diffeomorphism group of the circle $\mathrm{Diff}(S^1)$ equipped with the H^1 right-invariant metric [2,3]. Recently, it has been demonstrated [4] that the Degasperis–Procesi (DP) equation [5] also can be recast as a geodesic equation on $\mathrm{Diff}(S^1)$, although in this case the connection does not derive from an invariant metric [6]. Just like the CH, the DP equation is an approximation to the governing equations of motion for the classical water wave problem in the shallow-water regime; cf. [7]. Both the CH and DP equations are integrable and admit peakon solutions [8,9]. The integrability manifests itself in the existence of a Lax pair and a bi-Hamiltonian structure for each of the equations.

In this paper, we will develop the geometric picture for the following two-component generalizations of the CH and DP equations:

$$\begin{cases} m_t = -um_x - 2mu_x - \rho\rho_x, \\ \rho_t = -(\rho u)_x, \end{cases} \tag{2CH}$$

* Corresponding author.
E-mail addresses: escher@ifam.uni-hannover.de (J. Escher), kohlmann@ifam.uni-hannover.de (M. Kohlmann), Jonatan_Lenells@baylor.edu (J. Lenells).

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and

$$\begin{cases} m_t = -3mu_x - m_x u - \rho u_x + 2\rho\rho_x, \\ \rho_t = -2\rho u_x - \rho_x u, \end{cases} \tag{2DP}$$

where $u(x,t)$ and $\rho(x,t)$ are real-valued functions of $x \in S^1 \simeq \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$, and $m = u - u_{xx}$.

The system (2CH) was first derived in [10] using bi-Hamiltonian methods. The system admits a Lax pair formulation and is integrable. In fact, it is related to the first negative flow of the AKNS hierarchy via a reciprocal transformation [11,12]. A derivation of (2CH) in the context of shallow water waves appears in [13]. Well-posedness and blow-up results are obtained in [14,15].[1]

The system (2DP) was first proposed in [16] as a natural generalization of the DP equation in the context of supersymmetry. Although the approach of [16] automatically yields one Hamiltonian structure for (2DP), neither a second Hamiltonian structure nor a Lax pair could be found. The question of the integrability of (2DP) therefore remains open.

We will show that the two-component generalizations (2CH) and (2DP) can be regarded as geodesic equations on the semidirect product $\mathrm{Diff}(S^1)\circledS\mathcal{F}(S^1)$, where $\mathcal{F}(S^1)$ denotes a space of sufficiently smooth real-valued functions on the circle. For 2CH, the geodesic equation derives from a natural right-invariant Riemannian metric, whereas for 2DP the affine connection is not compatible with any such metric. The geometric construction will give immediate proofs of local well-posedness for both systems in $H^s(S^1) \times H^{s-1}(S^1)$ or $C^n(S^1) \times C^{n-1}(S^1)$ for sufficiently smooth initial data. Moreover, we will show that the local well-posedness can be extended to the Fréchet space $C^\infty(S^1) \times C^\infty(S^1)$. Our main result reads as follows.

Theorem 1.1. *There exist open intervals J_1 and J_2 centered at 0 and an open neighborhood U of $(0,0) \in C^\infty(S^1) \times C^\infty(S^1)$ such that for each $(u_0, \rho_0) \in U$ there exist a unique solution*

$$(u, \rho) \in C^\infty(J_1, C^\infty(S^1) \times C^\infty(S^1))$$

of (2CH) and a unique solution

$$(v, \eta) \in C^\infty(J_2, C^\infty(S^1) \times C^\infty(S^1))$$

of (2DP) with $(u(0), \rho(0)) = (v(0), \eta(0)) = (u_0, \rho_0)$. Furthermore, the solutions depend smoothly on the initial data in the sense that the local flows

$$\Phi_i : J_i \times U \to C^\infty(S^1) \times C^\infty(S^1),$$

for $i = 1, 2$, defined by $\Phi_1(t, u_0, \rho_0) = (u(t; u_0, \rho_0), \rho(t; u_0, \rho_0))$ and $\Phi_2(t, u_0, \rho_0) = (v(t; u_0, \rho_0), \eta(t; u_0, \rho_0))$ are smooth maps.

Although a geometric reformulation of the 2CH as a geodesic flow is presented in [17] (see also [18]), our work contains the following novel aspects: We apply the geometric picture to obtain local well-posedness results (in particular in the smooth category) and we provide a detailed discussion of the sectional curvature associated with the 2CH. A generalization of our approach for 2DP has previously not been presented in the literature.

Our paper is organized as follows: In Section 2, we introduce the relevant function spaces and semidirect products. In Section 3, we establish the geometric interpretation of 2CH as a geodesic equation with respect to a right-invariant metric and prove local well-posedness in various settings. The 2DP equation is considered in Section 4. In Section 5, we present some explicit computations of the sectional curvature for the 2CH equation. In an appendix, the geometric interpretations of 2CH, CH, and the rotating rigid body are compared in an attempt to emphasize the unifying features of the approach.

2. Function spaces and semidirect products

We will show that 2CH and 2DP are geodesic equations on the semidirect product

$$G = \mathrm{Diff}(S^1)\circledS\mathcal{F}(S^1), \tag{2.1}$$

where $\mathrm{Diff}(S^1)$ denotes the group of orientation-preserving diffeomorphisms of the circle $S^1 \simeq \mathbb{R}/\mathbb{Z}$ and $\mathcal{F}(S^1)$ denotes a space of sufficiently regular real-valued functions on S^1 (the exact regularity assumptions will be made precise below).

Let (φ, f) and (ψ, g) be two elements of G. The group product in G is defined by

$$(\varphi, f)(\psi, g) := (\varphi \circ \psi, g + f \circ \psi)$$

where $\circ$ denotes composition. The neutral element of G is $(\mathrm{id}, 0)$ and (φ, f) has the inverse $(\varphi^{-1}, -f \circ \varphi^{-1})$. Of particular interest to us will be the right translation operator $R_{(\psi,g)} : G \to G$ defined by

$$R_{(\psi,g)}(\varphi, f) = (\varphi, f)(\psi, g).$$

[1] In some of these references the term $-\rho\rho_x$ in (2CH) is chosen to have the opposite sign.

Several different regularity assumptions can be imposed on the elements of G. The structure of Eqs. (2CH) and (2DP) suggests that the function ρ should be allowed to have one spatial derivative less than u. This suggests the following choice for G:

$$H^sG := H^s\mathrm{Diff}(S^1)\circledS H^{s-1}(S^1), \tag{2.2}$$

where $H^s\mathrm{Diff}(S^1)$ denotes the space of orientation-preserving diffeomorphisms of S^1 of Sobolev class H^s. We will assume that $s > 5/2$. In this case, $H^s\mathrm{Diff}(S^1)$ is a Hilbert manifold and a topological group and the composition map

$$(\varphi, f) \mapsto f \circ \varphi : H^s\mathrm{Diff}(S^1) \times H^{s-1}(S^1) \to H^{s-1}(S^1)$$

is continuous; cf. [19]. Thus, H^sG is a topological group and a smooth manifold modeled on the Hilbert space $H^s(S^1) \times H^{s-1}(S^1)$.

Another natural choice for G is

$$C^nG := C^n\mathrm{Diff}(S^1)\circledS C^{n-1}(S^1), \tag{2.3}$$

where $C^n\mathrm{Diff}(S^1)$ denotes the space of orientation-preserving diffeomorphisms of S^1 of class C^n. We will assume that $n \geq 2$. In this case, C^nG is a topological group and a smooth manifold modeled on the Banach space $C^n(S^1) \times C^{n-1}(S^1)$. Note that H^sG and C^nG are *not* Lie groups, since left multiplication is only continuous and not smooth.

Finally, we may choose G as

$$C^\infty G := C^\infty\mathrm{Diff}(S^1)\circledS C^\infty(S^1). \tag{2.4}$$

This is a Lie group (the multiplication and inverse maps are smooth) and a Fréchet manifold modeled on $C^\infty(S^1) \times C^\infty(S^1)$. In contrast to H^sG and C^nG, it is *not* a Banach manifold.

The three choices (2.2)–(2.4) for G are all of interest due to their different advantages. We will first develop the theory for H^sG and then consider C^nG and $C^\infty G$.

We refer the reader to [20,21] for further information on geodesic flows on semidirect products.

3. The 2CH equation as a geodesic equation

Let G be the semidirect product defined in (2.1). We will define a metric $\langle\cdot,\cdot\rangle$ and a compatible covariant derivative ∇ on G and show that a curve $(\varphi(t), f(t))$ in G is a geodesic with respect to ∇ if and only if $(u(t), \rho(t)) \in T_{(\mathrm{id},0)}G$ defined by

$$(u, \rho) = TR_{(\varphi,f)^{-1}}(\varphi_t, f_t) = (\varphi_t \circ \varphi^{-1}, f_t \circ \varphi^{-1}) \tag{3.1}$$

satisfies the 2CH equation.

3.1. The H^s-category

We first consider the H^s-setting and let G be the group H^sG, $s > 5/2$, defined in (2.2). We define a bilinear operator $\Gamma_{(\mathrm{id},0)}$ on $H^s(S^1) \times H^{s-1}(S^1)$ by

$$\Gamma_{(\mathrm{id},0)}((u, \rho), (v, \tau)) = \begin{pmatrix} \Gamma^0_{\mathrm{id}}(u, v) - \dfrac{1}{2}A^{-1}\partial_x(\rho\tau) \\ -\dfrac{1}{2}(u_x\tau + v_x\rho) \end{pmatrix}, \tag{3.2a}$$

where $A = 1 - \partial_x^2$ and

$$\Gamma^0_{\mathrm{id}}(u, v) = -A^{-1}\partial_x\left(uv + \frac{1}{2}u_xv_x\right) \tag{3.2b}$$

is the Christoffel operator associated with the CH equation (cf. [22,23,2]). For vector fields X and Y on H^sG, we define

$$\Gamma_{(\varphi,f)}(X, Y) = \Gamma_{(\mathrm{id},0)}(X(\varphi, f) \circ \varphi^{-1}, Y(\varphi, f) \circ \varphi^{-1}) \circ \varphi. \tag{3.2c}$$

Then Γ is a right-invariant Christoffel map on H^sG, i.e.,

$$TR_{(\psi,g)}[\Gamma_{(\varphi,f)}(X, Y)] = \Gamma_{R_{(\psi,g)}(\varphi,f)}(TR_{(\psi,g)}X(\varphi, f), TR_{(\psi,g)}Y(\varphi, f)),$$

for all $(\varphi, f), (\psi, g) \in H^sG$. The associated covariant derivative ∇ is defined by

$$(\nabla_XY)(\varphi, f) = DY(\varphi, f) \cdot X(\varphi, f) - \Gamma_{(\varphi,f)}(Y(\varphi, f), X(\varphi, f)). \tag{3.3}$$

Observe that the Christoffel map Γ is the infinite-dimensional analog of the Christoffel symbols Γ^i_{jk} familiar from finite-dimensional differential geometry (see [24]; our Γ is denoted by B in [24]). Furthermore, it follows immediately from definition (3.3) that ∇ is a torsionless covariant derivative in the sense that

(i) $\nabla_{fX}Y = f\nabla_XY$,
(ii) $\nabla_XY - \nabla_YX = [X, Y]$,
(iii) $\nabla_X(fY) = (Xf)Y + f\nabla_XY$,

for all vector fields X, Y and functions f on H^sG.

We also define a Riemannian metric $\langle\cdot,\cdot\rangle$ on H^sG by letting

$$\begin{aligned}\langle(u,\rho),(v,\tau)\rangle_{(\mathrm{id},0)} &:= \langle u,v\rangle_{H^1}+\langle\rho,\tau\rangle_{L^2}\\ &= \int_{S^1}(uv+u_xv_x)\mathrm{d}x+\int_{S^1}\rho\tau\,\mathrm{d}x\end{aligned} \tag{3.4a}$$

at $(\mathrm{id},0)$ and extending it to all of H^sG by right-invariance:

$$\langle X,Y\rangle_{(\varphi,f)} := \langle X(\varphi,f)\circ\varphi^{-1}, Y(\varphi,f)\circ\varphi^{-1}\rangle_{(\mathrm{id},0)}, \tag{3.4b}$$

where X,Y are vector fields on H^sG. In the following, we will write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_{(\mathrm{id},0)}$.

It is a well-known fact that any Riemannian metric $\langle\cdot,\cdot\rangle$ on a finite-dimensional manifold M induces a unique compatible torsionless covariant derivative ∇ on M (*the Levi-Civita connection*); ∇_XY is defined by

$$2\langle\nabla_XY,Z\rangle = -\langle[Y,X],Z\rangle-\langle X,[Y,Z]\rangle-\langle Y,[X,Z]\rangle+X\langle Y,Z\rangle+Y\langle Z,X\rangle-Z\langle X,Y\rangle, \tag{3.5}$$

where X,Y,Z are vector fields on M. The bracket $\langle\cdot,\cdot\rangle$ establishes an isomorphism $T_mM\to T_m^*M$ for each $m\in M$, which guarantees the existence of $\nabla_XY(m)$ for all m. However, this approach fails for H^sG since the metric defined by (3.4) is only a *weak* Riemannian metric (i.e. the topology induced by the metric is weaker than the natural topology of any tangent space) and therefore the metric does not establish an isomorphism between the tangent space and its dual [22,19,23].

It is a first aim of this section to establish that ∇ as defined in (3.3) defines a smooth connection (i.e. Γ defines a smooth spray) on H^sG in the sense of Banach manifolds (see [24]) and that $\langle\cdot,\cdot\rangle$ is a compatible Riemannian metric. Note that this connection is unique; this can be deduced immediately from formula (3.5), as in the finite dimensional case.

In general, the Christoffel map for a Banach manifold is only defined locally. Henceforth, we will implicitly use the natural smooth identification

$$TH^sG\simeq H^sG\times(H^s(S^1)\times H^{s-1}(S^1)) \tag{3.6}$$

and view Γ as a map from H^sG to the space of bilinear symmetric maps from $H^s(S^1)\times H^{s-1}(S^1)$ to itself. Similarly, a vector field X on H^sG is viewed as a map $H^sG\to H^s(S^1)\times H^{s-1}(S^1)$. The identification (3.6) is given explicitly as follows. The map $\varphi\mapsto(\varphi(0),\varphi(x)-x-\varphi(0))$ is a diffeomorphism $\mathrm{Diff}(S^1)\to S^1\times U^s$, where

$$U^s := \{f\in H^s(S^1)|f(0)=0, f_x>-1\}.$$

Since U^s is an open subset of the closed linear subspace $E^s := \{f\in H^s(S^1)|f(0)=0\}\subset H^s(S^1)$, this map provides a local chart on $\mathrm{Diff}(S^1)$ with values in $I\times U^s\subset\mathbb{R}\times E^s$ for any open subinterval $I\subset S^1$. Moreover, using that $TS^1\simeq S^1\times\mathbb{R}$, we find

$$T\mathrm{Diff}(S^1)\simeq T(S^1\times U^s)\simeq S^1\times U^s\times\mathbb{R}\times E^s\simeq\mathrm{Diff}(S^1)\times H^s(S^1).$$

This yields the nontrivial part of (3.6).

For two Banach spaces E,F, we let $\mathcal{L}^2_{\mathrm{sym}}(E;F)$ denote the space of symmetric bilinear maps from E to F. For a manifold M, $\mathcal{L}^2_{\mathrm{sym}}(TM;F)$ denotes the bundle over M with fiber $\mathcal{L}^2_{\mathrm{sym}}(T_xM;F)$ over a point $x\in M$.

Proposition 3.1. *Let $s>5/2$. Let $H^sG := H^s\mathrm{Diff}(S^1)\circledS H^{s-1}(S^1)$ and let Γ be the Christoffel map defined in (3.2). Then Γ defines a smooth spray on H^sG, i.e., the map*

$$(\varphi,f)\mapsto\Gamma_{(\varphi,f)}: H^sG\to\mathcal{L}^2_{sym}(H^s(S^1)\times H^{s-1}(S^1);H^s(S^1)\times H^{s-1}(S^1)) \tag{3.7}$$

is smooth. Moreover, the metric $\langle\cdot,\cdot\rangle$ defined by (3.4) is a smooth (weak) Riemannian metric on H^sG, i.e., the map

$$(\varphi,f)\mapsto\langle\cdot,\cdot\rangle_{(\varphi,f)}: H^sG\to\mathcal{L}^2_{sym}(T_{(\varphi,f)}H^sG;\mathbb{R}) \tag{3.8}$$

is a smooth section of the bundle $\mathcal{L}^2_{\mathrm{sym}}(TH^sG;\mathbb{R})$. Finally, the connection ∇ and the metric $\langle\cdot,\cdot\rangle$ are compatible in the sense that

$$X\langle Y,Z\rangle=\langle\nabla_XY,Z\rangle+\langle Y,\nabla_XZ\rangle \tag{3.9}$$

for all vector fields X,Y,Z on H^sG.

Proof. In order to establish smoothness of (3.7), it is sufficient to show that the following map is smooth:

$$\begin{aligned}&((\varphi,f),w)\mapsto\Gamma_{(\varphi,f)}(w,w),\\ &H^sG\times[H^s(S^1)\times H^{s-1}(S^1)]\to H^s(S^1)\times H^{s-1}(S^1),\end{aligned}$$

where $w=(w_1,w_2)\in T_{(\varphi,f)}H^sG\simeq H^s(S^1)\times H^{s-1}(S^1)$ and

$$\Gamma_{(\varphi,f)}(w,w)=\begin{pmatrix}\Gamma^0_{\mathrm{id}}(w_1\circ\varphi^{-1},w_1\circ\varphi^{-1})-\frac{1}{2}A^{-1}\partial_x(w_2^2\circ\varphi^{-1})\\ -(w_1\circ\varphi^{-1})_xw_2\circ\varphi^{-1}\end{pmatrix}\circ\varphi.$$

We will show that the term $-\frac{1}{2}(A^{-1}\partial_x(w_2^2\circ\varphi^{-1}))\circ\varphi$ makes a smooth contribution to Γ; the other terms can be treated by similar arguments.

Consider the map

$$P : H^s\mathrm{Diff}(S^1)\times H^{s-1}(S^1)\to H^s\mathrm{Diff}(S^1)\times H^s(S^1)$$

defined by

$$P(\varphi, w) = (\varphi, (A^{-1}\partial_x(w^2\circ\varphi^{-1}))\circ\varphi).$$

We write P as the composition $P=\tilde{A}^{-1}\circ P_2\circ P_1$, where the maps

$$\begin{aligned}
&P_1 : H^s\mathrm{Diff}(S^1)\times H^{s-1}(S^1)\to H^s\mathrm{Diff}(S^1)\times H^{s-1}(S^1),\\
&P_2 : H^s\mathrm{Diff}(S^1)\times H^{s-1}(S^1)\to H^s\mathrm{Diff}(S^1)\times H^{s-2}(S^1),\\
&\tilde{A} : H^s\mathrm{Diff}(S^1)\times H^{s}(S^1)\to H^s\mathrm{Diff}(S^1)\times H^{s-2}(S^1)
\end{aligned}$$

are defined by

$$\begin{aligned}
&P_1(\varphi, w) = (\varphi, w^2),\\
&P_2(\varphi, w) = (\varphi, (w\circ\varphi^{-1})_x\circ\varphi) = \left(\varphi, \frac{w_x}{\varphi_x}\right),\\
&\tilde{A}(\varphi, w) = (\varphi, (A(w\circ\varphi^{-1}))\circ\varphi) = \left(\varphi, w-\frac{w_{xx}}{\varphi_x^2}+\frac{w_x\varphi_{xx}}{\varphi_x^3}\right).
\end{aligned}$$

The maps P_1, P_2, and $\tilde{A}$ are smooth since $H^s(S^1)$ is a Banach algebra under pointwise multiplication for $s>1/2$. To show that $\tilde{A}^{-1}$ is smooth, we compute

$$D\tilde{A}(\varphi, w) = \begin{pmatrix}\mathrm{id} & 0\\ * & \mathrm{id}-\frac{1}{\varphi_x^2}\partial_x^2+\frac{\varphi_{xx}}{\varphi_x^3}\partial_x\end{pmatrix}.$$

This is, for each $(\varphi, w)\in H^s\mathrm{Diff}(S^1)\times H^s(S^1)$, a bijective bounded linear map $H^s(S^1)\times H^s(S^1)\to H^s(S^1)\times H^{s-2}(S^1)$. The open mapping theorem implies that its inverse is also bounded. The inverse mapping theorem now implies that $\tilde{A}^{-1}$, and hence also P, is a smooth map.

We next establish the smoothness of (3.8). It is sufficient to show that the map

$$Q : H^sG\times[H^s(S^1)\times H^{s-1}(S^1)]\to\mathbb{R},$$

defined by

$$Q((\varphi,f), w) = \int_{S^1}(w_1\circ\varphi^{-1})A(w_1\circ\varphi^{-1})\mathrm{d}x+\int_{S^1}(w_2\circ\varphi^{-1})^2\mathrm{d}x \tag{3.10}$$

is smooth. The change of variables $y=\varphi^{-1}(x)$ yields

$$Q((\varphi,f), w) = \int_{S^1}\left(w_1^2\varphi_x+\frac{w_{1x}^2}{\varphi_x}+w_2^2\varphi_x\right)\mathrm{d}y,$$

and when Q is written in this form its smoothness is clear.

It remains to verify (3.9). Let X_i, Y_i, Z_i, $i=1,2$, denote the components of three vector fields X, Y, Z on H^sG. For $i=1,2$, let $u_i=X_i(\varphi,f)\circ\varphi^{-1}$, $v_i=Y_i(\varphi,f)\circ\varphi^{-1}$, $w_i=Z_i(\varphi,f)\circ\varphi^{-1}$. Let $\gamma(\epsilon)\in H^sG$ be a curve such that $\gamma(0)=(\varphi,f)$ and $\dot\gamma(0)=X(\varphi,f)$.

On the one hand,

$$\begin{aligned}
(X\langle Y,Z\rangle)(\varphi,f) &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\langle Y(\gamma(\epsilon)), Z(\gamma(\epsilon))\rangle_{\gamma(\epsilon)}\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left\langle Y_1(\gamma(\epsilon))\circ\gamma_1^{-1}, Z_1(\gamma(\epsilon))\circ\gamma_1^{-1}\right\rangle_{H^1}+\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left\langle Y_2(\gamma(\epsilon))\circ\gamma_1^{-1}, Z_2(\gamma(\epsilon))\circ\gamma_1^{-1}\right\rangle_{L_2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle\nabla_XY,Z\rangle_{(\varphi,f)} &= \langle DY_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}-\Gamma_\varphi^0(Y_1,X_1)\circ\varphi^{-1}, w_1\rangle_{H^1}+\frac{1}{2}\langle(v_{2x}u_2+u_{2x}v_2), w_1\rangle_{L_2}\\
&\quad+\left\langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}+\frac{1}{2}(v_{1x}u_2+u_{1x}v_2), w_2\right\rangle_{L_2}
\end{aligned}$$

and

$$\langle Y, \nabla_X Z\rangle_{(\varphi,f)} = \langle DZ_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - \Gamma^0_\varphi(Z_1,X_1)\circ\varphi^{-1}, v_1\rangle_{H^1} + \left\langle \frac{1}{2}(w_{2x}u_2 + u_{2x}w_2)v_1\right\rangle_{L_2}$$
$$+ \left\langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} + \frac{1}{2}(w_{1x}u_2 + u_{1x}w_2), v_2\right\rangle_{L_2}.$$

The calculations in [23] for the CH equation show that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \langle Y_1(\gamma(t))\circ\gamma_1^{-1}, Z_1(\gamma(t))\circ\gamma_1^{-1}\rangle_{H^1} = \langle DY_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - \Gamma^0_\varphi(Y_1,X_1)\circ\varphi^{-1}, w_1\rangle_{H^1}$$
$$+ \langle DZ_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - \Gamma^0_\varphi(Z_1,X_1)\circ\varphi^{-1}, v_1\rangle_{H^1},$$

so that it remains to check that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \langle Y_2(\gamma(t))\circ\gamma_1^{-1}, Z_2(\gamma(t))\circ\gamma_1^{-1}\rangle_{L_2} = \left\langle \frac{1}{2}(v_{2x}u_2 + u_{2x}v_2), w_1\right\rangle_{L_2} + \left\langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}\right.$$
$$\left. + \frac{1}{2}(v_{1x}u_2 + u_{1x}v_2), w_2\right\rangle_{L_2} + \left\langle \frac{1}{2}(w_{2x}u_2 + u_{2x}w_2), v_1\right\rangle_{L_2}$$
$$+ \left\langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} + \frac{1}{2}(w_{1x}u_2 + u_{1x}w_2), v_2\right\rangle_{L_2}. \tag{3.11}$$

Since

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \langle Y_2(\gamma(t))\circ\gamma_1^{-1}, Z_2(\gamma(t))\circ\gamma_1^{-1}\rangle_{L_2} = \langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - v_{2x}u_1, w_2\rangle_{L_2}$$
$$+ \langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - w_{2x}u_1, v_2\rangle_{L_2},$$

the condition in (3.11) is equivalent to

$$\int_{S^1}\left(u_1v_{2x}w_2 + u_1v_2w_{2x} + \frac{1}{2}u_2v_{2x}w_1 + \frac{1}{2}u_{2x}v_2w_1 + \frac{1}{2}u_2v_1w_{2x}\right.$$
$$\left. + \frac{1}{2}u_{2x}v_1w_2 + \frac{1}{2}u_2v_{1x}w_2 + u_{1x}v_2w_2 + \frac{1}{2}u_2v_2w_{1x}\right)\mathrm{d}x = 0.$$

Since the left-hand side is equal to

$$\int_{S^1}\left(\frac{1}{2}\partial_x(u_2v_1w_2) + \frac{1}{2}\partial_x(u_2v_2w_1) + \partial_x(u_1v_2w_2)\right)\mathrm{d}x = 0,$$

we are done. □

Remark 3.2. The crucial observation in the above proof is that $(P(w\circ\varphi^{-1}))\circ\varphi$ is a rational expression in w, φ, and their derivatives whenever P is a differential operator. This observation was already made on p. 154 of [19].

A geodesic in H^sG with respect to ∇ is a C^2-curve $(\varphi(t), f(t)) \in H^sG$ such that $\nabla_{(\varphi_t,f_t)}(\varphi_t, f_t) = 0$, i.e.

$$(\varphi_{tt}, f_{tt}) = \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t)). \tag{3.12}$$

Since the existence of a smooth connection on a Banach manifold immediately yields the local existence and uniqueness of a geodesic flow (see [24]), Proposition 3.1 implies the following result.

Theorem 3.3. *Let $s > 5/2$. Then there exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in H^s(S^1)\times H^{s-1}(S^1)$ such that for each $(u_0,\rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, H^sG)$ of (3.12) with $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi : J\times U \to H^sG$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

We write the Cauchy problem for 2CH in the form

$$\begin{pmatrix} u_t + uu_x \\ \rho_t + u\rho_x\end{pmatrix} = \begin{pmatrix} -A^{-1}\partial_x\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) \\ -\rho u_x\end{pmatrix}, \tag{3.13}$$
$$(u(0), \rho(0)) = (u_0, \rho_0).$$

This formulation of 2CH is suitable for the formulation of weak solutions. It follows from Theorem 3.3 that the 2CH equation is locally well-posed in $H^s(S^1) \times H^{s-1}(S^1)$ for $s > 5/2$.

Corollary 3.4 (*Local Well-Posedness in the H^s-Category*)**.** *Suppose $s > 5/2$. Then for any $(u_0, \rho_0) \in H^s(S^1) \times H^{s-1}(S^1)$ there exists an open interval J centered at 0 and a unique solution*

$$(u, \rho) \in C(J, H^s(S^1) \times H^{s-1}(S^1)) \cap C^1(J, H^{s-1}(S^1) \times H^{s-2}(S^1)) \tag{3.14}$$

of the Cauchy problem (3.13) which depends continuously on the initial data (u_0, ρ_0).

Proof. Theorem 3.3 yields the existence of a smooth curve $(\varphi(t), f(t)) \in H^sG$ such that $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Define $(u(t), \rho(t))$ by Eq. (3.1). Then, (u, ρ) has the regularity specified in (3.14) and depends continuously on (u_0, ρ_0). By right-invariance of Γ, the geodesic equation (3.12) can be written as

$$\begin{pmatrix} u_t + uu_x \\ \rho_t + u\rho_x \end{pmatrix} = \Gamma_{(\mathrm{id},0)}((u, \rho), (u, \rho)).$$

This is Eq. (3.13). □

Remark 3.5. The well-posedness result of Corollary 3.4 can also be proved using Kato's semigroup approach (see [14] for the case on the line).

3.2. The C^n-category

The results of the previous subsection hold with the obvious changes also in the C^n-category. Assuming $n \geq 2$, the proofs are the same with H^sG replaced with C^nG. In particular, Γ defines a smooth spray on $C^nG = C^n\mathrm{Diff}(S^1) Ⓢ C^{n-1}(S^1)$ compatible with the metric defined in (3.4). For the sake of brevity, we only state the analog of Theorem 3.3.

Theorem 3.6. *Let $n \geq 2$. Then there exists an open interval J centered at 0 and an open neighborhood U of $(0, 0) \in C^n(S^1) \times C^{n-1}(S^1)$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, C^nG)$ of (3.12) with $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi : J \times U \to C^nG$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

3.3. The smooth category

We now want to extend the above results for 2CH to the space $C^\infty G = C^\infty\mathrm{Diff}(S^1) Ⓢ C^\infty(S^1)$. Since $C^\infty G$ is not a Banach manifold, the local existence and uniqueness theorems for differential equations fail. We will therefore take an indirect approach and first consider the local geodesic flows on H^sG, $s > 5/2$. We will first show that the domains of definition of these flows do not shrink to zero as $s \to \infty$. By considering the limit as $s \to \infty$, the existence of a smooth local geodesic flow on $C^\infty G$ will then be established.

We will use the following blow-up result for 2CH.

Proposition 3.7. *Let $s > 5/2$. Let $(u_0, \rho_0) \in H^s(S^1) \times H^{s-1}(S^1)$ and let $T > 0$ be the maximal time of existence of the solution*

$$(u, \rho) \in C([0, T), H^s(S^1) \times H^{s-1}(S^1)) \cap C^1([0, T),\ H^{s-1}(S^1) \times H^{s-2}(S^1))$$

of the Cauchy problem (3.13). Then the solution (u, ρ) blows up in finite time if and only if

$$\lim_{t\to T} \inf_{x\in S^1}\{u_x(t, x)\} = -\infty \quad \text{or} \quad \limsup_{t\to T}\{\|\rho_x(t)\|_{L^\infty}\} = \infty. \tag{3.15}$$

Proof. A proof for the equation obtained from (2CH) by replacing $\rho\rho_x$ with $-\rho\rho_x$ in the case on the line is given in [14]; the same proof applies here. □

Let

$$\Phi_3 : [0, T_3) \times U_3 \to H^3G,$$

where $T_3 > 0$ and $U_3 \subset H^3(S^1) \times H^2(S^1)$, be the local geodesic flow on H^3G whose existence is guaranteed by Theorem 3.3. In the next proposition, we show that the restriction of Φ_3 to $H^s(S^1) \times H^{s-1}(S^1)$, $s \geq 3$, defines a smooth flow on H^sG for $t \in [0, T_3)$. Thus, the flow on H^sG exists for all $t \in [0, T_3)$ for any $s \geq 3$.

Proposition 3.8. *Suppose $s > 3$ and let Φ_s denote the restriction of Φ_3 to $[0, T_3) \times U_s$, where $U_s = U_3 \cap (H^s(S^1) \times H^{s-1}(S^1))$. Then Φ_s is a smooth local flow of the geodesic equation (3.12) on H^sG, that is,*

(a) *Φ_s is a smooth map from $[0, T_3) \times U_s$ to H^sG.*

(b) *For each* $(u_0, \rho_0) \in U_s$, $\Phi_s(\cdot, u_0, \rho_0)$ *is a smooth solution of Eq.* (3.12) *on* $[0, T_3)$ *satisfying* $\Phi_s(0, u_0, \rho_0) = (\mathrm{id}, 0)$ *and* $\partial_t \Phi_s(0, u_0, \rho_0) = (u_0, \rho_0)$.

Proof. Fix $(u_0, \rho_0) \in U_3$ and let $(u(t; u_0, \rho_0), \rho(t; u_0, \rho_0))$ be the corresponding solution in $H^3(S^1) \times H^2(S^1)$ of the Cauchy problem (3.13). This solution is defined at least on $[0, T_3)$. Since the criterion (3.15) is independent of $s \geq 3$, it follows from Proposition 3.7 that if $(u_0, \rho_0) \in U_s$ for some $s \geq 3$, then the curve $t \mapsto (u(t; u_0, \rho_0), \rho(t; u_0, \rho_0))$ belongs to the space

$$C([0, T_3), H^s(S^1) \times H^{s-1}(S^1)) \cap C^1([0, T_3),\ H^{s-1}(S^1) \times H^{s-2}(S^1)).$$

Let (φ, f) be the geodesic flow associated with the solution (u, ρ), defined on $[0, T_3)$.

Let $s > 3$. Suppose $(u_0, \rho_0) \in U_s$ and $\varphi \in C^1([0, T_3), H^r\mathrm{Diff}(S^1))$ for some r with $3 \leq r \leq s - 1$. We show that $\varphi \in C^1([0, T_3), H^{r+1}\mathrm{Diff}(S^1))$. Using

$$\varphi_{tx} = (u_x \circ \varphi)\varphi_x, \qquad \varphi_{txx} = (u_{xx} \circ \varphi)\varphi_x^2 + (u_x \circ \varphi)\varphi_{xx},$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = (u_{xx} \circ \varphi)\varphi_x.$$

Thus,

$$\varphi_{xx}(t) = \varphi_x(t) \int_0^t (u_{xx} \circ \varphi)\varphi_x \mathrm{d}s. \tag{3.16}$$

Since $\varphi_x \in C^1([0, T_3), H^{r-1}(S^1))$ and $u_{xx} \in C([0, T_3), H^{s-2}(S^1))$, Eq. (3.16) implies that

$$\varphi_{xx} \in C^1([0, T_3), H^{r-1}(S^1)). \tag{3.17}$$

This implies that $\varphi \in C^1([0, T_3), H^{r+1}\mathrm{Diff}(S^1))$. Indeed,

$$\left\| \frac{\varphi(t) - \varphi(s)}{t - s} - u \circ \varphi \right\|_{H^{r+1}}^2 = \left\| \frac{\varphi(t) - \varphi(s)}{t - s} - u \circ \varphi \right\|_{H^1}^2 + \left\| \frac{\varphi_{xx}(t) - \varphi_{xx}(s)}{t - s} - (u \circ \varphi)_{xx} \right\|_{H^{r-1}}^2.$$

As $t \to s$, the first term on the right-hand side vanishes because $\varphi \in C^\infty([0, T_3), H^3\mathrm{Diff}(S^1))$ and the second vanishes in view of (3.17). Induction shows that

$$\varphi \in C^1([0, T_3), H^s\mathrm{Diff}(S^1)). \tag{3.18}$$

We now show that in fact $(\varphi, f) \in C^\infty([0, T_3), H^s G)$. A computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\rho \circ \varphi)\varphi_x] = [(\rho_t + u\rho_x) \circ \varphi]\varphi_x + [(\rho u_x) \circ \varphi]\varphi_x = 0. \tag{3.19}$$

Thus, $f_t\varphi_x = (\rho \circ \varphi)\varphi_x = \rho_0$ and we infer that

$$f(t) = \rho_0 \int_0^t \frac{\mathrm{d}s}{\varphi_x(s)}. \tag{3.20}$$

It follows that

$$f \in C^2([0, T_3), H^{s-1}(S^1)). \tag{3.21}$$

Moreover, by Theorem 3.3, (φ, f) is a smooth solution of (3.12) in $H^s\mathrm{Diff}(S^1) \times H^{s-1}(S^1)$ for sufficiently small $t \geq 0$. Standard ODE results show that the only way this solution can cease to exist (Corollary IV.1.8 in [24]) is either that the condition $\varphi_x > 0$ ceases to hold or that one of the norms

$$\|(\varphi_t, f_t)\|_{H^s(S^1)\times H^{s-1}(S^1)}, \qquad \|\Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t))\|_{H^s(S^1)\times H^{s-1}(S^1)} \tag{3.22}$$

blows up. But we know that $\varphi_x > 0$ on $[0, T_3)$ and Eqs. (3.18) and (3.21) together with the smoothness of Γ imply that the norms in (3.22) remain bounded on $[0, T_3)$. This proves (b).

The standard ODE theorems on smooth dependence on initial data (Theorem IV.1.16 in [24]) imply (a). □

The Sobolev spaces $H^s(S^1)$ provide a Banach space approximation of the Fréchet space $C^\infty(S^1)$ in the following sense.

Definition 3.9. A *Banach space approximation* of a Fréchet space X is a sequence of Banach spaces $(X_n, \|\cdot\|_n)_{n\geq 0}$ such that

$$X_0 \supset X_1 \supset X_2 \supset \cdots \supset X \quad \text{and} \quad X = \cap_{n=0}^\infty X_n,$$

where $\{\|\cdot\|\}_{n\geq 0}$ is a sequence of norms inducing the topology on X such that

$$\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \cdots$$

for all $x \in X$.

The property of a Banach space approximation which is relevant for us is stated in the following lemma (a proof is given in [4]).

Lemma 3.10. *Let X and Y be Fréchet spaces with Banach space approximations $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$, respectively. Let $\Phi_0 : U_0 \to V_0$ be a smooth map between two open subsets $U_0 \subset X_0$ and $V_0 \subset Y_0$. Let $U = U_0 \cap X$, $V = V_0 \cap Y$, and, for each $n \geq 0$,*

$$U_n = U_0 \cap X_n, \qquad V_n = V_0 \cap Y_n.$$

Assume that, for each $n \geq 0$, the following properties are satisfied:

(1) *$\Phi_0(U_n) \subset V_n$,*
(2) *the restriction $\Phi_0|_{U_n} : U_n \to V_n$ is a smooth map.*

Then $\Phi_0(U) \subset V$ and the map $\Phi_0|_U : U \to V$ is smooth.

Proposition 3.8 together with Lemma 3.10 implies local well-posedness of the geodesic flow on $C^\infty G$.

Theorem 3.11. *There exists an open interval J centered at 0 and an open neighborhood U of $(0, 0) \in C^\infty(S^1) \times C^\infty(S^1)$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, C^\infty G)$ of (3.12) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi : J \times U \to C^\infty G$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

Since $C^\infty G$ is a Lie group with smooth multiplication and $(u, \rho) = (\varphi_t \circ \varphi^{-1}, f_t \circ \varphi^{-1})$, we have proved the first part of Theorem 1.1.

4. The 2DP equation as a geodesic equation

Most of the results for 2CH presented in the previous section have direct counterparts in the case of 2DP; the main exception being that the geodesic flow associated with 2DP is not induced by any right-invariant metric. (If this was the case, then, choosing the second component to be equal to zero, we would obtain a metric associated with the geodesic flow for DP which is not possible as shown in [25].)

4.1. The H^s-category

We define a bilinear operator $\Gamma_{(\mathrm{id},0)}$ on $H^s(S^1) \times H^{s-1}(S^1)$ by

$$\Gamma_{(\mathrm{id},0)}((u, \rho), (v, \tau)) = \begin{pmatrix} \Gamma^0_{\mathrm{id}}(u, v) - \frac{1}{2}A^{-1}(u_x\tau + v_x\rho) + A^{-1}\partial_x(\rho\tau) \\ -(u_x\tau + v_x\rho) \end{pmatrix}, \tag{4.1a}$$

where $A = 1 - \partial_x^2$ and

$$\Gamma^0_{\mathrm{id}}(u, v) = -\frac{3}{2}A^{-1}\partial_x(uv) \tag{4.1b}$$

is the Christoffel operator associated with the DP equation (cf. [4]). Γ is extended to all of H^sG by right-invariance; see Eq. (3.2c). The corresponding covariant derivative ∇ is defined by (3.3). The proof of the following proposition is similar to that of Proposition 3.1.

Proposition 4.1. *Let $s > 5/2$. Let $H^sG := H^s\mathrm{Diff}(S^1)Ⓢ H^{s-1}(S^1)$ and let Γ be the 2DP Christoffel map defined in (4.1). Then Γ defines a smooth spray on H^sG, i.e., the map*

$$(\varphi, f) \mapsto \Gamma_{(\varphi,f)} : H^sG \to \mathcal{L}^2_{sym}(H^s(S^1) \times H^{s-1}(S^1); H^s(S^1) \times H^{s-1}(S^1))$$

is smooth.

The existence of a smooth spray implies local existence and uniqueness of the geodesic flow.

Theorem 4.2. *Let $s > 5/2$. Let Γ be the 2DP Christoffel map defined in (4.1). Then there exists an open interval J centered at 0 and an open neighborhood U of $(0, 0) \in H^s(S^1) \times H^{s-1}(S^1)$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, H^sG)$ of the geodesic equation (3.12) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi : J \times U \to H^sG$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

We write the Cauchy problem for 2DP in the form

$$\begin{pmatrix} u_t + uu_x \\ \rho_t + u\rho_x \end{pmatrix} = \begin{pmatrix} -A^{-1}\left(\left(\frac{3}{2}u^2 - \rho^2\right)_x + \rho u_x\right) \\ -2\rho u_x \end{pmatrix}, \tag{4.2}$$
$$(u(0), \rho(0)) = (u_0, \rho_0).$$

It follows from Theorem 4.2 that 2DP is locally well-posed in $H^s(S^1) \times H^{s-1}(S^1)$ for $s > 5/2$.

Corollary 4.3 (*Local Well-Posedness in the H^s-Category*). *Suppose $s > 5/2$. Then for any $(u_0, \rho_0) \in H^s(S^1) \times H^{s-1}(S^1)$ there exists an open interval J centered at 0 and a unique solution*

$$(u, \rho) \in C(J, H^s(S^1) \times H^{s-1}(S^1)) \cap C^1(J, H^{s-1}(S^1) \times H^{s-2}(S^1))$$

of the Cauchy problem (4.2) which depends continuously on the initial data (u_0, ρ_0).

Proof. Let $(\varphi(t), f(t)) \in H^sG$ be the smooth curve with $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$ obtained in Theorem 4.2 and define $(u(t), \rho(t)) := (\varphi_t(t), f_t(t)) \circ \varphi^{-1}(t)$. Then, (u, ρ) has the regularity specified in the corollary and depends continuously on (u_0, ρ_0). By right-invariance of the 2DP Christoffel map Γ, the geodesic equation $(\varphi_{tt}, f_{tt}) = \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t))$ can be written as

$$\begin{pmatrix} u_t + uu_x \\ \rho_t + u\rho_x \end{pmatrix} = \Gamma_{(\mathrm{id},0)}((u, \rho), (u, \rho)).$$

This is Eq. (4.2). □

4.2. The C^n-category

The results of the previous subsection hold with the obvious changes also in the C^n-category, $n \geq 2$.

4.3. The smooth category

We have the following blow-up result for 2DP; the proof is similar to that of Proposition 3.7.

Proposition 4.4. *Let $s > 5/2$. Let $(u_0, \rho_0) \in H^s(S^1) \times H^{s-1}(S^1)$ and let $T > 0$ be the maximal time of existence of the solution*

$$(u, \rho) \in C([0, T), H^s(S^1) \times H^{s-1}(S^1)) \cap C^1([0, T),\ H^{s-1}(S^1) \times H^{s-2}(S^1))$$

of the Cauchy problem (4.2). Then the solution (u, ρ) blows up in finite time if and only if

$$\lim_{t\to T} \inf_{x\in S^1} \{u_x(t, x)\} = -\infty \quad \text{or} \quad \limsup_{t\to T} \{\|\rho_x(t)\|_{L^\infty}\} = \infty.$$

Let

$$\Phi_3 : [0, T_3) \times U_3 \to H^3G,$$

where $T_3 > 0$ and $U_3 \subset H^3(S^1) \times H^2(S^1)$, be the local geodesic flow on H^3G whose existence is guaranteed by Theorem 4.2.

Proposition 4.5. *Suppose $s > 3$ and let Φ_s denote the restriction of Φ_3 to $[0, T_3) \times U_s$, where $U_s = U_3 \cap (H^s(S^1) \times H^{s-1}(S^1))$. Let Γ be the 2DP Christoffel map defined in (4.1). Then Φ_s is a smooth local flow of the geodesic equation (3.12) on H^sG, that is,*

(a) *Φ_s is a smooth map from $[0, T_3) \times U_s$ to H^sG.*
(b) *For each $(u_0, \rho_0) \in U_s$, $\Phi_s(\cdot, u_0, \rho_0)$ is a smooth solution of Eq. (3.12) on $[0, T_3)$ satisfying $\Phi_s(0, u_0, \rho_0) = (\mathrm{id}, 0)$ and $\partial_t\Phi_s(0, u_0, \rho_0) = (u_0, \rho_0)$.*

Proof. The proof is identical to that of Proposition 3.8 except that Eq. (3.20) must be replaced with

$$f(t) = \rho_0 \int_0^t \frac{\mathrm{d}s}{\varphi_x^2(s)}. \tag{4.3}$$

Eq. (4.3) is proved by noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\rho \circ \varphi)\varphi_x^2] = [(\rho_t + u\rho_x) \circ \varphi]\varphi_x^2 + 2[(\rho u_x) \circ \varphi]\varphi_x^2 = 0,$$

and so $f_t\varphi_x^2 = (\rho \circ \varphi)\varphi_x^2 = \rho_0$. □

We find the following well-posedness results.

Theorem 4.6. *Let Γ be the 2DP Christoffel map. There exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in C^\infty(S^1) \times C^\infty(S^1)$ such that for each $(u_0,\rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, C^\infty G)$ of the geodesic equation (3.12) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow*

$$\Phi : J \times U \to C^\infty G, \qquad \Phi(t, u_0,\ \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$$

is a smooth map.

By the same arguments as in the previous section, this proves the second part of Theorem 1.1. Hence the proof of Theorem 1.1 is completed.

5. The sectional curvature for the 2CH equation

We have showed that both 2CH and 2DP are geodesic equations on $H^sG = H^s\mathrm{Diff}(S^1) \circledS H^{s-1}(S^1)$ with respect to a smooth affine connection. The existence of a smooth connection ∇ on a Banach manifold immediately implies the existence of a smooth curvature tensor R defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields on H^sG (cf. [24]). In the case of 2CH, since there exists a metric $\langle \cdot, \cdot \rangle$, we can also define an (unnormalized) sectional curvature tensor S by[2]

$$S(X, Y) := \langle R(X, Y)Y, X\rangle.$$

In this section, we will derive a convenient formula for S and use it to determine large subspaces of positive curvature for the 2CH equation.

We will work in the H^s-category; similar results are valid with H^sG replaced with C^nG. In view of the right-invariance of ∇, it is enough to consider the curvature at the identity $(\mathrm{id}, 0)$. We will write Γ for $\Gamma_{(\mathrm{id},0)}$.

Proposition 5.1. *Let $s > 5/2$. Let R be the curvature tensor on H^sG associated with the 2CH equation. Then $S(u, v) := \langle R(u, v)vu\rangle$ is given at the identity by*

$$S(u, v) = \langle \Gamma(u, v)\Gamma(u, v)\rangle - \langle \Gamma(u, u)\Gamma(v, v)\rangle, \quad u, v \in T_{(\mathrm{id},0)}H^sG. \tag{5.1}$$

Proof. Let $U, V, W \in T_pH^sG$ be three tangent vectors at a point $p \in H^sG$. The curvature tensor R is given locally by [24]

$$R_p(U, V)W = D_1\Gamma_p(W, U)V - D_1\Gamma_p(W, V)U + \Gamma_p(\Gamma_p(W, V), U) - \Gamma_p(\Gamma_p(W, U), V)$$

where Γ is the 2CH Christoffel map defined in (3.2) and D_1 denotes differentiation with respect to p:

$$D_1\Gamma_p(W, U)V = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \Gamma_{p+\epsilon V}(W, U).$$

Let $\mathfrak{g}_s := T_{(\mathrm{id},0)}H^sG$. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$, and $w = (w_1, w_2)$ be three vectors in $\mathfrak{g}_s \simeq H^s(S^1) \times H^{s-1}(S^1)$. Using the identity

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} u_1 \circ (\mathrm{id} + \varepsilon v_1)^{-1} = -u_{1x}v_1.$$

a long but straightforward computation shows that

$$D_1\Gamma(w, u)v = -\Gamma(w_xv_1, u) - \Gamma(u_xv_1, w) + \Gamma(w, u)_xv_1.$$

Thus,

$$\begin{aligned} S(u, v) = {} & \langle \Gamma(\Gamma(v, v), u)u\rangle - \langle \Gamma(\Gamma(v, u), v)u\rangle + \langle \Gamma(v, u)_xv_1, u\rangle - \langle \Gamma(v, v)_xu_1, u\rangle \\ & + \langle -\Gamma(v_xv_1, u) - \Gamma(v, u_xv_1) + 2\Gamma(v_xu_1, v)u\rangle. \end{aligned} \tag{5.2}$$

[2] Recall that the sectional curvature $\mathrm{Sec}(\sigma)$ of a subspace σ spanned by two tangent vectors u and v is defined by

$$\mathrm{Sec}(\sigma) = \frac{\langle R(u, v)v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}.$$

We define a bilinear operator $B = (B_1, B_2) : \mathfrak{g}_s \times \mathfrak{g}_s \to \mathfrak{g}_s$ by

$$\begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix} = \begin{pmatrix} -A^{-1}(2v_{1x}Au_1 + v_1Au_{1x} + u_2v_{2x}) \\ -(u_2v_1)_x \end{pmatrix}.$$

Then B satisfies $\langle B(u,v)w\rangle = \langle u[v,w]\rangle$ where $[v,w] = w_xv - v_xw$ and

$$\Gamma(u,v) = \frac{1}{2}\left[\begin{pmatrix} (u_1v_1)_x \\ u_{2x}v_1 + v_{2x}u_1 \end{pmatrix} + B(u,v) + B(v,u)\right]. \tag{5.3}$$

Let Γ_1 and Γ_2 denote the two components of Γ. With this notation, the first four terms on the right-hand side of (5.2) equal

$$\begin{aligned} &\frac{1}{2}\left\langle \begin{pmatrix} (\Gamma_1(v,v)u_1)_x \\ \Gamma_2(v,v)_xu_1 + u_{2x}\Gamma_1(v,v) \end{pmatrix} + B(\Gamma(v,v),u) + B(u,\Gamma(v,v)), u\right\rangle \\ &\quad -\frac{1}{2}\left\langle \begin{pmatrix} (\Gamma_1(v,u)v_1)_x \\ \Gamma_2(v,u)_xv_1 + v_{2x}\Gamma_1(v,u) \end{pmatrix} + B(\Gamma(v,u),v) + B(v,\Gamma(v,u)), u\right\rangle \\ &\quad + \langle \Gamma(v,u)_xv_1, u\rangle - \langle \Gamma(v,v)_xu_1, u\rangle. \end{aligned}$$

We rewrite this expression as

$$\frac{1}{2}\langle [v,\Gamma(v,u)]u\rangle + \langle u[\Gamma(v,v),u]\rangle - \frac{1}{2}\langle \Gamma(v,u)[v,u]\rangle - \frac{1}{2}\langle v[\Gamma(v,u),u]\rangle,$$

which in turn equals

$$\langle \Gamma(u,v)\Gamma(u,v)\rangle - \langle \Gamma(u,u)\Gamma(v,v)\rangle + \left\langle \begin{pmatrix} u_{1x}u_1 \\ u_{2x}u_1 \end{pmatrix}, \Gamma(v,v)\right\rangle - \left\langle \begin{pmatrix} u_{1x}v_1 \\ u_{2x}v_1 \end{pmatrix}, \Gamma(u,v)\right\rangle.$$

Hence, Eq. (5.2) becomes

$$\begin{aligned} S(u,v) &= \langle \Gamma(u,v)\Gamma(u,v)\rangle - \langle \Gamma(u,u)\Gamma(v,v)\rangle - \left\langle \begin{pmatrix} u_{1x}v_1 \\ u_{2x}v_1 \end{pmatrix}, \Gamma(u,v)\right\rangle + \left\langle \begin{pmatrix} u_{1x}u_1 \\ u_{2x}u_1 \end{pmatrix}, \Gamma(v,v)\right\rangle \\ &\quad + \langle -\Gamma(v_xv_1,u) - \Gamma(v,u_xv_1) + 2\Gamma(v_xu_1,v)u\rangle. \end{aligned} \tag{5.4}$$

We claim that the sum of the last three terms on the right-hand side of (5.4) is zero. Indeed, using the expression

$$\Gamma(u,v) = \begin{pmatrix} \Gamma^0(u_1,v_1) - \dfrac{1}{2}A^{-1}(u_2v_2)_x \\ -\dfrac{1}{2}(u_{1x}v_2 + v_{1x}u_2) \end{pmatrix} \tag{5.5}$$

for Γ, integration by parts shows that the terms in (5.4) involving Γ^0 cancel. A somewhat tedious computation involving further integration by parts shows that the remaining terms also vanish. This proves (5.1). □

A formula analogous to (5.1) for the CH equation was derived in [26]: If $S_{CH}(u_1, v_1)$ denotes the unnormalized sectional curvature on $H^s\mathrm{Diff}(S^1)$ associated with the CH equation, then

$$S_{CH}(u_1,v_1) = \langle \Gamma^0(u_1,v_1)\Gamma^0(u_1,v_1)\rangle - \langle \Gamma^0(u_1,u_1)\Gamma^0(v_1,v_1)\rangle,$$

for all $u_1, v_1 \in T_{\mathrm{id}}H^s\mathrm{Diff}(S^1)$. It was also shown in [26] that

$$S_{CH}(\cos kx, \cos lx) = \frac{1}{8}\left(\frac{\left(1+\frac{1}{2}kl\right)^2}{1+(k-l)^2}(k-l)^2 + \frac{\left(1-\frac{1}{2}kl\right)^2}{1+(k+l)^2}(k+l)^2\right) > 0, \tag{5.6}$$

whenever $k, l \in \{2\pi, 4\pi, \ldots\}$, $k \neq l$, establishing the existence of a large subspace of positive curvature for CH. Since

$$S\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right) = S_{CH}(u_1,v_1), \tag{5.7}$$

we conclude that the same example yields an infinite-dimensional subspace of positive curvature for 2CH. In the next proposition, we investigate the curvature of $\mathrm{Diff}(S^1) \circledS \mathcal{F}(S^1)$ in directions which are nontrivial along the second component.

Proposition 5.2. *Let $s > 5/2$. Let $S(u, v) := \langle R(u, v)vu\rangle$ be the unnormalized sectional curvature on H^sG associated with the 2CH equation. Then*

$$S(u, v) > 0$$

for all vectors $u, v \in T_{(\mathrm{id},0)}H^sG$, $u \neq v$, of the form

$$u = \begin{pmatrix}\cos k_1x\\ \cos k_2x\end{pmatrix}, \qquad v = \begin{pmatrix}\cos l_1x\\ \cos l_2x\end{pmatrix}, \quad k_1, k_2, l_1, l_2 \in \{2\pi, 4\pi, \ldots\}. \tag{5.8}$$

Moreover, the sectional curvature $\mathrm{Sec}(u, v)$ satisfies

$$\mathrm{Sec}(u, v) := \frac{S(u, v)}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2} \geq \frac{1}{8} \tag{5.9}$$

for all vectors $u, v \in T_{(\mathrm{id},0)}H^sG$, $u \neq v$, of the form

$$u = \begin{pmatrix}0\\ \cos k_2x\end{pmatrix}, \qquad v = \begin{pmatrix}0\\ \cos l_2x\end{pmatrix}, \quad k_2, l_2 \in \{2\pi, 4\pi, \ldots\}. \tag{5.10}$$

Proof. In view of (5.1), we have

$$\begin{aligned} S(u, v) = &\int_{S^1} \Gamma_1(u, v)A\Gamma_1(u, v)\mathrm{d}x + \int_{S^1} \Gamma_2(u, v)^2\mathrm{d}x \\ &- \int_{S^1} \Gamma_1(u, u)A\Gamma_1(v, v)\mathrm{d}x - \int_{S^1} \Gamma_2(u, u)\Gamma_2(v, v)\mathrm{d}x. \end{aligned}$$

Using the expression (5.5) for $\Gamma(u, v)$ and integrating by parts, we find

$$S(u, v) = S_{CH}(u_1, v_1) + \sum_{j=1}^{4} I_j, \tag{5.11}$$

where

$$\begin{aligned} I_1 &= \frac{1}{4}\int_{S^1} (u_2v_2)_xA^{-1}(u_2v_2)_x\mathrm{d}x \\ I_2 &= -\frac{1}{4}\int_{S^1} (u_2^2)_xA^{-1}(v_2^2)_x\mathrm{d}x, \\ I_3 &= \frac{1}{2}\int_{S^1} [\Gamma^0(u_1, u_1)(v_2^2)_x + \Gamma^0(v_1, v_1)(u_2^2)_x - 2\Gamma^0(u_1, v_1)(u_2v_2)_x]\mathrm{d}x, \\ I_4 &= \frac{1}{4}\int_{S^1} (u_{1x}^2v_2^2 + v_{1x}^2u_2^2)\mathrm{d}x - \frac{1}{2}\int_{S^1} u_{1x}u_2v_{1x}v_2\mathrm{d}x. \end{aligned}$$

Now suppose u and v have the form specified in (5.8). Then the terms $\{I_j\}_1^4$ can be computed explicitly using the trigonometric identities

$$\begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)), \\ \sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)), \\ \sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta)), \end{aligned}$$

the relations

$$\begin{aligned} &A^{-1}\cos\alpha x = \frac{1}{1+\alpha^2}\cos\alpha x, \quad \alpha \in \mathbb{R}, \\ &\int_0^1 \cos(\alpha x)\cos(\beta x)\mathrm{d}x = \frac{1}{2}(\delta_{\alpha,\beta} + \delta_{\alpha,-\beta}), \quad \alpha, \beta \in 2\pi\mathbb{Z}, \\ &\int_0^1 \sin(\alpha x)\sin(\beta x)\mathrm{d}x = \frac{1}{2}(\delta_{\alpha,\beta} - \delta_{\alpha,-\beta}), \quad \alpha, \beta \in 2\pi\mathbb{Z}, \\ &\int_0^1 \cos(\alpha x)\sin(\beta x)\mathrm{d}x = 0, \quad \alpha, \beta \in 2\pi\mathbb{Z}, \end{aligned}$$

and the identity

$$\Gamma^0(\cos\alpha x, \cos\beta x) = \partial_x\left[-\frac{\frac{1}{2}\left(1-\frac{1}{2}\alpha\beta\right)}{1+(\alpha+\beta)^2}\cos(\alpha+\beta)x - \frac{\frac{1}{2}\left(1+\frac{1}{2}\alpha\beta\right)}{1+(\alpha-\beta)^2}\cos(\alpha-\beta)x\right], \quad \alpha,\ \beta\in 2\pi\mathbb{Z}.$$

We find

$$\begin{aligned}
I_1 &= \frac{1}{32}\left(\frac{(k_2-l_2)^2}{1+(k_2-l_2)^2}+\frac{(k_2+l_2)^2}{1+(k_2+l_2)^2}\right),\\
I_2 &= -\frac{1}{8}\frac{k_2^2}{1+(2k_2)^2}\delta_{k_2,l_2},\\
I_3 &= \frac{1}{8}\frac{\left(1-\frac{1}{2}k_1l_1\right)(k_1+l_1)^2}{1+(k_1+l_1)^2}(\delta_{k_1+l_1,\ k_2-l_2}+\delta_{k_1+l_1,l_2-k_2}+\delta_{k_1+l_1,k_2+l_2})\\
&\quad+\frac{1}{8}\frac{\left(1+\frac{1}{2}k_1l_1\right)(k_1-l_1)^2}{1+(k_1-l_1)^2}(\delta_{k_1-l_1,k_2-l_2}+\delta_{k_1-l_1,l_2-k_2}+\delta_{k_1-l_1,k_2+l_2}+\delta_{l_1-k_1,k_2+l_2})\\
&\quad-\frac{k_1^2}{4}\frac{1-\frac{1}{2}k_1^2}{1+(2k_1)^2}\delta_{k_1,l_2}-\frac{l_1^2}{4}\frac{1-\frac{1}{2}l_1^2}{1+(2l_1)^2}\delta_{k_2,l_1},\\
I_4 &= \frac{1}{16}k_1^2\left(1-\frac{1}{2}\delta_{k_1,l_2}\right)+\frac{1}{16}l_1^2\left(1-\frac{1}{2}\delta_{l_1,k_2}\right)\\
&\quad-\frac{1}{16}k_1l_1(\delta_{k_1-l_1,k_2-l_2}+\delta_{k_1-l_1,l_2-k_2}+\delta_{k_1-l_1,k_2+l_2}+\delta_{l_1-k_1,k_2+l_2}\\
&\quad-\delta_{k_1+l_1,k_2-l_2}-\delta_{k_1+l_1,l_2-k_2}-\delta_{k_1+l_1,k_2+l_2}).
\end{aligned} \tag{5.12}$$

Together with expression (5.6) for $S_{CH}(u_1, v_1)$ this yields an expression for $S(u, v)$ in terms of k_1, k_2, l_1, l_2. The sum of the negative terms in this expression can be estimated as follows:

$$\begin{aligned}
&-\frac{1}{8}\frac{k_2^2}{1+(2k_2)^2}\delta_{k_2,l_2}-\frac{1}{16}k_1l_1\frac{(k_1+l_1)^2}{1+(k_1+l_1)^2}(\delta_{k_1+l_1,k_2-l_2}+\delta_{k_1+l_1,l_2-k_2}+\delta_{k_1+l_1,k_2+l_2})\\
&\quad-\frac{1}{16}k_1l_1(\delta_{k_1-l_1,k_2-l_2}+\delta_{k_1-l_1,l_2-k_2}+\delta_{k_1-l_1,k_2+l_2}+\delta_{l_1-k_1,k_2+l_2})\\
&\geq -\frac{1}{32}-\frac{k_1l_1}{16}-\frac{k_1l_1}{16},
\end{aligned} \tag{5.13}$$

because at most one delta function within each bracket can give a nonzero contribution for a given set of values of $k_1, k_2, l_1, l_2 \in 2\pi\mathbb{N}$.

On the other hand, the term $S_{CH}(u_1, v_1)$ contributes to $S(u, v)$ the positive term

$$\frac{1}{8}\frac{\left(1-\frac{1}{2}k_1l_l\right)^2}{1+(k_1+l_1)^2}(k_1+l_1)^2, \tag{5.14}$$

and the sum of the right-hand side of (5.13) and (5.14) is positive:

$$\begin{aligned}
&\frac{1}{8}\frac{\left(1-\frac{1}{2}k_1l_l\right)^2}{1+(k_1+l_1)^2}(k_1+l_1)^2-\frac{1}{32}-\frac{k_1l_1}{8}\geq\frac{1}{16}\left(1-\frac{1}{2}k_1l_l\right)^2-\frac{1}{32}-\frac{k_1l_1}{8}\\
&=\frac{k_1^2l_1^2}{16}\left[\frac{1}{k_1^2l_1^2}-\frac{1}{k_1l_l}+\frac{1}{4}-\frac{1}{2k_1^2l_1^2}-\frac{2}{k_1l_2}\right]>0,
\end{aligned}$$

where we used that $k_1, l_1 \geq 2\pi$. This shows that $S(u, v) > 0$.

In remains to prove (5.9). Suppose $u_1 = v_1 = 0$ and $u_2 \neq v_2$. It follows from (5.11) and (5.12) that

$$\begin{aligned}
S\left(\begin{pmatrix}0\\u_2\end{pmatrix},\begin{pmatrix}0\\v_2\end{pmatrix}\right) &= I_1+I_2\\
&=\frac{1}{32}\left(\frac{(k_2-l_2)^2}{1+(k_2-l_2)^2}+\frac{(k_2+l_2)^2}{1+(k_2+l_2)^2}\right)-\frac{1}{8}\frac{k_2^2}{1+(2k_2)^2}\delta_{k_2,l_2}\\
&\geq\frac{1}{64}+\frac{1}{64},
\end{aligned} \tag{5.15}$$

where we used that $k_2 \neq l_2$. On the other hand, for this choice of u and v,

$$\langle u, v\rangle = \frac{1}{2}\delta_{k_2,l_2},$$

and hence

$$\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2 = \frac{1}{4}. \tag{5.16}$$

Eqs. (5.15) and (5.16) yield (5.9). □

Remark 5.3. Although Proposition 5.2 establishes the existence of a large subspace of positive curvature, there are also directions for 2CH of strictly negative curvature. Indeed, it is shown in [26] that there exist directions of strictly negative sectional curvature for the CH equation. In view of (5.7), this implies that 2CH also admits directions of negative curvature.

Appendix A. Comparison with the rotating rigid body

In this appendix, the geometric interpretations of 2CH, CH, and the rotating rigid body are compared in an attempt to emphasize some unifying features of the approach pioneered by Arnold [1].

A.1. The rotating rigid body

The configuration space of a rigid body in $\mathbb{R}^3$ rotating around its center of mass is the Lie group $SO(3)$.[3] The corresponding Lie algebra is $\mathfrak{so}(3)$, the space of antisymmetric 3×3-matrices, which can be identified with $\mathbb{R}^3$ via the map

$$\hat{\ } : \mathbb{R}^3 \to \mathfrak{so}(3), \qquad x = (x_1, x_2, x_3) \mapsto \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Let $I : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ be the inertia matrix of the body. A *left*-invariant metric $\langle \cdot, \cdot\rangle$ on $SO(3)$ is defined by setting

$$\langle ab\rangle = a \cdot Ib, \quad a, b \in \mathbb{R}^3 \simeq \mathfrak{so}(3),$$

at the identity, and extending it to all of $SO(3)$ by left-invariance. The basic observation is that $R(t)$ is a geodesic on $(SO(3), \langle \cdot, \cdot\rangle)$ if and only if $\hat{\Omega}(t) := R(t)^{-1}\dot{R}(t)$ solves the classical Euler equation for the motion of a rotating rigid body,

$$I\dot{\Omega} = (I\Omega) \times \Omega.$$

Physically, $\hat{\Omega}(t)$ represents the angular velocity in a frame of reference fixed with respect to the body. The angular velocity in the spatially fixed frame is given by $\dot{R}(t)R(t)^{-1}$. In other words: Applying left and right translations to the material angular velocity $\dot{R}(t)$, one obtains the *body* and the *spatial* angular velocities, which are both elements of the Lie algebra $\mathfrak{so}(3)$. The body and spatial angular momenta, which are elements of the dual $\mathfrak{so}(3)^*$, are given by $\Pi(t) = I\Omega(t)$ and $\pi(t) = R(t)\Pi(t)$, respectively. The body and spatial quantities are related by the adjoint and coadjoint actions

$$\hat{\omega}(t) = \mathrm{Ad}_{R(t)}\hat{\Omega}(t) = R(t)\hat{\Omega}(t)R(t)^{-1}, \qquad \Pi(t) = \mathrm{Ad}^*_{R(t)}\pi(t). \tag{A.1}$$

Conservation of (spatial) angular momentum implies that π is in fact constant in time, i.e.

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} = 0. \tag{A.2}$$

A.2. The CH equation

For the CH equation

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad x \in S^1, \ t \in \mathbb{R}, \tag{A.3}$$

the configuration space is $G = \mathrm{Diff}(S^1)$ with multiplication $(\varphi, \psi) \mapsto \varphi \circ \psi$. Elements of the Lie algebra $\mathfrak{g}$ are identified with functions $S^1 \to \mathbb{R}$. A *right*-invariant metric is defined by setting

$$\langle u, v\rangle_{H^1} = \int_{S^1} uAv\mathrm{d}x = \int_{S^1}(uv + u_xv_x)\mathrm{d}x,$$

[3] See [27] for further details on the material of this subsection.

where $A = 1 - \partial_x^2 : \mathfrak{g} \to \mathfrak{g}^*$ is the inertia operator. The basic observation is that $\varphi(t)$ is a geodesic in $(\mathrm{Diff}(S^1), \langle\cdot,\cdot\rangle_{H^1})$ if and only if $u(t) = TR_{\varphi(t)^{-1}}\varphi_t(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ satisfies (A.3). In other words, the CH equation is the Euler equation on $(\mathrm{Diff}(S^1), \langle\cdot,\cdot\rangle_{H^1})$. Letting $U = TL_{\varphi^{-1}}\varphi_t = (u \circ \varphi)\varphi_x^{-1}$, U and u are the analogs of the body and spatial angular velocities: they are obtained by left and right translation, respectively, of the material velocity φ_t to the Lie algebra. The momentum in the spatial frame is $m = Au$. The analog of Eq. (A.1) is

$$u(t) = \mathrm{Ad}_{\varphi(t)}U(t), \qquad m_0(t) = \mathrm{Ad}^*_{\varphi(t)}m(t),$$

where $m_0 = (m \circ \varphi)\varphi_x^2$ is the momentum in the body frame. Since the metric now is right-invariant instead of left-invariant, the analog of the conservation law (A.2) is that the momentum m_0 in the body frame is conserved,

$$\frac{\mathrm{d}m_0}{\mathrm{d}t} = 0, \quad \text{i.e. } (m \circ \varphi)\varphi_x^2 = m_0.$$

A.3. The 2CH equation

For the 2CH equation (2CH) the configuration space is the semidirect product $G = \mathrm{Diff}(S^1) \circledS \mathcal{F}(S^1)$ introduced in Section 2. The Lie algebra $\mathfrak{g}$ is identified with $\mathcal{F}(S^1) \times \mathcal{F}(S^1)$. The inertia operator is $\mathrm{diag}(A, \mathrm{id})$ and the metric is the right-invariant metric $\langle\cdot,\cdot\rangle$ defined in (3.4). The basic observation is that $(\varphi(t), f(t))$ is a geodesic in $(\mathrm{Diff}(S^1) \circledS \mathcal{F}(S^1), \langle\cdot,\cdot\rangle)$ if and only if

$$(u(t), \rho(t)) = TR_{(\varphi(t),f(t))^{-1}}(\varphi_t(t), f_t(t))$$

satisfies (2CH). The analog of the body angular velocity is $(U_1, U_2) = TL_{(\varphi,f)^{-1}}(\varphi_t, f_t)$. The spatial momentum is $(m, \rho) = (Au, \rho)$. The analog of Eq. (A.1) is

$$(u(t), \rho(t)) = \mathrm{Ad}_{(\varphi(t),f(t))}(U_1(t), U_2(t))$$

and

$$(m_0(t), \rho_0(t)) = \mathrm{Ad}^*_{(\varphi(t),f(t))}(m(t), \rho(t))$$

where (m_0, ρ_0) is the momentum in the body frame. In order to find an explicit expression for (m_0, ρ_0), we need to compute the adjoint and coadjoint actions.

The adjoint action of G on $\mathfrak{g} := T_{(\mathrm{id},0)}G \simeq \mathcal{F}(S^1) \times \mathcal{F}(S^1)$ is defined by

$$\mathrm{Ad}_{(\varphi,f)}(v, \tau) := T_{(\mathrm{id},0)}I_{(\varphi,f)} \cdot (v, \tau), \quad (v, \tau) \in \mathfrak{g},$$

where $I_{(\varphi,f)} : G \to G$ denotes the inner automorphism defined by

$$I_{(\varphi,f)}(\psi, g) = (\varphi, f)(\psi, g)(\varphi, f)^{-1}.$$

A direct computation yields

$$\mathrm{Ad}_{(\varphi,f)}(v, \tau) = (\mathrm{Ad}_\varphi v, (f_x v + \tau) \circ \varphi^{-1}), \quad (v, \tau) \in \mathfrak{g},$$

where $\mathrm{Ad}_\varphi v = (\varphi_x v) \circ \varphi^{-1}$ is the adjoint action with respect to $\mathrm{Diff}(S^1)$. The L^2-pairing is used to identify the (regular part of the) dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathcal{F}(S^1) \times \mathcal{F}(S^1)$. Since

$$\begin{aligned}\langle (m, \rho)\mathrm{Ad}_{(\varphi,f)}(v, \tau)\rangle &= \int_{S^1} m\mathrm{Ad}_\varphi v\mathrm{d}x + \int_{S^1} \rho[(f_x v + \tau) \circ \varphi^{-1}]\mathrm{d}x \\ &= \left\langle \begin{pmatrix} (m \circ \varphi)\varphi_x^2 + (\rho \circ \varphi)f_x\varphi_x \\ (\rho \circ \varphi)\varphi_x \end{pmatrix}, \begin{pmatrix} v \\ \tau \end{pmatrix} \right\rangle,\end{aligned}$$

we find

$$\mathrm{Ad}^*_{(\varphi,f)}(m, \rho) = \begin{pmatrix} (m \circ \varphi)\varphi_x^2 + (\rho \circ \varphi)f_x\varphi_x \\ (\rho \circ \varphi)\varphi_x \end{pmatrix}, \quad (m, \rho) \in \mathfrak{g}^*.$$

The analog of the conservation law (A.2) is that the momentum (m_0, ρ_0) in the body frame is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix} = 0, \quad \text{i.e. } \begin{pmatrix} (m \circ \varphi)\varphi_x^2 + (\rho \circ \varphi)f_x\varphi_x \\ (\rho \circ \varphi)\varphi_x \end{pmatrix} = \begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix}.$$

This explains the origin of the conservation law (3.19) which was used in the proof of Proposition 3.8.

	Rigid body	CH	2CH
Configuration space	$SO(3)$	$\mathrm{Diff}(S^1)$	$\mathrm{Diff}(S^1)ⓈF(S^1)$
Material velocity	$\dot{R}$	φ_t	(φ_t, f_t)
Spatial velocity	$\hat{\omega} = \dot{R}R^{-1}$	$u = \varphi_t \circ \varphi^{-1}$	$(u, \rho) = (\varphi_t \circ \varphi^{-1}, f_t \circ \varphi^{-1})$
Body velocity	$\hat{\Omega} = R^{-1}\dot{R}$	$U = \frac{\varphi_t}{\varphi_x}$	$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\varphi_t}{\varphi_x} \\ f_t - \dfrac{f_x}{\varphi_x}\varphi_t \end{pmatrix}$
Inertia operator	I	$A = 1 - \partial_x^2$	$\begin{pmatrix} A & 0 \\ 0 & \mathrm{id} \end{pmatrix}$
Spatial momentum	$\pi = R\Pi$	$m = Au$	$(m, \rho) = (Au, \rho)$
Body momentum	$\Pi = I\Omega$	$m_0 = (m \circ \varphi)\varphi_x^2$	$\begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix} = \begin{pmatrix} (m \circ \varphi)\varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x \\ (\rho \circ \varphi)\varphi_x \end{pmatrix}$
Spatial velocity (Ad)	$\hat{\omega} = \mathrm{Ad}_R \hat{\Omega}$	$u = \mathrm{Ad}_\varphi U$	$(u, \rho) = \mathrm{Ad}_{(\varphi, f)}(U_1, U_2)$
Body momentum (Ad*)	$\Pi = \mathrm{Ad}_R^* \pi$	$m_0 = \mathrm{Ad}_\varphi^* m$	$(m_0, \rho_0) = \mathrm{Ad}_{(\varphi, f)}^*(m, \rho)$
Momentum conservation	$\pi = \text{const.}$	$m_0 = \text{const.}$	$(m_0, \rho_0) = \text{const.}$

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