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A left-symmetric algebraic approach to left invariant flat (pseudo-)metrics on Lie groups

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ABSTRACT

Left invariant flat metrics on Lie groups are revisited in terms of left-symmetric algebras which correspond to affine structures. There is a left-symmetric algebraic approach with an explicit formula to the classification theorem given by Milnor. When the positive definiteness of the metric is replaced by nondegeneracy, there are many more examples of left invariant flat pseudo-metrics, which play important roles in several fields in geometry and mathematical physics. We give certain explicit constructions of these structures. Their classification in low dimensions and some interesting examples in higher dimensions are also given.

1. Introduction

It is important and natural to study plenty of geometric structures on a Lie group G with a Riemannian metric invariant under left translation. In [1], Milnor studied the curvatures of left invariant metrics on Lie groups which outline "what is the Riemannian geometry of such a Lie group". In particular, as an extreme case, he gave a detailed structure theory on flat metrics, that is, the curvature is zero (Theorem 1.5 in [1]). Basically the study is pure Lie-algebraic and from "metrics" to "connections". Explicitly, in order to give the structure of the Lie algebra $\mathfrak{g}$ consisting of all smooth left invariant vector fields on a Lie group G with a left invariant flat Riemannian metric g, Milnor considered the left invariant Riemannian connection ∇ associated to g, that is, ∇ satisfies the "symmetry" condition (torsion free)

$$\nabla_x y - \nabla_y x = [x, y], \quad \forall\, x, y \in \mathfrak{g}, \tag{1.1}$$

and the compatibility condition (the parallel translation preserves the metric g [2])

$$g(\nabla_x y, z) + g(y, \nabla_x z) = 0, \quad \forall x, y \in \mathfrak{g}, \tag{1.2}$$

and in addition, the flatness of the connection ∇ or the metric g corresponds to

$$R_{xy} z = \nabla_{[x,y]} z - \nabla_x \nabla_y z + \nabla_y \nabla_x z = 0, \quad \forall x, y, z \in \mathfrak{g}, \tag{1.3}$$

that is, the Riemannian curvature tensor R is zero.

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Conversely, one may consider another approach from ''connections'' to ''metrics'' in the above study. That is, one can try to find a left invariant Riemannian metric g associated to a left invariant connection ∇ satisfying Eqs. (1.1) and (1.3) such that the compatibility (1.2) holds. In fact, manifolds (not necessarily Riemannian) or Lie groups with a connection ∇ satisfying Eqs. (1.1) and (1.3) have already been studied independently. Such structures are called affine manifolds or affine structures on Lie groups ([3–6], etc.). They are natural generalizations of Euclidean structures. Like most of geometric structures on Lie groups, the study of left invariant affine structures on a Lie group can be given through corresponding structures on its Lie algebra. Left invariant affine structures on a Lie group G bijectively correspond to left-symmetric algebra structures on the Lie algebra $\mathfrak{g}$ of G and the correspondence is given by

$$\nabla_x y = xy, \quad \forall x, y \in \mathfrak{g}. \tag{1.4}$$

Left-symmetric algebras also appear in many other fields in mathematics and mathematical physics (see [7] and the references therein).

So it is natural to study left invariant Riemannian metrics on a Lie group G by considering left invariant affine structures on G with a compatible Riemannian metric in terms of their corresponding left-symmetric algebras, which is one main motivation of this paper. We would like to point out that such an approach has already been appeared before. For example, there is also a proof of Milnor's classification theorem in terms of left-symmetric algebras in [8]. On the other hand, although the notion of left-symmetric algebra was not mentioned in [1], some essential results in the study of Milnor (see the proof of Theorem 1.5 in [1]) were in fact related to left-symmetric algebras. For example, in [1], Milnor proved that the linear map $x \to \nabla_x$ gives a homomorphism of Lie algebras which is exactly an essential property of the left-symmetric algebra given by Eq. (1.4). Furthermore, comparing with the study in [1,8], our left-symmetric algebraic approach cannot provide more newer results (in fact, the left-symmetric algebras appearing here are quite special since their products can be expressed through their sub-adjacent Lie algebras which explains why only the theory of Lie algebras is enough in [1]), but it can give a more practicable and explicit structure theory to Milnor's classification theorem, which leads to the classification in low dimensions easily.

Moreover, it is not all what we want to do in this paper so far. In fact, it is easy to know from Milnor's classification theorem or the left-symmetric algebraic approach that there are quite few Lie groups with a left invariant flat metric since the constraint condition of the positive definiteness of the metric seems a little strong. So it is natural to generalize the positive definiteness to be nondegenerate, that is, a left invariant flat pseudo-Riemannian metric.

Both left invariant flat metrics and pseudo-metrics on Lie groups are studied extensively due to their close relationships with many structures in geometry and mathematical physics. For example, left invariant flat metrics were applied to study Kähler structures on Lie groups [9–12] and then closely related to a theory of double extensions [13]. Left invariant flat pseudo-metrics are related to a special class of symplectic Lie groups ([14,12,15–17], etc.), namely, parakähler Lie groups [18–22], which are (non-abelian) phase spaces introduced by Kupershmidt in mathematical physics [23,24,20] when the base field is extended to the complex number field. Explicitly, a parakähler structure on a Lie algebra which is a symplectic Lie algebra with a decomposition into a direct sum of underlying vector spaces of two Lagrangian subalgebras, is equivalent to a bialgebra structure, namely, left-symmetric bialgebra. On the other hand, a left invariant flat pseudo-metric on a left-symmetric algebra is an invertible symmetric solution of an algebraic equation (namely, S-equation) in this left-symmetric algebra whose every symmetric solution induces a (coboundary) left-symmetric bialgebra and hence a parakähler structure [21].

Many examples are also given ([25,8], etc.). However, most results and examples are scattered in different presentations (mainly in terms of Lie algebras) under different backgrounds. In this paper, we give a systematic study in terms of left-symmetric algebras. Many examples are also given in a unified presentation (in terms of left-symmetric algebras) which can be a guide for further development, although some of them overlap with certain known results. We would also like to point out that there is a double construction of a Lie group with a left invariant flat pseudo-metric in terms of L-dendriform algebras, whose certain commutators are left-symmetric algebras [26].

The paper is organized as follows. In Section 2, we briefly recall some basic facts on left-symmetric algebras. In Section 3, we give a left-symmetric algebraic approach to Milnor's classification theorem. In Section 4, we give an explicit construction of left invariant flat metrics and their classification in low dimensions based on the study in Section 3. In Section 5, we give some properties of a left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form and a construction from linear functions. In Section 6, the classification of complex left-symmetric algebras with nondegenerate symmetric left invariant bilinear forms in low dimensions and some interesting examples in higher dimensions are given.

Throughout this paper, all algebras are finite-dimensional over the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.

2. Preliminaries on left-symmetric algebras

In order to be self-contained, we recall some basic facts on left-symmetric algebras (cf. [5,6], etc.)

Definition 2.1. Let A be a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \to xy$. A is called a *left-symmetric algebra* if for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \tag{2.1}$$

is symmetric in x, y, that is,

$$(x, y, z) = (y, x, z), \quad \text{or equivalently} \quad (xy)z - x(yz) = (yx)z - y(xz). \tag{2.2}$$

Let A be a left-symmetric algebra. For any $x, y \in A$, let $L(x)$ and $R(x)$ denote left and right multiplication operators respectively, that is, $L(x)(y) = xy$, $R(x)(y) = yx$.

Proposition 2.2. *Let A be a left-symmetric algebra.*

(1) *The commutator*

$$[x, y] = xy - yx, \quad \forall x, y \in A, \tag{2.3}$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called a compatible left-symmetric algebraic structure on the Lie algebra $\mathfrak{g}(A)$.

(2) *Let $L : A \to gl(A)$ be a linear map defined by $x \mapsto L(x)$ (for every $x \in A$). Then L gives a representation of the Lie algebra $\mathfrak{g}(A)$, that is,*

$$[L(x), L(y)] = L([x, y]), \quad \forall x, y \in A. \tag{2.4}$$

Proposition 2.3 ([27]). *If a real or complex Lie algebra $\mathfrak{g}$ has a compatible left-symmetric algebraic structure, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.*

Definition 2.4. Let A be a left-symmetric algebra. A bilinear form $f : A \times A \to \mathbb{F}$ is called *left invariant* if f satisfies

$$f(xy, z) + f(y, xz) = 0, \quad \forall x, y, z \in A. \tag{2.5}$$

A bilinear form $f : A \times A \to \mathbb{F}$ is called a 2-*cocycle* of A if

$$f(xy, z) - f(x, yz) = f(yx, z) - f(y, xz), \quad \forall x, y, z \in A. \tag{2.6}$$

In addition, a real left-symmetric algebra A is called *Hessian* if there exists a positive definite symmetric 2-cocycle f of A.

In geometry, a Hessian manifold M is a flat affine manifold provided with a Hessian metric g, that is, g is a Riemannian metric such that for each point $p \in M$ there exists a C^∞-function φ defined on a neighborhood of p such that $g_{ij} = \frac{\partial^2\varphi}{\partial x^i \partial x^j}$. A Hessian left-symmetric algebra corresponds to an affine Lie group G with a G-invariant Hessian metric [11].

Obviously, any left invariant bilinear form on a left-symmetric algebra is a 2-cocycle and hence a real left-symmetric algebra with a positive definite symmetric left invariant bilinear form is Hessian.

Proposition 2.5. *Let A be a left-symmetric algebra. Then there exists a nondegenerate left invariant bilinear form if and only if the representation L of the sub-adjacent Lie algebra $\mathfrak{g}(A)$ is isomorphic to its dual representation L^*, where $L^* : A \to gl(A^*)$ is given by*

$$\langle L^*(x)a^*, y\rangle = -\langle a^*, L(x)y\rangle = -\langle a^*, xy\rangle, \qquad \forall\, x, y \in A,\ a^* \in A^*. \tag{2.7}$$

Here $\langle , \rangle$ is the ordinary pair between A and A^*.

3. A left-symmetric algebraic approach to left invariant flat metrics on Lie groups

As we have indicated in the Introduction, the study of left invariant flat metrics on Lie groups can be reduced to the study of real left-symmetric algebras with positive definite symmetric left invariant bilinear forms.

In this section, we always let A be a real left-symmetric algebra and f be a positive definitive symmetric left invariant bilinear form on A. For any subspace V in A, set

$$V^\perp = \{x \in A \mid f(x, y) = 0,\ \forall\, y \in V\}. \tag{3.1}$$

Recall that the adjoint F^* of a linear transformation F on A with f is defined by the formula

$$f(F(x), y) = f(x, F^*(y)), \quad \forall\, x, y \in A. \tag{3.2}$$

The transformation F is self-adjoint if $F^* = F$ and skew-adjoint if $F^* = -F$. So Eq. (2.5) is equivalent to the fact that $L(x)$ is skew-adjoint for any $x \in A$. For any Lie algebra $\mathfrak{g}$, $\mathrm{ad}x$ is the linear transformation given by $\mathrm{ad}x(y) = [x, y]$ for any $x, y \in \mathfrak{g}$.

Lemma 3.1. $[A, A]^\perp = \{x \in A \mid R(x) = R(x)^*\}$.

Proof. For any $x, y, z \in A$, $x \in [A, A]^\perp$ if and only if $f(x, [y, z]) = 0$, if and only if $f(x, yz) - f(x, zy) = 0$, if and only if $f(R(x)y, z) = f(R(x)z, y)$, if and only if $R(x) = R(x)^*$. □

Lemma 3.2. $[A, A]^\perp = \{x \in A \mid R(x) = 0\}$.

Proof. For any $x \in [A, A]^{\perp}$ and $y \in A$, by Lemma 3.1, we show that

$$f(xx, y) = f(R(x)x, y) = f(x, R(x)y) = f(x, yx) = 0.$$

Hence $xx = 0$ due to the positive definiteness of f. For any $x \in [A, A]^{\perp}$, $R(x)$ is diagonalizable over the real number field $\mathbb{R}$ since it is self-adjoint. Let $\lambda \in \mathbb{R}$ be an arbitrary eigenvalue of $R(x)$ and $y \in A$ be a non-zero eigenvector associated to λ. Since $(xy)x - x(yx) = (yx)x - y(xx)$, we show that $(xy)x - \lambda xy = \lambda^2 y$. Hence

$$\lambda^2 f(y, y) = f((xy)x - \lambda xy, y) = f((xy)x, y) = f(xy, yx) = \lambda f(xy, y) = 0.$$

Therefore $\lambda = 0$ and then $R(x) = 0$. □

Lemma 3.3. $AA = [A, A]$.

Proof. In fact, $x \in (AA)^{\perp}$ if and only if $f(x, yz) = 0, \ \forall y, z \in A$, if and only if $f(yx, z) = 0, \ \forall y, z \in A$, if and only if $R(x)y = 0, \ \forall y \in A$, if and only if $R(x) = 0$. Then by Lemma 3.2, we show that $AA = [A, A]$. □

For any subalgebra V in A, we let $C_R(V) = \{x \in V \mid R(x)|_V = 0\}$. In particular, $C_R(A) = [A, A]^{\perp}$ due to Lemma 3.2.

Proposition 3.4. *As left-symmetric algebras, $[A, A]$ is a proper ideal of A with zero products.*

Proof. Obviously, AA is an ideal of the left-symmetric algebra A. Then $[A, A]$ is an ideal of A by Lemma 3.3. Moreover, A splits as an orthogonal direct sum

$$A = [A, A]^{\perp} \oplus [A, A] = C_R(A) \oplus [A, A].$$

By Proposition 2.3, we show that $[A, A] \neq A$. Hence $C_R(A) \neq 0$. As a special case in Corollary 5.8 in [22], A is solvable as a Lie algebra. Then there exists a positive integer m such that

$$A \supsetneqq A^{(1)} \supsetneqq \cdots \supsetneqq A^{(m-1)} \supsetneqq A^{(m)} = 0,$$

where $A = A^{(0)}$ and $A^{(i)} = [A^{(i-1)}, A^{(i-1)}]$. Assume that $m \geq 3$. Then we have

$$A^{(m-2)} = C_R(A^{(m-2)}) \oplus A^{(m-1)}. \tag{3.3}$$

Here $A^{(m-2)}$ is nilpotent as a Lie algebra since A is solvable and $m \geq 3$. Let $C(A^{(m-2)})$ be the center of the Lie algebra $A^{(m-2)}$. By a known result (for example, the lemma in page 13 of [28]), $C(A^{(m-2)}) \cap A^{(m-1)} \neq 0$. Let x be a non-zero element in $C(A^{(m-2)}) \cap A^{(m-1)}$. Since $x \in A^{m-1}$, by Lemma 3.3 and Eq. (3.3), $xy = 0$ for any $y \in A^{(m-2)}$. Hence $x \in C_R(A^{(m-2)})$ since $x \in C(A^{(m-2)})$. So $x = 0$, which is a contradiction. Then we have $m \leq 2$. Hence the conclusion holds. □

A key role for the structure theory of left-symmetric algebras is played by the notion of transitive algebra. A left-symmetric algebra A is called *transitive* if $R(x)$ is nilpotent for any $x \in A$. By Proposition 3.4, we have

Corollary 3.5. *Any left-symmetric algebra with a positive definite symmetric left invariant bilinear form is transitive.*

Lemma 3.6. *Let H be a subalgebra of A with zero products and V be a subspace of A such that $L(x)V \subseteq V$ for any $x \in H$. Then $\{L(x)|_V\}_{x \in H}$ is a family of commutative linear transformations on V.*

Proof. For any $x, y \in H, z \in V$,

$$L(x)(L(y)z) = x(yz) = x(yz) - (xy)z = y(xz) - (yx)z = y(xz) = L(y)(L(x)z)$$

since $xy = yx = 0$. Therefore $L(x)|_V L(y)|_V = L(y)|_V L(x)|_V$. □

Proposition 3.7. *The dimension of $[A, A]$ is even. As a consequence, $AA = 0$ if A is a 2-dimensional real left-symmetric algebra with a positive definitive symmetric left invariant bilinear form.*

Proof. By Lemma 3.6, $\{L(x)|_{[A,A]}\}_{x \in C_R(A)}$ is a family of commutative linear transformations on $[A, A]$. Then we get that $\dim\ [A, A]$ is even by the skew-adjoint property of $L(x)|_{[A,A]}$ for any $x \in C_R(A)$. Obviously, $AA = 0$ if and only if $\dim[A, A] = \dim\ AA = 0$. So if products of A are non-zero, then $\dim\ A > \dim[A, A] \geq 2$. □

From the above discussion, we have the following structure theory.

Theorem 3.8. *A positive definite bilinear form is left invariant with respect to a left-symmetric algebra A if and only if A splits as an orthogonal direct sum $A = [A, A] \oplus C_R(A)$, where $C_R(A)$ is a non-zero subalgebra with zero products, $[A, A]$ is an ideal with even dimension and zero products, and where the linear transformation $R(x) = 0$ and $L(x)$ is skew-adjoint for any $x \in C_R(A)$.*

Let $\mathrm{Ann}(A) = \{x \in A | xy = yx = 0, \ \forall y \in A\}$ be the annihilator of A. Obviously, $\mathrm{Ann}(A)$ is an ideal of A and $\mathrm{Ann}(A) \subseteq C_R(A)$. Set $\mathfrak{b} = \{x \in C_R(A) | f(x, y) = 0, \ \forall y \in \mathrm{Ann}(A)\}$. Therefore $C_R(A)$ splits as an orthogonal direct sum $C_R(A) = \mathfrak{b} \oplus \mathrm{Ann}(A)$.

Lemma 3.9. $\dim \mathfrak{b} \le \dim [A,A]/2$.

Proof. By Lemma 3.6 and the definition of $\mathfrak{b}$, $\{L(x)|_{[A,A]}\}_{x\in\mathfrak{b}}$ is a family of non-zero commutative skew-adjoint linear transformations on $[A,A]$. Then $\dim \mathfrak{b}$ is not greater than the rank of the semisimple Lie algebra $so(\dim[A,A])$. So $\dim \mathfrak{b} \le \dim[A,A]/2$. □

Theorem 3.8 can be rewritten as the following conclusion.

Theorem 3.10. *A positive definite bilinear form is left invariant with respect to a left-symmetric algebra A if and only if A splits as an orthogonal direct sum* $\mathfrak{b} \oplus \mathrm{Ann}(A) \oplus [A,A]$, *where* $\mathfrak{b}$ *is a subalgebra with zero products,* $[A,A] = AA$ *is an ideal with even dimension and zero products,* $\dim \mathfrak{b} \le \dim [A,A]/2$, *and where the linear transformation* $R(x) = 0$ *and* $L(x) \ne 0$ *is skew-adjoint for any non-zero* $x \in \mathfrak{b}$.

The following conclusion involving the corresponding structure of the sub-adjacent Lie algebra is obvious.

Theorem 3.11. *A Lie group G with a left invariant metric is flat if and only if the associated Lie algebra* $\mathfrak{g}$ *splits as an orthogonal direct sum* $\mathfrak{b} \oplus C(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$, *where* $\mathfrak{b}$ *is a commutative subalgebra,* $C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x,y] = 0,\ \forall y \in \mathfrak{g}\}$ *is the center of* $\mathfrak{g}$, $[\mathfrak{g},\mathfrak{g}]$ *is a commutative ideal (with even dimension),* $\dim \mathfrak{b} \le \dim [\mathfrak{g},\mathfrak{g}]/2$, *and where the linear transformation* $\mathrm{ad}b$ *is skew-adjoint for any* $b \in \mathfrak{b}$. *Furthermore, the compatible connection is given by*

$$\nabla_u = 0, \qquad \nabla_b = \mathrm{ad}(b) \ne 0, \quad \forall\, u \in C(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}],\ 0 \ne b \in \mathfrak{b}. \tag{3.4}$$

Set $\mathfrak{u} = C(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$. Then the Milnor's classification is obtained immediately.

Theorem 3.12 (*[1, Theorem 1.5]*)**.** *A Lie group G with a left invariant metric is flat if and only if the associated Lie algebra* $\mathfrak{g}$ *splits as an orthogonal direct sum* $\mathfrak{b} \oplus \mathfrak{u}$, *where* $\mathfrak{b}$ *is a commutative subalgebra,* $\mathfrak{u}$ *is a commutative ideal, and where the linear transformation* $\mathrm{ad}b$ *is skew-adjoint for any* $b \in \mathfrak{b}$. *Furthermore, if these conditions are satisfied, then*

$$\nabla_u = 0, \qquad \nabla_b = \mathrm{ad}(b), \quad \forall u \in \mathfrak{u},\ b \in \mathfrak{b}. \tag{3.5}$$

4. An explicit construction of left invariant flat metrics and their classification in low dimensions

From the study in the previous section, it is not difficult to give an explicit construction in any dimension of the Lie algebras in Theorem 3.12 or the left-symmetric algebras in Theorem 3.10 by a more convenient and direct way.

Theorem 4.1. *Let A be a real left-symmetric algebra with a positive definitive symmetric left invariant bilinear form f. Then there exist a basis* $\{e_1,\dots,e_m\}$ *of* $\mathfrak{b}$ *and a basis* $\{x_1,\dots,x_{2n}\}$ *of* $[A,A]$ *and integers* $1 = k_1 < k_2 < \cdots < k_m \le n$ *such that*

$$e_j x_{2i-1} = -\alpha_i(e_j)x_{2i}, \qquad e_j x_{2i} = \alpha_i(e_j)x_{2i-1}, \quad j = 1,\dots,m;\ i = 1,\dots,n,$$

where $\alpha_i(e_j) = 0$ *when* $1 \le i < k_j$, $\alpha_{k_j}(e_j) = 1$ *and* $1 = \alpha_1(e_1) \ge \alpha_2(e_1) \ge \cdots \ge \alpha_n(e_1) > 0$.

Proof. Let $\dim [A,A] = 2n$ and $\dim \mathfrak{b} = m$. By Lemma 3.6, $\{L(x)|_{[A,A]}\}_{x\in C_R(A)}$ is a family of commutative linear transformations on $[A,A]$. By the skew-adjoint property of $L(x)|_{[A,A]}$ for any $x \in C_R(A)$, there exists a basis $\{x_1,\dots,x_{2k}\}$ of $[A,A]$ such that $f(x_i,x_j) = \delta_{ij}$ and for any $y \in \mathfrak{b}$,

$$yx_{2i-1} = -\alpha_i(y)x_{2i}, \qquad yx_{2i} = \alpha_i(y)x_{2i-1}, \quad i = 1,\dots,n,$$

and for any i $(1 \le i \le n)$, there exists an element $y \in \mathfrak{b}$ such that $\alpha_i(y) \ne 0$.

Let $V_i = \{x \in \mathfrak{b} | \alpha_i(x) = 0\}$. Then for any i, V_i is a proper subspace of $\mathfrak{b}$. Hence $\bigcup_{i=1}^n V_i \ne \mathfrak{b}$. So there exists an element y_1 such that $\alpha_i(y_1) \ne 0$ for any i $(1 \le i \le n)$. We can suppose that $\alpha_i(y_1) > 0$ for any i since if there exists j such that $\alpha_j(y_1) < 0$, then after replacing x_{2j-1} by $-x_{2j-1}$, $\alpha_j(e_1)$ is replaced by $-\alpha_j(e_1)$, that is,

$$y_1(-x_{2j-1}) = -(-\alpha_j(y_1))x_{2j}, \qquad y_1x_{2j} = (-\alpha_j(y_1))(-x_{2j-1}).$$

Furthermore, after re-arranging the order of $x_1,\dots,x_{2n}$ which is still denoted by $x_1,\dots,x_n$, we can suppose that

$$\alpha_1(y_1) \ge \alpha_2(y_1) \ge \cdots \ge \alpha_n(y_1) > 0.$$

Let $e_1 = \frac{y_1}{\alpha_1(y_1)}$, then

$$1 = \alpha_1(e_1) \ge \alpha_2(e_1) \ge \cdots \ge \alpha_n(e_1) > 0.$$

Choose $y_2',\dots,y_m'$ such that $\{e_1,y_2',\dots,y_m'\}$ is a basis of $\mathfrak{b}$. Set $y_i = y_i' - \alpha_1(y_i')e_1$ for $i \ge 2$. Then $\{e_1,y_2,\dots,y_m\}$ is still a basis of $\mathfrak{b}$ and $\alpha_1(y_i) = 0$ for any $i \ge 2$. Let k_2 be the minimal l such that $\alpha_l(y_i) \ne 0$ for some i. Without loss of generality, assume that $\alpha_{k_2}(y_2) \ne 0$. Let $e_2 = \frac{y_2}{\alpha_{k_2}(y_2)}$. Then we have

$$e_2x_{2i-1} = e_2x_{2i} = 0, \quad i = 1,\dots,k_2-1; \qquad e_2x_{2k_2-1} = -x_{2k_2}, \qquad e_2x_{2k_2} = x_{2k_2-1}.$$

Therefore, by induction, there exist $e_3, \dots, e_m$ and integers $k_3, \dots, k_m$ such that $\{e_1, \dots, e_m\}$ is a basis of $\mathfrak{b}$ and $k_2 < k_3 < \cdots < k_m \leq n$ satisfying

$$e_j x_{2i-1} = -\alpha_i(e_j)x_{2i}, \qquad e_j x_{2i} = \alpha_i(e_j)x_{2i-1}, \quad j = 3, \dots, m; \ i = 1, \dots, n,$$

where $\alpha_i(e_j) = 0$ when $1 \leq i < k_j$ and $\alpha_{k_j}(e_j) = 1$ for $j \geq 3$. Hence the conclusion follows. $\square$

Remark 4.2. In fact, the converse of the above conclusion is still true. That is, if a left-symmetric algebra A has a basis $\{f_1, \dots, f_t, e_1, \dots, e_m, x_1, \dots, x_{2n}\}$ such that $m \leq n$,

$$L(f_i) = R(f_i) = R(e_j) = 0, \qquad x_k x_l = 0, \quad \forall i = 1, \dots, t; \ j = 1, \dots, m; \ k, l = 1, \dots, 2n,$$

and other products satisfy equations given in Theorem 4.1, then there exists a positive definitive symmetric left invariant bilinear form on A.

Corollary 4.3. *Let A be a real left-symmetric algebra with a positive definitive symmetric left invariant bilinear form. If* $\dim b = 1$ *and* $\mathrm{Ann}(A) = 0$, *then there exists a basis $\{e_1, e_2, \dots, e_{2n+1}\}$ of A satisfying*

$$e_1 e_{2k} = -\lambda_k e_{2k+1}, \qquad e_1 e_{2k+1} = \lambda_k e_{2k}, \quad k = 1, \dots, n, \qquad 1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0,$$

and the others are zero. Moreover, they are not mutually isomorphic for different $\{\lambda_1, \dots, \lambda_n\}$.

Proof. The first half follows immediately from Theorem 4.1. The second half follows from the fact that $\pm\lambda_1, \dots, \pm\lambda_n$ are all the eigenvalues of $L(e_1)|_{[A,A]}$. $\square$

Corollary 4.4. *Let A be a real left-symmetric algebra with a positive definitive symmetric left invariant bilinear form. Let* $\dim\, [A, A] = 2n \leq 6$ *and* $\mathrm{Ann}(A) = 0$. *Then A is isomorphic to one of the following (non-isomorphic mutually) left-symmetric algebras (only non-zero products are given).*

(1) $n = 0 : AA = 0$.
(2) $n = 1 : e_1e_2 = -e_3, \ e_1e_3 = e_2$.
(3) $n = 2$:
   (a) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, 1 \geq \alpha > 0$;
   (b) $e_1e_3 = -e_4, e_1e_4 = e_3, e_2e_5 = -e_6, e_2e_6 = e_5$.
(4) $n = 3$:
   (a) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, e_1e_6 = -\beta e_7, e_1e_7 = \beta e_6, 1 \geq \alpha \geq \beta > 0$;
   (b) $e_1e_3 = -e_4, e_1e_4 = e_3, e_1e_5 = -\alpha e_6, e_1e_6 = \alpha e_5, e_2e_7 = -e_8, e_2e_8 = e_7, 1 \geq \alpha > 0$;
   (c) $e_1e_3 = -e_4, e_1e_4 = e_3, e_1e_5 = -\alpha e_6, e_1e_6 = \alpha e_5, e_2e_5 = -e_6, e_2e_6 = e_5, e_2e_7 = -\beta e_8, e_2e_8 = \beta e_7, \ 1 \geq \alpha > 0, \beta > 0$;
   (d) $e_1e_4 = -e_5, e_1e_5 = e_4, e_2e_6 = -e_7, e_2e_7 = e_6, e_3e_8 = -e_9, e_3e_9 = e_8$.

Proof. Cases (1) ($n = 0$) and (2) ($n = 1$). The results are obvious by Theorem 4.1.

Case (3) ($n = 2$). There are two subcases.

Case (3–1) $\dim\ \mathfrak{b} = 1$. By Corollary 4.3, A is isomorphic to the type (a) of Case (3).

Case (3–2) $\dim\ \mathfrak{b} = 2$. By Theorem 4.1, there exist a basis $\{e_1, e_2\}$ of $\mathfrak{b}$ and a basis $\{e_3, e_4, e_5, e_6\}$ of $[A, A]$ such that

$$e_1e_3 = -e_4, \qquad e_1e_4 = e_3, \qquad e_1e_5 = -\alpha e_6, \qquad e_1e_6 = \alpha e_5, \qquad e_2e_5 = -e_6, \qquad e_2e_6 = e_5, \quad 1 \geq \alpha > 0.$$

Replacing e_1 by $e_1 - \alpha e_2$, we get the type (b) of Case (3).

Case (4) ($n = 3$). There are three subcases.

Case (4–1) $\dim\ \mathfrak{b} = 1$. By Corollary 4.3, A is isomorphic to the type (a) of Case (4).

Case (4–2) $\dim\ \mathfrak{b} = 2$. There are two additional subcases:

Case (4–2-i) $k_2 = 2$. By Theorem 4.1, there exist a basis $\{e_1, e_2\}$ of $\mathfrak{b}$ and a basis $\{e_3, e_4, e_5, e_6, e_7, e_8\}$ of $[A, A]$ such that

$$\begin{aligned} &e_1e_3 = -e_4, \qquad e_1e_4 = e_3, \qquad e_1e_5 = -\alpha e_6, \qquad e_1e_6 = \alpha e_5, \qquad e_1e_7 = -\beta e_8, \qquad e_1e_8 = \beta e_7, \\ &e_2e_5 = -e_6, \qquad e_2e_6 = e_5, \qquad e_2e_7 = -\gamma e_8, \qquad e_2e_8 = \gamma e_7, \quad 1 \geq \alpha \geq \beta > 0. \end{aligned}$$

If $\gamma = 0$ or $\beta = \alpha\gamma$, then we get the type (b) of Case (4) by replacing e_1 by $e_1 - \alpha e_2$. If $\gamma \neq 0$ and $\beta \neq \alpha\gamma$, then we get the type (c) of Case (4) by replacing e_1 by $e_1 - \frac{\beta}{\gamma}e_2$.

Case (4–2-ii) $k_2 = 3$. By Theorem 4.1, there exist a basis $\{e_1, e_2\}$ of $\mathfrak{b}$ and a basis $\{e_3, e_4, e_5, e_6, e_7, e_8\}$ of $[A, A]$ such that

$$\begin{aligned} &e_1e_3 = -e_4, \qquad e_1e_4 = e_3, \qquad e_1e_5 = -\alpha e_6, \qquad e_1e_6 = \alpha e_5, \qquad e_1e_7 = -\beta e_8, \qquad e_1e_8 = \beta e_7, \\ &e_2e_7 = -e_8, \qquad e_2e_8 = e_7, \quad 1 \geq \alpha \geq \beta > 0. \end{aligned}$$

Replacing e_1 by $e_1 - \beta e_2$, we get the type (b) of Case (4).

Case (4–3) $\dim\ \mathfrak{b} = 3$. By Theorem 4.1, there exist a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{b}$ and a basis $\{e_4, e_5, e_6, e_7, e_8, e_9\}$ of $[A, A]$ such that

$$\begin{aligned} &e_1e_4 = -e_5, && e_1e_5 = e_4, && e_1e_6 = -\alpha e_7, && e_1e_7 = \alpha e_6, && e_1e_8 = -\beta e_9, && e_1e_9 = \beta e_8,\\ &e_2e_6 = -e_7, && e_2e_7 = e_6, && e_2e_8 = -\gamma e_9, && e_2e_9 = \gamma e_8,\\ &e_3e_8 = -e_9, && e_3e_9 = e_8, && 1 \geq \alpha \geq \beta > 0. \end{aligned}$$

Replacing e_2 by $e_2 - \gamma e_3$ and e_1 by $e_1 - \alpha e_2 + (\alpha\gamma - \beta)e_3$, we get the type (d) of Case (4).

Furthermore, it is not difficult to show that they are not isomorphic mutually. □

Although it is not difficult at all, it is a little complicated to give the explicit classification for the cases $\dim[A, A] \geq 8$ (or $\dim\ \mathfrak{b} \geq 2$). Nevertheless, from above results, it is enough to give a complete classification of real left-symmetric algebras with positive definitive symmetric left invariant bilinear forms in dimension ≤ 9 as follows.

Corollary 4.5. *Let A be a real left-symmetric algebra in dimension ≤ 9 with a positive definitive symmetric left invariant bilinear form. Then A is isomorphic to one of the following left-symmetric algebras (only non-zero products are given).*

(1) $1 \leq n \leq 9, AA = 0$.
(2) $\dim\ A = 3 : e_1e_2 = -e_3, e_1e_3 = e_2$.
(3) $\dim\ A = 4 : e_1e_2 = -e_3, e_1e_3 = e_2$.
(4) $\dim\ A = 5$:
- (a) $e_1e_2 = -e_3, e_1e_3 = e_2$.
- (b) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, 1 \geq \alpha > 0$.

(5) $\dim\ A = 6$:
- (a) $e_1e_2 = -e_3, e_1e_3 = e_2$.
- (b) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, 1 \geq \alpha > 0$.
- (c) $e_1e_3 = -e_4, e_1e_4 = e_3, e_2e_5 = -e_6, e_2e_6 = e_5$.

(6) $\dim\ A = 7$:
- (a) $e_1e_2 = -e_3, e_1e_3 = e_2$.
- (b) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, 1 \geq \alpha > 0$.
- (c) $e_1e_3 = -e_4, e_1e_4 = e_3, e_2e_5 = -e_6, e_2e_6 = e_5$.
- (d) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, e_1e_6 = -\beta e_7, e_1e_7 = \beta e_6, 1 \geq \alpha \geq \beta > 0$.

(7) $\dim\ A = 8$:
- (a) $e_1e_2 = -e_3, e_1e_3 = e_2$.
- (b) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, 1 \geq \alpha > 0$.
- (c) $e_1e_3 = -e_4, e_1e_4 = e_3, e_2e_5 = -e_6, e_2e_6 = e_5$.
- (d) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, e_1e_6 = -\beta e_7, e_1e_7 = \beta e_6, 1 \geq \alpha \geq \beta > 0$.
- (e) $e_1e_3 = -e_4, e_1e_4 = e_3, e_1e_5 = -\alpha e_6, e_1e_6 = \alpha e_5, e_2e_7 = -e_8, e_2e_8 = e_7, 1 \geq \alpha > 0$.
- (f) $e_1e_3 = -e_4, e_1e_4 = e_3, e_1e_5 = -\alpha e_6, e_1e_6 = \alpha e_5, e_2e_5 = -e_6, e_2e_6 = e_5, e_2e_7 = -\beta e_8, e_2e_8 = \beta e_7, 1 \geq \alpha > 0, \beta > 0$.

(8) $\dim\ A = 9$:
- (a) $e_1e_2 = -e_3, e_1e_3 = e_2$.
- (b) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, 1 \geq \alpha > 0$.
- (c) $e_1e_3 = -e_4, e_1e_4 = e_3, e_2e_5 = -e_6, e_2e_6 = e_5$.
- (d) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, e_1e_6 = -\beta e_7, e_1e_7 = \beta e_6, 1 \geq \alpha \geq \beta > 0$.
- (e) $e_1e_3 = -e_4, e_1e_4 = e_3, e_1e_5 = -\alpha e_6, e_1e_6 = \alpha e_5, e_2e_7 = -e_8, e_2e_8 = e_7, 1 \geq \alpha > 0$.
- (f) $e_1e_3 = -e_4, e_1e_4 = e_3, e_1e_5 = -\alpha e_6, e_1e_6 = \alpha e_5, e_2e_5 = -e_6, e_2e_6 = e_5, e_2e_7 = -\beta e_8, e_2e_8 = \beta e_7, 1 \geq \alpha > 0, \beta > 0$.
- (g) $e_1e_4 = -e_5, e_1e_5 = e_4, e_2e_6 = -e_7, e_2e_7 = e_6, e_3e_8 = -e_9, e_3e_9 = e_8$.
- (h) $e_1e_2 = -e_3, e_1e_3 = e_2, e_1e_4 = -\alpha e_5, e_1e_5 = \alpha e_4, e_1e_6 = -\beta e_7, e_1e_7 = \beta e_6, e_1e_8 = -\gamma e_9, e_1e_9 = \gamma e_8, 1 \geq \alpha \geq \beta \geq \gamma > 0$.

5. Nondegenerate symmetric left invariant bilinear forms on left-symmetric algebras

Let A be a left-symmetric algebra and f be a symmetric left invariant bilinear form. When we extend the positive definiteness of f to be nondegenerate, we find that Lemma 3.1 still holds, but Lemma 3.2 may not be obtained any more. As a consequence, Lemma 3.3 may not hold. In fact, we have found that there exists a left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form such that $AA = A$ (see Corollary 5.5 or more examples in the next section). So it is far from giving a structure theory on such left-symmetric algebras like Theorem 3.10. Nevertheless, from the proof of Lemma 3.3, we still have the following result.

Proposition 5.1. *Let A be a left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form. Then $\dim\ C_R(A) + \dim\ AA = \dim\ A$.*

Proposition 5.2. *Let A be a left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form. If A is commutative, i.e., $xy = yx$ for any $x, y \in A$, then $AA = 0$.*

Proof. For any $x, y, z \in A$,

$$f(xy, z) = -f(y, xz) = f(zy, x) = -f(z, yx) = -f(xy, z).$$

Hence $xy = 0$ by the nondegeneracy of f. □

A very useful criterion is given as follows (also see [29]).

Proposition 5.3. *Let A be a left-symmetric algebra. If there exists an element $x \in A$ such that* $\dim \operatorname{Im}(L(x)) = 1$, *where* $\operatorname{Im}(L(x)) = \{y|L(x)(z) = y,$ *for some* $z \in A\}$, *then any symmetric left invariant bilinear form f is degenerate.*

Proof. If $\dim \operatorname{Im}(L(x)) = 1$, then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of A such that

$$L(x)e_1 = v \neq 0, \qquad L(x)e_i = 0, \quad \forall\, 2 \leq i \leq n.$$

Then by the left-invariance of f, we have:

$$f(v, e_1) = f(xe_1, e_1) = 0, \qquad f(v, e_i) = f(xe_1, e_i) = -f(e_1, xe_i) = 0, \quad \forall\, 2 \leq i \leq n.$$

So f is degenerate. □

Furthermore, in [30], certain left-symmetric algebras were constructed from linear functions. In particular, there is a class of left-symmetric algebras involving left invariant bilinear forms.

Proposition 5.4 ([30]). *Let A be a vector space over a field $\mathbb{F}$ and $c \in A$ be a non-zero element. Let $h : A \times A \to \mathbb{F}$ be a bilinear form on A such that $h(c, c) = 0$. Then the product*

$$x * y = -h(y, c)x + h(x, y)c, \quad \forall x, y \in A, \tag{5.1}$$

defines a left-symmetric algebra structure on A and h is left invariant on $(A, *)$. Moreover, the classification of such left-symmetric algebras with dimension $n \geq 2$ is given by the following matrices ($F = (h(e_i, e_j))$, where $\{e_1 = c, \ldots, e_n\}$ is a basis)

$$F^{(k)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & M^{(k)} \end{pmatrix}, \tag{5.2}$$

where $M^{(k)} = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$ is a $(n-2) \times (n-2)$ diagonal matrix with the first k elements are 1 and the others are zero on the diagonal, $k = 0, 1 \ldots, n-2$. The corresponding left-symmetric algebras are labeled as $A^{(k)}$ ($k = 0, 1 \ldots, n-2$) whose non-zero products are given by

$$e_2e_1 = -e_1, \qquad e_2e_2 = e_2, \qquad e_je_2 = e_j, e_le_l = e_1, \quad 3 \leq j \leq n,\ 3 \leq l \leq k+2. \tag{5.3}$$

Corollary 5.5. *With notations as above. For any $n \geq 2$, $A^{(n-2)}$ is a left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form h given by $F^{(n-2)}$. In this case, $A^{(n-2)}A^{(n-2)} = A^{(n-2)}$.*

6. Classification of complex left-symmetric algebras with nondegenerate symmetric left invariant bilinear forms in low dimensions and some examples in higher dimensions

In this section, we consider the base field to be the complex number field $\mathbb{C}$ since most of classification results on the left-symmetric algebras are over $\mathbb{C}$.

Let A be a left-symmetric algebra with a basis $\{e_1, e_2, \ldots, e_n\}$ over $\mathbb{C}$. Set $e_ie_j = \sum_{k=1}^n c_{ij}^k e_k$. Then the structure of A is determined by its (form) characteristic matrix

$$\begin{pmatrix} \sum\limits_{k=1}^n c_{11}^k e_k & \cdots & \sum\limits_{k=1}^n c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^n c_{n1}^k e_k & \cdots & \sum\limits_{k=1}^n c_{nn}^k e_k \end{pmatrix}. \tag{6.1}$$

Any bilinear form f on A is completely decided by the matrix $F = (f_{ij})$, where $f_{ij} = f(e_i, e_j)$. f is nondegenerate if and only if $\det F \neq 0$.

Obviously, products of any 1-dimensional left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form are zero since any 1-dimensional left-symmetric algebra is commutative.

Proposition 6.1. *Let A be a 2-dimensional complex left-symmetric algebra with a nondegenerate symmetric left invariant bilinear form. Then $AA = 0$ or A is isomorphic to $A^{(0)}$ given in Corollary 5.5 with $n = 2$.*

Proof. According to the classification of 2-dimensional complex left-symmetric algebras given in [31] or [32], if A is a 2-dimensional non-commutative left-symmetric algebra and there does not exist an element $x \in A$ such that $\dim \operatorname{Im}(L(x)) = 1$, then A is isomorphic to one of the following algebras given by their characteristic matrices

$$\begin{pmatrix} 0 & 0 \\ -e_1 & \lambda e_2 \end{pmatrix}, \quad \lambda \in \mathbb{C}; \qquad \begin{pmatrix} 0 & 0 \\ -e_1 & \lambda e_1 - e_2 \end{pmatrix}.$$

By a direct checking, one can find only the left-symmetric algebra given by $\mathcal{A} = \begin{pmatrix} 0 & 0 \\ -e_1 & e_2 \end{pmatrix}$ has a nondegenerate symmetric left invariant bilinear form, which is decided uniquely up to a scalar multiple by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In this case, it is just $A^{(0)}$ given in Corollary 5.5 with the bilinear form given by $F^{(0)}$ with $n = 2$. $\square$

It is a little more complicated in the 3-dimensional cases. In fact, the complete classification of complex left-symmetric algebras in dimension 3 are given in [33] (there are more than 100 classes). By a similar discussion as in the above proof (using Propositions 5.1–5.3 and then by a direct computation for the rest classes), we give the following classification.

Proposition 6.2. *Let A be a complex left-symmetric algebra in dimension 3 with a nondegenerate symmetric left invariant bilinear form f. Then A must be isomorphic to one of the following left-symmetric algebras given by their characteristic matrices (we also list their corresponding nondegenerate symmetric left invariant bilinear forms).*

(1) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and f is defined by* $\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$ *for* $\det(f_{ij}) \neq 0$.

(2) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ -e_3 & e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and f is defined by* $\begin{pmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & f_{11} \\ 0 & f_{11} & 0 \end{pmatrix}$ *for* $f_{11} \neq 0$.

(3) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & -e_3 \end{pmatrix}$ *and f is defined by* $\begin{pmatrix} f_{11} & 0 & 0 \\ 0 & 0 & f_{11} \\ 0 & f_{11} & 0 \end{pmatrix}$ *for* $f_{11} \neq 0$.

(4) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & -e_2 \\ e_1 & 0 & -e_3 \end{pmatrix}$ *and f is defined by* $\begin{pmatrix} 0 & 0 & f_{22} \\ 0 & f_{22} & 0 \\ f_{22} & 0 & 0 \end{pmatrix}$ *for* $f_{22} \neq 0$.

(5) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & -e_2 & 0 \end{pmatrix}$ *and f is defined by* $\begin{pmatrix} 0 & f_{12} & 0 \\ f_{12} & 0 & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$ *for* $f_{12}f_{33} \neq 0$.

(6) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & -e_1 - e_2 \\ e_1 & 0 & -e_2 - e_3 \end{pmatrix}$ *and f is defined by* $\begin{pmatrix} 0 & 0 & f_{22} \\ 0 & f_{22} & -f_{22} \\ f_{22} & -f_{22} & f_{22} \end{pmatrix}$ *for* $f_{22} \neq 0$.

Remark 6.3. In fact, the above type (4) is isomorphic to $A^{(1)}$ given in Corollary 5.5 with $n = 3$. Moreover, there are exactly two types ((4) and (6)) in Proposition 6.2 satisfying $AA = A$. On the other hand, comparing the above classification results and Corollary 4.5, we show that the left-symmetric algebra in Case (2) in Corollary 4.5 (considered as a complex left-symmetric algebra) is isomorphic to the type (5) in Proposition 6.2 by a linear transformation given by

$$e_1 \to ie_3, \qquad e_2 \to i(e_1 - e_2), \qquad e_3 \to e_1 + e_2.$$

Remark 6.4. There is a classification of the left invariant pseudo-metrics over the real number field $\mathbb{R}$ in dimension 2 or 3 in terms of Lie algebras in [8]. Note that we use a different (left-symmetric algebraic) approach here.

Since there is not a complete classification of complex left-symmetric algebras in dimension ≥ 4, it is far away from giving the classification of left-symmetric algebras with nondegenerate symmetric left invariant bilinear forms in dimension ≥ 4. But we can give some examples in higher dimensions. We would like to point out that for Lorentzian signature, there is a complete structure theory, at least for transitive cases in [25].

Example 6.5. In [6], Kim gave the classification of 4-dimensional transitive left-symmetric algebras on nilpotent Lie algebras over the real number field $\mathbb{R}$. Due to it and with a similar discussion as in the proof of Proposition 6.1, we show that a 4-dimensional real transitive left-symmetric algebra on a nilpotent Lie algebra with a nondegenerate symmetric left invariant bilinear form must be isomorphic to one of the following left-symmetric algebras given by their characteristic matrices (we also list their corresponding nondegenerate symmetric left invariant bilinear forms).

(1) $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and f is defined by $\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{12} & f_{22} & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & f_{34} \\ f_{14} & f_{24} & f_{34} & f_{44} \end{pmatrix}$ for $\det(f_{ij}) \neq 0$.

(2) $\mathcal{A}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&e_1&0&e_2\\0&0&e_1&e_3\end{pmatrix}$ and f is defined by $\begin{pmatrix}0&0&0&f_{14}\\0&-f_{14}&0&0\\0&0&-f_{14}&0\\f_{14}&0&0&f_{44}\end{pmatrix}$ for $f_{14}\neq 0$.

(3) $\mathcal{A}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&e_1&0&e_2\\0&0&-e_1&e_3\end{pmatrix}$ and f is defined by $\begin{pmatrix}0&0&0&f_{14}\\0&f_{14}&0&0\\0&0&f_{14}&0\\f_{14}&0&0&f_{44}\end{pmatrix}$ for $f_{14}\neq 0$.

(4) $\mathcal{A}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&e_1&0&e_3\end{pmatrix}$ and f is defined by $\begin{pmatrix}0&0&0&f_{14}\\0&0&-f_{14}&0\\0&-f_{14}&0&0\\f_{14}&0&0&f_{44}\end{pmatrix}$ for $f_{14}\neq 0$.

(5) $\mathcal{A}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&0&0\end{pmatrix}$ and f is defined by $\begin{pmatrix}0&0&0&f_{14}\\0&0&-f_{14}&0\\0&-f_{14}&f_{33}&f_{34}\\f_{14}&0&f_{34}&f_{44}\end{pmatrix}$ for $f_{14}\neq 0$.

(6) $\mathcal{A}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&e_1&e_3\end{pmatrix}$ and f is defined by $\begin{pmatrix}0&0&0&f_{14}\\0&f_{22}&0&f_{24}\\0&0&-f_{14}&0\\f_{14}&f_{24}&0&f_{44}\end{pmatrix}$ for $f_{14}f_{22}\neq 0$.

Note that over the complex number field $\mathbb{C}$, the type (3) is isomorphic to the type (2) by a linear transformation given by

$$e_1\to ie_1,\qquad e_2\to -ie_2,\qquad e_3\to -e_3,\qquad e_4\to ie_4.$$

Example 6.6. We can extend types (4) and (6) in Proposition 6.2 to be in any dimension $n\geq 4$. For the former, the generalization is given by Corollary 5.5 and for the latter, we give following constructions. Let A be an algebra in dimension n with a basis $\{e_1,e_2,\dots,e_n\}$ whose characteristic matrix is given by

$$\begin{pmatrix}0&0&\cdots&0&0&0\\0&0&\cdots&0&e_1&-e_1-e_2\\0&0&\cdots&e_1&0&-e_3\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&e_1&\cdots&0&0&-e_{n-1}\\e_1&0&\cdots&0&0&-e_{n-1}-e_n\end{pmatrix}.$$

Then A is a left-symmetric algebra satisfying $AA=A$. Moreover, there exists a unique (up to scalar multiple) nondegenerate symmetric left invariant bilinear form on A defined by

$$\begin{pmatrix}0&0&\cdots&0&1\\0&0&\cdots&1&-1\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&1&\cdots&0&0\\1&-1&\cdots&0&0\end{pmatrix}.$$

Example 6.7. Let A be left-symmetric algebra in dimension n with a basis $\{e_1,\dots,e_n\}$ whose products are given by

$$e_1e_1=0,\qquad e_1e_i=\lambda_ie_i,\qquad e_ie_j=0,\quad i=2,\dots,n,j=1,\dots,n$$

where $\lambda_i\neq 0$ and $\lambda_i\neq\lambda_j$ if $i\neq j$. Recall that a left-symmetric algebra A is called a *derivation algebra* if $L(x)$ is a derivation of its sub-adjacent Lie algebra $\mathfrak{g}(A)$ for any $x\in A$. Such structures correspond to flat left invariant connections adapted to the automorphism structure of a Lie group. The above left-symmetric algebra A is the unique compatible derivation algebra on its sub-adjacent Lie algebra [5]. Moreover there is a nondegenerate symmetric left invariant bilinear form on A if and only if for any $i\geq 2$ there is exactly one $j\geq 2$ such that $\lambda_i+\lambda_j=0$. As a consequence, n is odd and $f(e_i,e_j)\neq 0$ if and only if $\lambda_i+\lambda_j=0$ for any $i,j\geq 2$. In fact, assume that there is a nondegenerate symmetric left invariant bilinear form f on A. Then

$$\lambda_if(e_1,e_i)=f(e_1,e_1e_i)=-f(e_1e_1,e_i)=0,\quad i=2,\dots,n.$$

Thus $f(e_1,e_i)=0$ for any $i\geq 2$ and $f(e_1,e_1)\neq 0$ since f is nondegenerate. Moreover, $f(e_i,e_i)=0$ for any $i\geq 2$ since

$$\lambda_if(e_i,e_i)=f(e_1e_i,e_i)=-f(e_i,e_1e_i),\quad i=2,\dots,n.$$

Furthermore, since for any $i,j\geq 2$,

$$\lambda_if(e_i,e_j)=f(e_1e_i,e_j)=-f(e_i,e_1e_j)=-\lambda_jf(e_i,e_j),$$

we show that for any $i\geq 2$, there exists exactly one $j\geq 2$ such that $f(e_i,e_j)\neq 0$. Therefore $f(e_i,e_j)\neq 0$ if and only if $\lambda_i+\lambda_j=0$ and n is odd. Note that the above left-symmetric algebra is a generalization of the type (5) in Proposition 6.2 (also see Corollary 4.3).

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